Entropy-dissipative discretization of nonlinear diffusion equations and discrete Beckner inequalities

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ENTROPY-DISSIPATIVE DISCRETIZATION OF NONLINEAR DIFFUSION EQUATIONS AND DISCRETE BECKNER INEQUALITIES

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Abstract. The time decay of fully discrete finite-volume approximations of porous-medium and fast-diffusion equations with Neumann or periodic boundary conditions is proved in the entropy sense. The algebraic or exponential decay rates are computed explicitly. In particular, the numerical scheme dissipates all zeroth-order entropies which are dissipated by the continuous equation. The proofs are based on novel continuous and discrete generalized Beckner inequalities. Furthermore, the exponential decay of some first-order entropies is proved in the continuous and discrete case using systematic integration by parts. Numerical experiments in one and two space dimensions illustrate the theoretical results and indicate that some restrictions on the parameters seem to be only technical.

1. Introduction

This paper is concerned with the time decay of fully discrete finite-volume solutions to the nonlinear diffusion equation

$$u_t = \Delta(u^\beta) \quad \text{in } \Omega,\ t>0, \quad u(\cdot,0)=u_0 \quad \text{in } \Omega, \tag{1}$$

and with the relation to discrete generalized Beckner inequalities. Here, $\beta>0$ and $\Omega\subset\mathbb{R}^d$ ($d\geq 1$) is a bounded domain. When $\beta>1$, (1) is called the porous-medium equation, describing the flow of an isentropic gas through a porous medium [43]. Equation (1) with $\beta<1$ is referred to as the fast-diffusion equation, which appears, for instance, in plasma physics with $\beta=\frac{1}{2}$ [6] or in semiconductor theory with $0<\beta<1$ [32]. We impose homogeneous Neumann boundary conditions

$$\nabla(u^\beta)\cdot\nu = 0 \quad \text{on } \partial\Omega,\ t>0, \tag{2}$$

Date: April 22, 2015.

2000 *Mathematics Subject Classification.* 65M08, 65M12, 76S05.

Key words and phrases. Porous-medium equation, fast-diffusion equation, finite-volume method, entropy dissipation, Beckner inequality, entropy construction method.

The authors have been partially supported by the Austrian-French Project Amadeé of the Austrian Exchange Service (ÖAD). The first author acknowledges support from Labex CEMPI (ANR-11-LABX-0007-01) and from the Inria-Mephysto Team. The second and last author acknowledge partial support from the Austrian Science Fund (FWF), grants P22108, P24304, I395, and W1245. Part of this work was written during the stay of the second author at the University of Technology, Munich (Germany), as a John von Neumann Professor. The second author thanks the Department of Mathematics in Munich for the hospitality and Jean Dolbeault (Paris) for very helpful discussions on Beckner inequalities.

where ν denotes the unit normal exterior vector to $\partial\Omega$, or multiperiodic boundary conditions (i.e. Ω equals the torus $\mathbb{T}^d$). Let us denote by m the Lebesgue measure in $\mathbb{R}^d$ or $\mathbb{R}^{d-1}$; we assume for simplicity that $\mathrm{m}(\Omega) = 1$.

For existence and uniqueness results for the porous-medium equation in the whole space or under suitable boundary conditions, we refer to the monograph by Vázquez [43]. There are much less results for fast-diffusion equations (see [42]), and usually they hold for the whole-space problem. In particular, we are not aware of an existence result for fast-diffusion equations in bounded domains with homogeneous Neumann boundary conditions, but such a result can be easily established since there is a maximum principle.[1]

There exist also many results on the time decay of the *continuous* porous-medium or fast-diffusion equation, with optimal decay rates or in strong norms. For instance, by using invariance principles, the sharp decay rate $t^{-1/(\beta-1)}$ in the L^∞ norm was shown in [1]. Spectral methods applied to (1) with confinement were used in [16] for $\beta \in ((d-2)/d, 1)$ and in [40] for $\beta > 1$. It seems to be difficult to "translate" these techniques to the discrete case. Sharp time-decay results in L^∞ for the solutions to the porous-medium equation with homogeneous Neumann boundary conditions were shown in [9, 25, 26], based on regular Sobolev inequalities. The connection between logarithmic Sobolev inequalities and ultracontractivity-like bounds was investigated in [9], also proving short- and large-time asymptotics. These results imply the results of this paper in the *continuous* setting (in fact, the results of [9, 25, 26] are more general) but not in the *discrete* case. Optimal convergence rates to Barenblatt self-similar profiles for the fast-diffusion equation were derived in [8], employing entropy methods and Hardy-Poincaré inequalities. However, it is unclear to what extent the mentioned techniques can be "translated" to the discrete case, partially because certain Sobolev inequalities (like Gagliardo-Nirenberg inequalities) seem to be not available. We refer to [7] for special discrete Gagliardo-Nirenberg inequalities.

In the literature, there exist many numerical schemes for nonlinear diffusion equations related to (1). Numerical techniques include (mixed) finite-element methods [2, 19, 39], finite-volume approximations [23, 38], high-order relaxation ENO-WENO schemes [14], or particle methods [36]. In these references, stability and numerical convergence properties are proved. Also the preservation of the structure of diffusion equations is a very important property of a numerical scheme. For instance, ideas employed for hyperbolic conservation laws were extended to degenerate diffusion equations, like the porous-medium equation, which may behave like hyperbolic ones in the regions of degeneracy [37]. Positivity-preserving schemes for nonlinear fourth-order equations were thoroughly investigated in the context of lubrication-type equations [4, 45] and quantum diffusion equations [31]. Entropy-consistent finite-volume finite-element schemes for the fourth-order thin-film equation were suggested by Grün and Rumpf [28]. For quantum diffusion models, an entropy-dissipative

[1]First, take strictly positive initial data u_0. By the maximum principle, any solution to the fast-diffusion equation is strictly positive. Thus, the equation is no longer singular, and the existence of weak solutions follows by a standard procedure. For nonnegative functions u_0, we take $u_0 + \varepsilon$ for $\varepsilon > 0$ as initial data, apply the first step, and pass to the limit $\varepsilon \to 0$.

relaxation-type finite-difference discretization was investigated by Carrillo et al. [12]. Furthermore, entropy-dissipative schemes for electro-reaction-diffusion systems were derived by Glitzky and Gärtner [24]. However, it seems that there does not exist any systematic study on entropy-dissipative discretizations for (1) and the time decay of their discrete solutions.

The aim of this paper is to provide some results on the time decay of discrete solutions to (1)-(2) and to give estimates on the decay rates. To this end, we adapt the proofs for the continuous case to the discrete situation. The scheme under investigation is a backward Euler scheme in time and a finite-volume scheme in space, defined in (7). Only those proofs are chosen which can be directly "translated" in a finite-volume context.

Our main objective is to prove that the finite volume scheme for (1)-(2) dissipates the discrete versions of the functionals

$$E_\alpha[u] = \frac{1}{\alpha+1}\left(\int_\Omega u^{\alpha+1}dx - \Big(\int_\Omega u dx\Big)^{\alpha+1}\right), \tag{3}$$

$$F_\alpha[u] = \frac{1}{2}\int_\Omega |\nabla u^{\alpha/2}|^2 dx, \quad \alpha > 0. \tag{4}$$

In fact, we will prove (algebraic or exponential) convergence rates at which the discrete functionals converge to zero as $t \to \infty$. We call E_α a zeroth-order entropy and F_α a first-order entropy. The functional F_1 is known as the Fisher information, used in mathematical statistics and information theory [20]. Our analysis of the decay rates of the entropies will be guided by the entropy-dissipation method. An essential ingredient of this technique is a functional inequality relating the entropy to the entropy dissipation [3, 11]. For the diffusion equation (1), this relation is realized by the Beckner inequality [5].

The entropy-dissipation method was applied to (1) in the whole space to prove the decay of the solutions to the asymptotic self-similar profile in, e.g., [13, 15]. The convergence towards the constant steady state on the one-dimensional torus was proved in [10]. However, we are not aware of general entropy decay estimates for solutions to (1)-(2) to the constant steady state, even in the continuous case. The reason might be that generalizations to the Beckner inequality, needed to relate the entropy dissipation to the entropy, are missing. In this paper, we propose new Beckner-type inequalities which fill this gap. Moreover, our proof can be translated to the discrete case. These results will be presented in Section 3.

The proof of discrete time decay for solutions to the finite-volume approximation of (1) is inspired from entropy decay estimates in the continuous case, which we review first. Differentiating $E_\alpha[u(t)]$ with respect to time and employing a Beckner inequality, we show for $\beta > 1$ that

$$\frac{dE_\alpha}{dt}[u(t)] \le C E_\alpha[u(t)]^{(\alpha+\beta)/(\alpha+1)}, \quad t > 0,$$

where $C > 0$ depends on α, β, and $C_B(p,q)$. By a nonlinear Gronwall inequality, this implies the algebraic decay of $u(t)$ to equilibrium in the entropy sense; see Theorem 9. If

the solution is positive and $0 < \alpha \le 1$, the above inequality becomes

$$\frac{dE_\alpha}{dt}[u(t)] \le C(u_0)E_\alpha[u(t)], \quad t > 0,$$

which results in an exponential decay rate; see Theorem 10. We obtain similar results for a discrete version of E_α in Theorems 11 (algebraic decay) and 12 (exponential decay).

The first-order entropies $F_\alpha[u(t)]$ decay exponentially fast (for positive solutions) for all (α,β) lying in the strip $-2 \le \alpha - 2\beta \le 1$ (one-dimensional case) or in the region M_d, which is illustrated in Figure 1 below (multi-dimensional case); see Theorems 13 and 14. The proof is based on systematic integration by parts [29]. In order to avoid boundary integrals arising from the iterated integrations by parts, these results are valid only if $\Omega = \mathbb{T}^d$. It is very difficult to "translate" the iterated integrations by parts to iterated summations by parts since there is no discrete nonlinear chain rule. For the zeroth-order entropies, this is done by exploiting the convexity of the mapping $x \mapsto x^{\alpha+1}$. For the first-order entropies, we employ the concavity of the discrete version of dF_α/dt with respect to the time approximation parameter. We prove in Theorem 16 that for $1 \le \alpha \le 2$ and $\beta = \alpha/2$, the discrete first-order entropy is monotone (multi-dimensional case) and decays exponentially fast (one-dimensional case). We stress the fact that this is the first result in the literature on the decay of discrete first-order entropies.

Throughout this paper, we assume that the solutions to (1) are smooth and positive such that we can perform all the computations and integrations by parts. In particular, we avoid any technicalities due to the degeneracy ($\beta > 1$) or singularity ($\beta < 1$) in (1). Most of our results can be generalized to nonnegative weak solutions by using a suitable approximation scheme but details are left to the reader.

We stress the fact that we do not develop an efficient implementation and we do not perform a convergence analysis, since the scheme is rather standard. Our aim is of more theoretical interest. In fact, our results on the discrete decay rates contribute to the aim of developing and analyzing *structure-preserving* numerical schemes and this is the main originality of the present work.

The paper is organized as follows. Section 2 is devoted to the finite-volume setting: We introduce the numerical scheme under investigation and define discrete norms and discrete entropies. Then we prove some novel generalized Beckner inequalities in Section 3, at the continuous and discrete level. The algebraic and exponential decay of $E_\alpha[u]$ are studied in Section 4. W e first prove the results at the continuous level and then deduce similar results for the numerical scheme. Section 5 is devoted to the study of the exponential decay of the first-order entropies $F_\alpha[u]$. In Section 6, we illustrate the theoretical results by numerical experiments in one and two space dimensions. They indicate that some of the restrictions on the parameters (α,β) seem to be only technical. In the appendix, a discrete nonlinear Gronwall lemma and some auxiliary inequalities are proved.

2. The finite-volume setting

2.1. **Notations and finite-volume scheme.** Let Ω be an open bounded polyhedral subset of $\mathbb{R}^d$ ($d \geq 2$) with Lipschitz boundary and $\mathrm{m}(\Omega) = 1$. An admissible mesh of Ω is given by a family $\mathcal{T}$ of control volumes (open and convex polyhedral subsets of Ω with positive measure); a family $\mathcal{E}$ of relatively open parts of hyperplanes in $\mathbb{R}^d$ which represent the faces of the control volumes; and a family of points $(x_K)_{K\in\mathcal{T}}$ which satisfy Definition 9.1 in [21]. This definition implies that the straight line between two neighboring centers of cells (x_K, x_L) is orthogonal to the edge $\sigma = K|L$ between the two control volume K and L. For instance, triangular meshes in $\mathbb{R}^2$ satisfy the admissibility condition if all angles of the triangles are smaller than $\pi/2$ [21, Examples 9.1]. Voronoi meshes in $\mathbb{R}^d$ are also admissible meshes [21, Examples 9.2].

We distinguish the interior faces $\sigma \in \mathcal{E}_{\rm int}$ and the boundary faces $\sigma \in \mathcal{E}_{\rm ext}$. Then the union $\mathcal{E}_{\rm int} \cup \mathcal{E}_{\rm ext}$ equals the set of all faces $\mathcal{E}$. For a control volume $K \in \mathcal{T}$, we denote by $\mathcal{E}_K$ the set of its faces, by $\mathcal{E}_{{\rm int},K}$ the set of its interior faces, and by $\mathcal{E}_{{\rm ext},K}$ the set of edges of K included in $\partial\Omega$.

Let d be the distance in $\mathbb{R}^d$. We assume that the family of meshes satisfies the following regularity requirement: There exists $\xi > 0$ such that for all $K \in \mathcal{T}$ and all $\sigma \in \mathcal{E}_{{\rm int},K}$ with $\sigma = K|L$, it holds

$$\mathrm{d}(x_K, \sigma) \geq \xi \mathrm{d}(x_K, x_L). \tag{5}$$

This hypothesis is needed to apply a discrete Poincaré inequality; see Lemma 2. Introducing for $\sigma \in \mathcal{E}$ the notation

$$d_\sigma = \begin{cases} \mathrm{d}(x_K, x_L) & \text{if } \sigma \in \mathcal{E}_{\rm int},\ \sigma = K|L, \\ \mathrm{d}(x_K, \sigma) & \text{if } \sigma \in \mathcal{E}_{{\rm ext},K}, \end{cases}$$

we define the transmissibility coefficient

$$\tau_\sigma = \frac{\mathrm{m}(\sigma)}{d_\sigma}, \quad \sigma \in \mathcal{E}.$$

The size of the mesh is denoted by $\triangle x = \max_{K\in\mathcal{T}} \mathrm{diam}(K)$. Let $T > 0$ be some final time and M_T the number of time steps. Then the time step size and the time points are given by, respectively, $\triangle t = T/M_T$, $t^k = k\triangle t$ for $0 \leq k \leq M_T$. We denote by $\mathcal{D}$ an admissible space-time discretization of $\Omega_T = \Omega \times (0, T)$ composed of an admissible mesh $\mathcal{T}$ of Ω and the values $\triangle t$ and M_T.

We are now in the position to define the finite-volume scheme of (1)-(2) on $\mathcal{D}$. The initial datum is approximated by its L^2 projection on control volumes:

$$u^0 = \sum_{K\in\mathcal{T}} u_K^0 \mathbf{1}_K, \quad \text{where } u_K^0 = \frac{1}{\mathrm{m}(K)} \int_K u_0(x)dx, \tag{6}$$

and $\mathbf{1}_K$ is the characteristic function on K. Then it holds $\sum_{K\in\mathcal{T}} \mathrm{m}(K) u_K^0 = \int_\Omega u_0 dx$.

The numerical scheme reads as follows:

$$\mathrm{m}(K)\frac{u_K^{k+1}-u_K^k}{\triangle t}+\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u_K^{k+1})^\beta-(u_L^{k+1})^\beta\big)=0, \tag{7}$$

for all $K\in\mathcal{T}$ and $k=0,\ldots,M_T-1$. This scheme is based on a fully implicit Euler discretization in time and a finite-volume approach for the volume variable. The Neumann boundary conditions (2) are taken into account as the sum in (7) applies only on the interior edges. The implicit scheme allows us to establish discrete entropy-dissipation estimates which would not be possible with an explicit scheme.

We summarize in the next proposition the classical results of existence, uniqueness and stability of the solution to the finite-volume scheme (6)-(7).

Proposition 1. *Let $u_0\in L^\infty(\Omega)$, $m\ge 0$, $M\ge 0$ such that $m\le u_0\le M$ in Ω. Let $\mathcal{T}$ be an admissible mesh of Ω. Then the scheme* (6)-(7) *admits a unique solution $(u_K^k)_{K\in\mathcal{T},0\le k\le M_T}$ satisfying*

$$m\le u_K^k\le M,\quad \textit{for all}\ \ K\in\mathcal{T},\ 0\le k\le M_T,$$
$$\sum_{K\in\mathcal{T}}\mathrm{m}(K)u_K^k=\|u_0\|_{L^1(\Omega)},\quad \textit{for all}\ \ 0\le k\le M_T.$$

We refer, for instance, to [21] and [22] for the proof of this proposition.

2.2. **Discrete entropies.** A finite-volume scheme provides an approximate solution which is constant on each cell of the mesh and on each time interval. Let $X(\mathcal{T})$ be the linear space of functions $\Omega\to\mathbb{R}$ which are constant on each cell $K\in\mathcal{T}$:

$$X(\mathcal{T})=\left\{u=\sum_{K\in\mathcal{T}}u_K\mathbf{1}_K\right\}.$$

The set $X(\mathcal{T})$ is included in $L^p(\Omega)$ for $1\le p\le\infty$ and

$$\|u\|_{L^p(\Omega)}=\left(\int_\Omega|u|^p dx\right)^{1/p}=\left(\sum_{K\in\mathcal{T}}\mathrm{m}(K)|u_K|^p\right)^{1/p}\qquad\forall u\in X(\mathcal{T}),\ \forall 1\le p<+\infty.$$

Clearly, the set $X(\mathcal{T})$ is not included in $W^{1,p}(\Omega)$. However, for $1\le p<+\infty$, we can define a discrete $W^{1,p}$ seminorm and a discrete $W^{1,p}$ norm by, respectively,

$$|u|_{1,p,\mathcal{T}}=\left(\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\frac{\mathrm{m}(\sigma)}{d_\sigma^{p-1}}|u_K-u_L|^p\right)^{1/p}\qquad\forall u\in X(\mathcal{T}),$$
$$\|u\|_{1,p,\mathcal{T}}=\|u\|_{L^p(\Omega)}+|u|_{1,p,\mathcal{T}}\quad\forall u\in X(\mathcal{T}).$$

The zeroth-order entropies defined by (3) can be rewritten for $u \in X(\mathcal{T})$ as

$$E_\alpha[u] = \frac{1}{\alpha+1}\left(\sum_{K\in\mathcal{T}} \mathrm{m}(K) u_K^{\alpha+1} - \left(\sum_{K\in\mathcal{T}} \mathrm{m}(K) u_K\right)^{\alpha+1}\right). \tag{8}$$

Finally, we define the discrete first-order entropies, corresponding to (4), by

$$F_\alpha^d[u] = \frac{1}{2}|u^{\alpha/2}|_{1,2,\mathcal{T}}^2. \tag{9}$$

3. Generalized Beckner inequalities

The decay properties of the zeroth-order entropies rely on generalized Beckner inequalities which follow from the Poincaré-Wirtinger inequality. This section is devoted to the proof of these Beckner inequalities in the functional space $H^1(\Omega)$ and of their discrete counterpart in the functional space $X(\mathcal{T})$.

3.1. **Poincaré-Wirtinger inequalities.** We assume that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is a bounded domain such that the Poincaré-Wirtinger inequality

$$\|f - \bar{f}\|_{L^2(\Omega)} \leq C_P \|\nabla f\|_{L^2(\Omega)} \tag{10}$$

for all $f \in H^1(\Omega)$ holds, where $\bar{f} = \mathrm{m}(\Omega)^{-1}\int_\Omega f dx$ and $C_P > 0$ only depends on d and Ω. This is the case if, for instance, Ω has the cone property [35, Theorem 8.11] or if $\partial\Omega$ is locally Lipschitz continuous [44, Theorem 1.3.4]. We recall that $\mathrm{m}(\Omega) = 1$ in this paper (to shorten the proof). The discrete counterpart of (10) is stated in the following Lemma (see for instance [7, Theorem 5]):

Lemma 2 (Discrete Poincaré-Wirtinger inequality)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded polyhedral set and let $\mathcal{T}$ be an admissible mesh satisfying the regularity constraint* (5)*. Then there exists a constant $C_p > 0$, only depending on d and Ω, such that for all $f \in X(\mathcal{T})$,*

$$\|f - \bar{f}\|_{L^2(\Omega)} \leq \frac{C_p}{\xi^{1/2}}|f|_{1,2,\mathcal{T}}, \tag{11}$$

where $\bar{f} = \int_\Omega f dx$ (recall that $\mathrm{m}(\Omega) = 1$) and ξ is defined in (5)*.*

We present now a new inequality which can be seen as a generalized Poincaré inequality.

Lemma 3 (Generalized Poincaré-Wirtinger inequality)**.** *Let $0 < q \leq 2$ and $f \in H^1(\Omega)$. Then*

$$\|f\|_{L^2(\Omega)}^q \leq C_P^q \|\nabla f\|_{L^2(\Omega)}^q + \|f\|_{L^q(\Omega)}^q \tag{12}$$

holds, where $C_P > 0$ is the constant of the Poincaré-Wirtinger inequality (10)*.*

Proof. Let first $1 \leq q \leq 2$. The Poincaré-Wirtinger inequality (10) rewrites as

$$\|f\|_{L^2(\Omega)}^2 - \|f\|_{L^1(\Omega)}^2 = \|f - \bar{f}\|_{L^2(\Omega)}^2 \leq C_P^2 \|\nabla f\|_{L^2(\Omega)}^2 \tag{13}$$

and together with the Hölder inequality leads to

$$\|f\|_{L^2(\Omega)}^2 \leq C_P^2 \|\nabla f\|_{L^2(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2. \tag{14}$$

Here we use the assumption $\mathrm{m}(\Omega)=1$. Since $q/2\le 1$, it follows that

$$\|f\|_{L^2(\Omega)}^q \le \left(C_P^2\|\nabla f\|_{L^2(\Omega)}^2 + \|f\|_{L^q(\Omega)}^2\right)^{q/2} \le C_P^q\|\nabla f\|_{L^2(\Omega)}^q + \|f\|_{L^q(\Omega)}^q,$$

which equals (12).

Next, let $0<q<1$. We claim that

$$a^{q/2} - a^{q-1}b^{1-q/2} \le (a-b)^{q/2} \quad \text{for all } a\ge b>0. \tag{15}$$

Indeed, setting $c=b/a$, this inequality is equivalent to

$$1-c^{1-q/2} \le (1-c)^{q/2} \quad \text{for all } 0<c\le 1.$$

The function $g(c)=1-c^{1-q/2}-(1-c)^{q/2}$ for $c\in[0,1]$ satisfies $g(0)=g(1)=0$ and $g''(c)=(q/2)(1-q/2)(c^{-1-q/2}+(1-c)^{q/2-2})\ge 0$ for $c\in(0,1)$, which implies that $g(c)\le 0$, proving (15). Applying (15) to $a=\|f\|_{L^2(\Omega)}^2$ and $b=\|f\|_{L^1(\Omega)}^2$ and using (13), we find that

$$\|f\|_{L^2(\Omega)}^q - \|f\|_{L^2(\Omega)}^{2(q-1)}\|f\|_{L^1(\Omega)}^{2-q} \le \left(\|f\|_{L^2(\Omega)}^2 - \|f\|_{L^1(\Omega)}^2\right)^{q/2} \le C_P^q\|\nabla f\|_{L^2(\Omega)}^q. \tag{16}$$

In order to get rid of the L^1 norm, we employ the interpolation inequality

$$\|f\|_{L^1(\Omega)} = \int_\Omega |f|^\theta |f|^{1-\theta}dx \le \|f\|_{L^q(\Omega)}^\theta \|f\|_{L^2(\Omega)}^{1-\theta}, \tag{17}$$

where $\theta = q/(2-q)<1$. Since $(2-q)\theta = q$ and $(2-q)(1-\theta)=2(1-q)$, (16) becomes

$$\|f\|_{L^2(\Omega)}^q - \|f\|_{L^q(\Omega)}^q \le C_P^q\|\nabla f\|_{L^2(\Omega)}^q,$$

which is the desired inequality. □

Starting from the discrete Poincaré-Wirtinger inequality (11) instead of (10), we obtain the discrete analogue of (13):

$$\|f\|_{L^2(\Omega)}^2 - \|f\|_{L^1(\Omega)}^2 = \|f-\bar f\|_{L^2(\Omega)}^2 \le C_p^2\xi^{-1}|f|_{1,2,\mathcal{T}}^2 \quad \text{for all } f\in X(\mathcal{T}).$$

Then, following the lines of the proof of Lemma 3, we obtain the discrete counterpart of the generalized Poincaré-Wirtinger inequality (12)

$$: \|f\|_{L^2(\Omega)}^q \le C_p^q\xi^{-q/2}|f|_{1,2,\mathcal{T}}^q + \|f\|_{L^q(\Omega)}^q \quad \text{for all } f\in X(\mathcal{T}), \tag{18}$$

under the hypotheses of Lemma 2.

3.2. **First generalization of the Beckner inequality.** For the proof of the algebraic decay of the zeroth-order entropies, we need the following variant of the Beckner inequality.

Lemma 4 (Generalized Beckner inequality I)**.** *Let $d\ge 1$ and either $0<q<2$, $pq\ge 1$ or $q=2$, $\frac12-\frac1d\le p\le 1$ ($0<p\le 1$ if $d\le 2$), and let $f\in H^1(\Omega)$. Then the generalized Beckner inequality*

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p}dx\right)^{pq} \le C_B(p,q)\|\nabla f\|_{L^2(\Omega)}^q \tag{19}$$

holds, where

$$C_B(p,q) = \frac{2(pq-1)C_P^q}{2-q} \quad \text{if } q < 2, \quad C_B(p,2) = C_P^2 \quad \text{if } q = 2,$$

and $C_P > 0$ *is the constant of the Poincaré-Wirtinger inequality* (10).

Remark 5. The case $q = 2$ corresponds to the usual Beckner inequality [5]

$$\int_\Omega |f|^2 dx - \left(\int_\Omega |f|^{2/r} dx\right)^r \le C_B(p,2)\|\nabla f\|_{L^2(\Omega)}^2,$$

where $1 \le r = 2p \le 2$. It is shown in [18] that the constant $C_B(p,2)$ can be related to the lowest positive eigenvalue of a Schrödinger operator if Ω is convex. On the one-dimensonal torus, the generalized Beckner inequality (19) for $p > 0$ and $0 < q < 2$ has been derived in [10]. In the multi-dimensional situation, the special case $p = 2/q$ was proved in [17]. In this work, it was also shown that (19) with $q > 2$ and $p = 2/q$ cannot be true unless the Lebesgue measure dx is replaced by the Dirac measure. In the limit $pq \to 1$, (19) leads to a generalized logarithmic Sobolev inequality (see (21) below). If $q = 2$ in this limit, the usual logarithmic Sobolev inequality [27] is obtained. □

Proof of Lemma 4. Let first $q = 2$. Then the Beckner inequality is a consequence of the Poincaré-Wirtinger inequality (10) and the Jensen inequality:

$$C_P^2\|\nabla f\|_{L^2(\Omega)}^2 \ge \|f - \bar f\|_{L^2(\Omega)}^2 = \|f\|_{L^2(\Omega)}^2 - \|f\|_{L^1(\Omega)}^2 \ge \int_\Omega f^2 dx - \left(\int_\Omega |f|^{2/r} dx\right)^r,$$

where $1 - \frac{2}{d} \le r \le 2$ ($0 < r \le 2$ if $d \le 2$). The lower bound for r ensures that the embedding $H^1(\Omega) \hookrightarrow L^{2/r}(\Omega)$ is continuous. The choice $p = r/2 \in [\frac12 - \frac1d, 1]$ yields the formulation (19).

Next, let $0 < q < 2$. The first part of the proof is inspired by the proof of Proposition 2.2 in [17]. Taking the logarithm of the interpolation inequality

$$\|f\|_{L^r(\Omega)} \le \|f\|_{L^q(\Omega)}^{\theta(r)} \|f\|_{L^2(\Omega)}^{1-\theta(r)},$$

where $q \le r \le 2$ and $\theta(r) = q(2-r)/(r(2-q))$, gives

$$F(r) := \frac1r \log\int_\Omega |f|^r dx - \frac{\theta(r)}{q}\log\int_\Omega |f|^q dx - \frac{1-\theta(r)}{2}\log\int_\Omega |f|^2 dx \le 0.$$

The function $F : [q,2] \to \mathbb{R}$ is nonpositive, differentiable and $F(q) = 0$. Therefore, $F'(q) \le 0$, which equals

$$-\frac{1}{q^2}\log\int_\Omega |f|^q dx + \frac1q\left(\int_\Omega |f|^q dx\right)^{-1}\int_\Omega |f|^q \log|f| dx \\ + \theta'(q)\left(\frac12\log\int_\Omega |f|^2 dx - \frac1q\log\int_\Omega |f|^q dx\right) \le 0.$$

We multiply this inequality by $q^2\int_\Omega |f|^q dx$ to obtain

$$\int_\Omega |f|^q \log\frac{|f|^q}{\|f\|_{L^q(\Omega)}^q}dx \le \frac{2}{2-q}\|f\|_{L^q(\Omega)}^q \log\frac{\|f\|_{L^2(\Omega)}^q}{\|f\|_{L^q(\Omega)}^q}. \tag{20}$$

Then, we employ Lemma 3 and the inequality $\log(x+1)\le x$ for $x\ge 0$ to infer that

$$\|f\|_{L^q(\Omega)}^q \log\frac{\|f\|_{L^2(\Omega)}^q}{\|f\|_{L^q(\Omega)}^q} \le \|f\|_{L^q(\Omega)}^q \log\left(\frac{C_P^q\|\nabla f\|_{L^2(\Omega)}^q}{\|f\|_{L^q(\Omega)}^q}+1\right) \le C_P^q\|\nabla f\|_{L^2(\Omega)}^q.$$

Combining this inequality and (20), we conclude the generalized logarithmic Sobolev inequality

$$\int_\Omega |f|^q \log\frac{|f|^q}{\|f\|_{L^q(\Omega)}^q}dx \le \frac{2C_P^q}{2-q}\|\nabla f\|_{L^2(\Omega)}^q. \tag{21}$$

The generalized Beckner inequality (19) is derived by extending slightly the proof of [33, Corollary 1]. Let

$$G(r) = r\log\int_\Omega |f|^{q/r}dx, \quad r\ge 1.$$

The function G is twice differentiable with

$$G'(r) = \left(\int_\Omega |f|^{q/r}dx\right)^{-1}\left(\int_\Omega |f|^{q/r}dx\log\int_\Omega |f|^{q/r}dx - \frac{q}{r}\int_\Omega |f|^{q/r}\log|f|dx\right),$$
$$G''(r) = \frac{q^2}{r^3}\left(\int_\Omega |f|^{q/r}dx\right)^{-2}\left(\int_\Omega |f|^{q/r}dx\int_\Omega |f|^{q/r}(\log|f|)^2dx - \left(\int_\Omega |f|^{q/r}\log|f|dx\right)^2\right).$$

The Cauchy-Schwarz inequality shows that $G''(r)\ge 0$, i.e., G is convex. Consequently, $r\mapsto e^{G(r)}$ is also convex and $r\mapsto H(r) = -(e^{G(r)}-e^{G(1)})/(r-1)$ is nonincreasing on $(1,\infty)$, which implies that

$$H(r)\le \lim_{t\to 1}H(t) = -e^{G(1)}G'(1) = \int_\Omega |f|^q\log\frac{|f|^q}{\|f\|_{L^q(\Omega)}^q}dx.$$

This inequality is equivalent to

$$\frac{1}{r-1}\left(\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{q/r}dx\right)^r\right) \le \int_\Omega |f|^q\log\frac{|f|^q}{\|f\|_{L^q(\Omega)}^q}dx. \tag{22}$$

Combining this inequality and the generalized logarithmic Sobolev inequality (21), it follows that

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{q/r}dx\right)^r \le \frac{2(r-1)C_P^q}{2-q}\|\nabla f\|_{L^2(\Omega)}^q$$

for all $0<q<2$ and $r\ge 1$. Setting $p := r/q$, this proves (19) for all $pq = r\ge 1$. □

Lemma 6 (Discrete generalized Beckner inequality I)**.** *Let $0 < q < 2$, $pq > 1$ or $q = 2$ and $0 < p \le 1$, and $f \in X(\mathcal{T})$. Then*

$$\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} \le C_b(p,q)|f|_{1,2,\mathcal{T}}^q$$

holds, where

$$C_b(p,q) = \frac{2(pq-1)C_p^q}{(2-q)\xi^{q/2}} \quad \text{if } q < 2, \quad C_b(p,2) = \frac{C_p^2}{\xi} \quad \text{if } q = 2.$$

C_p is the constant in the discrete Poincaré-Wirtinger inequality, and ξ is defined in (5)*.*

Proof. The proof follows the lines of the proof of Lemma 4, noting that in the discrete (finite-dimensional) setting, we do not need anymore the lower bound on p. Indeed, if $q = 2$, the conclusion results from the discrete Poincaré-Wirtinger inequality (11) and the Jensen inequality. If $q < 2$, we first remark that (20) and (22) still holds for $f \in X(\mathcal{T})$, leading to

$$\begin{aligned}\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq} &\le (pq-1)\int_\Omega |f|^q \log\frac{|f|^q}{\|f\|_{L^q(\Omega)}^q} dx \\ &\le \frac{2(pq-1)}{2-q}\|f\|_{L^q(\Omega)}^q \log\frac{\|f\|_{L^2(\Omega)}^q}{\|f\|_{L^q(\Omega)}^q}. \end{aligned} \tag{23}$$

Then, inserting the discrete Poincaré-Wirtinger inequality (18) instead of (12) into (23) to replace $\|f\|_{L^2(\Omega)}$ and using $\log(x+1) \le x$ for $x \ge 0$, the lemma follows. □

3.3. **Second generalization of the Beckner inequality.** For the proof of exponential decay rates, we need the following variant of the Beckner inequality.

Lemma 7 (Generalized Beckner inequality II)**.** *Let $0 < q < 2$, $pq \ge 1$ and $f \in H^1(\Omega)$. Then*

$$\|f\|_{L^q(\Omega)}^{2-q}\left(\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p} dx\right)^{pq}\right) \le C_B'(p,q)\|\nabla f\|_{L^2(\Omega)}^2, \tag{24}$$

where

$$C_B'(p,q) = \begin{cases} \dfrac{q(pq-1)C_P^2}{2-q} & \text{if } 1 \le q < 2, \\ (pq-1)C_P^2 & \text{if } 0 < q < 1. \end{cases}$$

Proof. By (20), it holds that for all $0 < q < 2$,

$$\int_\Omega |f|^q \log\frac{|f|^q}{\|f\|_{L^q(\Omega)}^q} dx \le \frac{q}{2-q}\|f\|_{L^q(\Omega)}^q \log\frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2}.$$

Then, for $q > 1$, the Poincaré-Wirtinger inequality in the version (14) and the inequality $\log(x+1) \le x$ for $x \ge 0$ yield

$$\|f\|_{L^q(\Omega)}^q \log\frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2} \le \|f\|_{L^q(\Omega)}^q \log\left(C_P^2\frac{\|\nabla f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2}+1\right) \le C_P^2\|f\|_{L^q(\Omega)}^{q-2}\|\nabla f\|_{L^2(\Omega)}^2. \tag{25}$$

Taking into account (22), the conclusion follows for $q > 1$.

Let $0 < q \le 1$. Suppose that the following inequality holds:

$$\|f\|_{L^q(\Omega)}^2 + \frac{2-q}{q}C_P^2\|\nabla f\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)}^2 \ge 0. \tag{26}$$

This implies that, by (22) and for $r = pq$,

$$\begin{aligned}\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{q/r}dx\right)^r &\le \frac{(pq-1)q}{2-q}\|f\|_{L^q(\Omega)}^q \log\frac{\|f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2} \\ &\le \frac{(pq-1)q}{2-q}\|f\|_{L^q(\Omega)}^q \log\left(\frac{(2-q)C_P^2}{q}\,\frac{\|\nabla f\|_{L^2(\Omega)}^2}{\|f\|_{L^q(\Omega)}^2}+1\right) \\ &\le (pq-1)C_P^2\|\nabla f\|_{L^2(\Omega)}^2\|f\|_{L^q(\Omega)}^{q-2},\end{aligned}$$

which shows the desired Beckner inequality.

It remains to prove (26). For this, we employ the Poincaré-Wirtinger inequality (13)

$$C_P^2\|\nabla f\|_{L^2(\Omega)}^2 \ge \|f\|_{L^2(\Omega)}^2 - \|f\|_{L^1(\Omega)}^2$$

and the interpolation inequality (17) in the form

$$\|f\|_{L^q(\Omega)}^2 \ge \|f\|_{L^1(\Omega)}^{2/\theta}\|f\|_{L^2(\Omega)}^{2(\theta-1)/\theta}, \quad \theta = \frac{q}{2-q} \le 1,$$

to obtain

$$\begin{aligned}&\|f\|_{L^q(\Omega)}^2 + \frac{2-q}{q}C_P^2\|\nabla f\|_{L^2(\Omega)}^2 - \|f\|_{L^2(\Omega)}^2 \\ &\quad\ge \|f\|_{L^1(\Omega)}^{2/\theta}\|f\|_{L^2(\Omega)}^{2(\theta-1)/\theta} + \left(\frac{2-q}{q}-1\right)\|f\|_{L^2(\Omega)}^2 - \frac{2-q}{q}\|f\|_{L^1(\Omega)}^2.\end{aligned}$$

We interpret the right-hand side as a function G of $\|f\|_{L^1(\Omega)}^2$. Then, setting $A = \|f\|_{L^2(\Omega)}^2$,

$$\begin{aligned}G(y) &= y^{1/\theta}A^{1-1/\theta} + \frac{2(1-q)}{q}A - \frac{2-q}{q}y, \\ G'(y) &= \frac{1}{\theta}y^{1/\theta-1}A^{1-1/\theta} - \frac{2-q}{q}, \\ G''(y) &= \frac{1}{\theta}\left(\frac{1}{\theta}-1\right)y^{1/\theta-2}A^{1-1/\theta} \ge 0,\end{aligned}$$

Therefore, G is a convex function which satisfies $G(A) = 0$ and $G'(A) = 0$. This implies that G is a nonnegative function on $\mathbb{R}^+$ and in particular, $G(\|f\|_{L^1(\Omega)}^2) \ge 0$. This proves (26), completing the proof. □

The adaptation of the proof of Lemma 7 is straightforward, using the discrete Poincaré-Wirtinger inequality (11) instead of (10). This yields the following result.

Lemma 8 (Discrete generalized Beckner inequality II)**.** *Let $0 < q < 2$, $pq \geq 1$, and $f \in X(\mathcal{T})$. Then*

$$\|f\|_{L^q(\Omega)}^{2-q}\left(\int_\Omega |f|^q dx - \left(\int_\Omega |f|^{1/p}dx\right)^{pq}\right) \leq C_b'(p,q)|f|_{1,2,\mathcal{T}}^2$$

holds, where

$$C_b'(p,q) = \begin{cases} \dfrac{q(pq-1)C_p^2}{(2-q)\xi} & \text{if } 1 \leq q < 2, \\ \dfrac{(pq-1)C_p^2}{\xi} & \text{if } 0 < q < 1, \end{cases}$$

C_p is the constant in the discrete Poincaré-Wirtinger inequality, and ξ is defined in (5)*.*

4. Zeroth-order entropies: from the continuous to the discrete level

In this section, we prove the algebraic or exponential decay of the zeroth-order entropies. We first study the continuous case and then show how to extend the obtained result to the numerical scheme.

4.1. **The continuous case.** Let u be a smooth solution to (1)-(2) and let $u_0 \in L^\infty(\Omega)$, $\inf_\Omega u_0 \geq 0$ in Ω. By the maximum principle, $0 \leq \inf_\Omega u_0 \leq u(t) \leq \sup_\Omega u_0$ in Ω for $t \geq 0$. First, we prove algebraic decay rates for $E_\alpha[u]$, defined in (3).

Theorem 9 (Polynomial decay for E_α)**.** *Let $\alpha > 0$ and $\beta > 1$. Let u be a smooth solution to* (1)-(2) *and $u_0 \in L^\infty(\Omega)$ with $\inf_\Omega u_0 \geq 0$. Then*

$$E_\alpha[u(t)] \leq \frac{1}{(C_1 t + C_2)^{(\alpha+1)/(\beta-1)}}, \quad t \geq 0,$$

where

$$C_1 = \frac{4\alpha\beta(\beta-1)}{(\alpha+1)(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_B(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)}, \quad C_2 = E_\alpha[u_0]^{-(\beta-1)/(\alpha+1)},$$

and $C_B(p,q) > 0$ is the constant in the Beckner inequality for $p = (\alpha+\beta)/2$ and $q = 2(\alpha+1)/(\alpha+\beta)$.

Proof. We apply Lemma 4 with $p = (\alpha+\beta)/2$ and $q = 2(\alpha+1)/(\alpha+\beta)$. The assumptions on α and β guarantee that $0 < q < 2$ and $pq > 1$. Then, with $f = u^{(\alpha+\beta)/2}$,

$$E_\alpha[u] = \frac{1}{\alpha+1}\left(\int_\Omega u^{\alpha+1}dx - \left(\int_\Omega u dx\right)^{\alpha+1}\right) \leq \frac{C_B(p,q)}{\alpha+1}\left(\int_\Omega |\nabla u^{(\alpha+\beta)/2}|^2 dx\right)^{(\alpha+1)/(\alpha+\beta)}.$$

Now, computing the derivative,

$$\frac{dE_\alpha}{dt} = -\int_\Omega \nabla u^\alpha \cdot \nabla u^\beta dx = -\frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega |\nabla u^{(\alpha+\beta)/2}|^2 dx \tag{27}$$

$$\le -\frac{4\alpha\beta}{(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_B(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)} E_\alpha[u]^{(\alpha+\beta)/(\alpha+1)}. \tag{28}$$

An integration of this inequality gives the assertion. □

Next, we show exponential decay rates.

Theorem 10 (Exponential decay for E_α)**.** *Let u be a smooth solution to* (1)-(2)*, $0<\alpha\le 1$, $\beta>0$, $u_0\in L^\infty(\Omega)$ with $\inf_\Omega u_0\ge 0$. Then*

$$E_\alpha[u(t)]\le E_\alpha[u_0]e^{-\Lambda t},\quad t\ge 0.$$

The constant Λ is given by

$$\Lambda=\frac{4\alpha\beta}{C_B(\frac{1}{2}(\alpha+1),2)(\alpha+1)}\inf_\Omega\left(u_0^{\beta-1}\right)\ge 0,$$

for $\beta>0$ and

$$\Lambda=\frac{4\alpha\beta(\alpha+1)}{C_B'(p,q)(\alpha+\beta)^2}\|u_0\|_{L^1(\Omega)}^{\beta-1},$$

for $\beta>1$. Here, $C_B(\frac{1}{2}(\alpha+1),2)$ and $C_B'(p,q)$ are the constants in the Beckner inequalities (19) *and* (24)*, respectively, with $p=(\alpha+\beta)/2$ and $q=2(\alpha+1)/(\alpha+\beta)$.*

Proof. Let $\beta>0$. We compute

$$\begin{aligned}\frac{dE_\alpha}{dt}&=-\frac{4\alpha\beta}{(\alpha+1)^2}\int_\Omega u^{\beta-1}|\nabla u^{(\alpha+1)/2}|^2dx\\ &\le -\frac{4\alpha\beta}{(\alpha+1)^2}\inf_\Omega(u_0^{\beta-1})\int_\Omega|\nabla u^{(\alpha+1)/2}|^2dx.\end{aligned}\tag{29}$$

By the Beckner inequality (19) with $p=(\alpha+1)/2$, $q=2$, and $f=u^{(\alpha+1)/2}$, we find that

$$\frac{dE_\alpha}{dt}\le -\frac{4\alpha\beta}{C_B(p,2)(\alpha+1)}\inf_\Omega(u_0^{\beta-1})E_\alpha,\tag{30}$$

and Gronwall's lemma proves the claim. The restriction $p\le 1$ in Lemma 4 is equivalent to $\alpha\le 1$.

Next, let $\beta>1$. By Lemma 7, with $p=(\alpha+\beta)/2$, $q=2(\alpha+1)/(\alpha+\beta)$, and $f=u^{(\alpha+\beta)/2}$, it follows that

$$\|u\|_{L^{\alpha+1}(\Omega)}^{\beta-1}\left(\int_\Omega u^{\alpha+1}dx-\left(\int_\Omega udx\right)^{\alpha+1}\right)\le C_B'(p,q)\int_\Omega|\nabla u^{(\alpha+\beta)/2}|^2dx.$$

Hence, we can estimate

$$\begin{aligned}\frac{dE_\alpha}{dt}=-\frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega|\nabla u^{(\alpha+\beta)/2}|^2dx&\le -\frac{4\alpha\beta(\alpha+1)}{(\alpha+\beta)^2}\frac{\|u\|_{L^{\alpha+1}(\Omega)}^{\beta-1}}{C_B'(p,q)}E_\alpha[u]\\ &\le -\frac{4\alpha\beta(\alpha+1)}{(\alpha+\beta)^2}\frac{\|u_0\|_{L^1(\Omega)}^{\beta-1}}{C_B'(p,q)}E_\alpha[u],\end{aligned}$$

and Gronwall's lemma gives the conclusion. Note that in the last step of the inequality we used $\|u\|_{L^{\alpha+1}(\Omega)} \geq \|u\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$. □

4.2. **The discrete case.** We prove a result which is the discrete analogue of Theorem 9. The finite-volume scheme (7) permits to define uniquely a piecewise constant solution at each time step: $u^k = \sum_{K\in\mathcal{T}} u_K^k \mathbf{1}_K$. Then the discrete entropies at each time step $E_\alpha[u^k]$ are defined in (8).

Theorem 11 (Polynomial decay)**.** *Let $\alpha > 0$ and $\beta > 1$. Let $(u_K^k)_{K\in\mathcal{T},k\geq 0}$ be the solution to the finite-volume scheme (7) with $\inf_{K\in\mathcal{T}} u_K^0 \geq 0$. Then*

$$E_\alpha[u^k] \leq \frac{1}{(c_1 t^k + c_2)^{(\alpha+1)/(\beta-1)}}, \quad k \geq 0,$$

where

$$c_1 = (\beta-1)\left(\frac{(\alpha+1)(\alpha+\beta)^2}{4\alpha\beta}\left(\frac{C_b(p,q)}{\alpha+1}\right)^{(\alpha+\beta)/(\alpha+1)} + (\alpha+\beta)\triangle t E_\alpha[u^0]^{(\alpha+1)/(\beta-1)}\right)^{-1},$$

$$c_2 = E_\alpha[u^0]^{-(\beta-1)/(\alpha+1)},$$

and $C_b(p,q)$ for $p = (\alpha+\beta)/2$ and $q = 2(\alpha+1)/(\alpha+\beta)$ is defined in Lemma 6.

Proof. The idea is to "translate" the proof of Theorem 9 to the discrete case. To this end, we use the elementary inequality $y^{\alpha+1} - x^{\alpha+1} \leq (\alpha+1)y^\alpha(y-x)$, which follows from the convexity of the mapping $x \mapsto x^{\alpha+1}$. Using also the scheme (7), we obtain

$$\begin{aligned} E_\alpha[u^{k+1}] - E_\alpha[u^k] &= \frac{1}{\alpha+1}\sum_{K\in\mathcal{T}} \mathrm{m}(K)\big((u_K^{k+1})^{\alpha+1} - (u_K^k)^{\alpha+1}\big) \\ &\leq \sum_{K\in\mathcal{T}} \mathrm{m}(K)(u_K^{k+1})^\alpha(u_K^{k+1} - u_K^k) \\ &\leq -\triangle t \sum_{K\in\mathcal{T}} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma (u_K^{k+1})^\alpha\big((u_K^{k+1})^\beta - (u_L^{k+1})^\beta\big). \end{aligned}$$

Rearranging the sum leads to the discrete counterpart of (28):

$$E_\alpha[u^{k+1}] - E_\alpha[u^k] \leq -\triangle t \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1})^\alpha - (u_L^{k+1})^\alpha\big)\big((u_K^{k+1})^\beta - (u_L^{k+1})^\beta\big). \tag{31}$$

Then, employing the inequality in Lemma 19 (see the appendix), we deduce the discrete version of (28):

$$\begin{aligned} E_\alpha[u^{k+1}] - E_\alpha[u^k] &\leq -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma\big((u_K^{k+1})^{(\alpha+\beta)/2} - (u_L^{k+1})^{(\alpha+\beta)/2}\big)^2 \\ &\leq -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}|(u^{k+1})^{(\alpha+\beta)/2}|_{1,2,\mathcal{T}}^2. \end{aligned}$$

Applying the discrete Beckner inequality given in Lemma 6 with $p=(\alpha+\beta)/2$, $q=2(\alpha+1)/(\alpha+\beta)$, and $f=(u^{k+1})^{(\alpha+\beta)/2}$, we obtain the discrete counterpart of (28):

$$E_\alpha[u^{k+1}]-E_\alpha[u^k]\le -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_b(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)} E_\alpha[u^{k+1}]^{(\alpha+\beta)/(\alpha+1)}.$$

The discrete nonlinear Gronwall lemma (see Corollary 18 in the appendix) with

$$\tau=\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_b(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)},\quad \gamma=\frac{\alpha+\beta}{\alpha+1}>1,$$

implies that

$$E_\alpha[u^k]\le\frac{1}{(E_\alpha[u^0]^{1-\gamma}+c_1t^k)^{1/(\gamma-1)}},\quad k\ge 0,$$

where $c_1=(\gamma-1)/(1+\gamma\tau E_\alpha[u^0]^{\gamma-1})$. Finally, computing c_1 shows the result. □

The discrete analogue to Theorem 10 is as follows.

Theorem 12 (Exponential decay for E_α)**.** *Let $(u_K^k)_{K\in\mathcal{T},k\ge0}$ be a solution to the finite-volume scheme (7) and let $0<\alpha\le 1$, $\beta>0$, $\inf_{K\in\mathcal{T}}u_K^0\ge 0$. Then*

$$E_\alpha[u^k]\le E_\alpha[u^0]e^{-\lambda t^k},\quad k\ge 0.$$

The constant λ is given by

$$\lambda=\frac{4\alpha\beta}{C_b(\frac12(\alpha+1),2)(\alpha+1)}\inf_{K\in\mathcal{T}}\left((u_K^0)^{\beta-1}\right)\ge 0,$$

for $\beta>0$, and

$$\lambda=\frac{4\alpha\beta(\alpha+1)}{C_b'(p,q)(\alpha+\beta)^2}\|u_0\|_{L^1(\Omega)}^{\beta-1}$$

for $\beta>1$. Here $C_b'(p,q)>0$ is the constant from Lemma 8 with $p=(\alpha+\beta)/2$ and $q=2(\alpha+1)/(\alpha+\beta)$.

Proof. Let $\alpha\le 1$ and $\beta>0$. As in the proof of Theorem 11, we find that (see (31))

$$E_\alpha[u^{k+1}]-E_\alpha[u^k]\le-\triangle t\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u_K^{k+1})^\alpha-(u_L^{k+1})^\alpha\big)\big((u_K^{k+1})^\beta-(u_L^{k+1})^\beta\big).$$

Employing Corollary 20 (see the appendix), we obtain

$$\begin{aligned}E_\alpha[u^{k+1}]-E_\alpha[u^k]&\le-\frac{4\alpha\beta\triangle t}{(\alpha+1)^2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\min\left\{(u_K^{k+1})^{\beta-1},(u_L^{k+1})^{\beta-1}\right\}\\&\quad\times\left((u_K^{k+1})^{(\alpha+1)/2}-(u_L^{k+1})^{(\alpha+1)/2}\right)^2\\&\le-\frac{4\alpha\beta\triangle t}{(\alpha+1)^2}\inf_{K\in\mathcal{T}}(u_K^{k+1})^{\beta-1}|(u^{k+1})^{(\alpha+1)/2}|_{1,2,\mathcal{T}}^2,\end{aligned}$$

which is the discrete counterpart of (29). Then, applying the discrete Beckner inequality given in Lemma 6 with $p=(\alpha+1)/2$, $q=2$, and $f=u^{(\alpha+1)/2}$, we obtain the analogue of (30):

$$E_\alpha[u^{k+1}]-E_\alpha[u^k]\le -\frac{4\alpha\beta\triangle t}{C_b(\frac{1}{2}(\alpha+1),2)(\alpha+1)}\inf_{K\in\mathcal{T}}(u_K^0)^{\beta-1}E_\alpha[u^{k+1}],$$

and the Gronwall lemma shows the claim.

Next, let $\beta>1$. As in the proof of Theorem 11, we find that

$$E_\alpha[u^{k+1}]-E_\alpha[u^k]\le -\frac{4\alpha\beta\triangle t}{(\alpha+\beta)^2}|(u^{k+1})^{(\alpha+1)/2}|_{1,2,\mathcal{T}}^2.$$

We apply Lemma 8 with $p=(\alpha+\beta)/2$, $q=2(\alpha+1)/(\alpha+\beta)$, and $f=u^{(\alpha+\beta)/2}$ to obtain

$$\begin{aligned}E_\alpha[u^{k+1}]-E_\alpha[u^k]&\le -\frac{4\alpha\beta(\alpha+1)\triangle t}{(\alpha+\beta)^2}\frac{\|u^{k+1}\|_{L^{\alpha+1}(\Omega)}^{\beta-1}}{C_b'(p,q)}E_\alpha[u^{k+1}]\\&\le -\frac{4\alpha\beta(\alpha+1)\triangle t}{(\alpha+\beta)^2}\frac{\|u^{0}\|_{L^{1}(\Omega)}^{\beta-1}}{C_b'(p,q)}E_\alpha[u^{k+1}].\end{aligned}$$

Then Gronwall's lemma finishes the proof. □

5. First-order entropies: from the continuous to the discrete level

In this section, we consider the diffusion equation (1) on the torus $\Omega=\mathbb{T}^d$ and we first prove the exponential decay of the first-order entropies.

In the discrete setting, we consider the diffusion equation (1) on the half open unit cube $[0,1)^d\subset\mathbb{R}^d$ with multiperiodic boundary conditions (this is topologically equivalent to the torus $\mathbb{T}^d$). By identifying "opposite" faces on $\partial\Omega$, we can construct a family of control volumes and a family of edges in such a way that every face is an interior face. Then cells with such identified faces are neighboring cells. The numerical scheme we consider is similar to (7).

5.1. The continuous case.

The exponential decay for the first-order entropies (4) is given, for the one-dimensional case, in the following theorem.

Theorem 13 (Exponential decay of F_α in one space dimension)**.** *Let u be a smooth solution to (1) on the one-dimensional torus $\Omega=\mathbb{T}$. Let $u_0\in L^\infty(\Omega)$ with $\inf_\Omega u_0\ge 0$ and let $\alpha,\beta>0$ satisfy $-2\le\alpha-2\beta<1$. Then*

$$F_\alpha[u(t)]\le F_\alpha[u_0]e^{-\Lambda t},\quad 0\le t\le T,$$

where

$$\Lambda=\frac{2\beta}{C_P^2}\inf_\Omega(u_0^{\alpha+\beta-\gamma-1})\inf_\Omega(u_0^{\gamma-\alpha})\ge 0,\quad \gamma=\frac{2}{3}(\alpha+\beta-1),$$

where $C_P>0$ is the Poincaré constant in (10).

Proof. We extend slightly the entropy construction method of [29]. The time derivative of the entropy reads as

$$\begin{aligned}\frac{dF_\alpha}{dt} &= \frac{\alpha}{2}\int_\Omega (u^{\alpha/2})_x (u^{\alpha/2-1}u_t)_x dx = -\frac{\alpha}{2}\int_\Omega (u^{\alpha/2})_{xx} u^{\alpha/2-1}(u^\beta)_{xx} dx \\ &= -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}\left(\left(\frac{\alpha}{2}-1\right)(\beta-1)\xi_G^4 + \left(\frac{\alpha}{2}+\beta-2\right)\xi_G^2\xi_L + \xi_L^2\right)dx,\end{aligned}$$

where we introduced

$$\xi_G = \frac{u_x}{u}, \quad \xi_L = \frac{u_{xx}}{u}.$$

This integral is compared to

$$\int_\Omega u^{\alpha+\beta-\gamma-1}(u^{\gamma/2})_{xx}^2 dx = \frac{\gamma^2}{4}\int_\Omega u^{\alpha+\beta-1}\left(\left(\frac{\gamma}{2}-1\right)^2\xi_G^4 + (\gamma-2)\xi_G^2\xi_L + \xi_L^2\right)dx,$$

where, compared to the method of [29], $\gamma \neq 0$ gives an additional degree of freedom in the calculations. In the one-dimensional situation, there is only one relevant integration-by-parts rule:

$$0 = \int_\Omega (u^{\alpha+\beta-4}u_x^3)_x dx = \int_\Omega u^{\alpha+\beta-1}\big((\alpha+\beta-4)\xi_G^4 + 3\xi_G^2\xi_L\big)dx.$$

We introduce the polynomials

$$S_0(\xi) = \left(\frac{\alpha}{2}-1\right)(\beta-1)\xi_G^4 + \left(\frac{\alpha}{2}+\beta-2\right)\xi_G^2\xi_L + \xi_L^2, \tag{32}$$

$$D_0(\xi) = \left(\frac{\gamma}{2}-1\right)^2\xi_G^4 + (\gamma-2)\xi_G^2\xi_L + \xi_L^2, \tag{33}$$

$$T(\xi) = (\alpha+\beta-4)\xi_G^4 + 3\xi_G^2\xi_L,$$

where $\xi = (\xi_G, \xi_L)$. We wish to show that there exist numbers c, $\gamma \in \mathbb{R}$ ($\gamma \neq 0$) and $\kappa > 0$ such that

$$S(\xi) = S_0(\xi) + cT(\xi) - \kappa D_0(\xi) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^2.$$

The polynomial S can be written as $S(\xi) = a_1\xi_G^4 + a_2\xi_G^2\xi_L + (1-\kappa)\xi_L^2$, where

$$\begin{aligned}a_1 &= -\frac{1}{4}(\gamma-2)^2\kappa + (\alpha+\beta-4)c + \frac{1}{2}(\alpha-2)(\beta-1), \\ a_2 &= -(\gamma-2)\kappa + 3c + \frac{1}{2}(\alpha+2\beta-4).\end{aligned}$$

Therefore, the maximal value for κ is $\kappa = 1$. Let $\kappa = 1$. Then we need to eliminate the mixed term $\xi_G^2\xi_L$. The solution of $a_2 = 0$ is given by $c = -\frac{1}{6}(\alpha+2\beta-2\gamma)$, which leads to

$$a_1 = -\frac{1}{4}\left(\gamma - \frac{2}{3}(\alpha+\beta-1)\right)^2 - \frac{1}{18}(\alpha-2\beta-1)(\alpha-2\beta+2).$$

Choosing $\gamma = \frac{2}{3}(\alpha+\beta-1)$ to maximize a_1, we find that $a_1 \geq 0$ and hence $S(\xi) \geq 0$ if and only if $-2 \leq \alpha - 2\beta \leq 1$.

Using the Poincaré inequality (10) and the maximum principle, we obtain

$$\begin{aligned}\frac{dF_\alpha}{dt} &= -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}S_0(\xi)dx = -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}(S_0(\xi)+cT(\xi))dx \\ &\le -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}D_0(\xi)dx = -\frac{\alpha^2\beta}{\gamma^2}\int_\Omega u^{\alpha+\beta-\gamma-1}(u^{\gamma/2})_{xx}^2dx \\ &\le -\frac{\alpha^2\beta}{\gamma^2}\inf_{\Omega\times(0,\infty)}(u^{\alpha+\beta-\gamma-1})\int_\Omega (u^{\gamma/2})_{xx}^2dx \\ &\le -\frac{\alpha^2\beta}{\gamma^2 C_P^2}\inf_\Omega(u_0^{\alpha+\beta-\gamma-1})\int_\Omega (u^{\gamma/2})_x^2dx \\ &\le -\frac{2\beta}{C_P^2}\inf_\Omega(u_0^{\alpha+\beta-\gamma-1})\inf_\Omega(u_0^{\gamma-\alpha})F_\alpha.\end{aligned}$$

For the last inequality, we use the identity $(u^{\gamma/2})_x = \frac{\gamma}{\alpha}u^{(\gamma-\alpha)/2}(u^{\alpha/2})_x$, which cancels out the ratio α^2/γ^2. An application of the Gronwall's lemma finishes the proof. □

We turn to the multi-dimensional case.

Theorem 14 (Exponential decay of F_α in several space dimensions)**.** *Let u be a smooth solution to* (1) *on the torus $\Omega=\mathbb{T}^d$. Let $u_0\in L^\infty(\Omega)$ with $\inf_\Omega u_0>0$ and let*

$$(\alpha,\beta)\in M_d = \big\{(\alpha,\beta)\in\mathbb{R}^2 : (2-2\alpha+2\beta-d+\alpha d)(4-4\beta-2d+\alpha d+2\beta+2\beta d)>0$$
$$\text{and } (\alpha-2\beta-1)(\alpha-2\beta+2)<0\big\}$$

(see Figure 1). Then there exists $\Lambda>0$, depending on α, β, d, u_0, and Ω such that

$$F_\alpha[u(t)]\le F_\alpha[u_0]e^{-\Lambda t},\quad t\ge 0.$$

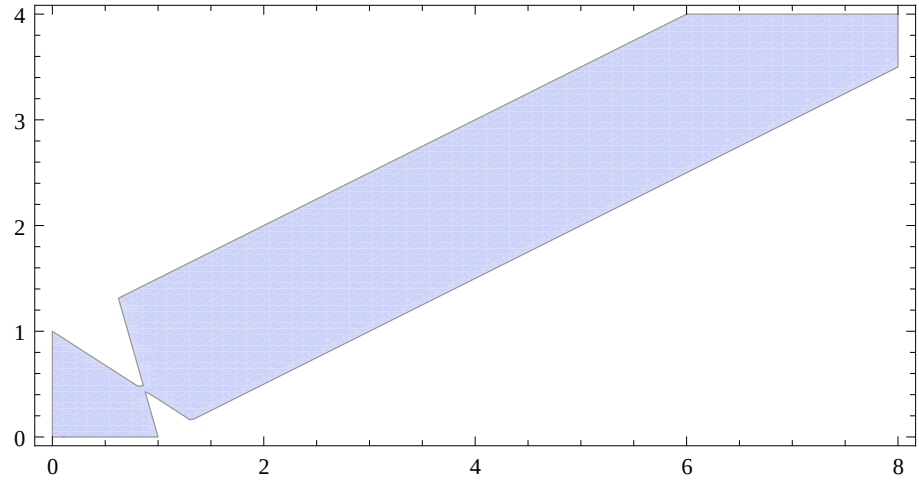

FIGURE 1. Illustration of the set M_d, defined in Theorem 14, for $d=9$.

Proof. The time derivative of the first-order entropy becomes

$$\frac{dF_\alpha}{dt} = -\frac{\alpha}{2}\int_\Omega u^{\alpha/2-1}\Delta(u^{\alpha/2})\Delta(u^\beta)dx = -\frac{\alpha^2\beta}{4}\int_\Omega u^{\alpha+\beta-1}S_0 dx, \tag{34}$$

where S_0 is defined in (32) with the (scalar) variables $\xi_G = |\nabla u|/u$, $\xi_L = \Delta u/u$. We compare this integral to

$$\int_\Omega u^{\alpha+\beta-\gamma-1}(\Delta(u^{\gamma/2}))^2 dx = \frac{\gamma^2}{4}\int_\Omega u^{\alpha+\beta-1}D_0 dx,$$

where D_0 is as in (33) and $\gamma \neq 0$. In contrast to the one-dimensional case, we employ two integration-by-parts rules:

$$0 = \int_\Omega \operatorname{div}\left(u^{\alpha+\beta-4}|\nabla u|^2\nabla u\right)dx = \int_\Omega u^{\alpha+\beta-1}T_1 dx,$$
$$0 = \int_\Omega \operatorname{div}\left(u^{\alpha+\beta-3}(\nabla^2 u - \Delta\mathbb{I})\cdot\nabla u\right)dx = \int_\Omega u^{\alpha+\beta-1}T_2 dx,$$

where

$$T_1 = (\alpha+\beta-4)\xi_G^4 + 2\xi_{GHG} + \xi_G^2\xi_L,$$
$$T_2 = (\alpha+\beta-3)\xi_{GHG} - (\alpha+\beta-3)\xi_G^2\xi_L + \xi_H^2 - \xi_L^2,$$

and $\xi_{GHG} = u^{-3}\nabla u^\top\nabla^2 u\nabla u$, $\xi_H = u^{-1}\|\nabla^2 u\|$. Here, $\|\nabla^2 u\|$ denotes the Frobenius norm of the Hessian.

In order to compare $\nabla^2 u$ and Δu, we employ Lemma 2.1 of [30]:

$$\|\nabla^2 u\|^2 \geq \frac{1}{d}(\Delta u)^2 + \frac{d}{d-1}\left(\frac{\nabla u^\top\nabla^2 u\nabla u}{|\nabla u|^2} - \frac{\Delta u}{d}\right)^2.$$

Therefore, there exists $\xi_R \in \mathbb{R}$ such that

$$\xi_H^2 = \frac{\xi_L^2}{d} + \frac{d}{d-1}\left(\frac{\xi_{GHG}}{\xi_G^2} - \frac{1}{d}\xi_L\right)^2 + \xi_R^2 = \frac{\xi_L^2}{d} + \frac{d}{d-1}\xi_S^2 + \xi_R^2,$$

where we introduced $\xi_S = \xi_{GHG}/\xi_G^2 - \xi_L/d$. Rewriting the polynomials T_1 and T_2 in terms of $\xi = (\xi_G, \xi_L, \xi_S, \xi_R) \in \mathbb{R}^4$ leads to:

$$T_1(\xi) = (\alpha+\beta-4)\xi_G^4 + \frac{2+d}{d}\xi_G^2\xi_L + 2\xi_G^2\xi_S,$$
$$T_2(\xi) = \frac{1-d}{d}(\alpha+\beta-3)\xi_G^2\xi_L + \frac{1-d}{d}\xi_L^2 + \xi_S\xi_G^2(\alpha+\beta-3) + \frac{d}{d-1}\xi_S^2 + \xi_R^2.$$

We wish to find c_1, c_2, $\gamma \in \mathbb{R}$ ($\gamma \neq 0$) and $\kappa > 0$ such that

$$S(\xi) = S_0(\xi) + c_1T_1(\xi) + c_2T_2(\xi) - \kappa D_0(\xi) \geq 0 \quad \text{for all } \xi \in \mathbb{R}^4.$$

The polynomial S can be written as

$$S(\xi) = a_1\xi_G^4 + a_2\xi_G^2\xi_L + a_3\xi_L^2 + a_4\xi_G^2\xi_S + a_5\xi_S^2 + c_2\xi_R^2, \text{ where}$$
$$a_1 = \left(\frac{\alpha}{2}-1\right)(\beta-1) + (\alpha+\beta-4)c_1 - \left(\frac{\gamma}{2}-1\right)^2\kappa,$$

$$
\begin{aligned}
a_2 &= \frac{\alpha}{2} + \beta - 2 + \left(\frac{2}{d} + 1\right) c_1 - (\alpha + \beta - 3)\frac{d-1}{d} c_2 - (\gamma - 2)\kappa, \\
a_3 &= 1 + \frac{1-d}{d} c_2 - \kappa, \\
a_4 &= 2c_1 + (\alpha + \beta - 3)c_2, \\
a_5 &= \frac{d}{d-1} c_2.
\end{aligned}
$$

We remove the variable ξ_R by requiring that $c_2 \geq 0$. The remaining polyomial can be reduced to a quadratic polynomial by setting $x = \xi_L/\xi_G^2$ and $y = \xi_S/\xi_G^2$:

$$
S(x,y) \geq a_1 + a_2 x + a_3 x^2 + a_4 y + a_5 y^2 \geq 0 \quad \text{for all } x,\, y \in \mathbb{R}. \tag{35}
$$

This quadratic decision problem can be solved by employing the computer algebra system `Mathematica`. The result of the command

```
Resolve[ForAll[{x, y}, Exists[{C1, C2, kappa, gamma},
a1 + a2*x + a3*x^2 + a4*y + a5*y^2 >= 0 && kappa > 0
&& gamma != 0]], Reals]
```

gives all $(\alpha, \beta) \in \mathbb{R}^2$ such that there exist c_1, c_2, $\gamma \in \mathbb{R}$ ($\gamma \neq 0$) and $\kappa > 0$ such that (35) holds. This region equals the set M_d, defined in the statement of the theorem.

Similar to the one-dimensional case, we infer that

$$
\frac{dF_\alpha}{dt} \leq -\frac{\alpha^2\beta\kappa}{4} \int_\Omega u^{\alpha+\beta-1} D_0(\xi) dx = -\frac{\alpha^2\beta\kappa}{\gamma^2} \int_\Omega u^{\alpha+\beta-\gamma-1} (\Delta u^{\gamma/2})^2 dx.
$$

Thus, proceeding as in the proof of Theorem 13 and using the identity

$$
\int_\Omega (\Delta f)^2 dx = \int_\Omega \|\nabla^2 f\|^2 dx
$$

for smooth functions f (which can be derived by integrating by parts twice), we obtain

$$
\frac{dF_\alpha}{dt} \leq -\frac{2\beta\kappa}{C_P^2} \inf_\Omega (u_0^{\alpha+\beta-\gamma-1}) \inf_\Omega (u_0^{\gamma-\alpha}) F_\alpha.
$$

Gronwall's lemma concludes the proof. □

Remark 15. Under the additional constraints $a_2 = a_3 = 0$, we are able to solve the decision problem (35) without the help of the computer algebra system. The solution set, however, is slightly smaller than M_d which is obtained from `Mathematica` without these constraints. Indeed, we can compute c_1 and c_2 from the equations $a_2 = a_3 = 0$ giving

$$
c_1 = \frac{d}{d+2}\left(\frac{\alpha}{2} - 1 + \kappa(1 + \gamma - \alpha - \beta)\right), \quad c_2 = \frac{d(1-\kappa)}{d-1}.
$$

The decision problem (35) reduces to

$$
a_1 + a_4 y + a_5 y^2 \geq 0 \quad \text{for all } y \in \mathbb{R}.
$$

If $\kappa < 1$, it holds $c_2 > 0$ and consequently, $a_5 > 0$. Therefore, the above polynomial is nonnegative for all $y \in \mathbb{R}$ if it has no real roots, i.e., if

$$0 \le 4a_1a_5 - a_4^2 = q_0 + q_1\gamma + q_2\gamma^2$$

for some $\gamma \neq 0$, where (for $d > 1$)

$$q_2 = -\frac{d^2\kappa}{(d+2)^2(d-1)^2}\big(3d(d-4)\kappa + (d+2)^2\big) < 0,$$

and q_0, q_1 are polynomials depending on d, α, β, and κ. The above problem is solvable if and only if there exist real roots, i.e. if

$$0 \le q_1^2 - 4q_0q_2 = \frac{4\kappa(1-\kappa)}{(d+2)^2(d-1)^2}(s_0 + s_1\kappa + s_2\kappa^2),$$

where

$$\begin{aligned}
s_0 &= -d(5d-8) + 6d(d-1)\alpha + 2d(d+2)\beta + 2(d+2)\alpha\beta - (2d^2+1)\alpha^2 - (d+2)^2\beta^2,\\
s_1 &= 2d(3d-4) - 2d(4d-3)\alpha - 4d(d+1)\beta + 2d(3d-5)\alpha\beta + 2d(d+1)\alpha^2\\
&\quad - 2d(d-6)\beta^2,\\
s_2 &= -d^2(\alpha+\beta-1)^2.
\end{aligned}$$

We set $f(\kappa) = s_0 + s_1\kappa + s_2\kappa^2$. We have to find $0 < \kappa < 1$ such that $f(\kappa) \ge 0$. Since $s_2 \le 0$, this is possible if $f(\kappa)$ possesses two (not necessarily distinct) real roots κ_0 and κ_1 and if at least one of these roots is between zero and one. Since $f(1) = -(d-1)^2(\alpha-2\beta)^2 \le 0$, there are only two possibilities for κ_0 and κ_1: either $\kappa_0 \le 0 \le \kappa_1 \le 1$ or $0 \le \kappa_0 \le \kappa_1 \le 1$. The first case holds if $f(0) = s_0 \ge 0$, the second one if

$$f'(0) = s_1 \ge 0, \quad f'(1) = s_1 + 2s_2 \le 0, \tag{36}$$

$$\begin{aligned}
s_1^2 - 4s_0s_2 &= -4d^2(\alpha-2\beta+2)(\alpha-2\beta-1)(4-2d+d\alpha+2d\beta)\\
&\quad \times (2-d+(d-2)\alpha+2\beta) \ge 0.
\end{aligned} \tag{37}$$

The set of all $(\alpha,\beta) \in \mathbb{R}^2$ fulfilling these conditions is illustrated in Figure 2. □

5.2. **The discrete case.** At the discrete level, we establish the decay of the first-order entropies in any dimension, with an exponential rate in one space dimension. We recall that the discrete first-order entropies are defined by (9).

Theorem 16 (Exponential decay of F_α^d)**.** *Let $(u_K^k)_{K\in\mathcal{T},k\ge0}$ be the solution to the finite-volume scheme (7) with $\Omega = \mathbb{T}^d$ and $\inf_{K\in\mathcal{T}} u_K^0 \ge 0$. Then, for all $1 \le \alpha \le 2$ and $\beta = \alpha/2$,*

$$F_\alpha^d[u^{k+1}] \le F_\alpha^d[u^k], \quad k \in \mathbb{N}.$$

Furthermore, if $d = 1$ and the grid is uniform with N subintervals,

$$F_\alpha^d[u^k] \le F_\alpha^d[u_0]e^{-\lambda t^k},$$

where $\lambda = 4\beta\sin^2(\pi/N)\min_i((u_i^0)^{2(\beta-1)}) \ge 0$.

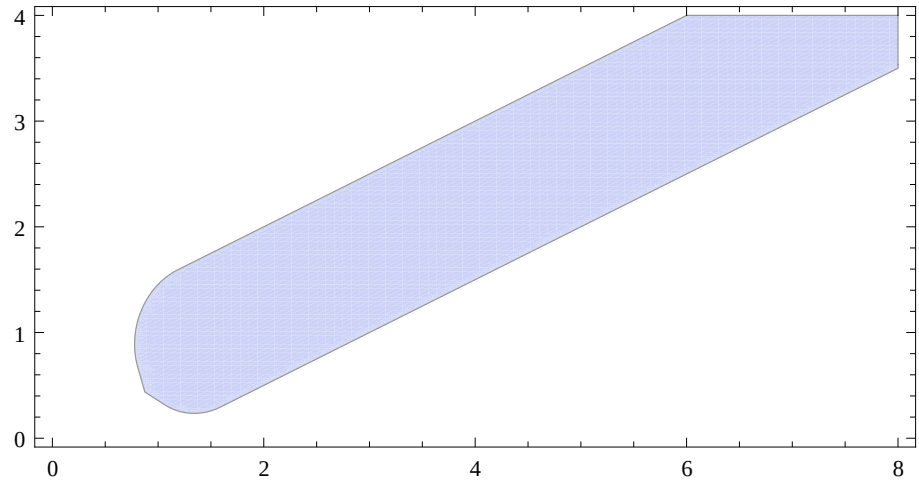

FIGURE 2. Set of all (α,β) fulfilling $s_0 \geq 0$, (36), and (37) for $d = 9$.

Proof. The difference $G_\alpha = F_\alpha^d[u^{k+1}] - F_\alpha^d[u^k]$ can be written as

$$G_\alpha = \frac{1}{2} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma \left(\left((u_K^{k+1})^{\alpha/2} - (u_L^{k+1})^{\alpha/2}\right)^2 - \left((u_K^k)^{\alpha/2} - (u_L^k)^{\alpha/2}\right)^2 \right).$$

Introducing $a_K = (u_K^{k+1} - u_K^k)/\tau$, we find that

$$G_\alpha = \frac{1}{2} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma \left(\left((u_K^{k+1})^{\alpha/2} - (u_L^{k+1})^{\alpha/2}\right)^2 - \left((u_K^{k+1} - \tau a_K)^{\alpha/2} - (u_L^{k+1} - \tau a_L)^{\alpha/2}\right)^2 \right).$$

We claim that G_α is concave with respect to τ. Indeed, we compute

$$\begin{aligned}
\frac{\partial G_\alpha}{\partial\tau} &= \frac{\alpha}{2} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma \big((u_K^{k+1} - \tau a_K)^{\alpha/2} - (u_L^{k+1} - \tau a_L)^{\alpha/2}\big) \\
&\quad \times \big((u_K^{k+1} - \tau a_K)^{\alpha/2-1} a_K - (u_L^{k+1} - \tau a_L)^{\alpha/2-1} a_L\big), \\
\frac{\partial^2 G_\alpha}{\partial\tau^2} &= -\frac{\alpha^2}{4} \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma \big((u_K^{k+1} - \tau a_K)^{\alpha/2-1} a_K - (u_L^{k+1} - \tau a_L)^{\alpha/2-1} a_L\big)^2 \\
&\quad - \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right) \sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}} \tau_\sigma \big((u_K^{k+1} - \tau a_K)^{\alpha/2} - (u_L^{k+1} - \tau a_L)^{\alpha/2}\big) \\
&\quad \times \big((u_K^{k+1} - \tau a_K)^{\alpha/2-2} a_K^2 - (u_L^{k+1} - \tau a_L)^{\alpha/2-2} a_L^2\big).
\end{aligned}$$

Replacing $u_K^{k+1}-\tau a_K$, $u_L^{k+1}-\tau a_L$ by u_K^k, u_L^k, respectively, the second derivative becomes

$$\begin{aligned}\frac{\partial^2 G_\alpha}{\partial\tau^2} &= -\frac{\alpha^2}{4}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u_K^k)^{\alpha/2-1}a_K-(u_L^k)^{\alpha/2-1}a_L\big)^2 \\ &\quad -\frac{\alpha}{2}\Big(\frac{\alpha}{2}-1\Big)\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)\big((u_K^k)^{\alpha/2-2}a_K^2-(u_L^k)^{\alpha/2-2}a_L^2\big) \\ &= -\frac{\alpha}{4}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma(c_1a_K^2+c_2a_Ka_L+c_3a_L^2),\end{aligned}$$

where

$$\begin{aligned}c_1 &= (\alpha-2)\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)(u_K^k)^{\alpha/2-2}+\alpha(u_K^k)^{\alpha-2}, \\ c_2 &= -2\alpha(u_K^k)^{\alpha/2-1}(u_L^k)^{\alpha/2-1}, \\ c_3 &= -(\alpha-2)\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)(u_L^k)^{\alpha/2-2}+\alpha(u_L^k)^{\alpha-2}.\end{aligned}$$

We show that the quadratic polynomial in the variables a_K and a_L is nonnegative for all u_K^k and u_L^k. This is the case if and only if $c_1\ge 0$ and $4c_1c_3-c_2^2\ge 0$. The former condition is equivalent to

$$2(\alpha-1)(u_K^k)^{\alpha-2}\ge(\alpha-2)(u_K^k)^{\alpha/2-2}(u_L^k)^{\alpha/2},$$

which is true for $1\le\alpha\le 2$. After an elementary computation, the latter condition becomes

$$4c_1c_3-c_2^2=8(\alpha-1)(2-\alpha)(u_K^k)^{\alpha/2-2}(u_L^k)^{\alpha/2-2}\big((u_K^k)^{\alpha/2}-(u_L^k)^{\alpha/2}\big)^2\ge 0$$

for $1\le\alpha\le 2$. This proves the concavity of $\tau\mapsto G_\alpha(\tau)$.

A Taylor expansion and $G_\alpha(0)=0$ leads to

$$\begin{aligned}G_\alpha(\tau) &\le G_\alpha(0)+\tau\frac{\partial G_\alpha}{\partial\tau}(0) \\ &= \frac{\alpha\tau}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u_K^{k+1})^{\alpha/2}-(u_L^{k+1})^{\alpha/2}\big)\big((u_K^{k+1})^{\alpha/2-1}a_K-(u_L^{k+1})^{\alpha/2-1}a_L\big) \\ &= \frac{\alpha\tau}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u_K^{k+1})^{\alpha/2}-(u_L^{k+1})^{\alpha/2}\big)(u_K^{k+1})^{\alpha/2-1}a_K \\ &\quad +\frac{\alpha\tau}{2}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\big((u_L^{k+1})^{\alpha/2}-(u_K^{k+1})^{\alpha/2}\big)(u_L^{k+1})^{\alpha/2-1}a_L.\end{aligned}$$

Replacing a_K and a_L by scheme (7) and rearranging the terms, we infer that

$$G_\alpha(\triangle t)=-\frac{\alpha\triangle t}{2{\rm m}(K)}\sum_{K\in\mathcal{T}}\sum_{\substack{\sigma\in\mathcal{E}_{\rm int},\\ \sigma=K|L}}\tau_\sigma\sum_{\substack{\widetilde{\sigma}\in\mathcal{E}_{\rm int},\\ \widetilde{\sigma}'=K|M}}\tau_{\widetilde{\sigma}}(u_K^{k+1})^{\alpha/2-1}$$

$$\times \big((u_K^{k+1})^\beta - (u_M^{k+1})^\beta\big)\big((u_K^{k+1})^{\alpha/2} - (u_L^{k+1})^{\alpha/2}\big). \tag{38}$$

Note that the expression on the right-hand side is the discrete counterpart of the integral

$$-\frac{\alpha}{2}\int_\Omega u^{\alpha/2-1}(u^\beta)_{xx}(u^{\alpha/2})_{xx}dx,$$

appearing in (34). The condition $\alpha = 2\beta$ implies immediately the monotonicity of $k \mapsto F_\alpha^d[u^k]$.

For the proof of the second statement, let $d = 1$ and decompose the interval Ω in N subintervals $K_1, \ldots, K_N$ of length $h > 0$. Because of the periodic boundary conditions, we may set $u_{N+1}^k = u_0^k$ and $u_{-1}^k = u_N^k$, where u_i^k is the approximation of the mean value of $u(\cdot, t^k)$ on the subinterval K_i, $i = 1, \ldots, N$. We rewrite (38) for $\alpha = 2\beta$ in one space dimension:

$$\begin{aligned} G_{2\beta}(\tau) &\le -\frac{\beta\tau}{2h}\sum_{i=1}^N\bigg(\sum_{j\in\{i-1,i+1\}}(u_i^{k+1})^{\beta-1}\big((u_i^{k+1})^\beta - (u_j{}^{k+1})^\beta\big)\bigg)^2 \\ &\le -\frac{\beta\tau}{2h}\min_{i=1,\ldots,N}\big((u_i^{k+1})^{2(\beta-1)}\big)\sum_{i=1}^N(z_i - z_{i-1})^2, \end{aligned}$$

where $z_i = (u_i^{k+1})^\beta - (u_{i+1}^{k+1})^\beta$. The periodic boundary conditions imply that $\sum_{i=1}^N z_i = 0$. Hence, we can employ the discrete Wirtinger inequality in [41, Theorem 1] to obtain

$$\begin{aligned} G_{2\beta}(\tau) &\le -\frac{2\beta\tau}{h}\sin^2\frac{\pi}{N}\min_{i=1,\ldots,N}\big((u_i^k)^{2(\beta-1)}\big)\sum_{i=1}^N z_i^2 \\ &= -\frac{4\beta\tau}{h}\sin^2\frac{\pi}{N}\min_{i=1,\ldots,N}\big((u_i^k)^{2(\beta-1)}\big)F_\alpha^d[u^{k+1}]. \end{aligned}$$

By the discrete maximum principle, $\max_i(u_i^{k+1})^{2(1-\beta)} \le \max_i(u_i^0)^{2(1-\beta)}$ which is equivalent to $\min_i(u_i^{k+1})^{\beta-1} \ge \min_i(u_i^0)^{\beta-1}$. Therefore,

$$F_\alpha^d[u^{k+1}] - F_\alpha^d[u^k] = G_{2\beta}(\triangle t) \le -\frac{4\beta\triangle t}{h}\sin^2\frac{\pi}{N}\min_{i=1,\ldots,N}\big((u_i^0)^{2(\beta-1)}\big)F_\alpha^d[u^{k+1}],$$

and Gronwall's lemma finishes the proof. □

6. Numerical experiments

We illustrate the time decay of the solutions to the discretized porous-medium ($\beta = 2$) and fast-diffusion equation ($\beta = 1/2$) in one and two space dimensions.

First, let $\beta = 2$. We recall that the Barenblatt profile

$$u_B(x,t) = (t+t_0)^{-A}\Big(C - \frac{B(\beta-1)}{2\beta}\,\frac{|x-x_0|^2}{(t+t_0)^{2B}}\Big)_+^{1/(\beta-1)}$$

is a special solution to the porous-medium equation in the whole space. (Here, z_+ denotes the positive part of a function $z_+ := \max\{0, z\}$.) The constants are given by

$$A = \frac{d}{d(\beta-1)+2}, \quad B = \frac{1}{d(\beta-1)+2},$$

and C is typically determined by the initial datum via $\int_\Omega u(x,t)dx = \int_\Omega u(x,0)dx$. We choose $C = B(\beta-1)(2\beta)^{-1}(t_1+t_0)^{-2B}|x_1-x_0|^2$, where $t_1 > 0$ is the smallest time for which $u(x_1,t_1) = 0$.

In the one-dimensional situation, we choose $\Omega = (0,1)$ with homogeneous Neumann boundary conditions and a uniform grid $(x_i, t^j) \in [0,1] \times [0,0.2]$ with $1 \le i \le 50$ and $0 \le j \le 1000$, i.e., the space grid size is $\triangle x = 0.02$ and the time step size equals $\triangle t = 2 \cdot 10^{-4}$. We have chosen a very small time step size for a smoother graphical presentation of the solution, but the implicit scheme clearly also works for time step sizes of the order of $\triangle x$ and for smaller values of $\triangle x$. The initial datum is given by the Barenblatt profile $u_B(\cdot, 0)$ with $x_0 = 0.5$, $x_1 = 1$ and $t_0 = 0.01$. The constant C is computed by using $t_1 = 0.1$, which yields $C \approx 0.091$. For $0 \le t \le 0.1$, the analytical solution corresponds to the Barenblatt profile.

The time decay of the zeroth- and first-order entropies are depicted in Figure 3 in semi-logarithmic scale for various values of α. The decay rates are exponential for sufficiently large times, even for $\alpha > 1$ (compare to Theorem 12) and for $\alpha \neq 2\beta$ (see Theorem 16), which indicates that the conditions imposed in these theorems are technical. For small times, the decay seems to be faster than the decay in the large-time regime. This fact has been already observed in [10, Remark 4]. There is a significant change in the decay rate of the first-order entropies F_α^d for times around $t_1 = 0.1$. Indeed, the positive part of the discrete solution, which approximates the Barenblatt profile u_B for $t < t_1$, arrives the boundary and does not approximate u_B anymore. The change is more apparent for $\alpha < 1$.

Next, we investigate the two-dimensional situation (still with $\beta = 2$). The domain $\Omega = (0,1)^2$ is divided into 144 quadratic cells each of which consists of four control volumes (see Figure 4). Again we employ the Barenblatt profile as the initial datum, choosing $t_0 = 0.01$, $t_1 = 0.1$, and $x_0 = (0.5, 0.5)$, and impose homogeneous boundary conditions. The time step size equals $\triangle t = 8 \cdot 10^{-4}$.

In Figure 5, the time evolution of the (logarithmic) zeroth- and first-order entropies are presented. Again, the decay seems to be exponential for large times, even for values of α not covered by the theoretical results. At time $t = t_1$, the profile reaches the boundary of the domain. Since the radially symmetric profile does not reach the boundary everywhere at the same time, the time decay rate of F_α^d does not change as distinct as in Figure 3.

Let $\beta = 1/2$. The one-dimensional interval $\Omega = (0,1)$ is discretized as before using 51 grid points and the time step size is $\triangle t = 2 \cdot 10^{-4}$. We impose homogeneous Neumann boundary conditions. As initial datum, we choose the following truncated polynomial $u_0(x) = C((x_0 - x)(x - x_1))_+^2$, where $x_0 = 0.3$, $x_1 = 0.7$, and $C = 3000$. In the two-dimensional box $\Omega = (0,1)^2$, we employ the discretization described above and the initial datum $u_0(x) = C(R^2 - |x - x_0|^2)_+^2$, where $R = 0.2$, $x_0 = (0.5, 0.5)$ and again $C = 3000$.

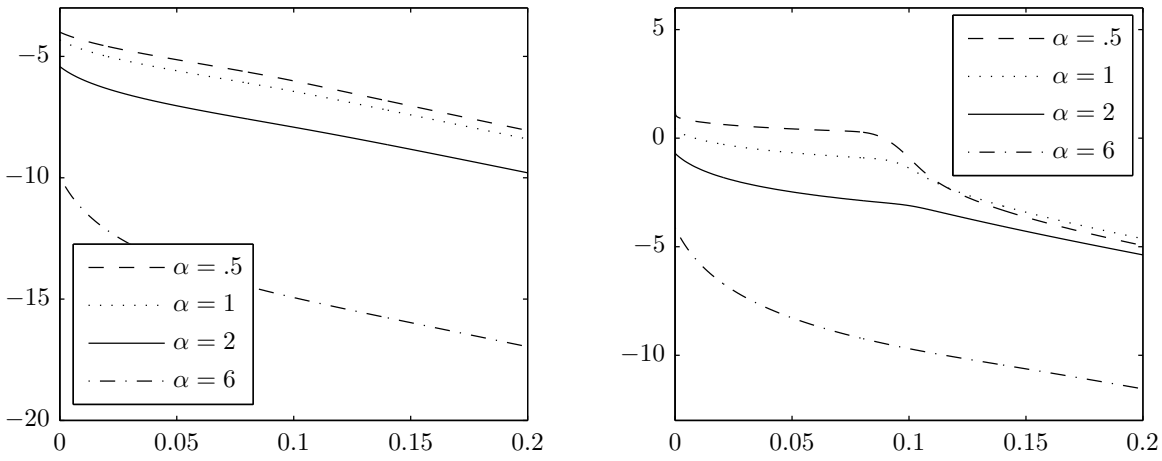

FIGURE 3. The natural logarithm of the entropies $\log(E^d_\alpha[u](t))$ (left) and $\log(F^d_\alpha[u](t))$ (right) versus time for different values of α ($\beta = 2$, $d = 1$).

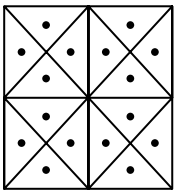

FIGURE 4. Four of the 144 cells used for the two-dimensional finite-volume scheme.

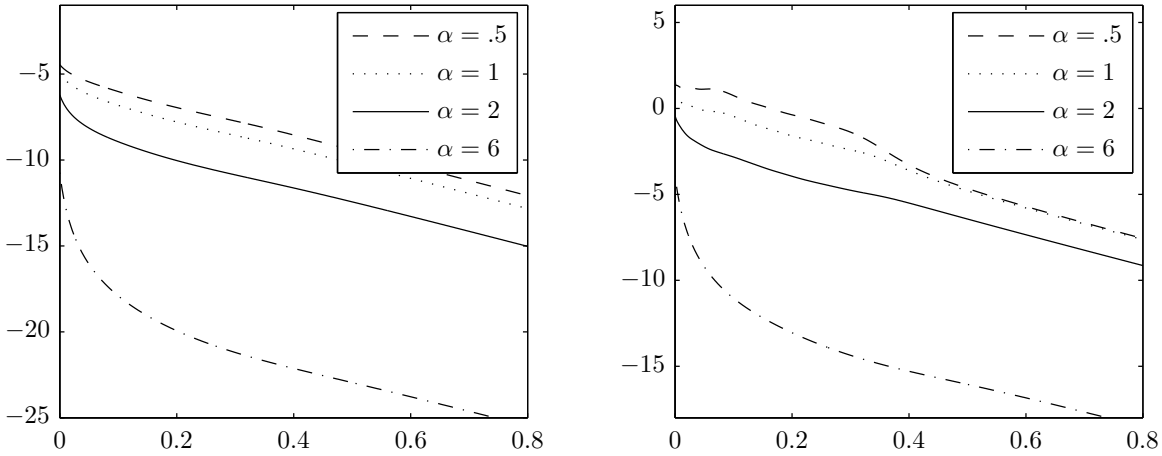

FIGURE 5. The natural logarithm of the entropies $\log(E^d_\alpha[u](t))$ (left) and $\log(F^d_\alpha[u](t))$ (right) versus time for different values of α ($\beta = 2$, $d = 2$).

In the fast-diffusion case $\beta < 1$, we do not expect significant changes in the decay rate since the initial values propagate with infinite speed. This expectation is supported by the numerical results presented in Figures 6 and 7. For a large range of values of α, the decay rate is exponential, at least for large times. Interestingly, the rate seems to approach almost the same value for $\alpha \in \{0.5, 1, 2\}$ in Figure 7.

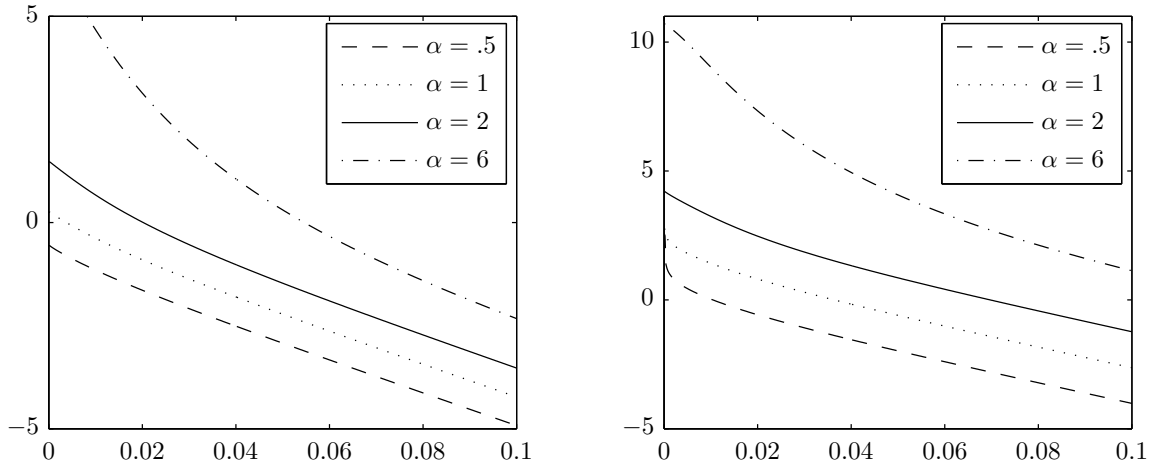

FIGURE 6. The natural logarithm of the entropies $\log(E^d_\alpha[u](t))$ (left) and $\log(F^d_\alpha[u](t))$ (right) versus time for different values of α ($\beta = 1/2$, $d = 1$).

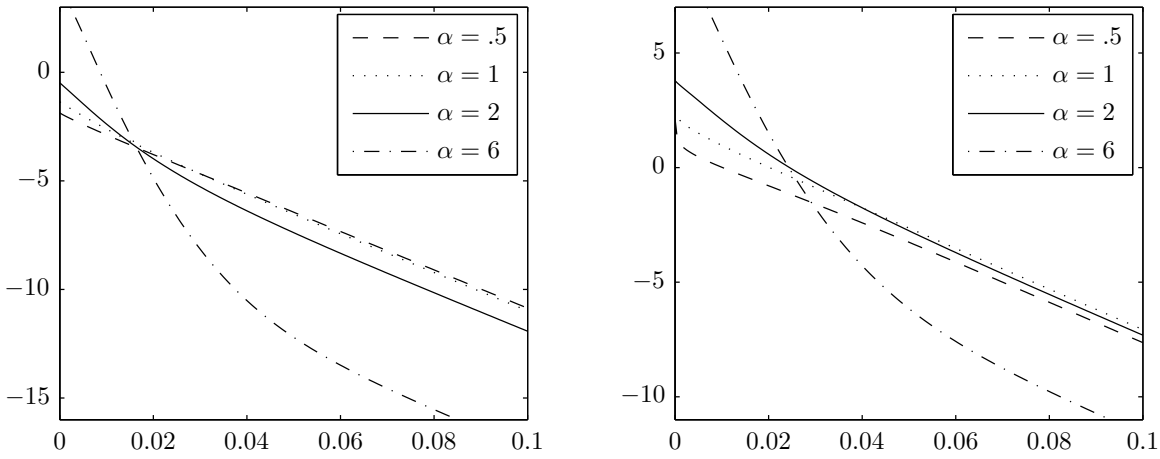

FIGURE 7. The natural logarithm of the entropies $\log(E^d_\alpha[u](t))$ (left) and $\log(F^d_\alpha[u](t))$ (right) versus time for different values of α ($\beta = 1/2$, $d = 2$).

Appendix A. Some technical lemmas

A.1. **Discrete Gronwall lemmas.** First, we prove a rather general discrete nonlinear Gronwall lemma.

Lemma 17 (Discrete nonlinear Gronwall lemma)**.** *Let $f \in C^1([0,\infty))$ be a positive, nondecreasing, and convex function such that $1/f$ is locally integrable. Define*

$$w(x) = \int_1^x \frac{dz}{f(z)}, \quad x \geq 0.$$

Let (x_n) be a sequence of nonnegative numbers such that $x_{n+1} - x_n + f(x_{n+1}) \leq 0$ for $n \in \mathbb{N}_0$. Then

$$x_n \leq w^{-1}\left(w(x_0) - \frac{n}{1+f'(x_0)}\right), \quad n \in \mathbb{N}.$$

Notice that the function w is strictly increasing such that its inverse is well defined.

Proof. Since f is nondecreasing and (x_n) is nonincreasing, we obtain

$$w(x_{n+1}) - w(x_n) = \int_{x_n}^{x_{n+1}} \frac{dz}{f(z)} \leq \frac{x_{n+1}-x_n}{f(x_n)}.$$

The sequence (x_n) satisfies $f(x_{n+1})/(x_{n+1}-x_n) \geq -1$. Therefore,

$$\begin{aligned} w(x_{n+1}) - w(x_n) &\leq \Big(\frac{f(x_{n+1})}{x_{n+1}-x_n} + \frac{f(x_n)-f(x_{n+1})}{x_{n+1}-x_n}\Big)^{-1} \\ &\leq \Big(-1 - \frac{f(x_n)-f(x_{n+1})}{x_n - x_{n+1}}\Big)^{-1}. \end{aligned}$$

By the convexity of f, $f(x_n) - f(x_{n+1}) \leq f'(x_n)(x_n - x_{n+1}) \leq f'(x_0)(x_n - x_{n+1})$, which implies that

$$w(x_{n+1}) - w(x_n) \leq (-1 - f'(x_0))^{-1}.$$

Summing this inequality from $n = 0$ to $N-1$, where $N \in \mathbb{N}$, yields

$$w(x_N) \leq w(x_0) - \frac{N}{1+f'(x_0)}.$$

Applying the inverse function of w shows the lemma. □

The choice $f(x) = \tau K x^\gamma$ for some $\gamma > 1$ in Lemma 17 lead to the following result.

Corollary 18. *Let (x_n) be a sequence of nonnegative numbers satisfying*

$$x_{n+1} - x_n + \tau x_{n+1}^\gamma \leq 0, \quad n \in \mathbb{N},$$

where $K > 0$ and $\gamma > 1$. Then

$$x_n \leq \frac{1}{\big(x_0^{1-\gamma} + c\tau n\big)^{1/(\gamma-1)}}, \quad n \in \mathbb{N},$$

where $c = (\gamma-1)/(1+\gamma\tau x_0^{\gamma-1})$.

A.2. **Some inequalities.** We show some inequalities in two variables.

Lemma 19. *Let α, $\beta > 0$. Then, for all x, $y \geq 0$,*

$$(y^\alpha - x^\alpha)(y^\beta - x^\beta) \geq \frac{4\alpha\beta}{(\alpha+\beta)^2}(y^{(\alpha+\beta)/2} - x^{(\alpha+\beta)/2})^2. \tag{39}$$

Proof. If $y = 0$, inequality (39) holds. Let $y \neq 0$ and set $z = (x/y)^\beta$. Then the inequality is proved if for all $z \geq 0$,

$$f(z) = (1 - z^{\alpha/\beta})(1 - z) - \frac{4\alpha\beta}{(\alpha+\beta)^2}(1 - z^{(\alpha+\beta)/2\beta})^2 \geq 0.$$

We differentiate f twice:

$$f'(z) = -1 - \frac{\alpha}{\beta}z^{\alpha/\beta-1} + \frac{(\alpha-\beta)^2}{\beta(\alpha+\beta)}z^{\alpha/\beta} + \frac{4\alpha}{\alpha+\beta}z^{(\alpha+\beta)/2\beta},$$

$$f''(z) = \frac{\alpha(\alpha-\beta)}{\beta}z^{\alpha/2\beta-3/2}\Big(-\frac{1}{\beta}z^{\alpha/2\beta-1/2} + \frac{\alpha-\beta}{\beta(\alpha+\beta)}z^{\alpha/2\beta+1/2} + \frac{2}{\alpha+\beta}\Big).$$

Then $f(1) = 0$ and $f'(1) = 0$. Thus, if we show that f is convex, the assertion follows. In order to prove the convexity of f, we define

$$g(z) = -\frac{1}{\beta}z^{\alpha/2\beta-1/2} + \frac{\alpha-\beta}{\beta(\alpha+\beta)}z^{\alpha/2\beta+1/2} + \frac{2}{\alpha+\beta}.$$

Then $g(1) = 0$ and it holds

$$g'(z) = \frac{\alpha-\beta}{2\beta^2}z^{\alpha/2\beta-3/2}(-1+z),$$

and therefore, $g'(1) = 0$. Now, if $\alpha > \beta$, $g(0) = 2/(\alpha+\beta) > 0$, and g is decreasing in $[0,1]$ and increasing in $[1,\infty)$. Thus, $g(z) \geq 0$ for all $z \geq 0$. If $\alpha < \beta$ then $g(0+) = -\infty$, and g is increasing in $[0,1]$ and decreasing in $[1,\infty)$. Hence, $g(z) \leq 0$ for $z \geq 0$. Independently of the sign of $\alpha - \beta$, we obtain

$$f''(z) = \frac{\alpha(\alpha-\beta)}{\beta}z^{\alpha/2\beta-3/2}g(z) \geq 0$$

for all $z \geq 0$, which shows the convexity of f. □

Corollary 20. *Let α, $\beta > 0$. Then, for all x, $y \geq 0$,*

$$(y^\beta - x^\beta)(y^\alpha - x^\alpha) \geq \frac{4\alpha\beta}{(\alpha+1)^2}\min\{x^{\beta-1}, y^{\beta-1}\}(y^{(\alpha+1)/2} - x^{(\alpha+1)/2})^2.$$

Proof. We assume without restriction that $y > x$. Then we apply Lemma 19 to $\beta = 1$:

$$(y^\beta - x^\beta)(y^\alpha - x^\alpha) = \frac{y^\beta - x^\beta}{y-x}(y^\alpha - x^\alpha)(y-x) \geq \frac{4\alpha}{(\alpha+1)^2}\frac{y^\beta - x^\beta}{y-x}(y^{(\alpha+1)/2} - x^{(\alpha+1)/2})^2.$$

Since

$$y^\beta - x^\beta = \beta\int_x^y t^{\beta-1}dt \geq \beta\min\{x^{\beta-1}, y^{\beta-1}\}(y-x),$$

the conclusion follows. □

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