



HAL
open science

Window double barrier options

Tristan Guillaume

► **To cite this version:**

Tristan Guillaume. Window double barrier options. Review of Derivatives Research, 2003, 6, pp.47-75.
hal-00924247

HAL Id: hal-00924247

<https://hal.science/hal-00924247>

Submitted on 6 Jan 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

WINDOW DOUBLE BARRIER OPTIONS

Revised version, 2005

Originally published in Review of Derivatives Research, 2003, (6), 47-75

TRISTAN GUILLAUME

Université de Cergy-Pontoise, Laboratoire Thema, 33 boulevard du port, F-95011 Cergy-Pontoise
Cedex, France

Abstract. This paper examines a path-dependent contingent claim called the window double barrier option, including standard but also more exotic features such as combinations of single and double barriers. Price properties and hedging issues are discussed, as well as financial applications. Explicit formulae are provided, along with simple techniques for their implementation. Numerical results show that they compare very favourably with alternative pricing approaches in terms of accuracy and efficiency.

Keywords : option, barrier, double barrier, window, pricing, hedging, numerical integration, dimension.

JEL classification : G13

Introduction

In conjunction with their growing popularity in the OTC markets, double barrier options have gradually moved to the forefront of derivatives research. The main developments in the literature pertaining to their analytical valuation can be briefly outlined. Based on a generalisation of the Levy formula, Kunitomo and Ikeda (1992) provide a valuation formula for double barrier options with payoff restricted by two curved absorbing boundaries assumed to be exponential functions of time. Geman and Yor (1996) make use of the Cameron-Martin theorem to derive the Laplace transform of the double barrier option price with respect to its expiry date. Inversion of this transform can be done numerically using the Fast Fourier transform (Geman and Eydeland, 1995), or analytically using the Cauchy Residue theorem (Schröder, 2000). Expressing double barrier option values as a linear combination of sine functions, Bhagavatula and Carr (1995) handle time-dependent parameters. Analytically solving the Black-Scholes partial differential equation with the appropriate boundary conditions, Hui (1997) prices front-end and rear-end double barrier options, featuring early-ending and forward-start monitoring respectively. Combining Laplace transform and contour integration, Pelsser (2000) studies binary double barrier options including a rebate paid when either one of the barriers is hit. Based on the first passage densities of Brownian motion derived by Sidenius (1998), Luo (2001) considers ordered double barrier options in which the payoff is contingent on whether the lower or the upper barrier is hit first.

Even though research on double barrier options has been growing steadily, there is still an important type of contract that admits no explicit solution : window double barrier options. In their standard form, they feature a double barrier whose monitoring starts after the contract initiation and terminates before the contract expiry. In this respect, they can be regarded as an extension of the forward-start and early-ending double barrier options studied by Hui (1997). A more exotic variation is the partial window double barrier option, featuring combinations of single barriers before and after the double barrier. The benefits of these contracts are manifold. Window double knock-out options are cheaper than vanilla options and less risky than standard double knock-out options. Window double knock-in options have greater leverage than standard double knock-in options. Whether they are knock-in or knock-out, window double barrier options are more flexible than standard double barrier options, allowing to match more closely the hedging needs or the speculative views of market participants.

However, the expansion of these contracts in the marketplace is contingent on the ability to obtain exact prices and hedging parameters in trading time. In this respect, closed form formulae would be most welcome, at least as benchmarks to test more general numerical schemes allowing to relax restrictive modeling assumptions. An analytical solution to such a valuation problem involves the calculation of several distributions of joint extrema of geometric Brownian motion that are currently unknown. It also implies to cope with a dimension issue, as the values of standard and partial window double barrier options can only be formulated in terms of multiple integrals.

This paper provides two exact formulae that suffice to span all kinds of standard and partial window double barrier options. It also shows how to implement them with high accuracy and efficiency. The valuation framework is the classical equivalent martingale measure one (Harrison and Pliska, 1981), unlike the partial differential equation approach used by Bhagavatula and Carr (1995) and Hui (1997), or the Laplace transform approach followed by Geman and Yor (1996) and Pelsser (2000). It is shown that by repeatedly conditioning and using the Markov property of Brownian motion, the appropriate discounted expectations can be rewritten in terms of tractable multiple integrals. The dimension issue is dealt with by using convolutions of the multivariate standard normal distribution that allow to dispose of most correlation coefficients.

Section 1 presents details on window double barrier options and their applications, along with the formula for standard-type contracts. Section 2 studies price behavior for various parameters, based on a comparison with other existing contracts. Section 3 discusses hedging issues. Appendix A gives detailed proof of the valuation formula for standard window double barrier options as well as a numerical implementation rule. Appendix B provides the valuation formula for partial window double barrier options, along with an appropriate numerical implementation technique.

1. The case for window double barrier options

Option users basically break down into hedgers and speculators. The former seek to reduce the uncertainty caused by the fluctuations in financial prices. It is well-known that, compared to alternative derivatives such as forward and futures contracts or swaps, options are very flexible and have the remarkable property to insure investors against adverse price changes while allowing them to benefit from favorable movements. The downside is that vanilla options are expensive. One way to cut the cost of hedging is to eliminate unlikely scenarios. This can be achieved by purchasing barrier options, especially those featuring a double barrier because only they allow not to pay for part of the

upward potential and part of the downward potential of the underlying. These contracts are now heavily traded, particularly in the foreign exchange markets. They are also embedded in a lot of popular structured derivatives in equity and interest rate markets, such as convertible/callable bonds and stock warrants. However, holders of knock-out options face the possibility of losing their insurance before expiry. Investors may even never become insured if their contract is contingent to a knock-in provision. These risks sometimes motivate the introduction of a rebate as a form of compensation. The danger of knocking-out could be reduced by setting the upper and the lower barrier far away from the underlying spot price. Conversely, the risk of never knocking-in could be diminished by locating the double barrier very near the spot price. But, in both cases, the premium then quickly rises to that of a vanilla option at a speed proportional to the volatility of the underlying. Given that hedgers do not often have precise views on the market direction over the entire option life (otherwise, they would not hedge !), an alternative way of limiting the risk of sudden death inherent in knock-out contracts consists in activating the double barrier during only a fraction of the option life, while avoiding exposure when there is greater uncertainty as to the volatility of the underlying. Partial double barrier options are supposed to meet this requirement, but they do so too rigidly, since the activation period must either start at the contract's inception or end at expiry. These conditions imposed on investors may not suit their needs. Window double barrier options, on the contrary, provide investors with all the flexibility they can expect from a customized exotic structure. In their standard form, these contracts are call or put options with a knock-out or a knock-in double barrier whose monitoring begins after the contract initiation and terminates before expiry. In other words, the location of the double barrier can be chosen anywhere during the option life.

Window double barrier options include partial and standard double barrier options as special cases. They enable investors to benefit from substantial premium discounts compared to vanilla or single barrier contracts, while allowing them to reduce and customize their risk exposure compared to standard or partial double barrier contracts. Suppose that an investor wants to hedge her or his portfolio of stocks at the lowest possible cost using options. If, for example, quarterly earnings are expected soon, she or he might prefer not to bet on the portfolio's worth in the short term. Furthermore, this cautious investor presumably does not want to risk losing her or his insurance at the end of the option life because of a short price spike through the barrier near expiry. Then, it is easily argued that only a window double barrier option can precisely match this investor's preferences.

Window-type contracts are attractive not only to investors concerned with hedging but also to those willing to speculate on market movements. Indeed, they provide outstanding leverage. Suppose, for example, that an investor is bullish on the currency of a country A in the medium/long term. She or he is willing to take a long position in a call option but finds it much too expensive in the current

market conditions. If there is a known event, during this period of time, that will almost certainly increase volatility in the currency, she or he can turn to a knock-in option. It could be, for instance, elections in country A scheduled in several months, especially if the contenders have opposite stances on monetary and fiscal policy. Purchasing a single barrier knock-in option is a very risky strategy because neither the results of the elections nor the reaction of the markets to them can be known ahead of time. The choice of a standard or partial double barrier knock-in option solves this problem, but not in an optimal manner, since the investor has to pay for activating rights during periods of time when she or he does not want them.

Window double barrier options also have a number of desirable properties for option writers. Some of them are shared by all double barrier contracts, such as the capacity to limit both downside and upside risk, in contrast to the unlimited liability typical of vanilla options or the semi-unlimited liability typical of single barrier options. Others are specific to window double barrier options. First, option writers receive a higher premium than that of a standard double knock-out contract. Second, hedging difficulties are mitigated since the possibility of breaching the barrier is monitored during only a fraction of the option life. Section 3 discusses this point in more detail. Third, linear combinations of window and forward-start/early-ending double barrier options could be used to replicate exotic structures that are popular but difficult to value and to hedge, such as the so-called “corridor” or “hot-dog” contracts, which involve sequences of double barriers in time.

As mentioned in the introduction of this paper, increased trading in window double barrier options in the marketplace crucially depends on the ability to obtain exact prices and hedge parameters in real time. Assuming that the underlying asset follows a geometric Brownian motion with riskless rate r , volatility σ and dividend rate δ , the following closed form formula then solves the valuation problem :

Proposition 1 :

The no-arbitrage value V of a standard window double knock-out option is given by:

$$V(S_0, K, H_1, H_2, r, \delta, \sigma, t_1, t_2, t_3) = \theta \left[e^{-rt_3} K \Phi(\tilde{\mu}) - e^{-\delta t_3} S_0 \Phi(\bar{\mu}) \right] \quad (1)$$

where :

H_1 is the lower part of the double barrier, H_2 is the upper part of the double barrier, S_0 is the underlying asset spot value, K is the strike price, $t_0 = 0$ is the contract's inception, t_1 is the time at which monitoring of the double barrier starts with $t_1 > t_0$, t_2 is the time at which monitoring of the double barrier ends with $t_2 > t_1$, t_3 is the option expiry with $t_3 > t_2$,

$$\theta = \begin{cases} -1 & \text{if the option is a call} \\ 1 & \text{if the option is a put} \end{cases} \quad \bar{\mu} = r - \delta - \sigma^2 / 2, \quad \tilde{\mu} = r - \delta + \sigma^2 / 2$$

$$\Phi(\mu) = \sum_{n=-\infty}^{\infty} e^{\frac{2\mu}{\sigma^2}nh} \left\{ \Psi^1(h_2, h_2) - \Psi^1(h_2, h_1) - \Psi^1(h_1, h_2) + \Psi^1(h_1, h_1) \right. \\ \left. - e^{\frac{2\mu}{\sigma^2}(h_1 - 2nh)} \left[\Psi^2(h_2, h_2 - 2h_1) - \Psi^2(h_2, -h_1) - \Psi^2(h_1, h_2 - 2h_1) + \Psi^2(h_1, -h_1) \right] \right\} \quad (2)$$

and :

$$\Psi^1(a, b) = N_3 \left[\frac{a - \mu t_1}{\sigma \sqrt{t_1}}; \frac{b - \mu t_2 - 2nh}{\sigma \sqrt{t_2}}; \theta \left(\frac{k - \mu t_3 - 2nh}{\sigma \sqrt{t_3}} \right); \rho_{12}; \theta \rho_{23} \right] \quad (3)$$

$$\Psi^2(a, b) = N_3 \left[\frac{a + \mu t_1}{\sigma \sqrt{t_1}}; \frac{b - \mu t_2 + 2nh}{\sigma \sqrt{t_2}}; \theta \left(\frac{k - \mu t_3 - 2h_1 + 2nh}{\sigma \sqrt{t_3}} \right); -\rho_{12}; \theta \rho_{23} \right] \quad (4)$$

$$h_i = \log(H_i / S_0), \quad h = h_2 - h_1, \quad k = \log(K / S_0), \quad \rho_{ij} = \sqrt{t_i / t_j}, t_i < t_j$$

$N_3[x_1, x_2, x_3; \pm\rho_{12}, \pm\rho_{23}]$ in proposition 1 is a special trivariate Gaussian convolution defined in Appendix A . Note that the price of a standard window double knock-in contract is obtained simply by subtracting the value of the corresponding standard window double knock-out contract from the value of a vanilla option.

For this formula to be well-defined, t_1 must never be strictly equal to t_0 , just as t_2 must never be strictly equal to t_3 . That, however, does not imply a loss of generality since, in the limit, t_1 can be made arbitrarily close to t_0 , just as t_2 can be made arbitrarily close to t_3 . This way, a formula for a forward-start knock-out double barrier option is nested by letting $t_2 \rightarrow t_3$, a formula for an early-ending knock-out double barrier option is recovered by letting $t_1 \rightarrow t_0$, a formula for a standard knock-out double barrier option is recovered by letting $t_2 \rightarrow t_3$ and $t_1 \rightarrow t_0$. By taking limits with respect to H_1 and H_2 , it is equally easy to nest formulae for standard and partial single barrier options. Appendix A gives proof of proposition 1, along with a formula for a rebate paid at expiry (eq. (35) in Appendix A). It also provides a straightforward implementation technique of this formula yielding extremely accurate and fast results.

To further improve flexibility, one can think of combining the benefits of single and double barriers into a unique structure. For example, resuming our previous scenario, the investor might very well have views about the path of the currency before or after the activation of the double barrier. She or he could believe that the market will display concern over the outcome of the elections before they

actually take place, putting the currency under strain. It would then be profitable to add a down-and-in provision at that moment. She or he could also anticipate that the newly elected party will be the one whose political and economic platform is the most likely to reassure market participants, which could trigger a rally immediately after the vote. It would then be profitable to add an up-and-in provision at that time. These additional features increase the chances of the option being activated. Besides, they do not necessarily result in a more expensive premium since the investor could offset the cost of these new opportunities by setting her or his double barrier further away from the underlying spot price.

In line with the usual terminology, these contracts could be called partial window double barrier options since only part of the time interval during which barrier crossing is monitored contains a double barrier, the rest of it containing single barriers. More specifically, let us divide the option life as follows : $t_0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_7$, where t_0 is the contract inception and t_7 is the contract expiry. If a single upper barrier is monitored within $[t_1, t_2]$, followed by a double barrier within $[t_3, t_4]$ and a single lower barrier within $[t_5, t_6]$, then one could speak of an up-and-down partial window double barrier option. Similarly, when monitoring starts with a single upper barrier within $[t_1, t_2]$, continues with a double barrier within $[t_3, t_4]$, and finishes with a single upper barrier within $[t_5, t_6]$, then one could speak of an up-and-up type of contract. A third configuration is when monitoring starts with a single lower barrier within $[t_1, t_2]$, continues with a double barrier within $[t_3, t_4]$, and finishes with a single upper barrier within $[t_5, t_6]$, which defines a down-and-up type of contract. Finally, if a single lower barrier is monitored within $[t_1, t_2]$, followed by a double barrier within $[t_3, t_4]$ and a single lower barrier within $[t_5, t_6]$, then one could speak of a down-and-down type of contract.

Note that monitoring of the double barrier may start immediately after monitoring the first single barrier, i.e. one may have : $t_2 \rightarrow t_3$. Likewise, there may be continuity in time between the monitoring of the double barrier and that of the second single barrier ($t_4 \rightarrow t_5$). Besides, there may of course be only one single barrier, either before or after the double barrier. Given all these possible specifications, as well as the choice between a knock-in and a knock-out provision and the choice between a call and a put, there is a very large number of possible partial window double barrier contracts. It is analytically feasible to obtain a unique closed form formula that can on its own provide the exact value of all of them. This formula, including a possible rebate paid at expiry, is given in appendix B, along with a simple numerical technique to implement it.

2. Numerical results

Let us now examine the behavior of prices for various contract specifications. Table 1 presents the values of standard window double knockout put options (SWDKOP) for several volatility and barrier levels, and compares them with the values of standard double knockout puts (SDKOP) and vanilla puts (VP). Three different SWDKOP prices are given : one applying the formula, one obtained from a binomial tree, and one performing Monte Carlo simulation. Details on how these values were obtained and how the different methods compare in terms of accuracy and efficiency can be found in Appendix A.

In Table 1, the premium of a SWDKOP is, on average, 55.6% lower than that of a VP; for a SDKOP, the premium discount amounts to 73.5%. Those figures point at the considerable savings investors can expect from acquiring long positions in window-type options instead of vanilla options. They are also in line with the fact that, by definition, the less valuable SDKOP contracts must be cheaper than the SWDKOP ones. There are, however, three situations in which the premium of a SWDKOP and that of a SDKOP tend to converge. The first one, quite obvious, is when double barrier monitoring extends over the whole option life; but then, the SWDKOP contract loses its specificity and, in the limit, cannot be distinguished from a SDKOP. The second one is when the distance between the upper and the lower barrier is short, or when either the upper or the lower barrier is located very near the underlying spot price, while volatility is high. Then, both the value of a SWDKOP and that of a SDKOP rapidly drop and, in the limit, tend to zero, reflecting the overwhelming impact of the likelihood of knocking-out, no matter how long the double barrier is monitored. When the double barrier range is (80,120) and volatility is 44%, the value of a SWDKOP is thus only 1.6% of that of a VP, the double barrier being active during one-third of the option life. The third configuration in which the premium of a SWDKOP and that of a SDKOP tend to converge is when the distance between the upper and the lower barrier is large and volatility is low. Then, the value of both a SWDKOP and a SDKOP tends to that of a VP, reflecting the rapidly decreasing likelihood of knocking-out.

Interestingly enough, there is a rather wide range of volatility, for each double barrier, in which SWDKOP values do not differ significantly from one another. For instance, we see in Table 1 that, with a double barrier set at (70,130), the SWDKOP value is quite the same whether volatility is 18% or 25%. Likewise, when the double barrier is set at (60,140), the SWDKOP value is quite the same whether volatility is 25% or 32%. The levels of volatility for which SWDKOP values do not differ significantly among each other become lower as the upper barrier and the lower barrier get closer to each other. Additional computations show that : the standard deviation of (80,120) SWDKOP prices is

0.046 when volatility ranges between 12% and 18% (for an average option value equal to 2.97), the standard deviation of (70,130) SWDKOP prices is 0.038 when volatility ranges between 18% and 25% (for an average option value equal to 5.34), and the standard deviation of (60,140) SWDKOP prices is 0.044 when volatility ranges between 25% and 32% (for an average option value equal to 8.07). This is because, in these volatility ranges, the chances of ending the option life in-the-money and the risks of knocking-out before expiry offset each other in a balanced manner. This non-monotonicity of the option value with respect to volatility can actually be observed, to a lesser or greater extent, in all kinds of knock-out barrier options. This is one of their main differences with vanilla options, which display non-linearity but monotonicity with respect to the volatility of the underlying asset.

Let us now move on to partial window double knockout puts (PWDKOP). Tables 2 and 3 present the prices of up-and-down (UDP), down-and-up (DUP), up-and-up (UUP) and down-and-down (DDP) partial window double knockout puts. They are an extension to the SWDKOP prices presented in Table 1 in the sense that the same contract specifications have been kept except for a time interval before and after the double barrier during which only the lower or the upper part of that double barrier is monitored. This allows to compare PWDKOP and SWDKOP prices. The difference between Table 2 and Table 3 lies in the extent of that time interval (long in Table 2, short in Table 3). With these specifications, the UDP, DUP, UUP and DDP values in Tables 2 and 3 necessarily lie between those of SDKOP (lower bound) and SWDKOP (upper bound) in Table 1. Again, Monte Carlo simulation and binomial estimators are provided along with analytical values. Details on how prices were obtained can be found in Appendix B.

By definition, since additional conditions are to be met, one would expect PWDKOP prices to be lower than SWDKOP ones, especially in Table 2 where those additional conditions are imposed for a longer period of time. Such an expectation is verified with UDP and DDP prices, but it is, to a surprisingly large extent, hardly validated in the case of DUP and UUP prices. In Table 2, UDP prices for instance are, on average, 51% lower than their SWDKOP counterparts, but DUP prices are only 4.2% lower than their SWDKOP counterparts. The closer the upper and the lower barrier to the spot value, the more difference between PWDKOP and SWDKOP prices : in Table 2, the (80,120) UDP premium is 78.9% lower than the corresponding (80,120) SWDKOP premium in Table 1, but the (60,140) UDP premium is only 24.9% lower than that of the corresponding (60,140) SWDKOP; for DUP options, the difference with SWDKOP prices is even almost negligible when the double barrier is (70,130) or (60,140).

Another surprising feature of DUP and UUP prices is that they are almost the same when the single barriers before and after the double barrier are monitored during a large (Table 2) or a short (Table 3)

time interval. The same phenomenon can be observed with up-and-down and down-and-down double window call options. This is not true of UDP and DDP prices; UDP prices, for instance, exhibit a substantial average difference of 32.1% between Table 2 and Table 3. Likewise, down-and-up and up-and-up double window call option values differ significantly from one another according to the amount of time during which single barriers are monitored, as one would expect intuitively. Actually, this is simply because the likelihood of knocking-out as a function of time exposure to barrier monitoring is more sensitive to the level of volatility in the case of UDP and DDP contracts : if volatility is raised to 44%, then DUP and UUP prices also begin to exhibit large differences between Table 2 and Table 3.

Finally, one last noticeable feature of both Tables 2 and 3 is the fact that UDP and DDP prices are very similar, such as DUP and UUP prices, whatever the barrier levels.

3. Hedging issues

To eliminate risk, option dealers need to hedge their positions. Delta hedging, exploiting the correlation between the option and its underlying, is the building block of dynamic hedging. The gamma parameter measures by how much or how often a position must be reheded in order to maintain a delta-neutral position. Vega measures volatility risk exposure. The following discussion briefly examines the delta, gamma and vega parameters of a number of window double knock-out options. Analytical formulae for these hedge parameters can be derived by differentiation. However, the derived formulae are cumbersome and it is easier and more efficient to look at finite-difference approximations by measuring the sensitivity of the option value to a slight change in the appropriate variable.

Let us compare the variations, with respect to the underlying asset price, of the hedge parameters of two different standard window double knock-out call options, SWC1 and SWC2, and those of a vanilla call, VC, as well as the variations, with respect to the underlying asset price too, of the hedge parameters of two down-and-up partial window double knock-out call options, DUC1 and DUC2. For all these options, the strike price is 100, volatility is 25%, the riskless rate is 5%, the dividend rate is 2%. The lower barrier and the upper barrier of both SWC1 and SWC2 are 70 and 130, respectively. The SWC1 expiry is 1.5 years, with barrier monitoring within the time interval [0.5-1]. The SWC2 expiry is 0.3 years, with barrier monitoring within the time interval [0.1-0.2]. Such contract specifications make it possible to gauge the effect of time to maturity on hedging. For DUC1 and DUC2 options, the single lower and upper barriers are 70 and 130, respectively, and the (70,130) double barrier is monitored within the time interval [0.5-1]. In the DUC1 case, the single lower barrier

is monitored within [0.2-0.3] and the single upper barrier is monitored within [1.2-1.3]. In the DUC2 case, the single lower barrier is monitored within [0-0.5] and the single upper barrier is monitored within [1-1.5]. These contract specifications allow to compare partial window double knock-out and standard window double knock-out hedge parameters, as well as to assess the impact of single barrier monitoring before and after the double barrier.

First, SWC1 and SWC2 deltas are always smaller than or equal to VC deltas, whatever the underlying spot value. This is because SWC1 and SWC2 options are cheaper than VC options, so that they stand to gain or lose less value. Both SWC1 and SWC2 deltas are positive for out-of-the-money contract specifications, even when the option is far out-of-the-money. But their value is small then, not only because the probability of expiring in-the-money is low, but also because of the significant risk of hitting the nearby lower barrier before expiry. Thus, the SWC1 delta is 9.5% at 71, which is less than half of the corresponding VC delta.

In the region between 71 and 100, the SWC2 delta curve is remarkably close to the VC delta one, increasing quite steeply from around zero to above 40%. In contrast, the SWC1 delta curve slowly decreases, from 9.5% to around zero, as if the higher risk of hitting the upper barrier prevailed over the higher chances of not hitting the lower barrier and expiring in-the-money. The region around 100 is interesting, since this is the area where the option is at-the-money and where the spot price is equally distant from the lower and the upper barrier. Around this point, the SWC2 delta reaches its peak (above 40%), and thus starts decreasing, while the SWC1 delta becomes negative. Beyond 100, the SWC1 delta curve continues to decrease smoothly, while the SWC2 delta one rapidly falls down to large negative values (around -50% near the upper barrier). This stands in complete opposition with the behavior of a VC delta, which increases regularly as the option becomes more and more in-the-money (because ending the option life in-the-money becomes more and more certain). This divergence is caused by the upper barrier, which raises the risk of knocking-out when the option is in-the-money. This effect is more pronounced when expiry is close (SWC2 case), leaving short time for the underlying asset to drift away from the upper barrier.

Overall, delta variations are steeper and more unstable for SWC2 than for SWC1, reflecting the significant impact of shorter time to maturity. As a result, SWC2 gamma values are larger than SWC1 gamma values : while the latter lie within a [0.6%, -0.8%] range, the former lie within a [3%, -5%] range. This makes SWC2 options less easy to hedge than SWC1 options using a delta-neutral dynamic strategy, since the hedging portfolio needs to be more frequently rebalanced. But when the barrier period is sufficiently distant from the beginning and the end of the option life (SWC1 case), gamma fluctuations are substantially smoother than with regular double knock-out barrier options, which is an advantage for an option dealer.

If we now examine the vegas, computations reveal that most SWC1 and SWC2 vega values are negative, unlike VC vega values, which are always positive whatever the underlying asset spot value. This divergence is caused by the ambivalent effect of volatility on knock-out option values : higher volatility increases the chances of expiring in-the-money, but also the risks of knocking-out. The former effect prevails over the latter when the SWC2 option is out-of-the-money, with vega topping a modest 2% when the underlying spot price is 90, but it goes reverse when the option is in-the-money. Thus, SWC2 vega values become more and more negative as the underlying spot price gets closer to the upper barrier (-1% at 110,- 4.4% at 120, -10% at 128). When time to maturity is long (SWC1 case), however, vega values are at their highest levels when the option is far out-of-the-money (1.9% at 71) or, to a lesser extent, when it is far in-the-money. This is because our contract specifications in this example locate the double barrier right in the middle of the option life. Had monitoring of the same barrier period (half a year) started soon after the beginning of the option life, vega values would have reached a peak for out-of-the-money contracts; had it ended soon before the option expiry (one year and a half), vega values would have been higher for far out-of-the-money contracts (with a peak only slightly greater than zero, though). This complex vega behavior is a reminder of the significance of time to maturity when it comes to measuring the impact of volatility on the risk of knocking-out. Overall, vega parameters are all the more difficult to interpret as both SWC1 and SWC2 do not have single-signed gamma everywhere. This leaves the option dealer quite exposed to volatility risk.

Next, one can turn to DUC1 and DUC2 hedge parameters. DUC2 prices are cheaper than DUC1 ones, because more stringent conditions are imposed on DUC2 payoffs. Consequently, DUC2 deltas are smaller than their DUC1 counterparts, although this gap shrinks as the underlying asset spot value approaches the upper knock-out barrier. Also, all DUC1 and DUC2 deltas remain positive, which is a noticeable difference with SWC1 and SWC2 deltas. They attain a peak when the option is slightly out-of-the-money (22% for DUC1 and 16% for DUC2 when the underlying spot price is 98). Quite typically, they reach bottom in the regions near the knock-out barriers, especially the lower barrier, which makes sense intuitively since it is the lower barrier that is first monitored (8.2% for DUC1 and 1.4% for DUC2 when the underlying spot price is 71). Had we valued down-and-down or up-and-up partial window double knock-out call options, however, we would have obtained a number of negative deltas.

DUC1 and DUC2 gamma fluctuations are quite moderate, lying within [1.2%,-0.3%] and [1%,-0.6%] ranges respectively. This is quite remarkable, especially in the DUC2 case where the whole option life is subject to barrier monitoring. However, gamma fluctuations could obviously be larger for other contract specifications, such as a shorter time to maturity or a narrower barrier range.

DUC1 and DUC2 vega values are quite similar. They are almost all negative, which is not surprising, given the number of knock-out conditions. Less evident is the fact that DUC1 and DUC2 vegas hit their lowest level when the option is slightly out-of-the-money (-5.2% at 99 for DUC1 and -3.5% at 98 for DUC2). Actually, this is simply because these are the regions where those options take their highest value; likewise, the largest delta parameters can be obtained in these regions.

Overall, it should be kept in mind that there is potentially a number of contract specifications for which dynamic hedging is either uneasy or relatively unreliable, due to gamma fluctuations and because vegas do not always provide a clear measure of volatility risk. Theoretically, gamma risk could be overcome by continuous rebalancing of the hedging portfolio, but, in practice, trading is discrete and transaction costs can accumulate to substantial amounts. More seriously, like all kinds of knock-out contracts, window double knock-out options are faced with a discontinuity of their delta at the barrier (with gamma possibly reaching infinite values in finite time). The problem of hedging close to the barrier is well described in Taleb (1997). As it is magnified near the option expiry, the option dealer would be better off with a contract in which monitoring of the double barrier ends sufficiently long before expiry. This is achievable with a window double knock-out contract, whereas it is by definition impossible with a regular double (or single) knock-out contract. Hedging will be made even easier if barrier monitoring starts sufficiently long after the beginning of the option life. Compared with partial double knock-out contracts (whether forward-start or early-ending), window double knock-out contracts allow to locate the double barrier away from both the beginning and the end of the option life. Thus, window double knock-out options do not eliminate hedging problems, but they can alleviate them, compared with other forms of double knock-out contracts.

The potential difficulties associated with dynamic hedging seem to call for a static hedging strategy (Carr, Ellis and Gupta, 1998) although this often merely shifts the problem to the vanilla options market, as static hedges need to be rebalanced too when the underlying spot value nears the knock-out barrier (Toft and Xuan, 1998). An interesting alternative is the superhedging strategy (Schmock, Shreve and Wystup, 1999; Wystup, 1999), which achieves continuity at the barrier by numerically solving a stochastic control problem under a constraint on the possible values of the gearing ratio of the option. This approach works well with regular double knock-out contracts. It could be extended to window double knock-out contracts.

4. Concluding remarks

This paper has studied standard and partial window double barrier options. These contracts are more flexible than regular double barrier ones, thus allowing to match more closely the hedging

needs or the speculative views of investors. As few as two formulae suffice to cover a very large number of complex payoffs. They are the basis for simple numerical integration schemes which compare favourably with alternative pricing approaches in terms of accuracy and efficiency. They also provide an easy and reliable way to obtain finite-difference approximations to hedge parameters.

Barriers have been assumed constant in this paper, but it would be an easy extension to make them deterministic exponential functions of time. Stochastic volatility and interest rates, however, would not be an easy extension, and there is no evidence that such a valuation problem would be analytically tractable.

Appendix A : proof and numerical implementation of the valuation formula for a standard window double barrier option

In this appendix, the following notations will be used :

- W_t is Brownian motion defined on a probability space (Ω, F_t, P) where $F_t = \sigma(W_s, s \leq t)$ is the natural filtration of W_t
- S_t is the value of the underlying asset at time t
- K is the strike price, H_1 is the lower barrier, H_2 is the upper barrier, r is the instantaneous riskless rate, σ is the constant volatility of the underlying asset, δ is the constant continuous dividend rate
- $X_t = \ln(S_t/S_0)$, $k = \ln(K/S_0)$, $h_i = \ln(H_i/S_0)$, $m_a^b = \inf_{t \in [t_a, t_b]}(X_t)$, $M_a^b = \sup_{t \in [t_a, t_b]}(X_t)$
- Q is the risk-neutral measure, $E^Q[\cdot]$ is the expectation operator under the Q – measure
- $\mathbf{1}_{\{\cdot\}}$ is the indicator function taking value 1 if the conditions inside the brackets are met and value zero otherwise
- $N[\cdot]$ refers to the univariate standard normal cumulative distribution function, $N_2[\cdot, \cdot; \rho]$ refers to the bivariate standard normal cumulative distribution function with correlation coefficient ρ

A.1 The multivariate normal distribution

In general, if $X = (X_1, \dots, X_n)$ is a vector of n joint standardized normal random variables with a symmetric, positive-definite $(n \times n)$ matrix of variances and covariances Σ , then the density of X is given by :

$$f(x_1, \dots, x_n) = \frac{e^{-\frac{X^T \Sigma^{-1} X}{2}}}{(2\pi)^{n/2} \sqrt{\text{Det}(\Sigma)}} \quad (5)$$

where X^T is the transpose of X , $\text{Det}(\Sigma)$ is the determinant of Σ and Σ^{-1} is the inverse of Σ (for a proof, see e.g., Tong, 1990).

For example, if we denote by ρ_{12}, ρ_{23} and ρ_{13} the correlation coefficients between three standardized normal random variables X_1, X_2 and X_3 , it is easily shown, by applying formula (5), that the joint density f of X_1, X_2 and X_3 is given by :

$$f(a, b, c; \rho_{12}, \rho_{23}, \rho_{13}) = \frac{e^{-\frac{1}{2}(\lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2 + 2\lambda_4 ab + 2\lambda_5 bc + 2\lambda_6 ac)}}{(2\pi)^{3/2} \sqrt{\text{Det}(\Sigma)}} \quad (6)$$

with : $\text{Det}(\Sigma) = 1 - \rho_{12}^2 - \rho_{23}^2 - \rho_{13}^2 + 2\rho_{12}\rho_{23}\rho_{13}$

$$\lambda_1 = \frac{1 - \rho_{23}^2}{\text{Det}(\Sigma)}; \lambda_2 = \frac{1 - \rho_{13}^2}{\text{Det}(\Sigma)}; \lambda_3 = \frac{1 - \rho_{12}^2}{\text{Det}(\Sigma)}; \lambda_4 = \frac{\rho_{13}\rho_{23} - \rho_{12}}{\text{Det}(\Sigma)};$$

$$\lambda_5 = \frac{\rho_{13}\rho_{12} - \rho_{23}}{\text{Det}(\Sigma)}; \lambda_6 = \frac{\rho_{12}\rho_{23} - \rho_{13}}{\text{Det}(\Sigma)}$$

However, using these general multinormal expressions becomes analytically cumbersome and computationally inefficient as the dimension of the integral rises. Actually, simplified expressions can be used when dealing with the finite-dimensional distributions of geometric Brownian motion GBM. Let t_1, t_2, t_3 be three different dates during the option life $[t_0, t_3]$, such that : $t_0 < t_1 < t_2 < t_3$. By conditioning with respect to X_{t_2} and applying the Markov property of Brownian motion, we have :

$$Q(X_{t_1} < a, X_{t_2} < b, X_{t_3} < c)$$

$$= E^Q \left[\mathbf{1}_{\{X_{t_1} < a, X_{t_2} < b\}} E^Q \left[\mathbf{1}_{\{X_{t_3} < c\}} \middle| X_{t_1} < a, X_{t_2} < b \right] \right] \quad (7)$$

$$= E^Q \left[\mathbf{1}_{\{X_{t_1} < a, X_{t_2} < b\}} E^Q \left[\mathbf{1}_{\{X_{t_3} < c\}} \middle| X_{t_2} < b \right] \right] \quad (8)$$

$$= \int_{-\infty}^{\frac{a - \mu t_1}{\sigma \sqrt{t_1}}} \int_{-\infty}^{\frac{b - \mu t_2}{\sigma \sqrt{t_2}}} e^{-\frac{x^2 - 2(\sqrt{t_1/t_2})xy + y^2}{2(1 - t_1/t_2)}} \frac{1}{2\pi \sqrt{(1 - t_1/t_2)}} E^Q \left[\mathbf{1}_{\{X_{t_3} < c\}} \middle| y \right] dy dx \quad (9)$$

$$= \int_{-\infty}^{\frac{a - \mu t_1}{\sigma \sqrt{t_1}}} \int_{-\infty}^{\frac{b - \mu t_2}{\sigma \sqrt{t_2}}} e^{-\frac{x^2 - 2(\sqrt{t_1/t_2})xy + y^2}{2(1 - t_1/t_2)}} N \left[\frac{c - y - \mu(t_3 - t_2)}{\sigma \sqrt{t_3 - t_2}} \right] dy dx \quad (10)$$

$$= \int_{-\infty}^{\frac{a-\mu t_1}{\sigma\sqrt{t_1}}} \int_{-\infty}^{\frac{b-\mu t_2}{\sigma\sqrt{t_2}}} \int_{-\infty}^{\frac{c-\mu t_3}{\sigma\sqrt{t_3}}} \frac{e^{-\frac{x^2}{2} - \frac{(y-\sqrt{t_1/t_2}x)^2}{2(1-t_1/t_2)} - \frac{(z-\sqrt{t_2/t_3}y)^2}{2(1-t_2/t_3)}}}{(2\pi)^{3/2} \sqrt{(1-t_1/t_2)(1-t_2/t_3)}} dz dy dx \quad (11)$$

where $\mu = r - \delta - \sigma^2/2$

This can be written in a more compact form as :

$$N_3 \left[\frac{a - \mu t_1}{\sigma\sqrt{t_1}}, \frac{b - \mu t_2}{\sigma\sqrt{t_2}}, \frac{c - \mu t_3}{\sigma\sqrt{t_3}}; \rho_{12}, \rho_{23} \right] \quad (12)$$

where $\rho_{12} = \sqrt{t_1/t_2}$ and $\rho_{23} = \sqrt{t_2/t_3}$

More generally, it is easily shown that :

$$\begin{aligned} &= P(X_{t_1} < x_1, X_{t_2} < x_2, \dots, X_{t_n} < x_n) \\ &= N_n \left[\frac{x_1 - \mu t_1}{\sigma\sqrt{t_1}}, \frac{x_2 - \mu t_2}{\sigma\sqrt{t_2}}, \dots, \frac{x_n - \mu t_n}{\sigma\sqrt{t_n}}; \rho_{12}, \rho_{23}, \dots, \rho_{n-1,n} \right] \\ &= \int_{-\infty}^{\frac{x_1 - \mu t_1}{\sigma\sqrt{t_1}}} \int_{-\infty}^{\frac{x_2 - \mu t_2}{\sigma\sqrt{t_2}}} \dots \int_{-\infty}^{\frac{x_n - \mu t_n}{\sigma\sqrt{t_n}}} \frac{e^{-\frac{y_1^2}{2} - \sum_{i=1}^{n-1} \frac{(y_{i+1} - \rho_{i,i+1} y_i)^2}{2(1-\rho_{i,i+1}^2)}}}{(2\pi)^{d/2} \sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2) \dots (1-\rho_{n-1,n}^2)}} dy_n dy_{n-1} \dots dy_1 \end{aligned} \quad (13)$$

with : $\rho_{i,j} = \sqrt{t_i/t_j}$, $t_i < t_j$

The larger dimension is, the more useful the Markovian convolution of the standard normal distribution introduced in (13), since it allows to dispose of the vast majority of correlation coefficients that would otherwise be required, making calculations tractable and computations efficient.

A.2 Proof of Proposition 1 in section 1

Equipped with the preliminary results (11) and (13), we can now start the proof of proposition 1 in section 1. Following the risk-neutral valuation approach, the value of a standard window double knock-out call option, C_{WDKO} , at the contract inception $t_0 = 0$, is given by the discounted expectation of its payoff under the equivalent martingale measure conditional on the information available at time t_0 :

$$C_{WDKO} = e^{-rt_3} E^Q \left[(S_{t_3} - K)^+ \mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2\}} \right] = e^{-rt_3} E^Q \left[(S_3 - K) \mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k\}} \right] \quad (14)$$

$$= e^{-rt_3} \left\{ S_0 e^{(r-\delta-\sigma^2/2)t_3 + \sigma W_{t_3}} E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k\}} \right] - K E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k\}} \right] \right\} \quad (15)$$

$$= e^{-rt_3} \left\{ F e^{(-\sigma^2/2)t_3 + \sigma W_{t_3}} E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k\}} \right] - K E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k\}} \right] \right\} \quad (16)$$

where $F = S_0 e^{(r-\delta)t_3}$ is the risk-neutral forward price

For a standard window double knock-out put option P_{WDKO} , one would have :

$$P_{WDKO} = e^{-rt_3} \left\{ K E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2, X_{t_3} < k\}} \right] - F e^{(-\sigma^2/2)t_3 + \sigma W_{t_3}} E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2, X_{t_3} < k\}} \right] \right\} \quad (17)$$

Let us define a new measure \bar{Q} such that :

$$\left. \frac{d\bar{Q}}{dQ} \right|_{F_t} = e^{-\frac{\sigma^2}{2}t + \sigma W_t} \quad (18)$$

Then, applying the Girsanov theorem :

$$C_{WDKO} = e^{-rt_3} \left\{ F \bar{Q}(m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k) - K Q(m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k) \right\} \quad (19)$$

It suffices to calculate the required probability under the Q – measure : a simple change of drift from $\mu = r - \delta - \sigma^2/2$ to $\bar{\mu} = r - \delta + \sigma^2/2$ will provide the required probability under the \bar{Q} – measure.

As a result of conditioning :

$$Q(m_1^2 > h_1, M_1^2 < h_2, X_{t_3} > k) = E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2\}} E^Q \left[\mathbf{1}_{\{X_{t_3} > k\}} \middle| m_1^2 > h_1, M_1^2 < h_2 \right] \right] \quad (20)$$

Now using the Markov property of Brownian motion, the right hand-side of eq. (20) becomes :

$$E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2\}} E^Q \left[\mathbf{1}_{\{X_{t_3} > k\}} \middle| h_1 < X_{t_2} < h_2 \right] \right] \quad (21)$$

To obtain (21), one first needs to find $Q(m_1^2 > h_1, M_1^2 < h_2)$, which can be expanded as the following nested expectations :

$$Q(m_1^2 > h_1, M_1^2 < h_2) = E^Q \left[\mathbf{1}_{\{h_1 < X_{t_1} < h_2\}} E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2\}} \middle| h_1 < X_{t_1} < h_2 \right] \right] \quad (22)$$

Since $Q(h_1 < X_1 < h_2) = N[(h_2 - \mu t_1)/\sigma\sqrt{t_1}] - N[(h_1 - \mu t_1)/\sigma\sqrt{t_1}]$, the right hand-side of eq. (22) can be reformulated as :

$$\int_{-\infty}^{h_2} E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2\}} \middle| X_{t_1} \in du \right] \frac{e^{-\frac{1}{2}\left(\frac{u-\mu t_1}{\sigma\sqrt{t_1}}\right)^2}}{\sigma\sqrt{2\pi t_1}} du - \int_{-\infty}^{h_1} E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2\}} \middle| X_{t_1} \in du \right] \frac{e^{-\frac{1}{2}\left(\frac{u-\mu t_1}{\sigma\sqrt{t_1}}\right)^2}}{\sigma\sqrt{2\pi t_1}} du \quad (23)$$

Then, using the classical formula for the distribution of the terminal value of generalized Brownian motion and its maximum and its minimum over an interval (see, e.g., Cox and Miller, 1965), (23) can be expanded as the following integral :

$$\int_{-\infty}^{h_2} \Theta(u) du - \int_{-\infty}^{h_1} \Theta(u) du \quad (24)$$

where :

$$\Theta(u) = \Lambda^{(1)} \left\{ \sum^{(1)} [\Phi^{(1)}(h_2) - \Phi^{(1)}(h_1)] - \sum^{(2)} [\Phi^{(2)}(h_2 - 2h_1) - \Phi^{(2)}(-h_1)] \right\} \quad (25)$$

$$\Lambda^{(1)} = \frac{e^{-\frac{1}{2} \left(\frac{u - \mu t_1}{\sigma \sqrt{t_1}} \right)^2}}{\sigma \sqrt{2\pi t_1}}, \quad \sum^{(1)} = \sum_{n=-\infty}^{\infty} e^{2\mu n h / \sigma^2}, \quad \sum^{(2)} = \sum_{n=-\infty}^{\infty} e^{2\mu(h_1 - u - nh) / \sigma^2}, \quad h = h_2 - h_1 \quad (26)$$

$$\Phi^{(1)}(a) = N \left[\frac{a - u - \mu(t_2 - t_1) - 2nh}{\sigma \sqrt{t_2 - t_1}} \right], \quad \Phi^{(2)}(a) = N_1 \left[\frac{a + u - \mu(t_2 - t_1) + 2nh}{\sigma \sqrt{t_2 - t_1}} \right] \quad (27)$$

Performing the necessary calculations, one can obtain the following closed form solution to (23) :

$$Q(m_1^2 > h_1, M_1^2 < h_2) = \sum_{n=-\infty}^{\infty} e^{2\mu n h / \sigma^2} \left\{ \Psi^1(h_2, h_2) - \Psi^1(h_2, h_1) - \Psi^1(h_1, h_2) + \Psi^1(h_1, h_1) \right. \\ \left. - e^{2\mu(h_1 - 2nh) / \sigma^2} [\Psi^2(h_2, h_2 - 2h_1) - \Psi^2(h_2, -h_1) - \Psi^2(h_1, h_2 - 2h_1) + \Psi^2(h_1, -h_1)] \right\} \quad (28)$$

where :

$$\Psi^1(a, b) = N_2 \left[\frac{a - \mu t_1}{\sigma \sqrt{t_1}}, \frac{b - \mu t_2 - 2nh}{\sigma \sqrt{t_2}}; \sqrt{\frac{t_1}{t_2}} \right], \quad \Psi^2(a, b) = N_2 \left[\frac{a + \mu t_1}{\sigma \sqrt{t_1}}, \frac{b - \mu t_2 + 2nh}{\sigma \sqrt{t_2}}; -\sqrt{\frac{t_1}{t_2}} \right] \quad (29)$$

Next, the formula in (28) enables to expand (21) as the following integration problem :

$$E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, M_1^2 < h_2\}} E^Q \left[\mathbf{1}_{\{X_{t_3} > k\}} \middle| h_1 < X_{t_2} < h_2 \right] \right] \\ = \sum_{n=-\infty}^{\infty} e^{2\mu n h / \sigma^2} \left\{ \int_{-\infty}^{h_2} \int_{-\infty}^{h_2} \Lambda^{(2)}(u, v) \Phi^{(3)}(v) dudv - \int_{-\infty}^{h_2} \int_{-\infty}^{h_1} \Lambda^{(2)}(u, v) \Phi^{(3)}(v) dudv \right. \\ \left. - \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \Lambda^{(2)}(u, v) \Phi^{(3)}(v) dudv + \int_{-\infty}^{h_1} \int_{-\infty}^{h_1} \Lambda^{(2)}(u, v) \Phi^{(3)}(v) dudv \right. \\ \left. - e^{2\mu(h_1 - 2nh) / \sigma^2} \left[\int_{-\infty}^{h_2} \int_{-\infty}^{h_2} \Lambda^{(3)}(u, v) \Phi^{(3)}(v) dudv - \int_{-\infty}^{h_2} \int_{-\infty}^{h_1} \Lambda^{(3)}(u, v) \Phi^{(3)}(v) dudv \right. \right. \\ \left. \left. - \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \Lambda^{(3)}(u, v) \Phi^{(3)}(v) dudv + \int_{-\infty}^{h_1} \int_{-\infty}^{h_1} \Lambda^{(3)}(u, v) \Phi^{(3)}(v) dudv \right] \right\} \quad (30)$$

where :

$$\Lambda^{(2)}(u, v) = \frac{e^{-\frac{1}{2}\left[\left(\frac{u-\mu t_1}{\sigma\sqrt{t_1}}\right)^2 + \left(\frac{v-u-\mu(t_2-t_1)-2nh}{\sigma\sqrt{t_2-t_1}}\right)^2\right]}}{2\pi\sigma^2\sqrt{t_1(t_2-t_1)}} \quad (31)$$

$$\Lambda^{(3)}(u, v) = \frac{e^{-\frac{1}{2}\left[\left(\frac{u+\mu t_1}{\sigma\sqrt{t_1}}\right)^2 + \left(\frac{v+u-2h_1-\mu(t_2-t_1)+2nh}{\sigma\sqrt{t_2-t_1}}\right)^2\right]}}{2\pi\sigma^2\sqrt{t_1(t_2-t_1)}} \quad (32)$$

$$\Phi^{(3)}(v) = N\left(\frac{-k + v + \bar{\mu}(t_3 - t_2)}{\sigma\sqrt{t_3 - t_2}}\right) \quad (33)$$

For a standard window double knock-out put option, the function $\Phi^{(3)}(v)$ would be :

$$\Phi^3(v) = N\left(\frac{k - v - \bar{\mu}(t_3 - t_2)}{\sigma\sqrt{t_3 - t_2}}\right) \quad (34)$$

A closed form solution can be found to the integration problem (30), which is precisely the formula for a standard window double knock-out option given in Proposition 1, section 1.

As mentioned in section 1, a rebate provision may be included in the contract, giving the option holder the right to receive an amount R at expiry if the option has been knocked-out. The value, V^R , of the standard window double knock-out option then becomes :

$$V^R = V + e^{-rt_3}R(1 - Q(m_1^2 > h_1, M_1^2 < h_2)) \quad (35)$$

where V is the option value without rebate as given by Proposition 1 in section 1, and $Q(m_1^2 > h_1, M_1^2 < h_2)$ is explicitly given by eq. (14) in this Appendix.

A.3 Numerical implementation of Proposition 1 in section 1

The trivariate normal integrals appearing in Proposition 1 of section 1, defined by eq. (11) in this appendix, must be numerically integrated. Several algorithms have already been designed to compute trivariate normal cumulative distribution functions (Genz, 2001). However, they do not fit the specific convolution used in (11). The following simple rule can be used instead :

$$N_3[a, b, c; \rho_{12}, \rho_{23}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^b e^{-x^2/2} N\left[\frac{a - \rho_{12}x}{\sqrt{1 - \rho_{12}^2}}\right] N\left[\frac{c - \rho_{23}x}{\sqrt{1 - \rho_{23}^2}}\right] dx \quad (36)$$

This numerical integration is very easy to perform. A level of at least 10^{-6} accuracy, which is more than enough for option pricing, can be achieved with a mere 16-point Gauss-Legendre rule (and a lower bound of -8.5 in the integral). Moreover, the integration rule in (36) is extremely efficient : the analytical values reported in Table 1 take a computational time of 0.4 second for the eight out of

twelve prices requiring the computation of three terms in the infinite series, and an average 0.6 second for the others (on a modest 2.4 Ghz-clock PC). Note that, at most, nine terms in the infinite series were required. In general, more and more terms are needed as the lower barrier and the upper barrier are closer to one another and as volatility increases. But, in the vast majority of cases, very few leading terms are required. Only for unrealistic contract specifications may significant errors arise from the truncation of the infinite series. This is in line with the findings of Kunitomo and Ikeda (1992) regarding standard double barrier options.

Along with analytical values, Table 1 reports numerical results obtained by binomial and Monte Carlo simulation methods. The binomial estimates are computed with 500 timesteps. The jump and probability parameters are set according to the Trigeorgis approach (1992), which has been proved to be slightly more accurate than that of Cox, Ross and Rubinstein (1979) or that of Jarrow and Rudd (1983). In addition, the lattice is constructed in such a way that there are horizontal layers of nodes as close as possible to the upper barrier and to the lower barrier, following the recommendations of Boyle and Lau (1994). Prices are obtained in less than one second. The binomial estimates diverge from the exact prices by 0.58% on average.

The Monte Carlo simulation estimates were obtained after running 1,000,000 simulations per option value and implementing a computationally demanding discretization of 16 monitoring times per business day for the time segment in which the double barrier is active. The efficiency of such a procedure is obviously very poor and definitely not suited to real time trading environments.

Appendix B : Valuation of partial window double barrier options

B.1 Analytical formula

Proposition 2 :

The value V of a partial window double knock-out option is given by :

$$V_{PDKO}(S_0, K, H_1, H_2, H_3, H_4, r, \delta, \sigma, t_1, t_2, t_3, t_4, t_5, t_6, t_7) = \theta \left[e^{-rt_7} K \Phi(\bar{\mu}) - e^{-\delta t_7} S_0 \Phi(\tilde{\mu}) \right] \quad (37)$$

where :

S_0 is the underlying asset spot value, K is the strike price, r is the constant instantaneous riskless rate, σ is the underlying asset's constant volatility, δ is its constant continuous dividend rate;

H_2 is the lower part of the double barrier, H_3 is the upper part of the double barrier;

H_1 is a single lower barrier if the priced contract is down-and-up or down-and-down; otherwise, it is a single upper barrier;

H_4 is a single lower barrier if the priced contract is down-and-down or up-and-down; otherwise, it is a single upper barrier;

assuming that the option life starts at $t_0 = 0$, $[t_1, t_2]$ is the time interval during which the possible crossing of H_1 is monitored, $[t_3, t_4]$ is the time interval during which the possible crossing of the double barrier (H_2, H_3) is monitored, $[t_5, t_6]$ is the time interval during which the possible crossing of H_4 is monitored, t_7 is the option expiry, with : $t_0 < t_1 < t_2 < t_3 < t_4 < t_5 < t_6 < t_7$

$$\bar{\mu} = r - \delta - \sigma^2 / 2 \quad , \quad \tilde{\mu} = r - \delta + \sigma^2 / 2$$

$$\begin{aligned} \Phi(\mu) = & \sum_{n=-\infty}^{\infty} e^{2\mu nh / \sigma^2} \left\{ \Psi^{(1)}(h_3, h_3) - \Psi^{(1)}(h_3, h_2) - \Psi^{(1)}(h_2, h_3) + \Psi^{(1)}(h_2, h_2) \right. \\ & - e^{2\mu h_1 / \sigma^2} \left[\Psi^{(2)}(h_3, h_3) - \Psi^{(2)}(h_3, h_2) - \Psi^{(2)}(h_2, h_3) + \Psi^{(2)}(h_2, h_2) \right] \\ & - e^{2\mu(h_2 - 2nh) / \sigma^2} \left[\Psi^{(3)}(h_3, h_3) - \Psi^{(3)}(h_3, h_2) - \Psi^{(3)}(h_2, h_3) + \Psi^{(3)}(h_2, h_2) \right] \\ & + e^{2\mu(h_2 - h_1 - 2nh) / \sigma^2} \left[\Psi^{(4)}(h_3, h_3) - \Psi^{(4)}(h_3, h_2) - \Psi^{(4)}(h_2, h_3) + \Psi^{(4)}(h_2, h_2) \right] \\ & - e^{2\mu(h_4 - 2nh) / \sigma^2} \left[\Psi^{(5)}(h_3, h_3) - \Psi^{(5)}(h_3, h_2) - \Psi^{(5)}(h_2, h_3) + \Psi^{(5)}(h_2, h_2) \right] \\ & + e^{2\mu(h_4 - h_1 - 2nh) / \sigma^2} \left[\Psi^{(6)}(h_3, h_3) - \Psi^{(6)}(h_3, h_2) - \Psi^{(6)}(h_2, h_3) + \Psi^{(6)}(h_2, h_2) \right] \\ & + e^{2\mu(h_4 - h_2) / \sigma^2} \left[\Psi^{(7)}(h_3, h_3) - \Psi^{(7)}(h_3, h_2) - \Psi^{(7)}(h_2, h_3) + \Psi^{(7)}(h_2, h_2) \right] \\ & \left. - e^{2\mu(h_4 - h_2 + h_1) / \sigma^2} \left[\Psi^{(8)}(h_3, h_3) - \Psi^{(8)}(h_3, h_2) - \Psi^{(8)}(h_2, h_3) + \Psi^{(8)}(h_2, h_2) \right] \right\} \end{aligned} \quad (38)$$

and :

$$\begin{aligned} \Psi^{(1)}(a, b) = & N_7 \left[\gamma \left(\frac{h_1 - \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{h_1 - \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a - \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2nh - \mu t_4}{\sigma \sqrt{t_4}}, \lambda \left(\frac{h_4 - 2nh - \mu t_5}{\sigma \sqrt{t_5}} \right), \right. \\ & \left. \lambda \left(\frac{h_4 - 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \theta \lambda \left(\frac{k - 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); \rho_{12}, \gamma \rho_{23}, \rho_{34}, \lambda \rho_{45}, \rho_{56}, \theta \rho_{67} \right] \\ \Psi^{(2)}(a, b) = & N_7 \left[\gamma \left(\frac{h_1 + \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{-h_1 - \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a - 2h_1 - \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2h_1 - 2nh - \mu t_4}{\sigma \sqrt{t_4}}, \right. \\ & \left. \lambda \left(\frac{h_4 - 2h_1 - 2nh - \mu t_5}{\sigma \sqrt{t_5}} \right), \lambda \left(\frac{h_4 - 2h_1 - 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \theta \lambda \left(\frac{k - 2h_1 - 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); \right. \\ & \left. -\rho_{12}, \gamma \rho_{23}, \rho_{34}, \lambda \rho_{45}, \rho_{56}, \theta \rho_{67} \right] \end{aligned} \quad (39)$$

$$\Psi^{(3)}(a, b) = N_7 \left[\gamma \left(\frac{h_1 + \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{h_1 + \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a + \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2h_2 + 2nh - \mu t_4}{\sigma \sqrt{t_4}}, \lambda \left(\frac{h_4 - 2h_2 + 2nh - \mu t_5}{\sigma \sqrt{t_5}} \right), \right. \\ \left. \lambda \left(\frac{h_4 - 2h_2 + 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \theta \lambda \left(\frac{k - 2h_2 + 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); \rho_{12}, \gamma \rho_{23}, -\rho_{34}, \lambda \rho_{45}, \rho_{56}, \theta \rho_{67} \right]$$

$$\Psi^{(4)}(a, b) = N_7 \left[\gamma \left(\frac{h_1 - \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{-h_1 + \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a - 2h_1 + \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2h_2 + 2h_1 + 2nh - \mu t_4}{\sigma \sqrt{t_4}}, \right. \\ \left. \lambda \left(\frac{h_4 - 2h_2 + 2h_1 + 2nh - \mu t_5}{\sigma \sqrt{t_5}} \right), \lambda \left(\frac{h_4 - 2h_2 + 2h_1 + 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \right. \\ \left. \theta \lambda \left(\frac{k - 2h_2 + 2h_1 + 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); -\rho_{12}, \gamma \rho_{23}, -\rho_{34}, \lambda \rho_{45}, \rho_{56}, \theta \rho_{67} \right]$$

$$\Psi^{(5)}(a, b) = N_7 \left[\gamma \left(\frac{h_1 + \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{h_1 + \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a + \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2nh + \mu t_4}{\sigma \sqrt{t_4}}, \lambda \left(\frac{h_4 - 2nh + \mu t_5}{\sigma \sqrt{t_5}} \right), \right. \\ \left. \lambda \left(\frac{-h_4 + 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \theta \lambda \left(\frac{k - 2h_4 + 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); \rho_{12}, \gamma \rho_{23}, \rho_{34}, \lambda \rho_{45}, -\rho_{56}, \theta \rho_{67} \right]$$

$$\Psi^{(6)}(a, b) = N_7 \left[\gamma \left(\frac{h_1 - \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{-h_1 + \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a - 2h_1 + \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2h_1 - 2nh + \mu t_4}{\sigma \sqrt{t_4}}, \right. \\ \left. \lambda \left(\frac{h_4 - 2h_1 - 2nh + \mu t_5}{\sigma \sqrt{t_5}} \right), \lambda \left(\frac{-h_4 + 2h_1 + 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \right. \\ \left. \theta \lambda \left(\frac{k - 2h_4 + 2h_1 + 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); -\rho_{12}, \gamma \rho_{23}, \rho_{34}, \lambda \rho_{45}, -\rho_{56}, \theta \rho_{67} \right]$$

$$\Psi^{(7)}(a, b) = N_7 \left[\gamma \left(\frac{h_1 - \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{h_1 - \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a - \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2h_2 + 2nh + \mu t_4}{\sigma \sqrt{t_4}}, \lambda \left(\frac{h_4 - 2h_2 + 2nh + \mu t_5}{\sigma \sqrt{t_5}} \right), \right. \\ \left. \lambda \left(\frac{-h_4 + 2h_2 - 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \theta \lambda \left(\frac{k - 2h_4 + 2h_2 - 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); \right. \\ \left. \rho_{12}, \gamma \rho_{23}, -\rho_{34}, \lambda \rho_{45}, -\rho_{56}, \theta \rho_{67} \right]$$

$$\Psi^{(8)}(a, b) = N_7 \left[\gamma \left(\frac{h_1 + \mu t_1}{\sigma \sqrt{t_1}} \right), \gamma \left(\frac{-h_1 - \mu t_2}{\sigma \sqrt{t_2}} \right), \frac{a - 2h_1 - \mu t_3}{\sigma \sqrt{t_3}}, \frac{b - 2h_2 + 2h_1 + 2nh + \mu t_4}{\sigma \sqrt{t_4}}, \right. \\ \left. \lambda \left(\frac{h_4 - 2h_2 + 2h_1 + 2nh + \mu t_5}{\sigma \sqrt{t_5}} \right), \lambda \left(\frac{-h_4 + 2h_2 - 2h_1 - 2nh - \mu t_6}{\sigma \sqrt{t_6}} \right), \right. \\ \left. \theta \lambda \left(\frac{k - 2h_4 + 2h_2 - 2h_1 - 2nh - \mu t_7}{\sigma \sqrt{t_7}} \right); -\rho_{12}, \gamma \rho_{23}, -\rho_{34}, \lambda \rho_{45}, -\rho_{56}, \theta \rho_{67} \right]$$

$$h_i = \log(H_i / S_0), h = h_3 - h_2, k = \log(K / S_0), \rho_{ij} = \sqrt{t_i / t_j}, t_i < t_j$$

$$\gamma = \begin{cases} -1 & \text{if the option is a down - and - up call or put or a down - and - down call or put} \\ 1 & \text{otherwise} \end{cases}$$

$$\lambda = \begin{cases} -1 & \text{if the option is an up - and - down call or put or a down - and - down call or put} \\ 1 & \text{otherwise} \end{cases}$$

$$\theta = \begin{cases} -1 & \text{if the option is an up - and - down put or an up - and - up call} \\ & \text{or a down - and - up call or a down - and - down put} \\ 1 & \text{otherwise} \end{cases}$$

This formula may appear cumbersome at first glance. Yet, it can rightfully be regarded as particularly concise in view of the very large variety of complex contracts it enables to value.

$N_7[x_1, x_2, x_3, x_4, x_5, x_6, x_7; \pm\rho_{12}, \pm\rho_{23}, \pm\rho_{34}, \pm\rho_{45}, \pm\rho_{56}, \pm\rho_{67}]$ is a special form of the normal cumulative distribution function, defined by eq. (13) in Appendix A. Note that the price of a partial window double knock-in option is obtained simply by subtracting the value of the corresponding partial window double knock-out option from the price of a vanilla option.

For proposition 2 to be well-defined, any t_i must never be strictly equal to t_j , where $i < j$. This, however, does not imply a loss of generality since, in the limit, t_i can be made arbitrarily close to t_j . This is how we use proposition 2 to obtain the analytical values reported in Table 2, section 2 : we let $t_1 \rightarrow t_0$, $t_3 \rightarrow t_2$, $t_5 \rightarrow t_4$ and $t_7 \rightarrow t_6$. More specifically, we take the following approximations of continuity: $t_1 = 0.0001$, $t_2 = 0.4999$, $t_5 = 1.0001$ and $t_6 = 1.4999$, recalling that $t_0 = 0$, $t_3 = 0.5$, $t_4 = 1$ and $t_7 = 1.5$. Further refinement of these approximations is numerically insignificant.

As mentioned in section 1, a rebate provision may be included in the contract, giving the option holder the right to receive an amount R at expiry if the option has been knocked-out. The value, V^R , of the partial window double knock-out option then becomes :

$$V^R = V + e^{-rt_7} R (1 - Q(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3, m_5^6 > h_4)) \quad (40)$$

if the option is up-and-down

$$V^R = V + e^{-rt_7} R (1 - Q(m_1^2 > h_1, m_3^4 > h_2, M_3^4 < h_3, M_5^6 < h_4)) \quad (41)$$

if the option is down-and-up

$$V^R = V + e^{-rt_7} R (1 - Q(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3, M_5^6 < h_4)) \quad (42)$$

if the option is up-and-up

$$V^R = V + e^{-rt_7} R (1 - Q(m_1^2 > h_1, m_3^4 > h_2, M_3^4 < h_3, m_5^6 > h_4)) \quad (43)$$

if the option is down-and-down

where V is the option value without rebate, and where we recall that $Q(\cdot)$ is the probability operator under the equivalent martingale measure, under which : $\mu = r - \delta - \sigma^2 / 2$. The probabilities involved in (40)-(43), which correspond to digital partial window double barrier options, are deduced from Proposition 2 above by taking the first six dimensions of each $\Psi^{(i)}, i \in [1, \dots, 8]$ in $\Phi(\mu)$, all other things being equal .

B.2 Sketch of proof

Using the same notations and following the same first steps as with standard window double knock-out options (Appendix A), the no-arbitrage value of an up-and-down partial window double knock-out call can be expressed as :

$$C_{PWUD} = e^{-rt_\tau} \left\{ \begin{array}{l} F \bar{Q} \left(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3, m_5^6 > h_4, X_{t_\tau} > k \right) \\ -K Q \left(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3, m_5^6 > h_4, X_{t_\tau} > k \right) \end{array} \right\} \quad (44)$$

Similarly, the no-arbitrage value of a down-and-up partial window double knock-out call can be expressed as :

$$C_{PWUDU} = e^{-rt_\tau} \left\{ \begin{array}{l} F \bar{Q} \left(m_1^2 > h_1, m_3^4 > h_2, M_3^4 < h_3, M_5^6 < h_4, X_{t_\tau} > k \right) \\ -K Q \left(m_1^2 > h_1, m_3^4 > h_2, M_3^4 < h_3, M_5^6 < h_4, X_{t_\tau} > k \right) \end{array} \right\} \quad (45)$$

The no-arbitrage value of an up-and-up partial window double knock-out call can be expressed as :

$$C_{PWUU} = e^{-rt_\tau} \left\{ \begin{array}{l} F \bar{Q} \left(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3, M_5^6 < h_4, X_{t_\tau} > k \right) \\ -K Q \left(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3, M_5^6 < h_4, X_{t_\tau} > k \right) \end{array} \right\} \quad (46)$$

Finally, the value of a down-and-down partial window double knock-out call can be expressed as :

$$C_{PWDD} = e^{-rt_\tau} \left\{ \begin{array}{l} F \bar{Q} \left(m_1^2 > h_1, m_3^4 > h_2, M_3^4 < h_3, m_5^6 > h_4, X_{t_\tau} > k \right) \\ -K Q \left(m_1^2 > h_1, m_3^4 > h_2, M_3^4 < h_3, m_5^6 > h_4, X_{t_\tau} > k \right) \end{array} \right\} \quad (47)$$

where \bar{Q} is a measure such that : $\left. \frac{d\bar{Q}}{dQ} \right|_{F_t} = e^{-\frac{\sigma^2}{2}t + \sigma W_t}$. Multiplying by (-1) and substituting

“ $X_{t_\tau} > k$ ” with “ $X_{t_\tau} < k$ ” in (44)-(47) provides the corresponding put option expressions.

Obtaining these probabilities involves long calculations that cannot be reproduced here. We simply outline how it works in the case of an up-and-down partial window double knock-out call option, knowing that the method is the same as with standard window double knock-out options (for which a

detailed proof is given in Appendix A). Basically, it boils down to repeatedly conditioning and making use of the Markov property of Brownian motion. Thus, the first stage is to calculate :

$$Q(M_1^2 < h_1) = E^Q \left[\mathbf{1}_{\{X_{t_1} < h_1\}} E^Q \left[\mathbf{1}_{\{M_1^2 < h_1\}} \middle| X_{t_1} \right] \right] \quad (48)$$

This result is then used to find :

$$Q(M_1^2 < h_1, h_2 < X_{t_3} < h_3) = E^Q \left[\mathbf{1}_{\{M_1^2 < h_1, X_{t_2} < h_1\}} E^Q \left[\mathbf{1}_{\{h_2 < X_{t_3} < h_3\}} \middle| X_{t_2} < h_1 \right] \right] \quad (49)$$

The next probability to calculate is :

$$Q(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3) = E^Q \left[\mathbf{1}_{\{M_1^2 < h_1, h_2 < X_{t_3} < h_3\}} E^Q \left[\mathbf{1}_{\{m_3^4 > h_2, M_3^4 < h_3\}} \middle| h_2 < X_{t_3} < h_3 \right] \right] \quad (50)$$

Conditioning goes on until the last stage where the final probability to work out is :

$$\begin{aligned} & Q(M_1^2 < h_1, m_3^4 > h_2, M_3^4 < h_3, m_5^6 > h_4, X_{t_7} > k) \\ &= E^Q \left[\mathbf{1}_{\{M_1^2 < h_1, h_2 < X_{t_3} < h_3, m_5^6 > h_4, X_{t_6} > h_4\}} E^Q \left[\mathbf{1}_{\{X_{t_7} > k\}} \middle| X_{t_6} > h_4 \right] \right] \end{aligned} \quad (51)$$

At each stage, each new conditional expectation can be written as a sum of multiple integrals of increasing dimension. Appropriate changes of variables and simplifications allow to reduce those multiple integrals to the convolutions of the standard normal distribution functions provided in Appendix A.

B.3 Numerical implementation

Proposition 2 in this Appendix consists of a sum of seven-dimensional integrals. In this case, there is no simple dimension reduction trick to get down to a one-dimensional integral similar to the one we used in Appendix A. However, the special convolution of the multivariate standard normal distribution introduced in eq. (4) of Appendix A allows to apply the following straightforward Monte Carlo integration algorithm :

- (i) To compute $N_7[x_1, x_2, \dots, x_7; \rho_{12}, \rho_{23}, \dots, \rho_{67}]$, first draw n samples of seven uniform numbers $u_i^j \sim \mathcal{U}(0,1), i \in [1, \dots, 7], j \in [1, \dots, n]$ and turn them into n samples of seven independent normal deviates $y_i^j, i \in [1, \dots, 7], j \in [1, \dots, n]$ with zero mean and unit variance $\{y_1^j \sim N(0,1), y_2^j \sim N(0,1), \dots, y_7^j \sim N(0,1)\}$, using, e.g., the polar rejection algorithm (Press et al., 1992)

(ii) Next, turn these n samples of seven independent normal deviates into n samples of seven correlated normal deviates :

$$\left\{ y_1^j, \rho_{12}y_1^j + \sqrt{(1 - \rho_{12}^2)}y_2^j, \rho_{23}y_2^j + \sqrt{(1 - \rho_{23}^2)}y_3^j, \dots, \rho_{67}y_6^j + \sqrt{(1 - \rho_{67}^2)}y_7^j \right\}, j \in [1, \dots, n]$$

(iii) Then, test the relevant conditions for each deviate in each sample :

$$y_1^j < x_1, \rho_{12}y_1^j + \sqrt{(1 - \rho_{12}^2)}y_2^j < x_2, \dots, \rho_{67}y_6^j + \sqrt{(1 - \rho_{67}^2)}y_7^j < x_7, \quad j \in [1, \dots, n]$$

If we denote by M the number of samples having passed the previous test, then the cumulative distribution function $N_7[x_1, x_2, \dots, x_7; \rho_{12}, \rho_{23}, \dots, \rho_{67}]$ can be approximated by M/n . By the strong law of large numbers, this sampling rule tends to the exact value of $N_7[x_1, x_2, \dots, x_7; \rho_{12}, \rho_{23}, \dots, \rho_{67}]$ as n goes to infinity. Note that this algorithm is readily extended to higher-dimensional cumulative distribution functions, as its convergence rate is independent of dimension.

In practice, the speed of convergence largely relies on the way the uniform deviates $u_i^j, i \in [1, \dots, 7], j \in [1, \dots, n]$ are drawn at stage (i) of the algorithm. Pseudo-random numbers can be used, along with classical variance reduction techniques such as antithetic variates, stratified or Latin hypercube sampling. However, convergence is achieved much faster by using quasi random numbers instead of pseudo random numbers, thanks to the greater uniformity of low discrepancy sequences (Niederreiter, 1992). To obtain the analytical values reported in Tables 2 and 3 in section 2, seven different sequences of 20,000 Sobol points (Sobol, 1967) have been used, applying the code provided by Press et al. (1992). This leads to computational times of 2.4 seconds for the 20 out of 24 option prices that require the computation of three terms in the infinite series, and 3.2 seconds for the others (that require five terms in the infinite series). The binomial and Monte Carlo simulation estimates provided in Tables 2 and 3 in section 2 are obtained in the same way as in Appendix A. Again, binomial results are excellent in terms of efficiency, with computational time as low as 0.8 second to obtain an option price (recalling that 500 timesteps are used). However, one should not jump to the conclusion that the binomial method is more efficient than the analytical one combined with Monte Carlo quasi-random sampling. First, for options written with a longer time-to-maturity than that used in Tables 2 and 3 (1.5 year), more timesteps would be required. More importantly, to assess the overall efficiency of a numerical technique, it is necessary to take into consideration the time needed to implement it in the first place, as well as the time that may be required to adapt it to future valuation problems. In that respect, once you have typed the formula given by Proposition 2, your job is done, with no need for subsequent trimming of your code, whatever the contract specifications or the model inputs. The same is not true with trees, which can lead to significant errors when applied to complex

path-dependent payoffs such as those of partial window double barrier options. Consequently, trees will have to be modified as new valuation problems arise.

Tables

Table 1. Comparison between standard window double knockout put prices (SWDKOP), standard double knockout put prices (SDKOP) and vanilla put prices (VP) ^a

Lower / Upper barrier	Volatility	SWDKOP (analytical)	SWDKOP (binomial)	SWDKOP (simulation)	SDKOP (analytical)	VP (analytical)
80/120	18 %	2.83	2.84	2.85	1.15	6.37
	25 %	1.94	1.93	1.96	0.32	9.54
	32 %	1.12	1.13	1.15	0.05	12.71
	44 %	0.30	0.30	0.31	0	18.09
70/130	18 %	5.26	5.26	5.28	3.88	6.37
	25 %	5.19	5.20	5.23	2.56	9.54
	32 %	4.22	4.21	4.22	1.24	12.71
	44 %	2.40	2.40	2.44	0.21	18.09
60/140	18 %	6.23	6.23	6.25	5.76	6.37
	25 %	7.97	7.98	7.98	5.91	9.54
	32 %	7.97	7.97	7.99	4.45	12.71
	44 %	6.16	6.14	6.19	1.83	18.09

^a This table presents the prices of standard window double knockout put options (SWDKOP), standard double knockout put options (SDKOP) and vanilla put options (VP) for four different levels of volatility. All contracts are at-the-money with strike price 100. The option life, measured in years, is [0-1.5]. The window double barrier is monitored within [0.5-1]. The standard double barrier, by definition, is monitored within [0-1.5]. The riskless rate is 5%, the dividend rate is 2%. There are no rebate provisions. SWDKOP analytical values were obtained applying the formula given in Proposition 1, section 1, by means of the implementation rule given in Appendix A, § 3. Binomial and Monte Carlo simulation estimates were obtained using the techniques described in Appendix A, § 3. SDKOP and VP analytical values were obtained using standard formulae (see, e.g., Zhang, 1998).

Table 2 . Comparison between up-and-down partial (UDP), down-and-up (DUP), up-and-up (UUP) and down-and-down (DDP) window double knockout put prices ^a

Lower / Upper barrier		analytical	binomial	simulation
80/120	UDP	0.42	0.41	0.43
70/130	UDP	2.63	2.62	2.66
60/140	UDP	5.95	5.96	5.97
80/120	DUP	1.70	1.72	1.71
70/130	DUP	5.11	5.12	5.15
60/140	DUP	7.94	7.96	7.96
80/120	UUP	1.62	1.63	1.61
70/130	UUP	5.03	5.04	5.08
60/140	UUP	7.92	7.92	7.88
80/120	DDP	0.44	0.44	0.46
70/130	DDP	2.68	2.70	2.70
60/140	DDP	5.27	5.29	5.27

^a This table presents the prices of up-and-down (UDP), down-and-up (DUP), up-and-up (UUP) and down-and-down (DDP) partial window double knockout put options for three different levels of the lower and upper barriers. All contracts are at-the-money with strike price 100. The option life, measured in years, is [0-1.5]. The window double barrier is monitored within [0.5-1]. The first single barrier is monitored within [0-0.5] and the second single barrier is monitored within [1-1.5]. Volatility is equal to 25%. The riskless rate is 5%, the dividend rate is 2%. There are no rebate provisions. Analytical prices were obtained applying the formula given by Proposition 2 in Appendix B, § 1, by means of the implementation rule given in Appendix B, § 3. Binomial and Monte Carlo simulation estimates were obtained using the techniques described in Appendix A, § 3.

Table 3 . Comparison between up-and-down partial (UDP), down-and-up (DUP), up-and-up (UUP) and down-and-down (DDP) window double knockout put prices ^a

Lower / Upper barrier		analytical	binomial	simulation
80/120 UDP		0.91	0.93	0.95
70/130 UDP		3.72	3.73	3.75
60/140 UDP		6.97	6.97	6.95
80/120 DUP		1.81	1.82	1.84
70/130 DUP		5.16	5.17	5.17
60/140 DUP		8.01	8	7.99
80/120 UUP		1.77	1.75	1.78
70/130 UUP		5.12	5.13	5.15
60/140 UUP		7.95	7.94	7.97
80/120 DDP		0.96	0.97	0.97
70/130 DDP		3.79	3.79	3.77
60/140 DDP		6.98	6.97	7.02

^a This table presents the prices of up-and-down (UDP), down-and-up (DUP), up-and-up (UUP) and down-and-down (DDP) partial window double knockout put options for three different levels of the lower and upper barriers. All contracts are at-the-money with strike price 100. The option life, measured in years, is [0-1.5]. The window double barrier is monitored within [0.5-1]. The first single barrier is monitored within [0.2-0.3] and the second single barrier is monitored within [1.2-1.3]. Volatility is equal to 25%. The riskless rate is 5%, the dividend rate is 2%. There are no rebate provisions. Analytical prices were obtained applying the formula given by Proposition 2 in Appendix B, § 1, by means of the implementation rule given in Appendix B, § 3. Binomial and Monte Carlo simulation estimates were obtained using the techniques described in Appendix A, § 3.

References

- Bhagavatula, R. S., and P. Carr, (1995), "Valuing Double Barrier Options with Time-Dependent Parameters", Working Paper, <http://www.math.columbia.edu/~pcarr/>
- Boyle, P.P., and S.H. Lau, (1994), "Bumping Up Against the Barrier with the Binomial Method", *Journal of Derivatives*, 1, 6-14
- Carr, P., (1995), "Two Extensions to Barrier Option Valuation", *Applied Mathematical Finance*, 173-209
- Carr, P., Ellis, K., and V. Gupta, (1998), "Static Hedging of Exotic Options", *Journal of Finance*, 53, 1165-90
- Cox, D.R., and Miller, H.D., (1965), *The Theory of Stochastic Processes*, Methuen, London
- Cox, J.C., Ross S.A., and M. Rubinstein, (1979), "Option Pricing : a Simplified Approach", *Journal of Financial Economics*, 7, 229-263
- Drezner, Z., and G.O. Wesolowsky, (1989), "On the Computation of the Bivariate Normal Integral", *Journal of Statistics and Computer Simulation*, 35, 101-107
- Eydeland, A., and H. Geman, (1995), "Domino Effect : Inverting the Laplace Transform", *RISK*, April, 65-7
- Geman, H., and M. Yor, (1996), "Pricing and Hedging Double-Barrier Options : a Probabilistic Approach", *Mathematical Finance*, 6 (4), 365-378
- Genz, A., (2001), "Numerical Computation of Bivariate and Trivariate Normal Probabilities", preprint, <http://www.sci.wsu.edu/math/faculty/genz/homepage>
- Harrison, J.M., and S.Pliska, "Martingales and Stochastic Integrals in the Theory of Continuous Trading", *Stochastic Processes and their Applications*, 11, 1981, 312-316
- Heynen, R.C., and H. Kat, (1995), "Partial Barrier Options", *Journal of Financial Engineering*, 3, 253-274
- Hui, C. H., (1997), "Time-Dependent Barrier Option Values", *The Journal of Futures Markets*, 17, 6, 667-688
- Jarrow, R., and A. Rudd, (1983), *Option Pricing*, Dow Jones-Irwin, Homewood, Illinois
- Kunitomo, N., and M. Ikeda, (1992), "Pricing Options with Curved Boundaries", *Mathematical Finance*, 2 (4), 275-298
- Luo, L. S.J., (2001), "Various Types of Double-Barrier Options", *Journal of Computational Finance*, 4 (3), 125-137
- Niederreiter, H., (1992), *Random Number Generation and Quasi Monte Carlo Methods*, SIAM,

Philadelphia

- Owen, D.B., (1956), "Tables for Computing Bivariate Normal Probability", *Annals of Mathematical Statistics* 27, 1075-1090
- Pelsser, A., (2000), "Pricing Double Barrier Options Using Laplace Transforms", *Finance and Stochastics*, 4, 95-104
- Press W., Teukolsky, W., Wetterling, W. and B. Flannery, (1992), *Numerical Recipes in C : The Art of Scientific Computing*, Cambridge University Press, Cambridge
- Schmock, U., Shreve, S., and U. Wystup, (1999), "Valuation of Exotic Options Under Short-Selling Constraints", Working Paper, Carnegie Mellon University
- Sobol, I.M., (1967), "On the Distribution of Points in a Cube and the Approximate Evaluation of Integrals", *USSR Computational Mathematics and Mathematical Physics*, 7, 86-112
- Schröder, M., (2000), "On the Valuation of Double-Barrier Options : computational aspects", *Journal of Computational Finance*, 3 (4), 5-33
- Sidenius, J., (1998), "Double Barrier Options : Valuation by Path Counting", *Journal of Computational Finance*, 1 (3), 63-79
- Taleb, N., (1997), *Dynamic Hedging*, J.Wiley & Sons, New York
- Toft, K., and C. Xuan, (1998), "How Well Can Barrier Options be Hedged by a Static Portfolio of Standard Options ?", *Journal of Financial Engineering*, 7, 147-75
- Tong, Y.L., (1990), *The Multivariate Normal Distribution*, Springer-Verlag, New York
- Trigeorgis, L., (1991), "A Log-transformed Binomial Numerical Analysis Method for Valuing Complex Multi-option Investments", *Journal of Financial and Quantitative Analysis*, 26 (3), 309-326
- Wystup, U., (1999), "Dealing with Dangerous Digitals", Working Paper, <http://www.mathfinance.de>
- Zhang, P.G., (1998), *Exotic Options*, World Scientific