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ON THE LACK OF COMPACTNESS ON STRATIFIED LIE GROUPS

CHIEH-LEI WONG

ABSTRACT. In \mathbb{R}^d , the characterization of the lack of compactness of the continuous Sobolev injection $\dot{H}^s \hookrightarrow L^p$, with $\frac{s}{d} + \frac{1}{p} = \frac{1}{2}$ and $0 < s < \frac{d}{2}$, can be rephrased as : a bounded nonzero sequence $f_n \in \dot{H}^s$ admits a decomposition as a sum of pairwise orthogonal h -oscillatory components - known as profiles - and a remainder term which is going strongly to 0 in L^p . The aim of this paper is to generalize this description due to Patrick Gérard [21] to stratified Lie groups. We shall obtain the collection of profiles from a wavelet decomposition by embracing the similar conceptual approach as in Jaffard [24] or Bahouri, Cohen and Koch [2].

1. INTRODUCTION

1.1. Lack of compactness.

Interest in the study of the lack of compactness in the Sobolev embedding $\dot{H}^s \hookrightarrow L^p$ originates from the works of Pierre-Louis Lions [30], [31], Luc Tartar [36] and Patrick Gérard [20]. Earlier decompositions of bounded sequences into a sum of profiles can be found in papers by Haïm Brézis and Jean-Michel Coron [9] and Michael Struwe [34]. In a paper by Patrick Gérard [21], the defect of compactness of the Sobolev embedding $\dot{H}^s \hookrightarrow L^p$ is described in terms of a sum of rescaled and translated orthogonal profiles, up to a small term in L^p . This was generalized to other Sobolev spaces by Stéphane Jaffard [24], to Besov spaces by Gabriel Koch [27], and finally to general critical embeddings $X \hookrightarrow Y$ including a wide range of functional spaces (Lebesgue L^p , Sobolev \dot{H}^s , Besov $\dot{B}_{p,q}^s$, Triebel-Lizorkin $\dot{F}_{p,q}^s$ only to name a few) by Hajer Bahouri, Albert Cohen and Gabriel Koch [2]. The interested reader can also refer to Kyril Tintarev and Karl-Heinz Fieseler [37] for an abstract, functional analytic presentation of the concept in various settings.

These profile decomposition techniques have been successfully used for studying nonlinear PDEs, namely :

- the description of bounded energy sequences of solutions of the defocusing semi-linear quintic wave equation, up to remainder terms negligible in energy norm by Hajer Bahouri and Patrick Gérard [5],
- the characterization of the defect of compactness for Strichartz estimates for the Schrödinger equation by Sahbi Keraani [26],
- the understanding of features of solutions of nonlinear wave equations with exponential growth by Hajer Bahouri, Mohamed Majdoub and Nader Masmoudi [7],
- the sharp estimates of the lifespan of the focusing critical semi-linear wave equation by means of the energy size of the Cauchy data by Carlos E. Kenig and Frank Merle [25],
- the study of bilinear Strichartz estimates for the wave equation by Terence Tao [35].

For more applications, refer for instance to Isabelle Gallagher and Patrick Gérard [18], Isabelle Gallagher [17], Camille Laurent [28], Hajer Bahouri and Isabelle Gallagher [4], Hajer Bahouri, Jean-Yves Chemin and Isabelle Gallagher [1], and multiple references therein.

Apart Jamel Benameur [8] who solved the case of the Heisenberg group (2008), publications were focused on the Euclidean case. This paper provides a positive answer to the natural question of extending the description of the lack of compactness in Sobolev embeddings to stratified Lie groups. Though we will further see a more precise definition, here are some examples of such Lie groups :

- Euclidean cases : Abelian fields such as \mathbb{R}^d or \mathbb{C}^d , (non Abelian) upper triangular groups,
- non-flat cases : Heisenberg groups \mathbb{H}^d , Carnot groups, Lie groups of polynomial growth.

We shall also assume that our stratified Lie groups G have a Hausdorff geometrical realization. Any of the above examples are likewise.

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1.2. Stratified Lie groups.

Definition 1.1 (FS [13]). A Lie group (G, \cdot) is called stratified if it is connected, simply connected and its Lie algebra \mathfrak{g} decomposes as a direct sum $\mathfrak{g} = V_1 \oplus \dots \oplus V_m$ with :

$$\begin{cases} [V_1, V_k] = V_{k+1} \text{ if } 1 \leq k < m \\ [V_1, V_m] = \{0\} \end{cases} .$$

Thus \mathfrak{g} is a m -step, nilpotent and finitely generated, as a Lie algebra, by the vector subspace V_1 . So as a manifold, G possesses a sub-Riemannian structure. The exponential map is a diffeomorphism from $\mathfrak{g} \rightarrow G$. When G is identified with \mathfrak{g} via \exp , the group law on G is a polynomial map provided by the Campbell-Baker-Hausdorff formula. As a result, the Lie correspondence endows G with a richer manifold structure, and subsequently all the classic notions of differentiable functions, Haar measure, functional spaces, and so on. The left-invariant Haar measure μ_G on G is induced by the Lebesgue measure on its Lie algebra \mathfrak{g} , and we then define the Lebesgue spaces on G as :

$$L^p(G) = \left\{ \text{Borel functions } f \mid \left(\int_G |f|^p d\mu_G \right)^{\frac{1}{p}} < +\infty \right\} ,$$

with the standard modification when $p = +\infty$. In particular, if left translations are defined by $\tau_{x'}(x) = x' \cdot x$, the property :

$$\forall f \in L^1(G), \forall y \in G, \int_G f(y \cdot x) d\mu_G(x) = \int_G f(x) dx \quad (1.1)$$

results from the left-invariance of μ_G . As a homogeneous group, there is a natural action of dilations $(\delta_\alpha)_{\alpha \in \mathbb{R}_+^*}$ on elements of G , given by :

$$\begin{aligned} \forall \alpha \in \mathbb{R}_+^*, \delta_\alpha(x) &= \alpha \odot x \\ &= \underbrace{(\alpha x_{1,1}, \dots, \alpha x_{1, \dim V_1})}_{\substack{\text{induced by the} \\ \text{canonical action on } V_1}}, \underbrace{(\alpha^2 x_{2,1}, \dots, \alpha^2 x_{2, \dim V_2})}, \dots, \underbrace{(\alpha^m x_{m,1}, \dots, \alpha^m x_{m, \dim V_m})} . \end{aligned}$$

In particular, it is immediate to see that :

$$\begin{aligned} \delta_1(x) &= 1_{\mathbb{R}} \odot x = x , \\ \text{and } (\delta_\alpha(x))^{-1} &= (\alpha \odot x)^{-1} = \alpha \odot x^{-1} . \end{aligned}$$

This family of nonisotropic dilations is a subgroup of $Aut(G)$. Given a dilation δ_α , a linear differential operator X is said homogeneous of degree ℓ if for any function f on G , $X(f \circ \delta_\alpha) = \alpha^\ell (Xf) \circ \delta_\alpha$.

Since we shall be dealing with a homogeneous group G endowed with a natural family of dilations, we rather define a homogeneous norm $|\cdot|_G : G \rightarrow [0, +\infty]$ which is a \mathcal{C}^∞ function on $G \setminus \{0\}$ such that :

$$\begin{cases} \forall x \in G, |x^{-1}|_G = |x|_G \\ \forall \alpha > 0, |\alpha \odot x|_G = \alpha |x|_G \\ |x|_G = 0 \text{ iff } x = 0 \end{cases} .$$

Let us denote by $Q = \sum_{k=1}^m k \dim_{\mathbb{R}} V_k$ the homogeneous dimension of G , and introduce the L^1 -normalized dilation $\underline{\delta}_t^1 f$ of a function f by :

$$\forall f, \forall x \in G, \forall t > 0, \underline{\delta}_t^1 f(x) = t^Q f(t \odot x) . \quad (1.2)$$

Note that, by definition, $\underline{\delta}_t^1$ preserves the norm L^1 , that is $\|\underline{\delta}_t^1 f\|_{L^1(G)} = \|f\|_{L^1(G)}$.

Remark 1.2. Depending on the context i.e. the functional spaces we are working with, we might use different normalizations for a given function. The superscript in $\underline{\delta}_t^\bullet$ is helpful to remind which normalization is chosen.

Elements of \mathfrak{g} can be identified with differential operators of length 1 of G which are invariant under left translations. Note that a vector field $X : G \rightarrow TG$ is said to be left-invariant when the following diagram

commutes for all $h \in G$:

$$\begin{array}{ccc} G & \xrightarrow{\tau_h} & G \\ X \downarrow & & \downarrow X \\ TG & \xrightarrow{d\tau_h} & TG \end{array}$$

where τ_h is the left translation on G defined by $\tau_h(x) = h \cdot x$. Then it follows that for all h in G , $X \circ \tau_h = d\tau_h \circ X$. This infinitesimal characterization is equivalent to say that for any smooth function f , one has $X(f \circ \tau_h) = (Xf) \circ \tau_h$.

Let $n \in \mathbb{N}$. Let I and k be multi-indexes in \mathbb{N}^n . For a differential operator $X^I = X_{k_1}^{I_1} X_{k_2}^{I_2} \dots X_{k_n}^{I_n}$ where the X_{k_i} 's are taken in \mathfrak{g} , we define two distinctive notions :

- its isotropic length (or order) : $|I| = I_1 + I_2 + \dots + I_n$,
- its homogeneous degree : $\deg X^I = \sum_{i=1}^n I_i \deg X_{k_i}$ where $\deg X_{k_i} = j$ if $X_{k_i} \in V_j$.

Under the identification $\mathfrak{g} \simeq G$, polynomials on G are polynomials on \mathfrak{g} . Let us denote by \mathcal{P}_k the vector space of homogeneous polynomials of degree k , and set $\mathcal{P} = \varinjlim \mathcal{P}_k$. One can also define the Schwartz space on G by $\mathcal{S}(G) = \mathcal{S}(\mathfrak{g})$. Let $\mathcal{S}'(G)$ be the space of tempered distributions on G and $\mathcal{S}'(G)/\mathcal{P}$ the space of tempered distributions modulo polynomials on G . Duality between the two spaces is achieved by the sesquilinear product $\langle \cdot, \cdot \rangle : \mathcal{S}'(G) \times \mathcal{S}(G) \rightarrow \mathbb{C}$ defined by $\langle f, g \rangle = \int_G f \bar{g} d\mu_G$.

It is common to use the spectral calculus of a suitable sub-Laplacian :

$$\Delta_G = \sum_{X_j \in V_1} X_j^2, \quad (1.3)$$

induced by the aforementioned sub-Riemannian structure of G in the intent to define a Littlewood-Paley decomposition for functions and tempered distributions on G . Restricted to $\mathcal{C}_c^\infty(G)$, Δ_G is a linear differential operator, homogeneous of degree 2 and formally self-adjoint :

$$\forall f, g \in \mathcal{C}_c^\infty(G), \quad \langle \Delta_G f, g \rangle = \langle f, \Delta_G g \rangle.$$

Its closure has a domain $\mathcal{D} = \left\{ u \in L^2(G) \mid \Delta_G u \in L^2(G) \right\}$ where $\Delta_G u$ is taken in the sense of distributions. It follows that its closure is also self-adjoint and it is actually the unique self-adjoint extension of $\Delta|_{\mathcal{C}_c^\infty(G)}$. We shall continue to denote this extension by Δ_G .

Moreover, the homogeneous Sobolev spaces $\dot{H}^s(G)$ are defined as :

$$\dot{H}^s(G) = \left\{ f \in \mathcal{S}'(G)/\mathcal{P} \mid (-\Delta_G)^{\frac{s}{2}} f \in L^2(G) \right\}. \quad (1.4)$$

1.3. Characterization of the lack of compactness of the Sobolev injection $\dot{H}^s(G) \hookrightarrow L^p(G)$.

Set $\frac{s}{Q} + \frac{1}{p} = \frac{1}{2}$ where $0 < s < \frac{Q}{2}$. In the framework of the Sobolev injection (see e.g. BFG [3], or BGX [6] for the specific case of the Heisenberg group \mathbb{H}^d) :

$$\dot{H}^s(G) \hookrightarrow L^p(G), \quad (1.5)$$

both spaces are homogeneous spaces with the same scaling properties since :

$$\begin{aligned} h^{\frac{Q}{p}} \|f \circ \delta_h\|_{\dot{H}^s(G)} &= \|f\|_{\dot{H}^s(G)}, \\ \text{and } h^{\frac{Q}{p}} \|f \circ \delta_h\|_{L^p(G)} &= \|f\|_{L^p(G)}. \end{aligned}$$

For any function $u \in \dot{H}^s(G)$, if we define the operators :

$$\begin{aligned} \tau_\kappa u &= u \circ \tau_{\kappa^{-1}}, \\ \text{and } \delta_h^p u &= h^{\frac{Q}{p}} u \circ \delta_h, \end{aligned}$$

both L^p and \dot{H}^s norms are preserved under translations $u \mapsto \tau_{\kappa}u$ and dilations $u \mapsto \delta_{\frac{1}{h}}^p u$. If u is a nonzero element of $\dot{H}^s(G)$, for any sequence of points $(\kappa_n)_{n \in \mathbb{N}}$ going to infinity in G (i.e. $|\kappa_n|_G \xrightarrow{n \rightarrow +\infty} +\infty$) and any sequence of positive real numbers $(h_n)_{n \in \mathbb{N}}$ converging to 0 or $+\infty$, the two sequences $\left(\tau_{\kappa_n}u\right)_{n \in \mathbb{N}}$ and $\left(\delta_{\frac{1}{h_n}}^p u\right)_{n \in \mathbb{N}}$ converge weakly to 0 in $\dot{H}^s(G)$, henceforth they are not relatively compact in $L^p(G)$. In this paper, we shall prove that these invariances under τ_{κ_n} and $\delta_{\frac{1}{h_n}}^p$ are single responsible for the lack of compactness of the continuous Sobolev injection $\dot{H}^s(G) \hookrightarrow L^p(G)$.

Before stating the main result, let us introduce the notions of scales and concentration cores.

Definition 1.3. We call a scale any sequence $(h_n)_{n \in \mathbb{N}}$ of positive real numbers, and a concentration core any sequence $(\kappa_n)_{n \in \mathbb{N}}$ of points in G . Pairs (h_ℓ, κ_ℓ) and $(h_{\ell'}, \kappa_{\ell'})$ are said orthogonal if :

$$(h_{\ell,n}) \perp (h_{\ell',n}) \iff \log \left| \frac{h_{\ell,n}}{h_{\ell',n}} \right| \xrightarrow{n \rightarrow +\infty} \pm\infty \quad \text{for the scales,} \quad (1.6)$$

$$\text{or } \left((h_{\ell,n}) = (h_{\ell',n}) \text{ and } \frac{1}{h_{\ell,n}} |\kappa_{\ell',n}^{-1} \cdot \kappa_{\ell,n}|_G \xrightarrow{n \rightarrow +\infty} +\infty \right) \quad \text{for the concentration cores.} \quad (1.7)$$

Theorem 1.4. Consider the continuous embedding $\dot{H}^s(G) \hookrightarrow L^p(G)$ with $\frac{s}{Q} + \frac{1}{p} = \frac{1}{2}$. Let $(u_n)_{n>0}$ be a sequence of bounded functions in $\dot{H}^s(G)$. Then, up to the possible extraction of a subsequence, there exist a family of functions $(\phi^\ell)_{\ell>0}$ - the so-called profiles - as well as sequences of scales (h_n) and concentration cores (κ_n) such that :

$$u_n(x) = \underbrace{\sum_{\ell=1}^L h_{\ell,n}^{s-\frac{Q}{p}} \phi^\ell \left(\frac{1}{h_{\ell,n}} \odot (\kappa_{\ell,n}^{-1} \cdot x) \right)}_{\substack{\text{superposition of} \\ h_{\ell,n}\text{-oscillatory components}}} + r_{n,L}(x) \quad \text{with} \quad \lim_{L \rightarrow +\infty} \overline{\lim}_{n \rightarrow +\infty} \|r_{n,L}\|_{L^p(G)} = 0. \quad (1.8)$$

The profile decomposition (1.8) is asymptotically orthogonal (or almost orthogonal) in the sense that :

$$\|u_n\|_{\dot{H}^s(G)}^2 = \sum_{\ell=1}^L \|\phi^\ell\|_{\dot{H}^s(G)}^2 + \|r_{n,L}\|_{\dot{H}^s(G)}^2 + o(1) \quad \text{as } n \rightarrow +\infty. \quad (1.9)$$

This profile decomposition recovers various versions of the concentration-compactness principle, as the ones in [21] or [30], [31].

Remark 1.5. The profile decomposition (1.8) is stable in the sense that :

$$\left\| \left(\|\phi^\ell\|_{\dot{H}^s(G)} \right)_{\ell>0} \right\|_{\ell^2(\mathbb{Z})} \leq C \sup_{n>0} \|u_n\|_{\dot{H}^s(G)},$$

where C is a constant that depends only on $\dot{H}^s(G)$.

Remark 1.6. For some sequences $(u_n)_{n>0}$, it may happen that the decomposition (1.8) contains only a finite number of profiles. In particular, the sequence $(u_n)_{n>0}$ is compact in $L^p(G)$ iff $\forall \ell \geq 1, \phi^\ell = 0$.

1.4. Layout of this paper.

We shall prove Theorem 1.4 by transposing to stratified Lie groups the method developed in [2] which is mainly based on wavelet decompositions. The authors considered critical Sobolev embeddings $X \hookrightarrow Y$ for generic functional spaces X and Y with same scaling properties and endowed with unconditional wavelet basis. They also emphasized on two key properties : the first one is tied to nonlinear approximation, and the other one is similar to Fatou's lemma.

To achieve our goal, we firstly exhibit a wavelet basis by using the spectral calculus of the sub-Laplacian. The mother wavelet ψ will feature some nice properties (in $\mathcal{S}(G)$, with infinite vanishing moments in the space variable, and compactly supported in its conjugate variable). Translated and dilated copies of ψ provide an unconditional wavelet basis for both homogeneous Besov spaces $\dot{B}_{p,q}^s(G)$ with $1 \leq p, q < +\infty$, and $L^p(G)$ with $1 < p < +\infty$. Consequently, the rest of this paper is divided into three parts :

- Section 2 deals with Littlewood-Paley (abbreviated as LP for short) -admissible functions ψ in order to characterize the homogeneous Besov spaces $\dot{B}_{p,q}^s(G)$,
- in Section 3, we will clarify the unconditional convergence of the wavelet expansion in both $L^p(G)$ and $\dot{B}_{p,q}^s(G)$ via discrete sampling techniques and Banach frames,
- in Section 4, we will perform the algorithm for the profiles' extraction on G , and eventually prove Theorem 1.4.

Note that in Sections 2 and 3, we shall make straightforward use of several results from FS [13] and FM [14]. We intentionally omit some technical proofs which can be easily found in these references.

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2. LP-ADMISSIBLE FUNCTIONS AND BESOV SPACES

The purpose of this section is to define the homogeneous Besov spaces $\dot{B}_{p,q}^s(G)$ via a Littlewood-Paley decomposition. In the Euclidean framework, a LP-admissible function ψ is constructed by defining a dyadic partition of unity in the Fourier side, and then apply inverse Fourier \mathcal{F}^{-1} . The standard method of transposing this construction to stratified Lie groups is to replace the Fourier transform by the spectral decomposition of the sub-Laplacian Δ_G (see further Lemma 2.6), and it is essentially based on convolution. Then this construction proves to be independent of the choice of the basis of the stratum V_1 . The proper framework for such Littlewood-Paley decompositions is actually the previously introduced space of tempered distributions modulo polynomials $\mathcal{S}'(G)/\mathcal{P}$. The convergence of these decompositions is obtained via duality with the subspace $\mathcal{V}(G) \subset \mathcal{S}(G)$ of functions with an infinite number of vanishing moments.

Let us recall that the noncommutative convolution product of two functions f and g on G is defined by :

$$f * g(x) = \int_G f(x \cdot y^{-1})g(y)d\mu_G(y) = \int_G f(y)g(y^{-1} \cdot x)d\mu_G(y) ,$$

and if X is a left-invariant vector field on G , then :

$$X(f * g)(x) = f * Xg(x) = \int_G f(y)(Xg)(y^{-1} \cdot x)d\mu_G(y) .$$

In what follows, for a function f on G , we define $\tilde{f}(x) = f(x^{-1})$ and set :

$$\forall x \in G, f^*(x) = \overline{f(x^{-1})} \iff f^* = \bar{\tilde{f}} . \quad (2.1)$$

Then one can easily verify that for f in $L^2(G) \cap L^1(G)$, the adjoint of the convolution operator $g \mapsto g * f$ is $g \mapsto g * f^*$.

2.1. The subspace $\mathcal{V}(G)$ of functions with an infinite number of vanishing moments.

Definition 2.1. *Let $n \in \mathbb{N}$, a function $f : G \rightarrow \mathbb{C}$ is said of polynomial decay of order n if there exists a constant $c > 0$ such that :*

$$\forall x \in G, |f(x)| \leq \frac{c}{(1 + |x|_G)^n} .$$

Definition 2.2. *Let $r \in \mathbb{N}$, a function f has vanishing moments of order r if :*

$$\forall p \in \mathcal{P}_{r-1}, \int_G f(x)p(x)dx = 0 .$$

Under the identification of G with \mathfrak{g} , the inverse map $x \mapsto x^{-1}$ is identified with the additive inverse map $x \mapsto -x$ on \mathfrak{g} . It follows that $\forall p \in \mathcal{P}_{r-1}$, $\tilde{p} \in \mathcal{P}_{r-1}$, where \tilde{p} is defined as above. If f has vanishing moments of order r then :

$$\forall p \in \mathcal{P}_{r-1}, \int_G \tilde{f}(x)p(x)dx = \int_G f(x)\tilde{p}(x)dx = 0 ,$$

which shows that \tilde{f} has also vanishing moments of order r . This property of having vanishing moments is central in the wavelet analysis due to the following principle : in a convolution product such as $g * \delta_t^1 f$, vanishing moments of one factor coupled with smoothness and regularity of the other factor lead to a rapidly decay at infinity of $g * \delta_t^1 f$. Thereafter, we will apply this property to functions f and g in the Schwartz

class which do not necessarily have an infinite number of vanishing moments. The next lemma formalizes a more general version of this principle.

Lemma 2.3 (FM [14] - Lemma 3.2). *Let $k, n \in \mathbb{N}$.*

- (i) *Let $f \in \mathcal{C}^k(G)$ such that $\forall I, \deg X^I \leq k$, $X^I f$ is of polynomial decay of order n . Let g have vanishing moments of order k and polynomial decay of order $n + k + Q + 1$, where Q is the homogeneous dimension of G . Then there exists a constant c_1 (depending only on the decay of $X^I f$ and g) such that :*

$$\forall p > \frac{Q}{n}, \forall 0 < t < 1, \|g * \underline{\delta}_t^1 f\|_{L^p(G)} \leq c_1 t^{k+Q(1-\frac{1}{p})}.$$

- (ii) *Assume now that $g \in \mathcal{C}^k(G)$ with $\forall I, \deg X^I \leq k$, $X^I \tilde{g}$ is of polynomial decay of order n . Let f have vanishing moments of order k and polynomial decay of order $n + k + Q + 1$. Then there exists a constant c_2 (depending only on the decay of f and $X^I \tilde{g}$) such that :*

$$\forall p > \frac{Q}{n}, \forall 1 < t < +\infty, \|g * \underline{\delta}_t^1 f\|_{L^p(G)} \leq \frac{c_2}{t^k}.$$

From now, we shall denote by $\mathcal{V}(G)$ the subspace in the Schwartz class $\mathcal{S}(G)$ of functions with an infinite number of vanishing moments. We sum up below some useful topological properties of $\mathcal{S}(G)$:

Properties (FS [13] - §1 D. The Schwartz class).

- $\mathcal{S}(G)$ is a Fréchet space whose topology is conveniently defined by any norm of the family of norms :

$$\forall n \in \mathbb{N}, \|f\|_n = \sup_{\substack{x \in G \\ |I| \leq n}} (1 + |x|_G)^{(n+1)(Q+1)} |X^I f(x)|.$$

Convolution is continuous from $\mathcal{S}(G) \times \mathcal{S}(G)$ to $\mathcal{S}(G)$. More precisely, for every $n \in \mathbb{N}$, there exists $c_n > 0$ such that :

$$\forall \varphi, \psi \in \mathcal{S}, \|\varphi * \psi\|_n \leq c_n \|\varphi\|_n \|\psi\|_{n+1}.$$

- $\mathcal{V}(G)$ is a closed subspace (in particular, it is complete) of $\mathcal{S}(G)$, with $\mathcal{S}(G) * \mathcal{V}(G) \subset \mathcal{V}(G)$. If $\mathcal{V}'(G)$ denotes the topological dual of $\mathcal{V}(G)$, then $\mathcal{V}'(G)$ can be canonically identified with $\mathcal{S}'(G)/\mathcal{P}$.
- For all $\psi \in \mathcal{S}(G)$, the map $\mathcal{S}'(G)/\mathcal{P} \rightarrow \mathcal{S}'(G)/\mathcal{P}$ defined by $u \mapsto u * \psi$ is a well-defined operator and it is continuous on $\mathcal{S}'(G)/\mathcal{P}$. If $\psi \in \mathcal{V}(G)$, the associated convolution operator is a well-defined and continuous operator from $\mathcal{S}'(G)/\mathcal{P}$ into $\mathcal{S}'(G)$.

2.2. The Calderón's reproducing formula.

From here, we shall use the classic notation $\hat{f} = \mathcal{F}f$ for the Fourier transform on \mathbb{R} . Suppose that Δ_G has a spectral resolution $\Delta_G = \int_0^{+\infty} \lambda dP_\lambda$, where dP_λ is the spectral projection measure i.e. a measure with values

in the spectrum of Δ_G . For any bounded Borel function \hat{f} on \mathbb{R}_+ , the operator $\hat{f}(\Delta_G) = \int_0^{+\infty} \hat{f}(\lambda) dP_\lambda$ which should be understood as :

$$\forall \phi, \eta \in L^2(G), \langle \hat{f}(\Delta_G) \phi, \eta \rangle = \int_0^{+\infty} \hat{f}(\lambda) dP_\lambda(\phi, \eta)$$

(where $dP_\lambda(\phi, \eta)$ is the unique Borel measure associated to the pair (ϕ, η)) is a bounded integral operator in \mathcal{L}^2 with a convolution distribution-kernel K_f in $L^2(G)$ satisfying $\forall \eta \in \mathcal{S}(G), \hat{f}(\Delta_G)\eta = \eta * K_f$. In the following, we shall denote this kernel K_f in some abusive way by f , that is :

$$\forall \eta \in \mathcal{S}(G), \hat{f}(\Delta_G)\eta = \eta * f. \quad (2.2)$$

An important property due to Andrzej Hulanicki [23] is that, for smooth and rapidly decaying functions $\hat{f} \in \mathcal{S}(\mathbb{R}_+)$, the kernel associated to $\hat{f}(\Delta_G)$ is a function in the Schwartz class, namely $f \in \mathcal{S}(G)$.

At this stage, we have all the ingredients for a Littlewood-Paley type decomposition :

$$f = \sum_{j \in \mathbb{Z}} f * \psi_j^* * \psi_j, \quad (2.3)$$

where $\forall j \in \mathbb{Z}, \psi_j = \underline{\delta}_{2^j}^1 \psi$ is a dilated of $\psi \in \mathcal{S}(G)$ (with the usual convention $\psi_0 = \psi$).

Definition 2.4. A function $\psi \in \mathcal{S}(G)$ is said LP-admissible if :

$$\forall g \in \mathcal{V}(G), g = \lim_{n \rightarrow +\infty} \sum_{|j| \leq n} g * \psi_j^* * \psi_j$$

holds, with convergence in $\mathcal{S}(G)$. Duality induces the convergence of the decomposition on $\mathcal{S}'(G)/\mathcal{P}$:

$$\forall u \in \mathcal{S}'(G)/\mathcal{P}, u = \lim_{n \rightarrow +\infty} \sum_{|j| \leq n} u * \psi_j^* * \psi_j .$$

Before stating Lemma 2.6 which is the cornerstone for the construction of LP-admissible functions, we need the next useful preliminary result.

Lemma 2.5 (FS [13] - Proposition 1.49). Let $\vartheta \in \mathcal{S}(G)$ and define $\vartheta_t = \delta_t^1 \vartheta$, then :

$$\begin{aligned} \forall \psi \in \mathcal{S}(G), \psi * \vartheta_{\frac{1}{t}} &\longrightarrow c_\vartheta \psi \text{ in } \mathcal{S}(G) , \\ \text{and } \forall f \in \mathcal{S}'(G), f * \vartheta_{\frac{1}{t}} &\longrightarrow c_\vartheta f \text{ in } \mathcal{S}'(G) , \end{aligned}$$

as $t \rightarrow 0$, where $c_\vartheta = \int_G \vartheta(x) dx$.

Lemma 2.6. Let $\hat{\phi}$ be a function in $\mathcal{C}^\infty(\mathbb{R})$ with support in $[0, 4]$ such that $\begin{cases} 0 \leq \hat{\phi} \leq 1 \\ \hat{\phi} \equiv 1 \text{ on } \left[0, \frac{1}{4}\right] \end{cases}$.

Set $\hat{\psi}(\xi) = \sqrt{\hat{\phi}(2^{-2}\xi) - \hat{\phi}(\xi)}$. Thus $\hat{\psi} \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ with support in the interval $\left[\frac{1}{4}, 4\right]$ and we get a dyadic partition of unity with $\hat{\psi} : \sum_{j \in \mathbb{Z}} |\hat{\psi}(2^{-2j}\xi)|^2 = 1$ a.e.

Let Δ_G be the sub-Laplacian. Let ψ be the convolution distribution-kernel as defined in (2.2) which is associated to the bounded left-invariant operator $\hat{\psi}(\Delta_G)$, then ψ is LP-admissible and belongs to $\mathcal{V}(G)$.

Proof. The spectral theorem applied to the dyadic partition of unity with $\hat{\psi}$ gives :

$$\sum_{j \in \mathbb{Z}} \left[\hat{\psi}(2^{-2j} \Delta_G) \right]^* \circ \left[\hat{\psi}(2^{-2j} \Delta_G) \right] = \mathbb{1} .$$

Let $g \in \mathcal{V}(G)$. First observe that due to the quadratic homogeneity of Δ_G , the convolution kernel associated to $\hat{\psi}(2^{-2j} \Delta_G)$ coincides with ψ_j . The decomposition :

$$g = \sum_{j \in \mathbb{Z}} \left[\hat{\psi}(2^{-2j} \Delta_G) \right]^* \circ \left[\hat{\psi}(2^{-2j} \Delta_G) \right] g = \sum_{j \in \mathbb{Z}} g * \psi_j^* * \psi_j \quad (2.4)$$

holds in norm L^2 . Note that, since $\hat{\psi}$ is a real-valued function, one has actually $\psi^* = \psi$.

For any integer $m \in \mathbb{N}$, one has :

$$\sum_{|j| \leq m} g * \psi_j^* * \psi_j = g * \underline{\delta_{2^{m+1}}^1} \phi - g * \underline{\delta_{2^{-m}}^1} \phi ,$$

where $\phi \in \mathcal{S}(G)$ is the convolution kernel of $\hat{\phi}(\Delta_G)$. Since ϕ is in the Schwartz class, it follows by Lemma 2.5 that $g * \underline{\delta_{2^{m+1}}^1} \phi \xrightarrow{m \rightarrow +\infty} c_\phi g$ in $\mathcal{S}(G)$ for some constant c_ϕ . Hence $\sum_{|j| \leq m} g * \psi_j^* * \psi_j \xrightarrow{m \rightarrow +\infty} c_\phi g$ in $\mathcal{S}(G)$, and

the identity (2.4) in $L^2(G)$ gives $c_\phi = 1$. ▀

The Calderón's decomposition $g = \sum_{j \in \mathbb{Z}} g * \psi_j^* * \psi_j$ converges strongly and unconditionally in norm L^2 because

of the unconditional convergence of the sum $\sum_{j \in \mathbb{Z}} \overline{\hat{\psi}(2^{-2j}\xi)} \hat{\psi}(2^{-2j}\xi)$.

Properties.

- $\psi \in \mathcal{S}(G)$ ¹ and has an infinite number of vanishing moments², hence $\psi \in \mathcal{V}(G)$.
- Any function LP-admissible built according to Lemma 2.6 satisfies the relation :

$$\forall j, \ell \in \mathbb{Z}, |j - \ell| > 1 \implies \psi_j^* * \psi_\ell = 0, \quad (2.5)$$

resulting from :

$$\left[\hat{\psi}(2^{-2j} \Delta_G) \right]^* \circ \left[\hat{\psi}(2^{-2\ell} \Delta_G) \right] = 0.$$

2.3. Homogeneous Besov spaces.

Definition 2.7. Let $\psi \in \mathcal{V}(G)$ be LP-admissible. Let $1 \leq p, q \leq +\infty$ and $s \in \mathbb{R}$. The homogeneous Besov space associated to ψ is defined by :

$$\mathring{B}_{p,q}^{s,\psi}(G) = \left\{ u \in \mathcal{S}'(G)/\mathcal{P} \mid \{2^{js} \|u * \psi_j^*\|_{L^p(G)}\}_{j \in \mathbb{Z}} \in \ell^q(\mathbb{Z}) \right\}, \quad (2.6)$$

with the associated norm :

$$\|u\|_{\mathring{B}_{p,q}^{s,\psi}(G)} = \left\| \left\{ 2^{js} \|u * \psi_j^*\|_{L^p(G)} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}. \quad (2.7)$$

The definition of homogeneous Besov spaces requires taking L^p norms of elements of $\mathcal{S}'(G)/\mathcal{P}$. We use the canonical embedding $L^p(G) \hookrightarrow \mathcal{S}'(G)$. For $p < +\infty$, by using that $\mathcal{P} \cap L^p(G) = \{0\}$, one has the embedding $L^p(G) \hookrightarrow \mathcal{S}'(G)/\mathcal{P}$. Given $u \in \mathcal{S}'(G)/\mathcal{P}$, we define :

- $\|u\|_{L^p(G)} = \|u + q\|_{L^p(G)}$ when $u + q \in L^p(G)$, for a suitable $q \in \mathcal{P}$; note that the decomposition $u + q$ is unique since $\mathcal{P} \cap L^p(G) = \{0\}$,
- $\|u\|_{L^p(G)} = +\infty$ otherwise.

By contrast, the norm $\|\cdot\|_{L^\infty(G)}$ can only be defined on $\mathcal{S}'(G)$ by assigning the value $+\infty$ to any $u \in \mathcal{S}'(G) \setminus L^\infty(G)$. Note that the Hausdorff-Young's inequality $\|u * f\|_{L^p(G)} \leq \|u\|_{L^p(G)} \|f\|_{L^1(G)}$ remains valid respectively :

$$\begin{aligned} \text{for } p < +\infty & : \forall f \in \mathcal{S}(G), \forall u \in \mathcal{S}'(G)/\mathcal{P}, \\ \text{and for } p = +\infty & : \forall f \in \mathcal{S}(G), \forall u \in \mathcal{S}'(G). \end{aligned}$$

For $p < +\infty$, if $u + q \in L^p(G)$ then $(u + q) * \psi = u * \psi + q * \psi \in L^p(G)$.

The combination of Lemma 2.6 and Definition 2.7 shows that we shall recover the usual notion of homogeneous Besov spaces built via the spectral calculus of sub-Laplacians. The definition turns out to be independent of the choice of ψ .

Theorem 2.8 (FM [14] - Theorem 3.11). Let $\psi_1, \psi_2 \in \mathcal{V}(G)$ be LP-admissible. Let $1 \leq p, q \leq +\infty$ and $s \in \mathbb{R}$. Then $\mathring{B}_{p,q}^{s,\psi_1}(G) = \mathring{B}_{p,q}^{s,\psi_2}(G)$ with equivalence of norms.

We will accordingly omit the ψ superscript and simply write $\mathring{B}_{p,q}^s(G)$ for any choice of LP-admissible $\psi \in \mathcal{V}(G)$.

Properties.

- $\mathring{B}_{p,q}^s(G)$ is a Banach space.
- For $1 \leq p, q \leq +\infty$ and $s \in \mathbb{R}$, one has the following continuous embeddings :

$$\begin{aligned} \mathcal{V}(G) &\hookrightarrow \mathring{B}_{p,q}^s(G) \hookrightarrow \mathcal{S}'(G)/\mathcal{P}, \\ \text{and } \mathcal{V}(G) &\hookrightarrow \left(\mathring{B}_{p,q}^s(G)\right)' = \mathring{B}_{p,q}^{-s}(G). \end{aligned}$$

For $p, q < +\infty$, $\mathcal{V}(G)$ is dense in $\mathring{B}_{p,q}^s(G)$.

- The space $\mathcal{S}(G) * \sum_{|j| \leq n} \psi_j^* * \psi_j$ is dense in $\mathcal{V}(G)$, as well as in $\mathring{B}_{p,q}^s(G)$ if $p, q < +\infty$. And in this space, the decomposition $g = \sum_{j \in \mathbb{Z}} g * \psi_j^* * \psi_j$ has a finite number of nonzero terms.

¹Andrzej Hulanicki : A functional calculus for Rockland operators on nilpotent Lie groups (1984)

²Daryl Geller, Azita Mayeli : Continuous wavelets and frames on stratified Lie groups (2006)

We now extend this Littlewood-Paley decomposition to other functional spaces.

Proposition 2.9 (FM [14] - Proposition 3.14). *Let $1 \leq p, q < +\infty$ and $\psi \in \mathcal{V}(G)$ be LP-admissible. Then for all g in $\dot{B}_{p,q}^s(G)$:*

$$g = \lim_{n \rightarrow +\infty} \sum_{|j| \leq n} g * \psi_j^* * \psi_j$$

holds in $\dot{B}_{p,q}^s(G)$.

Proof. Consider the operators $\Sigma_n : \dot{B}_{p,q}^s(G) \rightarrow \dot{B}_{p,q}^s(G)$ defined by :

$$\Sigma_n g = \sum_{|j| \leq n} g * \psi_j^* * \psi_j .$$

This family of operators $(\Sigma_n)_{n \in \mathbb{N}}$ is bounded in norm. The Σ_n 's converge strongly to the identity operator on a dense subspace of $\dot{B}_{p,q}^s(G)$. But by boundedness of the family, this implies the strong convergence everywhere. \blacksquare

Proposition 2.10 (FM [14] - Proposition 3.15). *Let $1 < p < +\infty$ and $\psi \in \mathcal{V}(G)$ be LP-admissible. Then for all g in $L^p(G)$:*

$$g = \lim_{n \rightarrow +\infty} \sum_{|j| \leq n} g * \psi_j^* * \psi_j$$

holds in $L^p(G)$.

Proof. Since $\Sigma_n g = g * \delta_{2^{n+1}}^1 \phi - g * \delta_{2^{-n}}^1 \phi$, Young's inequality implies that this sequence of operators is norm bounded. It is sufficient to show the convergence of the decomposition on the dense subspace $\mathcal{S}(G)$. We saw previously in Lemma 2.5 that $g * \delta_{2^{n+1}}^1 \phi \xrightarrow{n \rightarrow +\infty} c_\phi g$. Moreover, for $n \in \mathbb{N}$, one has :

$$g * \delta_{2^{-n}}^1 \phi(x) = \frac{1}{2^{nQ}} \int_G g(y) \phi\left(\frac{1}{2^n} \odot (y^{-1} \cdot x)\right) dy = \frac{1}{2^{nQ}} \delta_{2^n}^1 g * \phi\left(\frac{1}{2^n} \odot x\right),$$

and so :

$$\|g * \delta_{2^{-n}}^1 \phi\|_{L^p(G)} = \frac{1}{2^{nQ}} \left(\int_G \left| \delta_{2^n}^1 g * \phi\left(\frac{1}{2^n} \odot x\right) \right|^p dx \right)^{\frac{1}{p}} = \frac{1}{2^{nQ(1-\frac{1}{p})}} \|\delta_{2^n}^1 g * \phi\|_{L^p(G)} .$$

Again, $\delta_{2^n}^1 g * \phi \xrightarrow{n \rightarrow +\infty} c_\phi g$, and in particular : $\frac{1}{2^{nQ(1-\frac{1}{p})}} \|\delta_{2^n}^1 g * \phi\|_{L^p(G)} \xrightarrow{n \rightarrow +\infty} 0$.

Hence $\Sigma_n g \xrightarrow{n \rightarrow +\infty} c_\phi g$ and the case $p = 2$ determines that $c_\phi = 1$. \blacksquare

3. CHARACTERIZATION OF BESOV SPACES BY THE DISCRETE WAVELETS

We show in this section that the characterization of $\dot{B}_{p,q}^s(G)$ by a Littlewood-Paley theory can be discretized by sampling the convolution products $f * \psi_j^*$ over a discrete set $\Gamma \subset G$. This is equivalent to the study of the analysis operator $A_\psi : \mathcal{S}'(G)/\mathcal{P} \ni f \mapsto A_\psi f = \left\{ \langle f, \psi_{j,\gamma} \rangle \right\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}}$ associated to a discrete wavelet system

$\left\{ \psi_{j,\gamma} \right\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}}$ defined by :

$$\forall x \in G, \psi_{j,\gamma}(x) = \tau_\gamma \delta_{2^j}^1 \psi(x) = 2^{jQ} \psi(\gamma^{-1} \cdot 2^j \odot x), \quad (3.1)$$

where $\psi \in \mathcal{V}(G)$ is chosen as in Lemma 2.6 and $\psi^* = \psi$. The main goal of this section is the proof of the equivalence between Besov norms and some associated discrete norm in Theorem 3.6.

We cannot mention wavelets without some history. The development of wavelets started with Alfred Haar in 1909. Notable contributions can be attributed to George Zweig's discovery of the continuous wavelet transform (1975), Pierre Goupillaud, Alex Grossmann and Jean Morlet's formulation of the CWT (1982), Jan-Olov Strömberg's work on discrete wavelets (1983), Ingrid Daubechies' orthogonal wavelets with compact support (1988), Yves Meyer and Stéphane Mallat's MRA framework (1989), and many more since. In 1986, Yves Meyer built a wavelet basis that suits for the simultaneous characterization of any Sobolev space \dot{H}^s . Wavelet bases thus provide an elegant and unifying response to the problem of exhibiting unconditional bases for a wide range of classic functional spaces. Some useful references are [32] and [33].

The very first wavelet bases on stratified Lie groups were obtained in 1989 by Pierre-Gilles Lemarié [29] from a spline interpolation theory. Already the sub-Laplacian Δ_G played a major role. Later in 2006 came out continuous and discrete wavelet systems by Daryl Geller and Azita Mayeli [19] using the spectral theory of the sub-Laplacian. So it is rather natural to expect its involvement in a characterization of homogeneous Besov spaces. It was achieved in 2012 by Hartmut Führ and Azita Mayeli [14].

In order to discretize norms over elementary tiles of G and perform a multi resolution analysis (MRA), we shall introduce the notion of regular sampling sets.

3.1. Regular sampling sets.

Definition 3.1. *Let G be a Lie group. A subset $\Gamma \subset G$ is called a regular sampling set if there is a relatively compact Borel neighborhood $\mathcal{W} \subset G$ of the identity element of G satisfying $\bigcup_{\gamma \in \Gamma} \gamma \mathcal{W} = G$ (up to a set of measure zero), and for all $\alpha, \gamma \in \Gamma$, $\alpha \neq \gamma$, $\mu_G(\alpha \mathcal{W} \cap \gamma \mathcal{W}) = 0$. Such a set \mathcal{W} is called a Γ -tile.*

Definition 3.2. *For $\mathcal{U} \subset G$, a regular sampling set Γ is said \mathcal{U} -dense if there exists a Γ -tile $\mathcal{W} \subset \mathcal{U}$.*

For instance, lattices in \mathbb{R}^d are a special class of regular sampling sets which are also cocompact discrete subgroups. However, some Lie groups fail to admit lattices, whereas by contrast there are always sufficiently dense regular sampling sets as indicated in the following lemma.

Lemma 3.3. *Given a stratified Lie group G with a Hausdorff geometrical realization, for any neighborhood \mathcal{U} of the identity, there exists a \mathcal{U} -dense regular sampling set.*

Proof. There exists $\Gamma \subset G$ and a relatively compact set \mathcal{W} with non-empty interior, such that $\bigcup_{\gamma \in \Gamma} \gamma \mathcal{W}$ covers G (possibly up to a set of measure zero). Then $\mathcal{V} = \mathcal{W}x_0^{-1}$ is a Γ -tile for some point x_0 within \mathcal{W} . Finally, by choosing $\beta > 0$ small enough, we ensure that $\beta \mathcal{V} \subset \mathcal{U}$ and then $\beta \mathcal{V}$ is a $\beta \Gamma$ -tile. ■

Definition 3.4. *Let $\Gamma \subset G$. An automorphism $\varrho \in \text{Aut}(G)$ is said Γ -acceptable if it leaves Γ globally invariant, that is $\varrho \Gamma \subseteq \Gamma$.*

We shall of course choose Γ such that the dyadic dilations $\{\delta_{2^j}\}_{j \in \mathbb{Z}}$ and for any $\gamma \in \Gamma$, the translations τ_γ are acceptable automorphisms. This definition ensures the compatibility between the group law, the nonisotropic dilations, the sampling set and the iterated wavelet system.

3.2. Discretization of Besov norms.

Definition 3.5. *Fix a discrete set $\Gamma \subset G$. For any family $\{c_{j\gamma}\}_{j \in \mathbb{Z}, \gamma \in \Gamma}$ of complex numbers, we define :*

$$\left\| \{c_{j\gamma}\}_{j \in \mathbb{Z}, \gamma \in \Gamma} \right\|_{\mathring{b}_{p,q}^s} = \left(\sum_{j \in \mathbb{Z}} \left(\sum_{\gamma \in \Gamma} \left(2^{j(s-\frac{Q}{p})} |c_{j\gamma}| \right)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}. \quad (3.2)$$

We introduce the space of coefficients $\mathring{b}_{p,q}^s(\Gamma)$ defined as :

$$\mathring{b}_{p,q}^s(\Gamma) = \left\{ \left\{ c_{j\gamma} \right\}_{j \in \mathbb{Z}, \gamma \in \Gamma} \left| \left\| \{c_{j\gamma}\}_{j \in \mathbb{Z}, \gamma \in \Gamma} \right\|_{\mathring{b}_{p,q}^s} < +\infty \right. \right\}, \quad (3.3)$$

which will be sometimes denoted by $\mathring{b}_{p,q}^s$ when there is no possible misunderstanding on Γ .

The next theorem shows that Besov norms can be expressed in terms of discrete coefficients. The constants appearing in the norm equivalence depend on the functional spaces, but it does not matter if we use the same sampling set Γ simultaneously for all spaces.

Theorem 3.6 (FM [14] - Theorem 5.4). *There exists a neighborhood \mathcal{U} of the identity such that, for any \mathcal{U} -dense regular sampling set Γ , one has :*

$$\forall u \in \mathcal{S}'(G)/\mathcal{P}, \forall 1 \leq p, q \leq +\infty, u \in \mathring{B}_{p,q}^s(G) \iff \left\{ \langle u, 2^{-jQ} \psi_{j,\gamma} \rangle \right\}_{j \in \mathbb{Z}, \gamma \in \Gamma} \in \mathring{b}_{p,q}^s(\Gamma),$$

where $\psi_{j,\gamma}$ is defined as in (3.1). In addition, in $\dot{B}_{p,q}^s(G)$ we have the norm equivalence :

$$\|u\|_{\dot{B}_{p,q}^s(G)} \sim \left(\sum_{j \in \mathbb{Z}} \left(\sum_{\gamma \in \Gamma} \left(2^{j(s-\frac{Q}{p})} |\langle u, 2^{-jQ} \psi_{j,\gamma} \rangle| \right)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}, \quad (3.4)$$

with some constants depending only on p, q, s and the regular sampling set Γ .

The accuracy of the estimate improves as the tightness of the sampling set increases.

3.3. Banach wavelet frames for Besov spaces.

In Hilbert spaces, a norm equivalence such as (3.4) is sufficient to conclude that the wavelet basis is a frame, allowing the reconstruction of u from the discrete coefficients data. For Banach spaces, we need an extended definition of frames³, in order to show the invertibility of the associated frame operator.

After stating technical lemmas, we will prove that any linear wavelet combination converges unconditionally in any $L^p(G)$ in Theorem 3.9, and also in $\dot{B}_{p,q}^s(G)$ in Theorem 3.11 whenever its coefficients lie in $\dot{b}_{p,q}^s$. Then for a suitable choice of sufficiently dense regular sampling sets Γ , the wavelet system $\{2^{-jQ} \psi_{j,\gamma}\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}}$ is a Banach frame for $L^p(G)$ and $\dot{B}_{p,q}^s(G)$.

Definition 3.7. A basis (f_n) is an unconditional basis in a Banach space if any convergent series $\sum_n a_n f_n$ converges unconditionally, that is to say the series $\sum_n a_{\sigma(n)} f_{\sigma(n)}$ converges to the same limit for any permutation σ of the indexes.

Recall that the sampled convolution products can be interpreted as scalar products too, that is to assert :

$$f * \psi_j^*(2^{-j} \odot \gamma) = \langle f, \psi_{j,\gamma} \rangle,$$

where $\psi_{j,\gamma}(x) = 2^{jQ} \psi(\gamma^{-1} \cdot 2^j \odot x)$ denotes the wavelet of resolution $\frac{1}{2^j}$ and at position $\frac{1}{2^j} \odot \gamma$. The wavelet system is now used for synthesis.

3.3.1. Unconditionality in $L^p(G)$ with $1 < p < +\infty$.

To check the unconditionality of wavelet decompositions in both Lebesgue and homogeneous Besov spaces, we need the next preliminary estimate.

Lemma 3.8. Let $\eta, j \in \mathbb{Z}$ with $\eta \leq j$ and $n \geq Q + 1$. Let $\Gamma \subset G$ be a dense regular sampling set, in the sense of Definition 3.1. Then :

$$\forall x \in G, \sum_{\gamma \in \Gamma} \frac{2^{-jQ}}{(1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G)^n} \leq c 2^{-\eta Q},$$

where the constant c depends only on n and Γ .

Proof. By assumption, there exists a relatively compact open \mathcal{W} such that $\mu_G(\gamma \mathcal{W} \cap \gamma' \mathcal{W}) = 0$ when $\gamma \neq \gamma'$ in Γ . Then, in view of the left-invariance of the Haar measure μ_G in (1.1), we have :

$$\sum_{\gamma \in \Gamma} \frac{2^{-jQ}}{(1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G)^n} \leq \sum_{\gamma \in \Gamma} \frac{1}{|\mathcal{W}|} \int_{2^{-j} \odot (\gamma \mathcal{W})} \frac{1}{(1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G)^n} dy.$$

For any $y \in 2^{-j} \odot (\gamma \mathcal{W})$, the triangle inequality gives :

$$\begin{aligned} 1 + 2^\eta |y^{-1} \cdot x|_G &\leq 1 + 2^\eta c' (|y^{-1} \cdot 2^{-j} \odot \gamma|_G + |2^{-j} \odot \gamma^{-1} \cdot x|_G) \\ &\leq 1 + 2^\eta c' (2^{-j} \text{diam}(\mathcal{W}) + |2^{-j} \odot \gamma^{-1} \cdot x|_G) \\ &\leq c'' (1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G) \end{aligned}$$

³Karlheinz Gröchenig [22]

by using the fact that $\eta \leq j$. As a result :

$$\begin{aligned} \sum_{\gamma \in \Gamma} \frac{1}{|\mathcal{W}|} \int_{2^{-j} \odot (\gamma \mathcal{W})} \frac{1}{(1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G)^n} dy &\leq \left(\frac{1}{c''} \right)^n \sum_{\gamma \in \Gamma} \frac{1}{|\mathcal{W}|} \int_{2^{-j} \odot (\gamma \mathcal{W})} \frac{1}{(1 + 2^\eta |y^{-1} \cdot x|_G)^n} dy \\ &= 2^{-\eta Q} \left(\frac{1}{c''} \right)^n \frac{1}{|\mathcal{W}|} \int_G \frac{1}{(1 + |y|_G)^n} dy, \end{aligned}$$

since the $\gamma \mathcal{W}$'s are pairwise disjoint. For $n \geq Q + 1$, the latter integral is finite. Hence the lemma. \blacksquare

Theorem 3.9. *Let $1 \leq p \leq +\infty$. Let $\eta, j \in \mathbb{Z}$ be fixed with $\eta \leq j$. Suppose that $\Gamma \subset G$ is a regular sampling set. For all $\gamma \in \Gamma$, we consider functions $f_{j,\gamma}$ on G satisfying the following decay condition :*

$$\forall x \in G, \forall \eta, j \in \mathbb{Z}, \forall \gamma \in \Gamma, |f_{j,\gamma}(x)| \leq \frac{c_1}{(1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G)^{Q+1}},$$

with some constant $c_1 > 0$. We define $f_j = \sum_{\gamma \in \Gamma} c_{j\gamma} f_{j,\gamma}$ with $\{c_{j\gamma}\}_\gamma \in \ell^p(\Gamma)$. Then the series converges unconditionally in $L^p(G)$ with :

$$\|f_j\|_{L^p(G)} \leq c_2 2^{(j-\eta)Q} 2^{-\frac{jQ}{p}} \left(\sum_{\gamma \in \Gamma} |c_{j\gamma}|^p \right)^{\frac{1}{p}}, \quad (3.5)$$

for some constant c_2 independent of j, η, γ and the sequence of coefficients $\{c_{j\gamma}\}_\gamma$.

Proof. There exists a Γ -tile \mathcal{W} such that $G = \bigsqcup_{\alpha \in \Gamma} 2^{-j} \odot (\alpha \mathcal{W})$. Then :

$$\begin{aligned} \|f_j\|_{L^p(G)}^p &= \sum_{\alpha \in \Gamma} \int_{2^{-j} \odot (\alpha \mathcal{W})} \left| \sum_{\gamma \in \Gamma} c_{j\gamma} f_{j,\gamma}(x) \right|^p dx \\ &\leq c_1^p \sum_{\alpha \in \Gamma} \int_{2^{-j} \odot (\alpha \mathcal{W})} \left| \sum_{\gamma \in \Gamma} |c_{j\gamma}| \frac{1}{(1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G)^{Q+1}} \right|^p dx. \end{aligned}$$

On each integration domain $2^{-j} \odot (\alpha \mathcal{W})$, the triangle inequality yields the estimate :

$$1 + 2^\eta |2^{-j} \odot (\gamma^{-1} \cdot \alpha)|_G \leq c'' (1 + 2^\eta |2^{-j} \odot \gamma^{-1} \cdot x|_G).$$

Then the integrand can be majorized by the constant :

$$\left| \sum_{\gamma \in \Gamma} |c_{j\gamma}| \frac{c''}{(1 + 2^\eta |2^{-j} \odot (\gamma^{-1} \cdot \alpha)|_G)^{Q+1}} \right|^p.$$

Hence :

$$\begin{aligned} \|f_j\|_{L^p(G)}^p &\leq (c_1 c'')^p |\mathcal{W}| \sum_{\alpha \in \Gamma} 2^{-jQ} \left(\sum_{\gamma \in \Gamma} |c_{j\gamma}| \frac{1}{(1 + 2^\eta |2^{-j} \odot (\gamma^{-1} \cdot \alpha)|_G)^{Q+1}} \right)^p \\ &= (c_1 c'')^p |\mathcal{W}| \sum_{\alpha \in \Gamma} 2^{-jQ} \left(\sum_{\gamma \in \Gamma} |c_{j\gamma}| a_{\alpha\gamma} \right)^p, \end{aligned}$$

with $a_{\alpha\gamma} = \frac{1}{(1 + 2^\eta |2^{-j} \odot (\gamma^{-1} \cdot \alpha)|_G)^{Q+1}}$. Lemma 3.8 now ensures that the Schur lemma's conditions are

fulfilled for the coefficients $\{a_{\alpha\gamma}\}$ with $\max \left(\sup_{\alpha} \sum_{\gamma \in \Gamma} |a_{\alpha\gamma}|, \sup_{\gamma} \sum_{\alpha \in \Gamma} |a_{\alpha\gamma}| \right) \leq 2^{(j-\eta)Q}$. Therefore, there exists

a constant c_2 such that :

$$\|f_j\|_{L^p(G)} \leq 2^{-\frac{jQ}{p}} c_1 c'' |\mathcal{W}|^{\frac{1}{p}} \left(\sum_{\alpha \in \Gamma} \left(\sum_{\gamma \in \Gamma} |c_{j\gamma}| a_{\alpha\gamma} \right)^p \right)^{\frac{1}{p}} \leq 2^{-\frac{jQ}{p}} 2^{(j-\eta)Q} c_2 \left(\sum_{\gamma \in \Gamma} |c_{j\gamma}|^p \right)^{\frac{1}{p}}.$$

\blacksquare

3.3.2. *Unconditionality in $\mathring{B}_{p,q}^s(G)$ with $1 \leq p, q < +\infty$.*

Lemma 3.10. *With the above notations, there exists a constant $c > 0$ such that $\forall j, \ell \in \mathbb{Z}, \forall \gamma \in \Gamma, \forall x \in G$, the following estimate holds :*

$$|\psi_{j,\gamma} * \psi_\ell^*(x)| \leq \begin{cases} \frac{c 2^{jQ}}{(1 + 2^j |2^{-j} \odot \gamma^{-1} \cdot x|_G)^{Q+1}} & \text{if } |j - \ell| \leq 1 \\ 0 & \text{otherwise} \end{cases}. \quad (3.6)$$

Proof. Let us compute :

$$\begin{aligned} \psi_{j,\gamma} * \psi_\ell^*(x) &= \int_G 2^{jQ} \psi(\gamma^{-1} \cdot 2^j \odot y) 2^{\ell Q} \overline{\psi(2^\ell \odot (x^{-1} \cdot y))} dy \\ &= \int_G 2^{jQ} \psi(y) \overline{\psi_{\ell-j}((\gamma^{-1} \cdot 2^j \odot x)^{-1} \cdot y)} dy \\ &= 2^{jQ} (\psi * \psi_{\ell-j}^*)(\gamma^{-1} \cdot 2^j \odot x). \end{aligned}$$

By (2.5), this convolution product vanishes when $|j - \ell| > 1$.

In the other case i.e. $\ell - j \in \{-1, 0, +1\}$, the convolution products $\psi * \psi_{\ell-j}$ are functions in the Schwartz class and henceforth, $|\psi_{j,\gamma} * \psi_\ell^*(x)| \leq \frac{c 2^{jQ}}{(1 + 2^j |2^{-j} \odot \gamma^{-1} \cdot x|_G)^{Q+1}}$ by Lemma 2.3. \blacksquare

The next result is crucial for the upcoming Section 4.

Theorem 3.11. *Let $1 \leq p, q < +\infty$. If Γ is the same regular sampling set as in Theorem 3.9, then for any sequence of coefficients $\{c_{j\gamma}\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}} \in \mathring{b}_{p,q}^s(\Gamma)$, the sum $f = \sum_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}} 2^{-jQ} c_{j\gamma} \psi_{j,\gamma}$ converges unconditionally in Besov norm with :*

$$\|f\|_{\mathring{B}_{p,q}^s(G)} \leq c \left(\sum_{j \in \mathbb{Z}} \left(\sum_{\gamma \in \Gamma} \left(2^{j(s-\frac{Q}{p})} |c_{j\gamma}| \right)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}, \quad (3.7)$$

for some constant c independent of $\{c_{j\gamma}\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}}$.

Proof. It is sufficient to prove the norm estimate for finite sequences of coefficients $\{c_{j\gamma}\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}}$. The full statement follows from the completeness of $\mathring{B}_{p,q}^s(G)$ and the property that the Kronecker symbols δ form an unconditional basis of $\mathring{b}_{p,q}^s$ (this is why $p, q < +\infty$ is required). By definition :

$$\begin{aligned} \|f\|_{\mathring{B}_{p,q}^s(G)} &= \left\| \left\{ 2^{\ell s} \|f * \psi_\ell^*\|_{L^p(G)} \right\}_{\ell \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} = \left\| \left\{ 2^{\ell s} \left\| \sum_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}} 2^{-jQ} c_{j\gamma} \psi_{j,\gamma} * \psi_\ell^* \right\|_{L^p(G)} \right\}_{\ell \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \\ &= \left\| \left\{ 2^{\ell s} \left\| \sum_{\gamma \in \Gamma} \sum_{j=\ell-1}^{\ell+1} 2^{-jQ} c_{j\gamma} \psi_{j,\gamma} * \psi_\ell^* \right\|_{L^p(G)} \right\}_{\ell \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \\ &\leq \left\| \left\{ 2^{\ell s} \sum_{j=\ell-1}^{\ell+1} \left\| \sum_{\gamma \in \Gamma} 2^{-jQ} c_{j\gamma} \psi_{j,\gamma} * \psi_\ell^* \right\|_{L^p(G)} \right\}_{\ell \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \end{aligned}$$

by using (2.5). For instance, let us consider the middle term when $j = \ell$. Applying successively (3.6) and Theorem 3.9 where $j - \eta = 0$, we obtain that :

$$2^{\ell s} \left\| \sum_{\gamma \in \Gamma} 2^{-\ell Q} c_{\ell\gamma} \psi_{\ell,\gamma} * \psi_\ell^* \right\|_{L^p(G)} \leq 2^{\ell(s-\frac{Q}{p})} c_2 \left(\sum_{\gamma \in \Gamma} |c_{\ell\gamma}|^p \right)^{\frac{1}{p}}.$$

Therefore, we get :

$$\left\| \left\{ 2^{\ell s} \left\| \sum_{\gamma \in \Gamma} 2^{-\ell Q} c_{\ell\gamma} \psi_{\ell,\gamma} * \psi_\ell^* \right\|_{L^p(G)} \right\}_{\ell \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \leq c_2 \left(\sum_{\ell \in \mathbb{Z}} \left(\sum_{\gamma \in \Gamma} \left(2^{\ell(s-\frac{Q}{p})} |c_{\ell\gamma}| \right)^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} = c_2 \left\| \{c_{j\gamma}\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}} \right\|_{\mathring{b}_{p,q}^s}.$$

Of course, by applying similar calculations when $j = \ell \pm 1$, we will find that $\|f\|_{\mathring{B}_{p,q}^s(G)} \leq c \left\| \left\{ c_{j\gamma} \right\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}} \right\|_{\mathring{b}_{p,q}^s} \cdot$ ■

Generally, for sufficiently dense regular sampling sets Γ , the invertibility of the synthesis operator $S_{\psi,\Gamma} : f \mapsto S_{\psi,\Gamma} f = \sum_{j,\gamma} 2^{-jQ} \langle f, \psi_{j,\gamma} \rangle \psi_{j,\gamma}$ ensures that the discrete wavelet basis obtained by dilations by

powers of 2 and translations provide an universal Banach frame simultaneously for all Besov spaces $\mathring{B}_{p,q}^s(G)$ with $s \in \mathbb{R}$ and $1 \leq p, q < +\infty$. In other words, the wavelet basis induces the unconditional convergence of the decomposition $f = \sum_{j,\gamma} 2^{-jQ} c_{j\gamma} \psi_{j,\gamma}$ in $\mathring{B}_{p,q}^s(G)$ whenever $f \in \mathring{B}_{p,q}^s(G)$ with coefficients $\{c_{j\gamma}\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}} \in \mathring{b}_{p,q}^s(\Gamma)$

and $\|f\|_{\mathring{B}_{p,q}^s(G)} \sim \left\| \left\{ c_{j\gamma} \right\}_{\substack{j \in \mathbb{Z} \\ \gamma \in \Gamma}} \right\|_{\mathring{b}_{p,q}^s} \cdot$

4. CONSTRUCTION OF THE PROFILES ON STRATIFIED LIE GROUPS

At this stage, we have enough tools to prove Theorem 1.4 by adapting the method in [2]. To sum up, we have constructed LP-admissible wavelets $\psi \in \mathcal{V}(G)$, and then confirmed the existence of unconditional bases (spanned as an iterated functions system by ψ) for both $L^p(G)$ and $\mathring{B}_{p,q}^s(G)$ in Theorems 3.9 and 3.11.

Let $\Gamma \subset G$ be a regular sampling set such that $\{\tau_\gamma\}_{\gamma \in \Gamma}$ and $\{\delta_{2^j}\}_{j \in \mathbb{Z}}$ are Γ -acceptable automorphisms. We introduce a new index $\lambda = (j, \gamma)$ for the wavelet basis, where the component $j = j_\lambda$ is a scale index (actually the dyadic exponent) while the other component $\gamma = \gamma_\lambda$ is a space index. For any element f of $\mathring{H}^s(G) = \mathring{B}_{2,2}^s(G)$, its wavelet decomposition can be written as :

$$f = \sum_{\lambda \in \mathbb{Z} \times \Gamma} d_\lambda \psi_\lambda ,$$

where ψ_λ is now L^p -normalized according to :

$$\forall \lambda \in \mathbb{Z} \times \Gamma, \psi_\lambda(x) = 2^{\frac{j_\lambda Q}{p}} \psi(\gamma^{-1} \cdot 2^{j_\lambda} \odot x) = \underline{\tau}_\gamma \underline{\delta}_{2^{j_\lambda}}^p \psi(x) . \quad (4.1)$$

By Theorem 2.8, the homogeneous Besov spaces are independent of the choice of $\psi \in \mathcal{V}(G)$. By picking $\hat{\psi}$ with support in the interval $\left[\frac{1}{2}, 1\right]$, the ψ_λ 's shall form an orthonormal basis. From (4.1), one has readily :

$$\begin{cases} \|\psi_\lambda\|_{L^p(G)} = \|\psi\|_{L^p(G)} \\ \|\psi_\lambda\|_{\mathring{H}^s(G)} = \|\psi\|_{\mathring{H}^s(G)} \end{cases} , \quad (4.2)$$

$$\text{as well as } \|f\|_{\mathring{H}^s(G)} = \|\{d_\lambda\}\|_{\ell^2(\mathbb{Z} \times \Gamma)} . \quad (4.3)$$

The unconditionality of this basis implies the existence of a constant D such that for any finite subset $E \subset \mathbb{Z} \times \Gamma$, any coefficients $(c_\lambda)_{\lambda \in E}$ and $(d_\lambda)_{\lambda \in E}$ satisfying $\forall \lambda, |c_\lambda| \leq |d_\lambda|$, one has :

$$\left\| \sum_{\lambda \in E} c_\lambda \psi_\lambda \right\|_{\mathring{H}^s(G)} \leq D \left\| \sum_{\lambda \in E} d_\lambda \psi_\lambda \right\|_{\mathring{H}^s(G)} . \quad (4.4)$$

For $M > 0$, let us now consider the nonlinear projector Q_M defined by :

$$\forall f \in \mathring{H}^s(G), Q_M(f) = \sum_{\lambda \in E_M(f)} d_\lambda \psi_\lambda , \quad (4.5)$$

where $E_M(f)$ is the subset of $\mathbb{Z} \times \Gamma$ of cardinality M corresponding to the M largest values of $|d_\lambda|$. Note that such a set always exists, but the nonlinear projection (4.5) may not be unique when some $|d_\lambda|$ are equal. In which case, any realization of such set suits for $E_M(f)$. Additionally, it has been proven in RDV [11] or DVJP [12] that :

$$\lim_{M \rightarrow +\infty} \sup_{\|f\|_{\mathring{H}^s(G)} \leq 1} \|f - Q_M(f)\|_{L^p(G)} = 0 , \quad (4.6)$$

with $\frac{s}{Q} + \frac{1}{p} = \frac{1}{2}$. Let us underline that the uniform convergence (4.6) of $Q_M(f)$ to f in $L^p(G)$ is tied to the nonlinear nature of the operator Q_M . The nonlinear projection $Q_M(f)$, sometimes called the best M -term approximation of f , has been extensively studied - refer for instance to RDV [11] and the numerous references therein.

The proof of Theorem 1.4 is based on a diagonal subsequence, and it is structured in three main steps. We work of course under the theorem's assumptions and consider a sequence of bounded functions $(u_n)_{n>0} \subset \mathring{H}^s(G)$. Then let us define :

$$K = \sup_{n>0} \|u_n\|_{\mathring{H}^s(G)} < +\infty .$$

4.1. Reordering of the wavelet decomposition.

From the wavelet decomposition $u_n = \sum_{\lambda \in \mathbb{Z} \times \Gamma} d_{\lambda,n} \psi_\lambda$, the summands are reordered by decreasing moduli $|d_{\lambda,n}|_{\lambda \in \mathbb{Z} \times \Gamma}$ such that :

$$u_n = \sum_{m>0} d_{m,n} \psi_{\lambda(m,n)} .$$

Using the nonlinear projector defined by (4.5), one gets :

$$u_n = \sum_{m=1}^M d_{m,n} \psi_{\lambda(m,n)} + (u_n - Q_M(u_n)) ,$$

with, in light of (4.6) :

$$\lim_{M \rightarrow +\infty} \sup_{n>0} \|u_n - Q_M(u_n)\|_{L^p(G)} = 0 . \quad (4.7)$$

Since wavelets are normalized in $\mathring{H}^s(G)$, we know that : $\sup_{m>0,n} |d_{m,n}| \leq DK$, where D is the constant of (4.4).

Up to a possible diagonal extraction in n of a subsequence, we can assume that, for $m > 0$, the sequence $(d_{m,n})_{n>0}$ converges to a finite limit depending only on m :

$$d_m = \lim_{n \rightarrow +\infty} d_{m,n} .$$

We can then write :

$$u_n = \sum_{m=1}^M d_m \psi_{\lambda(m,n)} + \sum_{m=1}^M (d_{m,n} - d_m) \psi_{\lambda(m,n)} + (u_n - Q_M(u_n)) . \quad (4.8)$$

4.2. Extraction of the approximate profiles.

The exact profiles ϕ^ℓ involved in (1.8) are inferred as limits in $\mathring{H}^s(G)$ of some approximate profiles $\phi^{\ell,i}$ obtained by the following procedure :

- (1) Initialize $\phi^{1,1} = d_1 \psi$, $\lambda_1(n) = \lambda(1, n)$ and $\varphi_1(n) = n$.
- (2) At step $i - 1$, assume that we have obtained $\nu(i - 1)$ functions $(\phi^{1,i-1}, \phi^{2,i-1}, \dots, \phi^{\nu(i-1),i-1})$, scale-space indexes $(\lambda_1(n), \lambda_2(n), \dots, \lambda_{\nu(i-1)}(n))$, as well as an increasing sequence of positive integers $\varphi_{i-1}(n)$ such that :

$$\sum_{m=1}^{i-1} d_m \psi_{\lambda(m, \varphi_{i-1}(n))} = \sum_{\ell=1}^{\nu(i-1)} \phi_{\lambda_\ell(\varphi_{i-1}(n))}^{\ell, i-1} .$$

- (3) Add the i -th term $d_i \psi_{\lambda(i, \varphi_{i-1}(n))}$ either to build a new function, either to modify slightly one of the previous functions according to the next dichotomy :

- Case 1 : Assume that we can extract $\varphi_i(n)$ from $\varphi_{i-1}(n)$ such that for any $\ell \in \llbracket 1, \nu(i - 1) \rrbracket$, at least one of the two conditions below is satisfied :

$$\lim_{n \rightarrow +\infty} |j(\lambda_\ell(\varphi_i(n))) - j(\lambda(i, \varphi_i(n)))| = +\infty ,$$

or

$$\lim_{n \rightarrow +\infty} \left| \frac{2^{j(\lambda(i, \varphi_i(n)))}}{2^{j(\lambda_\ell(\varphi_i(n)))}} \odot \gamma(\lambda_\ell(\varphi_i(n)))^{-1} \cdot \gamma(\lambda(i, \varphi_i(n))) \right|_G = +\infty .$$

In that case, with $\nu(i) = \nu(i-1) + 1$, we add a new function $\phi^{\nu(i),i}$ such that :

$$\phi^{\nu(i),i} = d_i \psi, \quad \lambda_{\nu(i)}(n) = \lambda(i, n),$$

and we keep every previous approximate profiles, by setting : $\forall \ell \in \llbracket 1, \nu(i-1) \rrbracket$, $\phi^{\ell,i} = \phi^{\ell,i-1}$.

- Case 2 : Suppose that for some subsequence $\varphi_i(n)$ of $\varphi_{i-1}(n)$ and some $\ell \in \llbracket 1, \nu(i-1) \rrbracket$, none of the two above conditions is satisfied. Then one can check that $j(\lambda_\ell(\varphi_i(n))) - j(\lambda(i, \varphi_i(n)))$ and $\left| \frac{2^{j(\lambda(i, \varphi_i(n)))}}{2^{j(\lambda_\ell(\varphi_i(n)))}} \odot \gamma(\lambda_\ell(\varphi_i(n)))^{-1} \cdot \gamma(\lambda(i, \varphi_i(n))) \right|_G$ only take a finite number of values as n varies. Therefore, up to an additional subsequence extraction, we can infer the existence of finite values $\tilde{j} \in \mathbb{Z}$ and $\tilde{\gamma} \in G$ such that $\forall n > 0$:

$$\begin{aligned} j(\lambda(i, \varphi_i(n))) - j(\lambda_\ell(\varphi_i(n))) &= \tilde{j}, \\ \text{and } \frac{2^{j(\lambda(i, \varphi_i(n)))}}{2^{j(\lambda_\ell(\varphi_i(n)))}} \odot \gamma(\lambda_\ell(\varphi_i(n)))^{-1} \cdot \gamma(\lambda(i, \varphi_i(n))) &= \tilde{\gamma}. \end{aligned}$$

Set $\nu(i) = \nu(i-1)$. The function $\phi^{\ell,i-1}$ is now replaced by :

$$\phi^{\ell,i}(x) = \phi^{\ell,i-1}(x) + 2^{\frac{\tilde{j}Q}{p}} d_i \psi(\tilde{\gamma}^{-1} \cdot 2^{\tilde{j}} \odot x),$$

whereas the other profiles remain unchanged i.e. for all $\llbracket 1, \nu(i-1) \rrbracket \ni \ell' \neq \ell$, $\phi^{\ell',i} = \phi^{\ell',i-1}$.

This algorithm shows that, for any $M > 0$, there exists $\nu(M) \leq M$ such that :

$$\sum_{m=1}^M d_m \psi_{\lambda(m,n)} = \sum_{\ell=1}^{\nu(M)} \phi_{\lambda_\ell(n)}^{\ell,M}.$$

More explicitly, for every $\ell \in \llbracket 1, \nu(M) \rrbracket$, we have :

$$\phi_{\lambda_\ell(n)}^{\ell,M} = \sum_{m \in E(\ell, M)} d_m \psi_{\lambda(m,n)},$$

where the sets $E(\ell, M)$ form a disjoint partition of $\llbracket 1, M \rrbracket = \bigsqcup_{\ell=1}^{\nu(M)} E(\ell, M)$.

It is clear that $E(\ell, M) \subseteq E(\ell, M+1)$, and the number of approximate profiles $\nu(M)$ increases by at most one unit when going from M to $M+1$.

4.3. End of the proof.

To finish the proof of Theorem 1.4, we need to construct the exact profiles ϕ^ℓ , and to control the error terms in (4.8).

By a diagonal extraction on n encoded in the function φ_i , the exact profiles ϕ^ℓ are obtained as limits in $\tilde{H}^s(G)$ of the approximate profiles $\phi^{\ell,M}$ as $M \rightarrow +\infty$, the same way as in [2] by means of the invariance by scaling (4.2) and the unconditionality of the wavelet basis.

Let us now estimate the error terms.

For a given $0 \leq L \leq M$, according to Subsections 4.1 and 4.2, we can rewrite u_n as :

$$u_n = \sum_{\ell=1}^L \phi_{\lambda_\ell(n)}^\ell + r_{n,L}, \quad (4.9)$$

where the remainder $r_{n,L}$ can be split into :

$$r_{n,L} = \underbrace{\sum_{\ell=1}^L \left(\phi_{\lambda_\ell(n)}^{\ell,M} - \phi_{\lambda_\ell(n)}^\ell \right)}_{r_1(n,L,M)} + \underbrace{\sum_{\ell=1}^L \sum_{m \in E(\ell, M)} (d_{m,n} - d_m) \psi_{\lambda(m,n)} + \sum_{\ell=L+1}^{\nu(M)} \sum_{m \in E(\ell, M)} d_{m,n} \psi_{\lambda(m,n)} + (u_n - Q_M(u_n))}_{r_2(n,L,M)}.$$

Observe that each of these summands depends on the chosen value of M , but their total sum $r_{n,L}$ is actually independent of M .

Under the norm invariance by scaling (4.2) of the L^p -normalized $\{\psi_\lambda\}$ basis, we infer that

$$\left\| \sum_{\ell=1}^L \left(\phi_{\lambda_{\ell(n)}}^{\ell, M} - \phi_{\lambda_{\ell(n)}}^\ell \right) \right\|_{\dot{H}^s(G)} \leq \sum_{\ell=1}^L \left\| \left(\phi_{\lambda_{\ell(n)}}^{\ell, M} - \phi_{\lambda_{\ell(n)}}^\ell \right) \right\|_{\dot{H}^s(G)} = \sum_{\ell=1}^L \left\| \phi^{\ell, M} - \phi^\ell \right\|_{\dot{H}^s(G)}.$$

Since for all $\ell \geq 1$, $\phi^{\ell, M} \xrightarrow{M \rightarrow +\infty} \phi^\ell$ in $\dot{H}^s(G)$, one deduces that for any fixed $L > 0$:

$$\overline{\lim}_{n \rightarrow +\infty} \left\| \sum_{\ell=1}^L \left(\phi_{\lambda_{\ell(n)}}^{\ell, M} - \phi_{\lambda_{\ell(n)}}^\ell \right) \right\|_{\dot{H}^s(G)} \xrightarrow{M \rightarrow +\infty} 0.$$

Now combining (4.4) and the norm invariance (4.2), for all M and $L \leq \nu(M)$ fixed, one has :

$$\begin{aligned} \left\| \sum_{\ell=1}^L \sum_{m \in E(\ell, M)} (d_{m,n} - d_m) \psi_{\lambda(m,n)} \right\|_{\dot{H}^s(G)} &\leq D \left\| \sum_{m=1}^M (d_{m,n} - d_m) \psi_{\lambda(m,n)} \right\|_{\dot{H}^s(G)} \\ &\leq D \sum_{m=1}^M |d_{m,n} - d_m| \|\psi\|_{\dot{H}^s(G)}. \end{aligned}$$

Consequently :

$$\forall L, M > 0, \left\| \sum_{\ell=1}^L \sum_{m \in E(\ell, M)} (d_{m,n} - d_m) \psi_{\lambda(m,n)} \right\|_{\dot{H}^s(G)} \xrightarrow{n \rightarrow +\infty} 0.$$

So we get :

$$\forall L > 0, \overline{\lim}_{n \rightarrow +\infty} \|r_1(n, L, M)\|_{\dot{H}^s(G)} \xrightarrow{M \rightarrow +\infty} 0,$$

which in view of (1.5) ensures that the same holds for $\|r_1(n, L, M)\|_{L^p(G)}$.

Moreover, the term $r_2(n, L, M)$ can be viewed as the partial sum $\sum_{\ell \geq L+1} d_{m,n} \psi_{\lambda(m,n)}$. So :

$$\|r_2(n, L, M)\|_{L^p(G)} \leq D \left\| \sum_{\ell \geq L+1} d_{m,n} \psi_{\lambda(m,n)} \right\|_{L^p(G)},$$

and by (4.7), its convergence to 0 when $L \rightarrow +\infty$ is assured. Since $M \geq L$, one obtains :

$$\lim_{L \rightarrow +\infty} \sup_{n > 0} \|r_2(n, L, M)\|_{L^p(G)} = 0.$$

Lastly, the property :

$$\begin{aligned} \|u_n\|_{\dot{H}^s(G)}^2 &= \sum_{\ell=1}^L \|\phi^\ell\|_{\dot{H}^s(G)}^2 + \|r_{n,L}\|_{\dot{H}^s(G)}^2 + o(1) \text{ as } n \rightarrow +\infty, \\ \text{with } \lim_{L \rightarrow +\infty} \overline{\lim}_{n \rightarrow +\infty} \|r_{n,L}\|_{L^p(G)} &= 0, \end{aligned}$$

follows as a corollary of the wavelets' mutual orthogonality and their well-defined L^2 -normalization in (1.8). That eventually concludes Theorem 1.4's proof.

Remark 4.1. *As a final observation, note that due to multiple extractions of subsequences - and the use of the underlying axiom of choice (AC), it turns out that the profile decomposition may yet not be unique.*

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