Interconnecting a System Having a Single Input-to-State Gain With a System Having a Region-Dependent Input-to-State Gain

Humberto Stein Shiromoto, Vincent Andrieu, Christophe Prieur

To cite this version:

Humberto Stein Shiromoto, Vincent Andrieu, Christophe Prieur. Interconnecting a System Having a Single Input-to-State Gain With a System Having a Region-Dependent Input-to-State Gain. CDC 2013 - 52nd IEEE Conference on Decision and Control, Dec 2013, Florence, Italy. pp.n/c. hal-00924633

HAL Id: hal-00924633
https://hal.science/hal-00924633
Submitted on 7 Jan 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Interconnecting a System Having a Single Input-to-State Gain With a System Having a Region-Dependent Input-to-State Gain

Humberto Stei Shiromoto, Vincent Andrieu, Christophe Prieur

Abstract— For an ISS system, by analyzing local and non-local properties, it is obtained different input-to-state gains. The interconnection of a system having two input-to-state gains with a system having a single ISS gain is analyzed. By employing the Small Gain Theorem for the local (resp. non-local) gains composition, it is concluded about the local (resp. global) stability of the origin (resp. of a compact set). Additionally, if the region of local stability of the origin strictly includes the region attraction of the compact set, then it is shown that the origin is globally asymptotically stable. An example illustrates the approach.

I. INTRODUCTION

The use of nonlinear input-output gains for the study of the stability of nonlinear was introduced in [22], [23] by considering a system as an input-output operator. The condition that ensures stability, called Small Gain Theorem, of the resulting interconnected system is based on the contraction principle ([23]).

The works [15] and [16] introduce a new concept of gain relating the input to system states. This notion of stability, called Input-to-State Stability (ISS), combines Zames and Lyapunov approaches ([18], [19]). Characterizations in terms of dissipation and Lyapunov functions are given in [20] and [21].

In [11], the contraction principle is used in the ISS notion to obtain an equivalent Small Gain Theorem. A formulation of this criteria in terms of Lyapunov functions may be found in [10] and [12].

Besides stability analysis, the Small Gain Theorem may also be used for the design of dynamic feedback laws in [10] and [12]. See also [1], [8] and [9] for the interconnection of possibly non-ISS systems.

In order to apply the Small Gain Theorem, it is required that the composition of the nonlinear gains is smaller than the argument for all of its positive values ([10], [12]). Such a condition, called Small Gain Condition, restricts the application of the Small Gain Theorem to a composition of well chosen gains.

In this work, it is made use of the Small Gain Theorem in a less conservative way. This new condition ensures the asymptotic stability of a system by showing that if there exist two different gains compositions such that they satisfy the Small Gain Condition, not for all values of the arguments, but in two different regions, and if these regions cover the set of all positive values, then the resulting interconnected system is globally asymptotically stable. Thus, this approach may be seen as a composition of two different small gain conditions that hold in different regions: a local and a global.

The use of a unifying approach is well known in the literature, see [1] for the combination of control Lyapunov functions and [4] for a stability concept uniting ISS and the integral variant of ISS (namely, iISS [17]) properties.

This paper is organized as follows. In Section II, the basic concepts of Input-to-State Stability and the Dini derivative are presented. Also, the system under consideration, the problem statement and a motivational example are presented. In Section III, the assumptions to solve the problem under consideration, as well as the main results are presented. Section IV presents an example that illustrates the assumptions and main results. Section V contains the proofs of the main results. Concluding remarks are given in Section VI. Finally, in Section VII, auxiliary results are stated. Due to space limitations, some of the proofs were omitted.

Notation. Let $S$ be a subset of $\mathbb{R}^n$ containing the origin, the notation $\mathbb{S}_{\neq 0}$ stands for $S \setminus \{0\}$.

II. BACKGROUND AND PROBLEM STATEMENT

Consider the system

$$\dot{x}(t) = f(x(t), u(t)),$$  

where, $\forall t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, for some positive integers $n$ and $m$. The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be continuous, locally Lipschitz on $x$ and uniformly in $u$ on compact sets. A solution of (1) with initial condition $x$, and input $u$ at time $t$ is denoted by $X(t, x, u)$.

Assume that the origin is an equilibrium point for the system (1), i.e., $f(0, 0) = 0$.

Definition 1. Consider the function $\xi : [a, b) \to \mathbb{R}$, the limit

$$D^+_\tau \xi(t) : = \limsup_{\tau \searrow 0} \frac{\xi(t+\tau)-\xi(t)}{\tau}$$

The authors are with 1 Control Systems Department, GIPSA-lab, Grenoble, France; 2 LAGEP, Université de Lyon 1, France. E-mail: humberto.shiromoto@gmail.com. The works of the first and the third authors are partly supported by HYCON2 Network of Excellence Highly-Complex and Networked Control Systems, grant agreement 257462.
(if it exists) is called Dini derivative. Let \( k \) be a positive integer. Consider the functions \( \varphi : \mathbb{R}^k \to \mathbb{R} \) and \( h : \mathbb{R}^k \to \mathbb{R}^k \), the limit
\[
D^+_h \varphi(y) = \limsup_{\tau \to 0} \frac{\varphi(y + \tau h(y)) - \varphi(y)}{\tau},
\]
(if it exists) is called Dini derivative of \( \varphi \) in the \( h \)-direction at \( y \).

**Definition 2.** A continuous locally Lipschitz function \( V : \mathbb{R}^n \to \mathbb{R} \) is called an ISS-Lyapunov function for system (1) if
1. there exist class \( K_\infty \) functions \( \alpha \) and \( \overline{\alpha} \) such that, \( \forall x \in \mathbb{R}^n \),
   \[\alpha(|x|) \leq V(x) \leq \overline{\alpha}(|x|);\]
2. there exist a class \( K \) function \( \alpha_x \), called ISS gain, and a continuous positive definite function \( \lambda_x : \mathbb{R}^n \to \mathbb{R} \) such that,
   \[\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, |x| \geq \alpha_x(|u|) \Rightarrow D^+_x V(x, u) \leq -\lambda_x(x) \]
holds.

From now on, \( V \) will be assumed to be an ISS-Lyapunov function for (1).

Consider the system
\[
\dot{z}(t) = g(v(t), z(t)),
\]
where, \( \forall t \in \mathbb{R}_{\geq 0}, v(t) \in \mathbb{R}^n \) and \( z(t) \in \mathbb{R}^m \), for some positive integers \( n \) and \( m \). The map \( g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m \) is assumed to be continuous, locally Lipschitz on \( z \) and uniformly in \( v \) on compact sets. A solution of (3) with initial condition \( z \), and input \( v \) at time \( t \) is denoted by \( Z(t, z, v) \). Assume that the origin is an equilibrium point for the system (3), i.e., \( g(0, 0) = 0 \). Consider also the following

**Assumption 1.** There exists a continuous locally Lipschitz function \( W : \mathbb{R}^m \to \mathbb{R} \) that is an ISS-Lyapunov function for the \( z \)-subsystem. More precisely, there exist class \( K_\infty \) functions \( \beta \) and \( \overline{\beta} \) satisfying, \( \forall z \in \mathbb{R}^m, \beta(|z|) \leq W(z) \leq \overline{\beta}(|z|) \). Furthermore, there exist a class \( K \) function \( \delta \) and a continuous positive definite function \( \lambda_z : \mathbb{R}^m \to \mathbb{R} \) such that,
\[\forall (x, z) \in \mathbb{R}^n \times \mathbb{R}^m, W(z) \geq \delta(V(x)) \Rightarrow D^+_x W(x, z) \leq -\lambda_z(x),\]
whose \( V \) is the ISS-Lyapunov function of \( x \)-subsystem.

**System under consideration.** Interconnecting systems (1) and (3) by linking the state of the (1) with the input of (3) and vice versa leads to the following system
\[
\begin{cases}
\dot{x} = f(x, z) \\
\dot{z} = g(x, z).
\end{cases}
\]
(5)

Since \( f(0, 0) = 0 \) and \( g(0, 0) = 0 \), the origin is an equilibrium point for (5). Considering the ISS-Lyapunov inequalities, after the interconnection the following implications
\[V(x) \geq \gamma(W(z)) \Rightarrow D^+_f V(x, z) \leq -\lambda_x(x),\]
\[W(z) \geq \delta(V(x)) \Rightarrow D^+_g W(x, z) \leq -\lambda_z(x)\]
are obtained with suitable class \( K \) functions \( \gamma \) and \( \delta \).

A sufficient condition that ensures stability of (5) is given by the following

**Theorem 1.** [10] If
\[
\forall s \in \mathbb{R}_{>0}, \quad \gamma \circ \delta(s) < s.
\]
(6)

Then, the origin is globally asymptotically stable for (5).

**Problem statement.** At this point, it is possible to explain the problems that are dealt with, in this work.

* ISS gains computation. Although the use of ISS gains renders the analysis of stability easy to work with, it is not a trivial task to compute those gains;
* Small gain condition. Since the ISS gain is not unique, it might not be an easy task to find two ISS gains: one for the \( x \)-subsystem of (5) and another for the \( z \)-subsystem of (5) such that their composition satisfies (6), for all positive values of the argument. An illustration of the problem that is dealt with is presented in the following

**Example 1.** Let the functions \( f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and consider the system
\[
\begin{cases}
\dot{x} = f(x, z) = -\rho(x) + z \\
\dot{z} = g(x, z) = -\text{sign}(z) \delta(|z|) + x,
\end{cases}
\]
(7)
where \( \delta \) will be defined below. Let, \( \forall x \in \mathbb{R}, V(x) = |x|, \rho(x) = 5x^4 - 2x^2 + x^3 \) and, \( \forall z \in \mathbb{R}, W(z) = |z| \).

Taking the Dini derivative of \( V \) in the \( f \)-direction, \( \forall (x, z) \in \mathbb{R} \times \mathbb{R} \), it yields
\[D^+_f V(x, z) \leq -\rho(V(x)) + W(z).
\]
(8)
This implies that \( \exists \varepsilon_x \in (0, 1) \) such that, \( \forall (x, z) \in \mathbb{R} \times \mathbb{R}, \rho(V(x)) \geq \frac{\varepsilon_x |V(x)|}{1 + |x|} \Rightarrow D^+_f V(x, z) \leq -\lambda_x(x) \)
(9)
where \( \lambda_x(\cdot) = \varepsilon_x \rho(V(\cdot)). \) From now on, let \( \varepsilon_x = 0.05 \).

Note that, in the interval \([1/2, 5/6], \rho \) is decreasing.

Consider the piecewise continuous function \( \Gamma \) defined by
\[\Gamma(s) = \begin{cases}
\rho^{-1}(\frac{s}{5}) & \text{if } s \in [0, 0.95 \rho(\frac{5}{6})], \\
\rho_{+}^{-1}(\frac{s}{5}) & \text{if } s \in (0.95 \rho(\frac{5}{6}), \infty),
\end{cases}
\]
(10)
where \([5/6, \infty) \ni s \Rightarrow \rho(s) = \rho(s) \in [\rho(5/6), \infty)\).

**Remark 1.** The function \( \Gamma \) can be viewed as a discontinuous input-to-state gain of the \( z \)-subsystem of (7). More precisely, \( \forall (x, z) \in \mathbb{R} \times \mathbb{R}, V(x) \geq \Gamma(W(z)) \Rightarrow D^+_f V(x, z) \leq -\lambda_z(x) \).
Furthermore, the function \( \Gamma \) is “optimal”, in the sense that if there exist a function \( \Gamma^* : \mathbb{R} \to \mathbb{R} \) and a value \( s^* \in \mathbb{R}_{>0} \) such that \( \Gamma^*(s^*) < \Gamma(s) \), then \( \exists (x^*, z^*) \neq (0, 0) \) such that \( V(x^*) \geq \Gamma^*(W(z^*)) \) and \( D^+_f V(x^*, z^*) > 0 \).

It follows from Remark 1 that an ISS gain for the \( x \)-subsystem of (7) is any class \( K \) function \( \gamma \) such that, \( \forall s \in \mathbb{R}_{>0}, \Gamma(s) \leq \gamma(s) \).

A local gain. Consider the function \( [0, 1/2] \ni s \Rightarrow \rho_{-}(s) = \rho(s) \in [0, \rho(1/2)]. \) Since \( \rho_{-} \) is strictly increasing on its domain, it is invertible. Let \( \gamma_{\epsilon} \) be a class \( K \) function such that, \( \forall s \in [0, 0.95 \rho(1/2)], \gamma_{\epsilon}(s) = \rho_{-}^{-1}\left( \frac{s}{0.95} \right) \).

(11)
Note that, \( \gamma_{\epsilon} \) satisfies the following inequalities
\[\forall s \in [0, 0.95 \rho(\frac{1}{2})], \gamma_{\epsilon}(s) \leq \Gamma(s),
\]
\[\forall s \in (0.95 \rho(\frac{1}{2}), 0.95 \rho(\frac{5}{6})), \gamma_{\epsilon}(s) < \Gamma(s).
\]
Moreover, \( \forall (x, z) \in \Omega(1/2) \times \mathbb{R} \),
\[V(x) \geq \gamma_{\epsilon}(W(z)) \Rightarrow D^+_f V(x, z) \leq -\lambda_z(x) \).
(12)
Let the constant values \( M_\rho = 0.236 \) and \( M_\delta = 0.245 \).
At this point, it is possible to define the function \( \delta \) of the \( z \)-subsystem of (7). It is a function of class \( K_{\infty} \) satisfying the following inequalities
\[\forall s \in [0, M_\rho], \gamma_{\epsilon}(s) < \delta(s),
\]
(13)
\[\forall s \in [M_\rho, \infty), \Gamma(s) < \delta(s),
\]
(14)
\[\forall s \in (\rho(\frac{1}{2}), M_\rho, \delta), \delta(s) < \Gamma(s).
\]
(15)
Equations (13) and (14) correspond to two different small gain conditions, the first may be seen as a small gain condition for small values of the argument while the last as a small gain condition for large values of the argument. Note that (15) implies that Theorem 1 cannot be applied.\footnote{To see this fact, note that $M_t < 0.95\rho(\frac{1}{2})$. Since $\delta$ is of class $K_{\infty}$ and from (15), \(\forall s \in (0.95\rho(\frac{5}{6}), M_t), s < \delta^{-1} \circ \Gamma(s)\). Thus, there exists no class $K$ function $\gamma$ such that (6) holds.}

Fig. 1 shows a plot of the functions $\rho$, $\text{id}$, $\Gamma$, $\gamma_\ell$ and $\delta$. \bullet

In this work, it will be shown that, if

\begin{itemize}
  \item there exist two ISS gains $\gamma_\ell$ and $\gamma_g$, for the $x$-subsystem of (5);
  \item there exists one ISS gain $\delta$, for the $z$-subsystem of (5);
  \item the compositions $\gamma_\ell \circ \delta$ and $\gamma_g \circ \delta$ satisfy the Small Gain Condition, not for all values of the arguments, but for two different intervals ($I_1, I_2 \subset \mathbb{R}_{\geq 0}$). In other words, $\forall s \in I_1 \setminus \{0\}, \gamma_\ell \circ \delta(s) < s$ and $\forall s \in I_2 \setminus \{0\}, \gamma_g \circ \delta(s) < s$;
  \item these intervals are such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \mathbb{R}_{\geq 0}$; then, the origin is globally asymptotically stable for (5). See Theorem 2 below for a precise statement of this result.
\end{itemize}

III. ASSUMPTIONS AND MAIN RESULTS

In this section, it is specified the assumptions on the system (5) necessary to solve the problem under consideration. The proof of the stabilization results are provided from Section V-A to Section V-C.

A. Local set of assumptions on the $x$-subsystem

In this section, it is introduced the set of assumptions to ensure that the origin is locally asymptotically stable for (5).

Assumption 2. There exist a class $K$ function $\gamma_\ell$ and a strictly positive constant $M_\ell$ such that

$$M_\ell < \lim_{s \rightarrow \infty} \gamma_\ell(s) = b_\ell.$$ \hspace{1cm} (16)

Moreover, $\forall (x, z) \in \Omega_{M_\ell}(V) \times \mathbb{R}^m$, $V(x) \geq \gamma_\ell(W(z)) \Rightarrow D_x^+ V(x, z) \leq -\lambda_x(x)$. \hspace{1cm} (17)

Assumption 3. The composition of the functions $\gamma_\ell$ and $\delta$ is such that

$$\forall s \in [0, M_\ell], \quad \gamma_\ell \circ \delta(s) < s.$$ \hspace{1cm} (18)

Proposition 1. Under Assumptions 1, 2 and 3 the origin is locally asymptotically stable for system (5).

B. Non-local set of assumptions on the $x$-subsystem

In this section, it is introduced the set of assumptions to ensure that a neighborhood of the origin is globally attractive for (5).

Assumption 4. There exist a class $K$ function $\gamma_g$ and a strictly positive constant $M_g$ such that

$$M_g < \lim_{s \rightarrow \infty} \gamma_g(s) = b_g.$$ \hspace{1cm} (19)

Moreover, $\forall (x, z) \in (\mathbb{R}^n \setminus \Omega_{M_g}(V)) \times \mathbb{R}^m$, $V(x) \geq \gamma_g(W(z)) \Rightarrow D_x^+ V(x, z) \leq -\lambda_x(x)$. \hspace{1cm} (20)

Assumption 5. The composition of the functions $\gamma_\ell$ and $\delta$ is such that

$$\forall s \in [M_g, \infty), \quad \gamma_g \circ \delta(s) < s.$$ \hspace{1cm} (21)

Proposition 2. Under Assumptions 1, 4 and 5, there exist a proper definite positive function $U_g$ and a positive constant $M_g$ such that the set $\Omega_{M_g}(U_g)$ is globally asymptotically stable for system (5).

C. Main result

In this section, it is introduced the assumption to ensure that the origin is globally asymptotically stable for (5).

Assumption 6. The positive constants $M_\ell$ and $M_g$ given, respectively, by Assumptions 2 and 4 satisfy $M_g < M_\ell$. \hspace{1cm} \Box

Theorem 2. Under Assumptions 1-6, the origin is globally asymptotically stable for system (5).

IV. ILLUSTRATION

Example 2. [Example 1 revisited.]

Verifying Assumption 1. Let the function $\delta$ be given by the inverse of $\varphi$. It follows that, $\forall (x, z) \in \mathbb{R} \times \mathbb{R}, W(z) \geq \delta(V(x)) \Rightarrow D_x^+ W(z) \leq -\lambda_z(z)$, for a given $\varepsilon_z \in (0, 1)$ and $\forall z \in \mathbb{R}$, $\lambda_z(z) = \varepsilon_z W(z)$. Thus, Assumption 1 holds.

Verifying Assumption 2. The function $\gamma_\ell$ is given by (11) and $M_\ell = 0.236$. Moreover, it follows from (12) that Assumption 2 holds.

Verifying Assumption 3. It follows from inequality (13) that Assumption 3 holds.

From Proposition 1, it follows that the origin is locally asymptotically stable for (7). Figure 2 shows some solutions of (7). \bullet

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Plot of the functions $\rho$, $\text{id}$, $\Gamma$, $\gamma_\ell$ and $\delta$.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Solutions to (7) for initial condition starting at a ball centered at the origin with radius, respectively, given by $M_\ell$ and 5.}
\end{figure}

Example 3. [Example 1 revisited.]

Verifying Assumption 4. Let a class $K_{\infty}$ function $\gamma_g$ be such that, $\forall s \in [0.95\rho(\frac{5}{6}), \infty), \gamma_g(s) = \Gamma(s)$. Moreover,
\[ M_0 = 0.245. \] It follows from Remark 1 that, \( \forall (x, z) \in (\mathbb{R} \setminus \Omega_{M_t}(V)) \times \mathbb{R}, \ V(x) \geq \gamma_d(W(z)) \Rightarrow D^+_f V(x, z) \leq -\lambda_z(x). \) Thus, Assumption 4 holds.

Verifying Assumption 5. It follows from inequality (14) that Assumption 5 holds.

From Proposition 2, it follows that a neighborhood of the origin is globally asymptotically stable for (7). Figure 2 shows some solutions of (7).

\[ \bullet \]

V. PROOFS

A. Proof of Proposition 1

Proof. This proof is divided into three parts. In the first one, it is obtained a function \( \sigma_\ell \) that is class \( \mathcal{K}_\infty \) and \( \mathcal{C}^1 \) with strictly positive derivative. This function is used in the second part, where a class \( \mathcal{C}^0 \) proper positive definite function \( U_\ell \) is defined and its Dini derivative is studied. In the last part, it is shown that \( U_\ell \) is locally Lipschitz and the local asymptotical stability of the origin is concluded by using Lemma 2.

First part. Consider the class \( \mathcal{K} \) functions \( \delta \) and \( \gamma_\ell \) from Assumptions 1 and 2. Under Assumption 3, \( \delta \) and \( \gamma_\ell \) are such that, \( \forall s \in (0, M_\ell), \delta(s) < \gamma_\ell^{-1}(s) \).

Since \( \gamma_\ell \) is class of \( \mathcal{K}_\infty \), from Lemma 1, there exists a class \( \mathcal{K}_\infty \) function \( \gamma_\ell(s) \) such that, \( \forall s \in \mathbb{R}_{>0}, \delta(s) < \gamma_\ell^{-1}(s) \). (22) and, \( \forall s \in (0, M_\ell), \)

\[ \gamma_\ell(s) < \gamma_\ell^{-1}(s). \] (23)

Since \( \delta \) is of class \( \mathcal{K} \) and \( \gamma_\ell \) is of class \( \mathcal{K}_\infty \) satisfying, \( \forall s \in \mathbb{R}_{>0}, \) inequality (22), from Lemma 10, Lemma A.1, there exists a class \( \mathcal{K}_\infty \) and \( \mathcal{C}^1 \) function \( \sigma_\ell \) whose derivative is strictly positive and satisfies, \( \forall s \in \mathbb{R}_{>0}, \)

\[ \delta(s) < \sigma_\ell(s) < \gamma_\ell^{-1}(s). \] (24)

Second part. Let, \( \forall (x, z) \in \mathbb{R}^n \times \mathbb{R}^m, \ U_\ell(x, z) := \max\{\sigma_\ell(V(x)), W(z)\}. \) Note that the function \( U_\ell \) is proper positive definite. Pick \( x, z \in \mathbb{R}^n \times \mathbb{R}^m \), one of three cases is possible: \( \sigma_\ell(V(x)) < W(z), W(z) < \sigma_\ell(V(x)), \) or \( W(z) = \sigma_\ell(V(x)). \) The proof follows by showing that the Dini derivative of \( U_\ell \) is negative. For each case, assume that \( (x, z) \in \overline{\Omega_{M_t}(V)} \times \mathbb{R}^m. \)

Case 1. Suppose that \( \sigma_\ell(V(x)) < W(z) \). This implies that \( U_\ell(x, z) = W(z) \) and \( D^+_g U_\ell(x, z) = \sigma_\ell'(V(x)) D^+_f V(x, z). \) From (24), the following inequality \( \delta(V(x)) < \sigma_\ell'(V(x)) D^+_f V(x, z) \) holds. Together with (4), it follows that \( D^+_g W(z) \leq -\lambda_z(x). \) This concludes Case 1.

Case 2. Suppose that \( W(z) < \sigma_\ell(V(x)) \). This implies that \( U_\ell(x, z) = \sigma_\ell(V(x)) \) and \( D^+_g U_\ell(x, z) = \sigma_\ell'(V(x)) D^+_f V(x, z). \) From (24), the following inequality \( W(z) < \sigma_\ell(V(x)) \) holds. Since \( V(x) \leq M_\ell \), it follows that

\[ W(z) < \sigma_\ell(V(x)) < \gamma_\ell^{-1}(V(x)), \] (25)

where the last inequality follows from (23). Equation (17) together with (25) yields \( D^+_f V(x, z) \leq -\lambda_z(x). \)

Since, \( \forall s \in \mathbb{R}_{>0}, s_\ell(s) > 0, \) it follows that

\[ D^+_g U_\ell(x, z) = \sigma_\ell'(V(x)) D^+_f V(x, z) \leq -s_\ell'(V(x)) \lambda_z(x). \]

This concludes Case 2.

Case 3. Let \( W(z) = \sigma_\ell(V(x)) := U_\ell^+(x, z). \) This implies

\[ D^+_g U_\ell^+(x, z) = \limsup_{\tau \to 0} \frac{1}{\tau} \{\max\{\sigma_\ell(V(X(x, z, t))), W(Z(x, x, z, t)) - U_\ell^+(x, z)\} \} \]

\[ = \limsup_{\tau \to 0} \max \left\{ \sigma_\ell(V(X(x, z, t))) - \sigma_\ell(V(x)), \frac{W(Z(x, x, z, t)) - W(z)}{\tau} \right\} \]

\[ = \max\{\sigma_\ell'(V(x)) D^+_f V(x, z), D^+_g W(z)(x)\}. \]

The analysis of \( D^+_g U_\ell^+ \) is divided in two sub cases. In the first one, the function \( D^+_g W \) is analyzed while in the last, the function \( D^+_f V \) is analyzed.

Case 3.a. The analysis of \( D^+_g W \). From (24), the following inequality \( \delta(V(x)) < \sigma_\ell(V(x)) = W(z) \) holds. Together with Equation (4), it yields \( D^+_g W(z, x) \leq -\lambda_z(x). \)

Case 3.b. The analysis of \( D^+_f V \). From (24), the following inequality \( W(z) = \sigma_\ell(V(x)) < \gamma_\ell^{-1}(V(x)) \) holds. Since \( V(x) \leq M_\ell \), it follows that

\[ W(z) = \sigma_\ell(V(x)) < \gamma_\ell^{-1}(V(x)), \] (26)

where the last inequality is due to (23).

Equation (17) together with (26) yields \( D^+_f V(x, z) \leq -\lambda_z(x). \)

To conclude Case 3, \( W(z) = \sigma_\ell(V(x)) \Rightarrow D^+_g U_\ell^+(x, z) \leq -\min\{\sigma_\ell(V(x)), \lambda_z(x)\} \). Since \( (x, z) \in \overline{\Omega_{M_t}(V)} \times \mathbb{R}^m. \)

Let \( \tilde{\ell} := \max\{c \in \mathbb{R}_{>0} : \overline{\Omega_{c}(U_\ell)} \subset \overline{\Omega_{M_t}(V)} \times \{0\} \} \) and \( \overline{\mathcal{L}_\ell(U_\ell)} \) is connected. To sum up all the above cases, \( \forall (x, z) \in \overline{\Omega_{\tilde{\ell}}(U_\ell)}. \)

\[ U_\ell(x, z) \leq M_\ell \Rightarrow D^+_g U_\ell(x, z) \leq -E_\ell(x, z), \] (27)

where \( E_\ell(\cdot, \cdot) := \min\{\sigma_\ell(V(\cdot)), \lambda_z(\cdot)\} \) is continuous and positive definite.

Third part. To conclude local asymptotical stability of the origin, it remains to show that \( U_\ell \) is locally Lipschitz. Since \( \sigma_\ell(V(\cdot)) \) is locally Lipschitz, \( U_\ell \) is locally Lipschitz in the region \( W(\cdot) \leq \sigma_\ell(V(\cdot)) \). Since the hypotheses of Lemma 2 (in Section VII) below are verified with \( U(\cdot) = U_\ell(\cdot) \) and \( E(\cdot) = E_\ell(\cdot) \), the origin is locally asymptotically stable for (5). This concludes the proof of Proposition 1.

B. Proof of Proposition 2

Proof. This proof is analogous to the proof of Proposition 1 and divided into three parts. In the first one, it is obtained a function \( \sigma_\ell \) that is class \( \mathcal{K}_\infty \) and \( \mathcal{C}^1 \) with strictly positive derivative. This function is used in the second part, where a class \( \mathcal{C}^0 \) proper and positive definite function \( U_\ell \) is defined and its Dini derivative is studied. In the last part, it is used Lemma 4 to show that the set \( \Omega_{M_\ell}(V) \times \{0\} \) is globally asymptotically stable.

First part. Consider the class \( \mathcal{K} \) functions \( \delta \) and \( \gamma_\ell \) from Assumptions 1 and 4. The function \( \gamma_\ell^{-1} \) is defined on \( [0, b_g) \) and satisfies \( \lim s_n \gamma_\ell^{-1}(s) = \infty. \) Assumption 5 implies that, \( \forall s \in [M_g, b_g], \delta(s) < \gamma_\ell^{-1}(s). \) Since \( \gamma_\ell \) is of class \( \mathcal{K}, \) from Lemma 1 (in Section VII), there exists a class \( \mathcal{K}_\infty \) function \( \tilde{\gamma}_g \) such that, \( \forall s \in \mathbb{R}_{>0}, \)

\[ \tilde{\gamma}_g(s) < \gamma_\ell^{-1}(s). \] (28)

and, \( \forall s \in [M_g, b_g], \)

\[ \delta(s) < \tilde{\gamma}_g(s). \] (29)
Suppose that $\Omega$ is positive definite. Pick $(x, z) \in (\mathbb{R}^n \times \mathbb{R}^m)$, one of three cases is possible: $\sigma_g(V(x)) < W(z)$, $W(z) < \sigma_g(V(x))$ or $W(z) = \sigma_g(V(x))$. The proof follows by showing that the Dini derivative of $U_g$ is negative. For each case, assume that $(x, z) \in (\mathbb{R}^n \setminus \Omega_M)^g(V) \times \mathbb{R}^m$.

**Case 1.** Suppose that $\sigma_g(V(x)) < W(z)$. Analogously to the Case 1 of proof of Proposition 1, $\sigma_g(V(x)) < W(z)$ implies that $U_g(x, z) = \sigma_g(V(x)) - D^T U_g(x, z)$. From (30), the following inequality holds, since $\forall D^T U_g(x, z)$, it will be proven that $\sigma_g(V(x)) < W(z)$.

$$W(z) < \sigma_g(V(x)) < \gamma^{-1}(V(x)).$$

Equation (20) together with (32) yields $-\lambda_x(x)$. This concludes Case 1.

**Case 2.** Suppose that $W(z) < \sigma_g(V(x))$. This implies that $U_g(x, z) = \sigma_g(V(x)) - D^T U_g(x, z)$, and $\sigma_g(V(x)) - D^T U_g(x, z) < -\lambda_x(x)$. Since, $\forall s \in \mathbb{R}^n$, $\sigma_x^s(s) > 0$, it follows that $D^T U_g(x, z) = \sigma_g(V(x)) - D^T U_g(x, z) < -\lambda_x(x)$. This concludes Case 2.

**Case 3.** Let $W(z) = \sigma_g(V(x)) := U^g_{st}$. Analogously to the Case 3 of proof Proposition 1 and together with the analysis of Cases 1 and 2, the implication $W(z) = \sigma_g(V(x)) \Rightarrow D^T U_g(x, z) = -\min\{\sigma_x^s(s) \lambda_x(z), \lambda_x(z)\}$ holds, since $(x, z) \in (\mathbb{R}^n \times \Omega_M)^g(V) \times \mathbb{R}^m$.

Let $M_g = \min\{c \in \mathbb{R}^n : \Omega_M(V) \times \{0\} \subset \Omega_{c}(U_g)$$ and $$\Omega_{\gamma}(U_g)$$ is connected\}$. To sum up above the cases, $\forall(x, z) \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \Omega_{M_g}^g(U_g)$, $M_g < U_g(x, z) \Rightarrow D^T U_g(x, z) < -E_g(x, z)$.

$$E_g(x, z) = \min\{\sigma_x^s(V(x)) \lambda_x(z), \lambda_x(z)\}$$ is continuous and positive definite.

**Third part.** Analogously to the third part of the proof of Proposition 1, it follows that $U_g$ is locally Lipschitz. From Lemma 3 and (33), it follows that, $\forall(x, z) \in \mathbb{R}^n \times \mathbb{R}^m$ and $\forall t \in \mathbb{R}^n$, along solutions of (5), $D^T U_g(X(t, x, z), Z(t, x), x) = D^T U_g(X(t, x, z), Z(t, x, z))$.

Since solutions of (5) are absolutely continuous functions and, along solutions of (5), $E_g$ is a continuous positive definite function, from Lemma 4, $\forall(x, z) \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \Omega_{M_g}^g(U_g)$ and $\forall t \in \mathbb{R}^n$, the function $t \mapsto U_g(X(t, x, z), Z(t, x, z))$ is strictly decreasing. Pick $(x, z) \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \Omega_{M_g}^g(U_g)$, it will be proven that $U_g := \lim_{t \to \infty} U_g(X(t, x, z), Z(t, x, z)) \leq M_g$.

To see the above suppose, by contradiction, that $U_g > M_g$.

From the continuity of $U_g$, $\exists \tau > 0$ such that $U_g > \epsilon \geq M_g$ and $U_g - \epsilon \leq U_g(x, z) < U_g + \epsilon$. Since $U_g$ is proper, the constant $\xi = \min\{E_g(x, z) > 0 : (x, z) \in U_g(x, z) \setminus U_g(x, z) \leq U_g(x, z) + \epsilon$ exists. Recalling the definition of $U_g$, $\exists T > 0$ such that, $\forall t \geq T$, $U_g(X(t, x, z), Z(t, x, z)) - U_g < \epsilon$. Moreover, from the definition of the constant $\xi$, $U_g(X(t, x, z), Z(t, x, z)) - U_g(X(T, x, z), Z(T, x, z)) = \int_0^T D^T U_g(X(s, x, z), Z(s, x, z)) ds \leq -\xi \cdot (t - T)$.

Then, $U_g = \lim_{t \to \infty} U_g(X(t, x, z), Z(t, x, z)) = U_g(X(T, x, z), Z(T, x, z)) + \lim_{t \to \infty} \int_0^T D^T U_g(X(s, x, z), Z(s, x, z)) ds \leq -\infty$

which contradicts the fact that $U_g$ is proper positive definite. Therefore, $U_g \leq M_g$.

In summary, the following facts hold for the function $U_g$: 1) $U_g$ is a proper positive definite function; 2) $U_g$ decreases along solutions of (5) having initial conditions in $(\mathbb{R}^n \times \mathbb{R}^m) \setminus \Omega_{M_g}^g(U_g)$. From facts 1) and 2), the set $\Omega_{M_g}^g(U_g)$ is globally asymptotically stable for (5). This concludes the proof of Proposition 2.
From the above, combining the local asymptotical stability of the origin with its global attraction it is concluded that the origin is globally asymptotically stable for (5).  

VI. CONCLUSION AND PERSPECTIVES

In this work, the authors shown that it is possible to make use of local and non-local input-to-state properties of an ISS system, in order to derive an “optimal” ISS gain. As a result of such approach, it is possible to apply the Small Gain Theorem in a less conservative way by deriving local and non-local small gain conditions to ensure the stability of an interconnected system.

In a future work, the authors will generalize the above results for the case in which there exist four ISS gains: two for each subsystem. Moreover, the authors also intend to use the region-dependent gain condition to develop a methodology for the design of feedback laws under different gains constraints.

VII. AUXILIARY RESULTS

**Lemma 1.** Let $\beta$ be a class $K$ function with
\[ b = \lim_{s \to 0} \beta(s). \]  
(36)

Let also $p, q$ be two constants and $\alpha$ be a class $K$ function such that, $0 < p < q$ and, $\forall s \in [p, q],$
\[ \beta \circ \alpha(s) < s. \]  
(37)

Then, the class $K_{\alpha}$ function given by
\[ \gamma(s) = \max\{s, \beta^{-1}(s)\}, \]  
(38)

\[ \beta^{-1}(s) < \gamma(s), \]  
(39)

\[ \gamma(s) < s < \beta^{-1}(s), \]  
(40)

Due to space constraints, the proof of Lemma 1 is not provided in this paper.

**Lemma 2.** [13, Théorème 2.133] Let $S \subset \mathbb{R}^k$ be a be a neighborhood of the origin. Let also the class $C^0$ function $h : \mathbb{R}^k \to \mathbb{R}$ and consider the system $\dot{y} = h(y)$. If there exist a positive definite and locally Lipschitz function $U : S \to \mathbb{R}$ and a positive definite function $E : S \to \mathbb{R}$ such that, $\forall y \in S, D_k^+ U(y) \leq -E(y)$. Then, the origin is locally asymptotically stable for $y = 0$.

**Lemma 3.** [13, Lemme 1.28] Let the measurable and essentially bounded function $d : \mathbb{R} \to \mathbb{R}^p$ and the class $C^0$ function $h : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}$. If $U : \mathbb{R}^k \to \mathbb{R}$ is locally Lipschitz, then, for all maximal solutions $Y(t, y, d)$ of the system $\dot{y} = h(y, d(t))$ defined in the interval $(t_-, t_+)$, the function $t \mapsto U(Y(t, y, d))$, defined over $(t_-, t_+)$, is locally Lipschitz and, for almost every $t \in (t_-, t_+)$,
\[ \frac{dU(Y)}{dt}(t, y, d) = D^+ U(Y(t, y, d)) = D_k^+ U(Y(t, y, d)). \]

Moreover, if $d$ is continuous, the above equality holds, $\forall t \in (t_-, t_+)$.  

**Lemma 4.** Let $Y : \mathbb{R} \to \mathbb{R}^k$ be an absolutely continuous function, $U : \mathbb{R}^k \to \mathbb{R}$ be a locally Lipschitz proper positive definite function and $E : \mathbb{R}^k \to \mathbb{R}$ be a continuous positive definite function. Define, $\forall t \in \mathbb{R}, U(t) = U \circ Y(t)$ and $E(t) = E \circ Y(t)$. If, $\forall t \in \mathbb{R}, D^+ U(t) \leq -E(t)$, then, $\forall t \in \mathbb{R}, U(t)$ is strictly decreasing.

Due to space constraints, the proof of Lemma 4 is not provided in this paper.

REFERENCES


