Discretely monitored lookback and barrier options : a semi-analytical approach

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Abstract

All the explicit formulae for the valuation of lookback and barrier options available in the financial literature assume continuous monitoring of the underlying asset. In practice, however, monitoring is always discrete, and the gap between continuously and discretely monitored option values can be very large. In this paper, we provide explicit formulae for discretely monitored lookback and barrier options. They allow for non-constant volatility, interest rate, dividend rate and barrier parameters that vary as step functions of time. They can deal with any number and spacing of monitoring dates. They are not restricted to particular payoffs or strike price specifications. We also provide a simple rule for the numerical integration of these high-dimensional formulae, as well as an efficient interpolation method.

Keywords : option, lookback option, barrier option, discrete monitoring, numerical integration, dimension.

JEL classification : G13

1. Introduction

Lookback and barrier options are among the most heavily traded OTC derivatives, particularly in the foreign exchange markets. They are also embedded in a lot of popular structured products in equity and interest rate markets. Lookback options allow investors to « buy at the lowest » and « sell at the highest », but at the cost of a substantially increased premium. Barrier options owe their success to their low price and their strong leverage effect, as well as to the precision and the flexibility with which they can adapt to the needs or views of market participants.

In a stylised Black-Scholes framework, closed form formulae can be obtained for the value of these contracts. They assume continuous monitoring of the underlying asset. In the real markets, however, monitoring is always discrete, for practical but also financial reasons : discretely monitored lookback options are more affordable, thus contributing to eliminate the main obstacle to their commercial success ; discretely monitored knock-out barrier options bear a diminished risk of ending worthless ; and discretely monitored knock-in barrier options benefit from enhanced leverage. Not to mention, in the trader's perspective, the case for facilitated hedging.

As the pricing bias caused by the assumption of continuous monitoring can be very large, alternative approaches are needed to tackle discrete monitoring. Monte Carlo simulation, which is rather slow and inaccurate when it comes to valuing path-dependent contracts with continuous monitoring, is a better choice when extrema are discretely monitored. Nevertheless, the use of advanced variance reduction techniques is uneasy. In particular, closed form lookback and barrier option formulae that might be used as control variates are available only for simple contract specifications; and even when they do exist, they can hardly be used if the number of fixing dates is moderate or low. Likewise, conditional Monte Carlo cannot be easily implemented to reduce the number of random number samplings at each simulation. Lattice-based methods, using either binomial trees (Cheuk and Vorst, 1997) or, better, trinomial trees (Cheuk and Vorst, 1996 ; Ahn et al, 1999) can be accurate and fast. However, they entail stability and convergence issues, especially when parameters and barrier levels are time-varying, and they do not easily cope with complex payoffs. Broadie et al (1996 ; 1999) propose an approximation formula

that involves a correction to the continuous-monitoring closed form formulae. This approach is simple to use and efficient. But, as mentioned earlier, continuous-monitoring closed form formulae are scarce. Furthermore, the accuracy of this approach deteriorates as the number of fixing dates decreases or as the fixing dates become more and more unevenly spaced (a limitation shared by the extrapolation method developed by Levy and Mantion, 1997). Sullivan (2000) introduces an efficient method combining Gaussian quadrature and Chebyshev polynomial approximation, but his analysis covers only the most simple barrier option payoffs and assumes all parameters are constant. Finally, Öhgren (2001) develops an interesting approach based on a result known as the Spitzer identity but, as the author points out himself, it is too restrictive inasmuch as it can be used only if the option is in-the-money and for specific forms of lookback payoff.

In this paper, we provide explicit formulae for discretely monitored European lookback and barrier options that aim to avoid the above limitations :

- their accuracy is not contingent on the spacing of monitoring dates
- they are not restricted to particular payoffs or strike price specifications
- they allow for non-constant volatility, interest rate, dividend rate and barrier parameters that vary as step functions of time, without loss of convergence or stability

Besides, those formulae can show us how the various parameters affect the solution and bring out the interrelationships among them by mere differentiation. However, they have to be numerically integrated, in dimensions that may look daunting as the number of monitoring dates increases. We address this issue by providing simple numerical integration schemes. Furthermore, we show that, as long as monitoring dates are evenly spaced, option values with an arbitrarily large number of monitoring dates can be easily interpolated by means of cubic and quintic splines within our analytical framework, which drastically cuts computational time.

This paper is organized as follows. Section 2 derives explicit formulae for discretely monitored European lookback options. Section 3 derives explicit analytical representations for discretely monitored European barrier options in the form of multiple integrals. Section 4 dicusses the numerical implementation of the analytical results provided in Section 2 and Section 3.

2. Valuation formulae for discretely monitored lookback options

The option life is divided into n intervals, n being the number of fixing dates. We start by showing how to value a fixed-strike discrete lookback option with three fixing dates. The method will then be easily extended to a greater number of fixing dates.

The option life starts at time t_0 and ends at time t_n. Thus, our three fixing dates are : t_1, t_2, t_3 with : t_0 = contract inception $< t_1 < t_2 < t_3$ = expiry. Any spacing can be chosen between the fixing dates.

S_t is the value of the underlying asset at time t. In each interval $[t_{i-1}, t_i]$, it is assumed that the underlying asset follows a geometric brownian motion under the equivalent martingale measure denoted by Q :

$$dS_t = (r_i - \delta_i) S_t dt + \sigma_i S_t dW_t \ , \ t \in [t_{i-1}, t_i]$$

where :

- r_i is the risk-free interest rate in $[t_{i-1}, t_i]$
- δ_i is the dividend rate paid out by the underlying asset in $[t_{i-1}, t_i]$
- σ_i is the volatility parameter in $[t_{i-1}, t_i]$
- W_t is Brownian motion defined on a probability space (Ω, F, P), with $F_t = \sigma(W_s, s \le t)$ being the natural filtration of W_t.

Following the risk-neutral valuation approach, at the contract inception t_0, the undiscounted value of a fixed-strike discrete lookback call option with three fixing dates, is given by :

$$C(S_0; K; t_0, t_1, t_2, t_3) = E^Q\left[\left(\left(S_{t_1} \vee S_{t_2} \vee S_{t_3}\right) - K\right)^+ \mid S_0\right] \qquad (1)$$

where E^Q is the expectation operator under the equivalent martingale measure and K is the strike price.

Let $X_t = \ln(S_t / S_0)$ and $k = \ln(K / S_0)$. Then, denoting the indicator function by $\mathbf{1}_{(.)}$, the conditional expectation in (1) can be expanded as :

$$
\begin{aligned}
&E^Q\left[S_0 e^{(r_1-\delta_1-\sigma_1^2/2)t_1+\sigma_1 W_{t_1}}\mathbf{1}_{\left\{S_{t_1}>K,S_{t_1}>S_{t_2},S_{t_1}>S_{t_3}\right\}}\right]\\
&+E^Q\left[S_0 e^{(r_1-\delta_1-\sigma_1^2/2)t_1+(r_2-\delta_2-\sigma_2^2/2)(t_2-t_1)+\sigma_1 W_{t_1}+\sigma_2\left(W_{t_2}-W_{t_1}\right)}\mathbf{1}_{\left\{S_{t_2}>K,S_{t_2}>S_{t_1},S_{t_2}>S_{t_3}\right\}}\right] \qquad (2)\\
&+E^Q\left[\begin{matrix}S_0 e^{(r_1-\delta_1-\sigma_1^2/2)t_1+(r_2-\delta_2-\sigma_2^2/2)(t_2-t_1)+(r_3-\delta_3-\sigma_3^2/2)(t_3-t_2)+\sigma_1 W_{t_1}+\sigma_2\left(W_{t_2}-W_{t_1}\right)+\sigma_3\left(W_{t_3}-W_{t_2}\right)}\\ \mathbf{1}_{\left\{S_{t_3}>K,S_{t_3}>S_{t_1},S_{t_3}>S_{t_1}\right\}}\end{matrix}\right]\\
&-K\,E^Q\left[\mathbf{1}_{\left\{S_{t_1}>K,S_{t_1}>S_{t_2},S_{t_1}>S_{t_3}\right\}}+\mathbf{1}_{\left\{S_{t_2}>K,S_{t_2}>S_{t_1},S_{t_2}>S_{t_3}\right\}}+\mathbf{1}_{\left\{S_{t_3}>K,S_{t_3}>S_{t_1},S_{t_3}>S_{t_1}\right\}}\right]\\
&=S_0 e^{(r_1-\delta_1)t_1}\tilde{Q}\left(X_{t_1}>k\right)Q\left(X_{t_1}>X_{t_2},X_{t_1}>X_{t_3}\right)\\
&+S_0 e^{(r_1-\delta_1)t_1+(r_2-\delta_2)(t_2-t_1)}\tilde{Q}\left(X_{t_2}>k,X_{t_2}>X_{t_1}\right)Q\left(X_{t_2}>X_{t_3}\right) \qquad (3)\\
&+S_0 e^{(r_1-\delta_1)t_1+(r_2-\delta_2)(t_2-t_1)+(r_3-\delta_3)(t_3-t_2)}\tilde{Q}\left(X_{t_3}>k,X_{t_3}>X_{t_1},X_{t_3}>X_{t_2}\right)\\
&-K\begin{bmatrix}Q\left(X_{t_1}>k\right)Q\left(X_{t_1}>X_{t_2},X_{t_1}>X_{t_3}\right)+Q\left(X_{t_2}>k,X_{t_2}>X_{t_1}\right)Q\left(X_{t_2}>X_{t_3}\right)\\ +Q\left(X_{t_3}>k,X_{t_3}>X_{t_1},X_{t_3}>X_{t_2}\right)\end{bmatrix}
\end{aligned}
$$

To get to (3), we have used the property of independence of Brownian increments on non-overlapping time intervals, so that :

$$\mathbf{1}_{\left\{S_{t_1}>K,S_{t_1}>S_{t_2},S_{t_1}>S_{t_3}\right\}}=\mathbf{1}_{\left\{S_{t_1}>K\right\}}\mathbf{1}_{\left\{S_{t_1}>S_{t_2},S_{t_1}>S_{t_3}\right\}} \qquad (4)$$

and :

$$\mathbf{1}_{\left\{S_{t_2}>K,S_{t_2}>S_{t_1},S_{t_2}>S_{t_3}\right\}}=\mathbf{1}_{\left\{S_{t_2}>K,S_{t_2}>S_{t_1}\right\}}\mathbf{1}_{\left\{S_{t_2}>S_{t_3}\right\}} \qquad (5)$$

We have also applied Girsanov's theorem by defining a new measure $\tilde{Q}$ such that :

$$\left.\frac{d\tilde{Q}}{dQ}\right|_{F_{t_n}}\exp\left(\sum_{i=1}^{n}\sigma_i\left(W_{t_i}-W_{t_{i-1}}\right)-\frac{\sigma_i^2}{2}\left(t_i-t_{i-1}\right)\right) \qquad (6)$$

Let us show how to obtain such probabilities under the $Q-$measure. A change of drift from :

$$\mu_n=\sum_{i=1}^{n}\left(r_i-\delta_i-\sigma_i^2/2\right)\left(t_i-t_{i-1}\right) \qquad (7)$$

to :

$$\tilde{\mu}_n = \sum_{i=1}^{n} (r_i - \delta_i + \sigma_i^2/2)(t_i - t_{i-1}) \tag{8}$$

provides the corresponding probabilities under the $\tilde{Q}$ – measure.

First, $Q\left(X_{t_1} > X_{t_2}, X_{t_1} > X_{t_3}\right) = Q\left(X_{t_2} - X_{t_1} < 0, X_{t_3} - X_{t_1} < 0\right)$ is the joint cumulative distribution function of two correlated increments of a geometric Brownian motion with parameters that vary as step functions of time . Both increments are normally distributed :

$$\left(X_{t_2} - X_{t_1}\right) \sim \mathcal{N}\left(\mu_2(t_2 - t_1); \sigma_2^2(t_2 - t_1)\right) \tag{9}$$

$$\left(X_{t_3} - X_{t_1}\right) \sim \mathcal{N}\left(\mu_2(t_2 - t_1) + \mu_3(t_3 - t_2); \sigma_2^2(t_2 - t_1) + \sigma_3^2(t_3 - t_2)\right) \tag{10}$$

The pair $\left(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_1}\right)$ is bivariate normal. The correlation coefficient between $\left(X_{t_2} - X_{t_1}\right)$ and $\left(X_{t_3} - X_{t_1}\right)$ is :

$$\rho = \frac{\operatorname{cov}\left(X_{t_2} - X_{t_1}, X_{t_3} - X_{t_1}\right)}{\sqrt{\operatorname{var}(X_2 - X_1)\operatorname{var}\left(X_{t_3} - X_{t_1}\right)}} = \sqrt{\frac{\sigma_2^2(t_2 - t_1)}{\sigma_2^2(t_2 - t_1) + \sigma_3^2(t_3 - t_2)}} \tag{11}$$

Thus :

$$Q\left(X_{t_1} > X_{t_2}, X_{t_1} > X_{t_3}\right) = N_2\left[-\frac{\mu_2(t_2 - t_1)}{\sigma_2\sqrt{t_2 - t_1}}, -\frac{\mu_2(t_2 - t_1) + \mu_3(t_3 - t_2)}{\sqrt{\sigma_2^2(t_2 - t_1) + \sigma_3^2(t_3 - t_2)}}; \rho\right] \tag{12}$$

where $N_2[.,.;\rho]$ refers to the bivariate standard normal cumulative distribution function.

Next, the probability $Q\left(X_{t_2} > k, X_{t_2} > X_{t_1}\right)$ can be decomposed as :

$$Q\left(X_{t_2} > k, X_{t_2} > X_{t_1}\right) = E^Q\left[\mathbf{1}_{\left\{X_{t_2} > k, X_{t_1} < k\right\}}\right] + E^Q\left[\mathbf{1}_{\left\{X_{t_1} > k, X_{t_2} > X_{t_1}\right\}}\right]$$

$$= Q\left(X_{t_1} < k, X_{t_2} > k\right) + Q\left(X_{t_1} > k\right)Q\left(X_{t_2} - X_{t_1} > 0\right) \tag{13}$$

The correlation between X_{t_1} and X_{t_2} is equal to $\rho = \sqrt{\frac{\sigma_1^2 t_1}{\sigma_1^2 t_1 + \sigma_2^2(t_2 - t_1)}}$, so we have:

$$Q\left(X_{t_2} > k, X_{t_2} > X_{t_1}\right) =$$

$$N_2\left[\frac{k-\mu_1 t_1}{\sigma_1\sqrt{t_1}}, \frac{-k+\mu_2\left(t_2-t_1\right)+\mu_1 t_1}{\sqrt{\sigma_1^2 t_1+\sigma_2^2\left(t_2-t_1\right)}}; -\rho\right] + N\left[\frac{-k+\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right] N\left[\frac{\mu_2\left(t_2-t_1\right)}{\sigma_2\sqrt{t_2-t_1}}\right] \qquad (14)$$

Next, we deal with $Q\left(X_{t_3} > k, X_{t_3} > X_{t_1}, X_{t_3} > X_{t_2}\right)$. The events $\left\{X_{t_3} > k\right\}, \left\{X_{t_3} - X_{t_1} > 0\right\}$ and $\left\{X_{t_3} - X_{t_2} > 0\right\}$ are all correlated, so there is no way to get round a triple integral. The correlation between X_{t_3} and $\left(X_{t_3} - X_{t_1}\right)$ is :

$$\rho_1 = \sqrt{\frac{\sigma_3^2\left(t_3-t_2\right)+\sigma_2^2\left(t_2-t_1\right)}{\sigma_3^2\left(t_3-t_2\right)+\sigma_2^2\left(t_2-t_1\right)+\sigma_1^2 t_1}} \qquad (15)$$

while the correlation between $\left(X_{t_3} - X_{t_2}\right)$ and $\left(X_{t_3} - X_{t_1}\right)$ is :

$$\rho_2 = \sqrt{\frac{\sigma_3^2\left(t_3-t_2\right)}{\sigma_3^2\left(t_3-t_2\right)+\sigma_2^2\left(t_2-t_1\right)}} \qquad (16)$$

Denoting by $N_3[.,.,.;\rho_1,\rho_2]$ the special form of the trivariate normal cumulative distribution function defined in Appendix 1, we obtain :

$$Q\left(X_{t_3} > k, X_{t_3} > X_{t_1}, X_{t_3} > X_{t_2}\right)$$

$$= N_3\left[\begin{array}{l} \dfrac{-k+\mu_3\left(t_3-t_2\right)+\mu_2\left(t_2-t_1\right)+\mu_1 t_1}{\sqrt{\sigma_3^2\left(t_3-t_2\right)+\sigma_2^2\left(t_2-t_1\right)+\sigma_1^2 t_1}}, \dfrac{\mu_3\left(t_3-t_2\right)+\mu_2\left(t_2-t_1\right)}{\sqrt{\sigma_3^2\left(t_3-t_2\right)+\sigma_2^2\left(t_2-t_1\right)}}, \\ \dfrac{\mu_3\left(t_3-t_2\right)}{\sigma_3\sqrt{t_3-t_2}}; \rho_1, \rho_2 \end{array}\right] \qquad (17)$$

Retracing our steps, the value of a fixed-strike lookback discrete option with three fixing dates t_1, t_2 and t_3 within the option life $[t_0; t_3]$ is given by :

PROPOSITION 1

$$V\left(S_0; K; t_0, t_1, t_2, t_3\right)$$

$$= \varepsilon_3\left\{\theta\left[F_1\left(\tilde{P}_{11}P_{12}\right) + F_2\left(\tilde{P}_{21}P_{22}\right) + F_3\left(\tilde{P}_3\right)\right] - \theta K\left[\left(P_{11}P_{12}\right) + \left(P_{21}P_{22}\right) + \left(P_3\right)\right]\right\} \qquad (18)$$

where :

$$P_{11} = N\left[\frac{\theta(-k+f(\mu_1))}{f(\sigma_1)}\right];\ P_{12} = N_2\left[-\frac{\theta f(\mu_2)}{f(\sigma_2)}, -\frac{\theta f(\mu_{2\to 3})}{f(\sigma_{2\to 3})}; \frac{f(\sigma_2)}{f(\sigma_{2\to 3})}\right]$$

$$P_{21} = N_2\left[\frac{\theta(k-f(\mu_1))}{f(\sigma_1)}, \frac{\theta(-k+f(\mu_{1\to 2}))}{f(\sigma_{1\to 2})}; -\frac{f(\sigma_1)}{f(\sigma_{1\to 2})}\right]$$

$$+N\left[\frac{\theta(-k+f(\mu_1))}{f(\sigma_1)}\right]N\left[\frac{\theta f(\mu_2)}{f(\sigma_2)}\right]; P_{22} = N\left[-\frac{\theta f(\mu_3)}{f(\sigma_3)}\right];$$

$$P_3 = N_3\left[\frac{\theta(-k+f(\mu_{1\to 3}))}{f(\sigma_{1\to 3})}, \frac{\theta f(\mu_{2\to 3})}{f(\sigma_{2\to 3})}, \frac{\theta f(\mu_3)}{f(\sigma_3)}; \frac{f(\sigma_{2\to 3})}{f(\sigma_{1\to 3})}, \frac{f(\sigma_3)}{f(\sigma_{2\to 3})}\right]$$

$$\theta = \begin{cases} 1 & \textit{if the option is a call} \\ -1 & \textit{if the option is a put} \end{cases} \qquad F_n = S_0 \exp\left(\sum_{i=1}^{n}(r_i-\delta_i)(t_i-t_{i-1})\right)$$

$$\varepsilon_n = \exp\left(-\sum_{i=1}^{n} r_i(t_i-t_{i-1})\right)$$

$$f(\sigma_i) = \sqrt{\sigma_i^2(t_i-t_{i-1})};\ \ f(\sigma_{m\to n}) = \sqrt{\sum_{i=m}^{n}\sigma_i^2(t_i-t_{i-1})}$$

$$f(\mu_i) = \mu_i(t_i-t_{i-1});\ \ f(\mu_{m\to n}) = \sum_{i=m}^{n}\mu_i(t_i-t_{i-1})$$

$$f(\tilde{\mu}_i) = \tilde{\mu}_i(t_i-t_{i-1});\ \ f(\tilde{\mu}_{m\to n}) = \sum_{i=m}^{n}\tilde{\mu}_i(t_i-t_{i-1})$$

$\tilde{P}_{11}, \tilde{P}_{21}$ and $\tilde{P}_3$ are the same as P_{11}, P_{21} and P_3 respectively, except for the drift coefficients, given by : $\tilde{\mu}_i = r_i - \delta_i + \sigma_i^2/2$, instead of : $\mu_i = r_i - \delta_i - \sigma_i^2/2$.

A formula for a fixed-strike discrete lookback call option with four fixing dates can be derived by following the same steps as with three fixing dates. The option life starts at time t_0 and ends at time t_4, so that our four fixing dates are : t_1, t_2, t_3, t_4, with : $t_0 < t_1 < t_2 < t_3 < t_4$. Performing the necessary calculations, it can be shown that the value of such an option is given by :

PROPOSITION 2

$$V(S_0;K;t_0,t_1,t_2,t_3,t_4)$$
$$= \varepsilon_4\{\theta[F_1(\tilde{P}_{11}P_{12}) + F_2(\tilde{P}_{21}P_{22}) + F_3(\tilde{P}_{31}P_{32}) + F_4(\tilde{P}_4)] - \theta K[(P_{11}P_{12}) + (P_{21}P_{22}) + (P_{31}P_{32}) + P_4]\} \quad (19)$$

where :

$$P_{11} = N\left[\frac{\theta(-k+f(\mu_1))}{f(\sigma_1)}\right]; P_{12} = N_3\left[-\frac{\theta f(\mu_2)}{f(\sigma_2)}, -\frac{\theta f(\mu_{2\to3})}{f(\sigma_{2\to3})}, -\frac{\theta f(\mu_{2\to4})}{f(\sigma_{2\to4})}; \frac{f(\sigma_2)}{f(\sigma_{2\to3})}, \frac{f(\sigma_{2\to3})}{f(\sigma_{2\to4})}\right]; P_{21} = N_2\left[\frac{\theta(k-f(\mu_1))}{f(\sigma_1)}, \frac{\theta(-k+f(\mu_{1\to2}))}{f(\sigma_{1\to2})}; -\frac{f(\sigma_1)}{f(\sigma_{1\to2})}\right]$$
$$+N\left[\frac{\theta(-k+f(\mu_1))}{f(\sigma_1)}\right]N\left[\frac{\theta f(\mu_2)}{f(\sigma_2)}\right]; P_{22} = N_2\left[-\frac{\theta f(\mu_3)}{f(\sigma_3)}, -\frac{\theta f(\mu_{3\to4})}{f(\sigma_{3\to4})}; \frac{f(\sigma_3)}{f(\sigma_{3\to4})}\right]$$
$$P_{31} = N_3\left[\frac{\theta(-k+f(\mu_{1\to3}))}{f(\sigma_{1\to3})}, \frac{\theta f(\mu_{2\to3})}{f(\sigma_{2\to3})}, \frac{\theta f(\mu_3)}{f(\sigma_3)}; \frac{f(\sigma_{2\to3})}{f(\sigma_{1\to3})}, \frac{f(\sigma_3)}{f(\sigma_{2\to3})}\right];$$
$$P_{32} = N\left[-\frac{\theta f(\mu_4)}{f(\sigma_4)}\right]; P_4 = N_4\left[\frac{\theta(-k+f(\mu_{1\to4}))}{f(\sigma_{1\to4})}, \frac{\theta f(\mu_{2\to4})}{f(\sigma_{2\to4})}, \frac{\theta f(\mu_{3\to4})}{f(\sigma_{3\to4})}, \frac{\theta f(\mu_4)}{f(\sigma_4)}; \frac{f(\sigma_{2\to4})}{f(\sigma_{1\to4})}, \frac{f(\sigma_{3\to4})}{f(\sigma_{2\to4})}, \frac{f(\sigma_4)}{f(\sigma_{3\to4})}\right]$$

$\tilde{P}_{11}, \tilde{P}_{21}, \tilde{P}_{31}$ and $\tilde{P}_4$ are the same as P_{11}, P_{21}, P_{31} and P_4 respectively, except for the drift coefficients, given by : $\tilde{\mu}_i = r_i - \delta_i + \sigma_i^2/2$, instead of : $\mu_i = r_i - \delta_i - \sigma_i^2/2$. The other notations are defined in Proposition 1.

Appendix 2 provides a compact expression for the value of a fixed-strike discrete lookback option with n fixing dates, for any n such that $n \in \mathbb{N}$ and $n > 4$.

Knowing a formula for the value of a fixed-strike discrete lookback option, we can easily deduce the value of a floating-strike discrete lookback option. Indeed, the well known parity relation between fixed-strike and floating-strike lookback options in a continuous framework is readily transposed into a discrete framework. If we denote by $C(S_0;K;t_0,t_1,t_2,...,t_n)$ the value of a discrete lookback call and by

$P\left(S_0;K;t_0,t_1,t_2,...,t_n\right)$ the value of a discrete lookback put, we have the following parity relations :

$$C^{Fixed}\left(S_0;K;t_0,t_1,t_2,...,t_n\right) = P^{floating}\left(S_0;K;t_0,t_1,t_2,...,t_n\right)$$
$$+S_0\exp\left(-\sum_{i=1}^{n}\delta_i\left(t_i-t_{i-1}\right)\right)-K\exp\left(-\sum_{i=1}^{n}r_i\left(t_i-t_{i-1}\right)\right) \quad (20)$$

$$P^{Fixed}\left(S_0;K;t_0,t_1,t_2,...,t_n\right) = C^{floating}\left(S_0;K;t_0,t_1,t_2,...,t_n\right)$$
$$-S_0\exp\left(-\sum_{i=1}^{n}\delta_i\left(t_i-t_{i-1}\right)\right)+K\exp\left(-\sum_{i=1}^{n}r_i\left(t_i-t_{i-1}\right)\right) \quad (21)$$

3. Analytical valuation of discretely monitored barrier options

The same model assumptions and notations as in section 2 are used here. In particular, in each interval $\left[t_{i-1},t_i\right]$, the underlying asset follows a geometric brownian motion with a given set of piecewise constant parameters. We first seek an analytical formula for the value of a European-style up-and-out barrier option, with two different barrier levels H_1 and H_2 discretely monitored at times t_1 and t_2 respectively ; t_0 is the option contract inception, t_3 is the contract expiry and we have : $t_0 < t_1 < t_2 < t_3$. Following the risk-neutral valuation approach, the value of such an up-and-out call option is given by :

$$e^{-r_1t_1-r_2(t_2-t_1)-r_3(t_3-t_2)}E^Q\left[\left(S_{t_3}-K\right)^+\mathbf{1}_{\left\{S_{t_1}<H_1,S_{t_2}<H_2\right\}}\middle|S_0\right]$$
$$= S_0e^{-\delta_1t_1-\delta_2(t_2-t_1)-\delta_3(t_3-t_2)}\tilde{Q}\left(X_{t_1}<h_1,X_{t_2}<h_2,X_{t_3}>k\right)$$
$$-e^{-r_1t_1-r_2(t_2-t_1)-r_3(t_3-t_2)}K\,Q\left(X_{t_1}<h_1,X_{t_2}<h_2,X_{t_3}>k\right) \quad (22)$$

with $h_i = \ln\left(H_i/S_0\right)$; the other notations are the same as in section 1.

By conditioning and applying the Markov property of Brownian motion, simple calculations yield :

$$Q\left(X_{t_1} < h_1, X_{t_2} < h_2, X_{t_3} > k\right)$$

$$= E^Q\left[1_{\left\{X_{t_1} < h_1, X_{t_2} < h_2\right\}} E^Q\left[X_{t_3} > k \middle| X_{t_2} < h_2\right]\right] \tag{23}$$

$$= \int_{-\infty}^{h_1}\int_{-\infty}^{h_2} \frac{e^{-\left(\frac{x-\mu_1 t_1}{\sigma_1\sqrt{t_1}}\right)^2}}{\sigma_1\sqrt{t_1}\sqrt{2\pi}} \frac{e^{-\left(\frac{y-x-\mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2-t_1}}\right)^2}}{\sigma_2\sqrt{t_2-t_1}\sqrt{2\pi}} N\left[\frac{-k+\mu_3(t_3-t_2)+y}{\sigma_3\sqrt{t_3-t_2}}\right] dy dx \tag{24}$$

$$= \int_{-\infty}^{\frac{h_1-\mu_1 t_1}{\sigma_1\sqrt{t_1}}} \int_{-\infty}^{\frac{h_2-\mu_1 t_1-\mu_2(t_2-t_1)}{\sigma_2\sqrt{t_2}}} \int_{-\infty}^{\frac{-k+\mu_1 t_1+\mu_2(t_2-t_1)+\mu_3(t_3-t_2)}{\sigma_3\sqrt{t_3}}} \Phi(x,y,z)\, dz\, dy\, dx \tag{25}$$

where :

$$\Phi(x,y,z) = \frac{e^{-\frac{x^2}{2} - \frac{\left(y-\frac{\sigma_1}{\sigma_2}\sqrt{\frac{t_1}{t_2}}x\right)^2}{2(1-t_1/t_2)} - \frac{\left(z-\frac{\sigma_2}{\sigma_3}\sqrt{\frac{t_2}{t_3}}y\right)^2}{2(1-t_2/t_3)}}}{(2\pi)^{3/2}\sqrt{(1-t_1/t_2)(1-t_2/t_3)}} \tag{26}$$

The function $\Phi(x,y,z)$ in (26) is the same as the integrand in the cumulative distribution function $N_3[.,.,.;\rho_{12},\rho_{23}]$ defined in Appendix 1, except for the correlation coefficients ρ_{ij}, $(i,j) \in \mathbb{N}^2$. When all volatility coefficients on every sub-interval $[t_{i-1}, t_i]$ are the same, then both functions are identical.

The extension of the integral in (25) to a greater number of monitoring dates, as well as to put options, is analytically straightforward. In general, for a European-style up-and-out option with $n-1$ barrier levels $H_1, H_2, ..., H_{n-1}$ monitored at times $t_1, t_2, ..., t_{n-1}$ respectively, with : $t_0 < t_1 < ...t_{n-1} < t_n$, t_n being the expiry date, we have :

$$Q\left(X_{t_1} < h_1, X_{t_2} < h_2, ..., X_{t_{n-1}} < h_{n-1}, X_{t_n} <> k\right)$$

$$= \int_{-\infty}^{\Lambda_1} \int_{-\infty}^{\Lambda_2} \dots \int_{-\infty}^{\Lambda_{n-1}} \int_{-\infty}^{\Lambda_n} \Phi\left(x_1, x_2, \dots, x_{n-1}, x_n\right) dx_n dx_{n-1} \dots dx_2 dx_1 \tag{27}$$

where :

$$\Lambda_1 = \frac{h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}};\ \Lambda_2 = \frac{h_2 - \mu_1 t_1 - \mu_2 \left(t_2 - t_1\right)}{\sigma_2 \sqrt{t_2}};\ \dots;$$

$$\Lambda_{n-1} = \frac{h_{n-1} - \mu_1 t_1 - \mu_2 \left(t_2 - t_1\right) - \dots - \mu_{n-1} \left(t_{n-1} - t_{n-2}\right)}{\sigma_{n-1} \sqrt{T_{n-1}}}$$

$$\Lambda_n = \frac{\theta(-k) + \theta\mu_1 t_1 + \theta\mu_2 \left(t_2 - t_1\right) - \dots + \theta\mu_n \left(t_n - t_{n-1}\right)}{\sigma_n \sqrt{t_n}}$$

$$\Phi\left(x_1, x_2, \dots, x_{n-1}, x_n\right) = \tag{28}$$

$$\frac{e^{-\frac{x_1^2}{2} - \frac{\left(x_2 - \frac{\sigma_1}{\sigma_2}\sqrt{\frac{t_1}{t_2}} x_1\right)^2}{2\left(1 - t_1/t_2\right)} - \dots - \frac{\left(x_{n-1} - \frac{\sigma_{n-2}}{\sigma_{n-1}}\sqrt{\frac{t_{n-2}}{t_{n-1}}} x_{n-2}\right)^2}{2\left(1 - t_{n-2}/t_{n-1}\right)} - \frac{\left(x_n - \frac{\sigma_{n-1}}{\sigma_n}\sqrt{\frac{t_{n-1}}{t_n}} x_{n-1}\right)^2}{2\left(1 - t_{n-1}/t_n\right)}}}{(2\pi)^{n/2} \sqrt{\left(1 - t_1/t_2\right) \dots \left(1 - t_{n-2}/t_{n-1}\right)\left(1 - t_{n-1}/t_n\right)}}$$

$$\theta = \begin{cases} 1 & \text{if the option is a call} \\ -1 & \text{if the option is a put} \end{cases} \qquad <>\equiv \begin{cases} > & \text{if the option is a call} \\ < & \text{if the option is a put} \end{cases}$$

The value of an up-and-in option is easily deduced by subtracting the value of the corresponding up-and-out option from that of a plain vanilla option. As for the values of down-and-out or double knock-out options, they can be written in a similar manner simply by modifying the integration bounds.

4. Numerical implementation

The formulae provided in Section 2 and Section 3 raise the question of multidimensional integration. Indeed, as the number of monitoring dates increases, so does the dimension of the integrals involved in the computation of the multivariate distribution functions that appear in these formulae.

In low dimensions, this problem can be analytically tackled. The following results provide two exact integration rules that reduce the trivariate integral to a univariate one and the quadrivariate one to a bivariate one. Let $N_3[a,b,c;\rho_{12},\rho_{23}]$ and $N_4[a,b,c,d;\rho_{12},\rho_{23},\rho_{34}]$ denote the trivariate and quadrivariate integrals defined in Appendix 1 respectively. Then,

$$N_3[a,b,c;\rho_{12},\rho_{23}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{b} e^{-x^2/2} N\left[\frac{a-\rho_{12}x}{\sqrt{1-\rho_{12}^2}}\right] N\left[\frac{c-\rho_{23}x}{\sqrt{1-\rho_{23}^2}}\right] dx \qquad (29)$$

$N_4[a,b,c,d;\rho_{12},\rho_{23},\rho_{34}]$

$$\int_{-\infty}^{b}\int_{-\infty}^{c} \frac{e^{\left(-\frac{x^2}{2}-\frac{(y-\rho_{23}x)^2}{2\left(1-\rho_{23}^2\right)}\right)}}{2\pi\sqrt{1-\rho_{23}^2}} N\left[\frac{a-\rho_{12}x}{\sqrt{1-\rho_{12}^2}}\right] N\left[\frac{d-\rho_{34}y}{\sqrt{1-\rho_{34}^2}}\right] dy\,dx \qquad (30)$$

The numerical integrations implied by (29) and (30) are very easy to perform. A level of at least 10^{-6} accuracy, which is more than enough for option pricing, can be achieved with a mere 16-point Gauss-Legendre rule (and a lower bound of – 8.5 in the integral), which is extremely efficient.

However, as dimension rises, such analytical simplifications seem more difficult to carry out. To adress the issue of multidimensional integration in high dimension, one can turn to Monte Carlo integration, the convergence rate of which is independent of dimension.

To compute the $m-$dimensional integral $N_m[x_1,x_2,...,x_m;\rho_1,\rho_2,...,\rho_{m-1}]$, one can draw n samples of m independent normal deviates with zero mean and unit variance, denoted by $y_i^j, i\in[1,...,m], j\in[1,...,n]$, then turn them into n samples of m correlated normal deviates, denoted by w_i^j, through the following transformation :

$$\left\{w_1^j = y_1^j, w_2^j = \rho_1 w_1^j + \sqrt{(1-\rho_1^2)}y_2, ..., w_m^j = \rho_{m-1} w_{m-1}^j + \sqrt{(1-\rho_{m-1}^2)}y_m^j\right\} \qquad (31)$$

and then test the relevant conditions for each deviate in each sample :

$w_1^j < x_1,\ w_2^j < x_2,\ ...,\ w_m^j < x_m, \quad j\in[1,...,n]$

By the law of large numbers, repeating this procedure will achieve convergence to the exact value of $N_m[x_1,x_2,...,x_m;\rho_1,\rho_2,...,\rho_{m-1}]$ as n tends to infinity. The convergence

rate is in the order of $1/\sqrt{n}$. Besides, this method is easily combined with classical variance reduction techniques such as antithetic variates, or stratified and Latin hypercube sampling.

Let us now implement the formula for a discretely monitored fixed strike lookback call option. For this first example, we use the same volatility, riskless rate and dividend rate parameters during the entire option life. This will enable us to draw a comparison with the exact value of a fixed strike lookback call with continuous monitoring, which admits a well-known explicit formula. We assume volatility is equal to 32%, while the riskless rate is 5%, the dividend rate is 1.5% and the option expiry is one year. The spot value of the underlying asset at the contract's inception is 100 and the option is at-the-money. Under those assumptions, the exact value of a plain vanilla call is 14.074 (in a Black-Scholes framework, and to four decimal places), while the exact value of a fixed strike lookback call with continuous monitoring is equal to 28.634. Let us now examine how a finite number of fixing dates affects the option value compared with the case in which there is no fixing date at all before expiry (i.e., the case of a plain vanilla call), which is a lower bound, and the case in which there is an infinite number of fixing dates (i.e., the case of a fixed strike lookback call under the assumption of continuous monitoring), which is an upper bound. If we set four evenly spaced fixing dates during the option life (that is, the first fixing occurs at t = 0.25), the option value increases from to 14.074 to 19.732, applying the analytical formula given by Proposition 2, and following the implementation rule given by eq. (30). This value is obtained in less than one second [1], which is much faster than the 26 seconds required to obtain an approximate value of 19.761 after performing 2,000,000 Monte Carlo simulations using antithetic variates and stratified sampling. This is a large increase in value (roughly 40%), which suggests that the bulk of the price transformation is concentrated in the first few fixing dates. This observation is confirmed by computing the option value with eight evenly spaced fixing dates (first fixing at $t = 0.125$), which yields 22.045. Thus, setting only eight evenly spaced fixing dates accounts for 54.5% of the total increase in value that results from the transformation of a plain vanilla contract into a contract with an infinite number of fixing dates. This is

[1] All reported values were obtained on a 2.4 Ghz –clock single computer

obviously linked to the fact that our fixing dates uniformly fill the option life. If we set all eight fixing dates during, say, the sixth month of the option life, the option value diminishes to 18.7 (and to 17.2 if we set all eight fixing dates during the first month of the option life).

Let us now turn to higher monitoring frequencies of the underlying asset. Monte Carlo integration will tend slowly to exact option values when the number of fixing dates becomes large. This efficiency issue is more acute for discretely monitored lookback options than for discretely monitored barrier options. Indeed, the formulae for discretely monitored lookback options are more complicated and they involve the computation of many more high dimensional integrals. Implementing the formula for a one-year fixed strike lookback call with daily monitoring or simply weekly monitoring would be slow and quite cumbersome to code. Fortunately, all that is needed to obtain a fast approximation for the value for our fixed strike lookback call with an arbitrarily large number of fixing dates is to compute its value with four and eight fixing dates, which is easy and accurate using the analytical formulae provided in this paper, and to know the value of the two limiting cases (plain vanilla call and fixed strike lookback call with continuous monitoring), for which there are explicit formulae available. Then, all fixed strike lookback call values with a number of fixing dates greater than eight can be interpolated by means of quintic splines.

Let us denote by $s(x)$ the logarithm of the value of a fixed strike lookback call with e^x fixing dates (with a given set of spot, strike, volatility, riskless rate, dividend rate and expiry parameters). Let us approximate continuous monitoring by 12500 fixing dates in a year (which corresponds to 50 monitoring times per business day [2]). Resuming the inputs of our first example above, we have :

$s(0) = 2.644351543$, $s(1.386294361) = 2.982647$, $s(2.079441542) = 3.09285898$, $s(9.433483923) = 3.35461126$

If we construct a quintic spline out of these data, we obtain [3], for $x > 2.079441542$ (that is, for more than eight fixing dates), the following polynomial of degree 5 :

[2] Adding more fixing dates to the approximation has insignificant effet on the spline value

[3] All the spline polynomials reported in this section were obtained using the spline function in the Maple software .

$$s(x)=2.631426028+0.3324445005x-0.659113161x^2+0.006854474588x^3 \\ -0.0003633055742x^4+0.000007702468719x^5 \quad (32)$$

To obtain the value of a fixed strike lookback call with daily monitoring, for example, it suffices to take the exponential of $s(\ln(250))$. The following table reports a few numerical results.

TABLE 1 : Fixed strike lookback call option
Spot = 100 ; strike = 100 ; expiry = 1 year ; volatility = 32% ; riskless rate = 5% ; dividend rate = 1.5%

Number of fixing dates	Quintic spline interpolation	Monte Carlo simulation 2,000,000 sample
12	23.16	23.22
24	24.72	24.76
50	25.94	25.97
100	26.75	26.79
150	27.11	27.16
250	27.47	27.58

The average discrepancy between interpolated values and Monte Carlo simulation values is equal to 0.36%. Note that it is extremely time-consuming to obtain a reliable Monte Carlo simulation estimator of the fixed strike lookback call with 250 fixing dates, not only because of the number of pseudo random numbers to be drawn, but also because of the very large number of conditions to be tested.

Our second example involving discretely monitored lookback options provides a basic illustration of the advantage of using a model that allows for different parameters over various sub-intervals of the option life. We have observed that setting all fixing dates in the sixth month of the option life substantially decreases the option value in a framework in which volatility remains constant during the entire option life. Now suppose that volatility jumps to 0.75% in the sixth month. Applying the formula provided in section 2, our option value rises from 18.7 (case where volatility is constantly 32%) to 24.1, while the option value when all fixing dates are uniformly spread throughout the option life is 26.5 with this new term structure of volatility. This stylised example hints at the

mispricings we are exposed to if our model cannot allow for volatility, interest rate and dividend rate parameters to vary.

Next, let us move on to discretely monitored barrier options. We focus on an up-and-out put option written on an underlying with spot value $S = 100$, and four different barriers $H_1 = 112, H_2 = 116, H_3 = 118, H_4 = 120$, monitored over four different intervals of the option life (four-step up-and-out put option). We model the motion of the underlying asset by standard geometric Brownian motion because, in this particular case, the option value admits a closed form formula in continuous time (Guillaume, 2001), which, again, allows us to draw a comparison between discrete monitoring and continuous monitoring. Our parameters are the following : volatility = 32% ; riskless rate = 5% ; dividend rate = 1.5% ; expiry = 1 year ; barrier H_1 is monitored within $[t_0 = 0; t_1 = 0.25[$, barrier H_2 is monitored within $[t_1 = 0.25; t_2 = 0.5[$, barrier H_3 is monitored within $[t_2 = 0.5; t_3 = 0.75[$ and barrier H_4 is monitored within $[t_3 = 0.75; t_4 = \text{expiry} = 1[$. Again, we cut computational time by interpolating option values when the number of monitoring dates is large. All we need to do is to compute the option value with four and eight monitoring dates, which is fast and accurate, and then, given the value of a plain vanilla put and the value of the corresponding four-step up-and-out put with continuous monitoring, to construct a cubic spline that will provide option values when the number of monitoring dates is greater than eight. If we denote by $s(x)$ the logarithm of the value of our four-step up-and-out put with e^x fixing dates, we obtain, for $x > 2.079441542$:

$$s(x) = 2.429982994 - 0.117511439x + 0.1134826262x^2 - 0.0004009922778x^3$$

if the option is at-the-money (33)

$$s(x) = 2.834021593 - 0.134440963x + 0.01314524630x^2 - 0.0004644889206x^3$$

if the option is struck at 110 (34)

$$s(x) = 1.919908568 - 0.100517865x + 0.00956291970x^2 - 0.0003379069626x^3$$

if the option is struck at 90 (35)

Those splines are of degree 3, instead of 5 in the lookback option case first examined. This is because the gradual loss in the knock-out option value as the number of monitoring dates increases is smoother than the increase in the lookback option value as the number of fixing dates increases. The following tables, where all monitoring dates are uniformly spread throughout the option life, report a few numerical results :

TABLE 2 : Step up-and-out put option struck at 100 (at-the-money)
Plain vanilla put option value = 10.881
Step up-and-out put option value with continuous monitoring = 7.35

Number of monitoring dates	Cubic spline interpolation	Monte Carlo integration 2,000,000 sample
12	9.04	9.01
24	8.65	8.61
50	8.33	8.28
100	8.08	8.03
150	7.96	7.92
250	7.84	7.75

TABLE 3 : Step up-and-out put option struck at 110 (in-the-money)
Plain vanilla put option value = 16.453
Step up-and-out put option value with continuous monitoring = 10.44

Number of monitoring dates	Cubic spline interpolation	Monte Carlo integration 2,000,000 sample
12	13.11	13.09
24	12.48	12.44
50	11.96	11.91
100	11.56	11.49
150	11.38	11.26
250	11.18	11.07

TABLE 4 : Step up-and-out put option struck at 90 (out-of-the-money)

Plain vanilla put option value = 6.485
Step up-and-out put option value with continuous monitoring = 4.66

Number of monitoring dates	Cubic spline interpolation	Monte Carlo integration 2,000,000 sample
12	5.60	5.58
24	5.40	5.36
50	5.22	5.17
100	5.08	5.02
150	5.02	4.96
250	4.95	4.86

A significant advantage of the analytical representations provided in section 3 is their flexibility : they enable to value a broad range of exotic barrier options that admit no explicit solution under the assumption of continuous monitoring because their implied distribution of joint extrema is analytically untractable. This is the case for the so-called « corridor » or « hot-dog » options, which feature combinations of various double knock-in or knock-out barriers during the option life. Consider, for example, a one-year expiry at-the-money double knock-out call option written on an underlying asset with spot value 100, featuring four up-barriers : $U_1 = 116, U_2 = 122, U_3 = 132, U_4 = 150$, and four down-barriers : $D_1 = 85, D_2 = 80, D_3 = 74, D_4 = 65$; U_1 and D_1 are simultaneously monitored within $[t_0 = 0, t_1 = 0.25[$, U_2 and D_2 are simultaneouly monitored within $[t_1 = 0.25; t_2 = 0.5[$, U_3 and D_3 are simultaneously monitored within $[t_2 = 0.5; t_3 = 0.75[$, U_4 and D_4 are simultaneously monitored within $[t_3 = 0.75; t_4 = \text{expiry} = 1[$. Volatility is equal to : 18% during $[t_0; t_1[$; 25% during $[t_1; t_2[$; 35% during $[t_2; t_3[$; 42% during $[t_3; t_4]$. The riskless rate is equal to 5% during $[t_0; t_2[$ and 5.5% during $[t_2; t_4]$. The dividend rate is equal to 2% during $[t_0; t_2[$ and 1% during $[t_2; t_4]$. Those contract specifications are designed so that the distance between the upper barrier and the lower barrier widens as volatility increases. This option better

protects investors when volatility is high, by reducing the likelihood of knocking-out, while it diminishes protection when volatility is low, thus reducing the overall cost of the option. This ability to adapt to the fluctuations of volatility is one of the main attractions of « corridor » options, so that it would be particularly inappropriate to value them with a model that would set all parameters constant during the entire option life. Table 5 presents a few numerical results, obtained by implementing the analytical representation provided in Section 3 along with the Monte Carlo integration scheme explained at the beginning of this Section.

TABLE 5 : Knock-out corridor call option
Plain vanilla call option value = 13.96

Number of monitoring dates	Monte Carlo integration 2,000,000 sample
8	5.58
12	5.02
24	4.06
50	3.59
250	2.94

We could have constructed a polynomial interpolation again. But, as this option admits no known explicit solution under the assumption of continuous monitoring, we would have had to begin by numerically integrating the formula with discrete monitoring for a very large number of monitoring dates, making it an approximation to the continuous monitoring limit, which would have been very slow. In practice, this will be a worthy time investment only if a lot of option values with discrete monitoring are subsequently required.

Eventually, we illustrate the case of a double knock-in call option, with an up-barrier set at 150 and a down-barrier set at 60, and all monitoring dates set within the last quarter of the option life, during which volatility jumps to 45% after being equal to 15% during the first three quarters (riskless rate = 5%, dividend rate = 1.5%, spot value = 100, strike price = 100, expiry = one year). As mentioned in section 3, this option value must be equal to

the value of a plain vanilla call minus the value of a double knock-out call with the same specifications. Implementing our semi-analytical approach, we find a value of 1.60 with eight monitoring dates, 3.13 with sixteen monitoring dates, and 3.77 with twenty-four monitoring dates.

5. Conclusion

In view of the variety of valuation methods dealing with discretely monitored lookback and barrier options, it is not always an easy task to choose the most appropriate one for a given problem. We should like to close this paper with the following recommendations :

If you need to price a contract with complex payoff specifications (such as a many different single or double barriers) or non-standard monitoring specifications (such as unequally spaced monitoring dates), use :

(i) Monte Carlo simulation if you need to incorporate several stochastic factors into the equation modelling the motion of the underlying asset

(ii) Analytical formulae if you can model the motion of the underlying asset as a continuous process with piecewise constant parameters (in particular, if you want to use the market information contained in volatility surfaces and in the term structure of interest rates)

If you want to price a contract with simple payoff and monitoring specifications :

(i) Continuity adjustment is the fastest and most straightforward method available, but its accuracy depends on the type of option being valued as well as on the number of monitoring dates

(ii) Sequential quadrature is extremely efficient too, but its implementation is lengthy and rather complicated

(iii) Robust lattice methods and explicit formulae combined with spline interpolation appear to us to be a good trade-off between accuracy and efficiency

If you need to price an American-style contract with early exercise features, use robust lattice methods.

Appendix 1

In general, if $X = (X_1,...,X_n)$ is a vector of n joint standardized normal random variables with a symmetric, positive-definite $(n \times n)$ matrix of variances and covariances Σ, then the density of X is given by :

$$f(x_1,...,x_n) = \frac{e^{-\frac{X^T \Sigma^{-1} X}{2}}}{(2\pi)^{n/2}\sqrt{Det(\Sigma)}} \tag{36}$$

where X^T is the transpose of X, $Det(\Sigma)$ is the determinant of Σ and Σ^{-1} is the inverse of Σ (for a proof, see e.g., Tong, 1990).

However, using these general multinormal expressions becomes analytically cumbersome and computationally inefficient as the dimension of the integral rises. Actually, simplified expressions can be used when dealing with the finite-dimensional distributions of geometric Brownian motion GBM. Let $\{t_1, t_2,..., t_n\}$ be n successive points in a time interval. Then, it is easily shown (Guillaume, 2003) that :

$$\begin{aligned} & P\left(X_{t_1} < x_1, X_{t_2} < x_2,..., X_{t_n} < x_n\right) \\ &= \int_{-\infty}^{\frac{x_1-\mu t_1}{\sigma\sqrt{t_1}}} \int_{-\infty}^{\frac{x_2-\mu t_2}{\sigma\sqrt{t_2}}} \cdots \int_{-\infty}^{\frac{x_n-\mu t_n}{\sigma\sqrt{t_n}}} \frac{e^{-\frac{y_1^2}{2} - \sum_{i=1}^{n-1} \frac{\left(y_{i+1}-\rho_{i,i+1}y_i\right)^2}{2\left(1-\rho_{i,i+1}^2\right)}}}{(2\pi)^{d/2}\sqrt{\left(1-\rho_{12}^2\right)\left(1-\rho_{23}^2\right)...\left(1-\rho_{n-1,n}^2\right)}} dy_n dy_{n-1}...dy_1 \\ &\triangleq N_n\left[\frac{x_1-\mu t_1}{\sigma\sqrt{t_1}}; \frac{x_2-\mu t_2}{\sigma\sqrt{t_2}},..., \frac{x_n-\mu t_n}{\sigma\sqrt{t_n}}; \rho_{12}, \rho_{23},..., \rho_{n-1,n}\right] \end{aligned} \tag{37}$$

where :

$\rho_{i,j} = \sqrt{t_i / t_j}, t_i < t_j$

Appendix 2

The no-arbitrage value of a fixed-strike discrete lookback option with n fixing dates, for any n such that $n \in \mathbb{N}$ and $n > 4$, is given by :

$$
\begin{aligned}
&V(S;K;t_0,t_1,t_2,...,t_n) \\
&= \varepsilon_n \left\{ \theta \left[F_1 \tilde{P}_{11} P_{12} + F_2 \tilde{P}_{21} P_{22} + F_3 \tilde{P}_{31} P_{32} + F_4 \tilde{P}_{41} P_{42} + ... + F_{n-1} \tilde{P}_{(n-1)1} P_{(n-1)2} + F_n \tilde{P}_n \right] \right. \\
&\left. -\theta K \left[P_{11} P_{12} + P_{21} P_{22} + P_{31} P_{32} + P_{41} P_{42} + ... + P_{(n-1)1} P_{(n-1)2} + P_n \right] \right\}
\end{aligned}
\tag{38}
$$

where :

$$P_{11} = N\left[\frac{\theta(-k + f(\mu_1))}{f(\sigma_1)}\right]; P_{12} = N_{n-1}\left[-\frac{\theta f(\mu_2)}{f(\sigma_2)}, -\frac{\theta f(\mu_{2\to 3})}{f(\sigma_{2\to 3})}, ...,\right.$$

$$\left. -\frac{\theta f(\mu_{2\to n})}{f(\sigma_{2\to n})}; \frac{f(\sigma_2)}{f(\sigma_{2\to 3})}, \frac{f(\sigma_{2\to 3})}{f(\sigma_{2\to 4})}, ..., \frac{f(\sigma_{2\to n-1})}{f(\sigma_{2\to n})}\right]$$

$$P_{21} = N_2\left[\frac{\theta(k - f(\mu_1))}{f(\sigma_1)}, \frac{\theta(-k + f(\mu_{1\to 2}))}{f(\sigma_{1\to 2})}; -\frac{f(\sigma_1)}{f(\sigma_{1\to 2})}\right]$$

$$+N\left[\frac{\theta(-k + f(\mu_1))}{f(\sigma_1)}\right] N\left[\frac{\theta f(\mu_2)}{f(\sigma_2)}\right]$$

$$P_{22} = N_{n-2}\left[\begin{array}{l} -\frac{\theta f(\mu_3)}{f(\sigma_3)}, -\frac{\theta f(\mu_{3\to 4})}{f(\sigma_{3\to 4})}, ..., -\frac{\theta f(\mu_{3\to n})}{f(\sigma_{3\to n})}; \frac{f(\sigma_3)}{f(\sigma_{3\to 4})}, \frac{f(\sigma_{3\to 4})}{f(\sigma_{3\to 5})}, ..., \\ \frac{f(\sigma_{3\to n-1})}{f(\sigma_{3\to n})} \end{array}\right]$$

$$P_{31} = N_3\left[\frac{\theta(-k + f(\mu_{1\to 3}))}{f(\sigma_{1\to 3})}, \frac{\theta f(\mu_{2\to 3})}{f(\sigma_{2\to 3})}, \frac{\theta f(\mu_3)}{f(\sigma_3)}; \frac{f(\sigma_{2\to 3})}{f(\sigma_{1\to 3})}, \frac{f(\sigma_3)}{f(\sigma_{2\to 3})}\right]$$

$$P_{32} = N_{n-3}\left[\begin{array}{l} -\frac{\theta f(\mu_4)}{f(\sigma_4)}, -\frac{\theta f(\mu_{4\to 5})}{f(\sigma_{4\to 5})}, \ldots, -\frac{\theta f(\mu_{4\to n})}{f(\sigma_{4\to n})}; \frac{f(\sigma_4)}{f(\sigma_{4\to 5})}, \frac{f(\sigma_{4\to 5})}{f(\sigma_{4\to 6})}, \ldots, \\ \frac{f(\sigma_{4\to n-1})}{f(\sigma_{4\to n})} \end{array}\right]$$

$$P_{41} = N_4\left[\begin{array}{l} \frac{\theta(-k+f(\mu_{1\to 4}))}{f(\sigma_{1\to 4})}, \frac{\theta f(\mu_{2\to 4})}{f(\sigma_{2\to 4})}, \frac{\theta f(\mu_{3\to 4})}{f(\sigma_{3\to 4})}, \frac{\theta f(\mu_4)}{f(\sigma_4)}; \frac{f(\sigma_{2\to 4})}{f(\sigma_{1\to 4})}, \\ \frac{f(\sigma_{3\to 4})}{f(\sigma_{2\to 4})}, \frac{f(\sigma_4)}{f(\sigma_{3\to 4})} \end{array}\right]$$

$$P_{42} = N_{n-4}\left[\begin{array}{l} -\frac{\theta f(\mu_5)}{f(\sigma_5)}, -\frac{\theta f(\mu_{5\to 6})}{f(\sigma_{5\to 6})}, \ldots, -\frac{\theta f(\mu_{5\to n})}{f(\sigma_{5\to n})}; \frac{f(\sigma_5)}{f(\sigma_{5\to 6})}, \frac{f(\sigma_{5\to 6})}{f(\sigma_{5\to 7})}, \ldots, \\ \frac{f(\sigma_{5\to n-1})}{f(\sigma_{5\to n})} \end{array}\right]$$

…

$$P_{(n-1)1} = N_{n-1}\left[\begin{array}{l} \frac{\theta(-k+f(\mu_{1\to n-1}))}{f(\sigma_{1\to n-1})}, \frac{\theta f(\mu_{2\to n-1})}{f(\sigma_{2\to n-1})}, \ldots, \frac{\theta f(\mu_{n-2\to n-1})}{f(\sigma_{n-2\to n-1})}, \\ \frac{\theta f(\mu_{n-1})}{f(\sigma_{n-1})}; \frac{f(\sigma_{2\to n-1})}{f(\sigma_{1\to n-1})}, \frac{f(\sigma_{3\to n-1})}{f(\sigma_{2\to n-1})}, \ldots, \frac{f(\sigma_{n-1})}{f(\sigma_{n-2\to n-1})} \end{array}\right]$$

$$P_{(n-1)2} = N\left[-\frac{\theta f(\mu_n)}{f(\sigma_n)}\right]$$

$$P_n = N_n\left[\begin{array}{l} \frac{\theta(-k+f(\mu_{1\to n}))}{f(\sigma_{1\to n})}, \frac{\theta f(\mu_{2\to n})}{f(\sigma_{2\to n})}, \ldots, \frac{\theta f(\mu_{n-1\to n})}{f(\sigma_{n-1\to n})}, \frac{\theta f(\mu_n)}{f(\sigma_n)}; \\ \frac{f(\sigma_{2\to n})}{f(\sigma_{1\to n})}, \frac{f(\sigma_{3\to n})}{f(\sigma_{2\to n})}, \ldots, \frac{f(\sigma_n)}{f(\sigma_{n-1\to n})} \end{array}\right]$$

The $\tilde{P}_{mn}$ terms are the same as the P_{mn} terms, except for the drift coefficients, which are: $\tilde{\mu}_i = r_i - \delta_i + \sigma_i^2/2$, instead of: $\mu_i = r_i - \delta_i - \sigma_i^2/2$. The other notations are defined in Proposition 1.

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