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ANALYTICAL VALUATION OF OPTIONS ON JOINT MINIMA AND MAXIMA

Tristan Guillaume¹

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Abstract

In this paper, we show how to obtain explicit formulae for a variety of popular path-dependent contracts with complex payoffs involving joint distributions of several extrema. More specifically, we give formulae for standard step-up and step-down barrier options, as well as partial and outside step-up and step-down barrier options, involving multiple integrals of dimensions ranging between three and five. Our method can be extended to other exotic path-dependent payoffs as well as to higher dimensions. Numerical results show that the quasi random integration of these formulae involving multivariate distributions of correlated Gaussian random variables provides option values more quickly and more accurately than Monte Carlo simulation.

Keywords : option, barrier, step barrier, pricing, dimension

¹ Université de Cergy-Pontoise, Laboratoire Thema, 33 boulevard du port, F-95011 Cergy-Pontoise Cedex, France
Email : tristan.guillaume@u-cergy.fr

Introduction

Barrier options are one of the oldest types of options. They are also one of the most popular, due to their low cost and their strong leverage effect, as well as to the precision and the flexibility with which they can adapt to investors' needs or views about the market. As a result, barrier options are now heavily traded, particularly in the foreign exchange markets. They are also embedded in a lot of popular structured derivatives in equity and interest rate markets.

There has been quite extensive research dealing with the analytical pricing of barrier options. A closed form formula for a down-and-out European call option was published by Merton as early as 1973 (Merton, 1973). An exhaustive list of formulae for standard barrier options was published by Rich (1994). In the mid-90's, contributions started to focus on more exotic types of barrier options, such as partial barrier options (Heynen & Kat, 1995 ; Carr, 1995), outside barrier options (Bermin, 1995), double barrier options (Kunitomo and Ikeda, 1992 ; Bhagavatula and Carr, 1995 ; Geman and Yor, 1996). Heynen and Kat (1995) derive formulae for partial barrier options in a partial-differential-equation framework. Carr (1995) includes American-type rebates. Bermin (1995) develops a decorrelation approach to reduce the dimension of the partial outside barrier option pricing problem. Kunitomo and Ikeda (1992) draw on Levy's seminal works (1948) to provide a formula for a double barrier option with curved boundaries. Bhagavatula and Carr (1995) deal with double barrier options by means of Fourier series, whereas Geman and Yor (1996) apply Laplace transformation.

So far, there are no available formulae for a special type of barrier option that attracts strong interest among practitioners : step barrier options. These contracts feature sequences of constant one-sided barriers. They are called step barrier options, because barriers are step functions of time. When barriers are monotonically decreasing or increasing, we have a standard step barrier option. There can be gaps during which there is no barrier monitoring. In this case, the step barrier option is said to be partial. When the asset involved in the payoff is not the same as the one involved in barrier monitoring, we have an outside step barrier option. The main

difficulty associated with an analytical valuation of such contracts, in a risk-neutral framework, is that they involve joint distributions of extrema over various time intervals that are currently not explicitly known. Furthermore, the dimension of the pricing problem quickly increases. Consequently, it is assumed that these contracts have to be numerically priced. When the dimension of the problem is greater than three, Monte Carlo simulation is thought to be the only tractable method, as its convergence rate is independent of dimension, contrary to a grid-based approach.

The purpose of this paper is to show that this valuation problem can be analytically solved, by providing explicit formulae for all kinds of step barrier options, whether they are standard, non-standard, partial or outside contracts, in dimensions that range between three and five, knowing that the method we use can be applied to higher dimensions (sections 1, 2, 3). Then, by using simple low discrepancy sequences to perform quasi random integration, this paper aims at showing that the numerical implementation of these exact formulae is both more efficient and more accurate than alternative advanced or plain Monte Carlo simulation methods (section 4).

1. Analytical framework

It is assumed that the underlying asset follows a geometric brownian motion :

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (1)$$

The parameters μ and σ are constant, and σ is positive. The option life starts at time 0 and that its expiry is T. W_t is Brownian motion defined on a probability space (Ω, F_t, P) , where $F_t = \sigma(W_s, s \leq t)$ is the natural filtration of W_t .

According to the risk-neutral valuation approach, at the contract inception $t = 0$, the no-arbitrage value of a European call option with path-dependent conditions A_1, A_2, \dots, A_m is given by the discounted expectation of its payoff under the equivalent martingale measure, conditional on the information available at time $t = 0$:

$$e^{rT} C(S, T, K, r, \sigma, \delta; A_1, \dots, A_m) = E^Q \left[(S_T - K)^+ \mathbf{1}_{\{A_1, \dots, A_m\}} \mid S_0 \right] \quad (2)$$

where K is the strike price, r is the riskless rate, δ is a dividend rate, E^Q is the expectation operator under the equivalent martingale measure, Q , and $\mathbf{1}_{\{\cdot\}}$ is the indicator function taking value 1 if the conditions inside the brackets are met and value zero otherwise.

Equivalently, we have :

$$\begin{aligned}
e^{rT} C(S, T, K, r, \sigma, \delta; A_1, \dots, A_m) &= E^Q \left[(S_T - K) \mathbf{1}_{\{A_1, \dots, A_m, S_T > K\}} \mid S_0 \right] \\
&= E^Q \left[S_0 e^{\left(r - \delta - \frac{\sigma^2}{2} \right) T + \sigma W_T} \mathbf{1}_{\{A_1, \dots, A_m, S_T > K\}} \right] - K E^Q \left[\mathbf{1}_{\{A_1, \dots, A_m, S_T > K\}} \right] \\
&= F E^Q \left[e^{-\frac{\sigma^2}{2} T + \sigma W_T} \mathbf{1}_{\{A_1, \dots, A_m, S_T > K\}} \right] - K E^Q \left[\mathbf{1}_{\{A_1, \dots, A_m, S_T > K\}} \right]
\end{aligned} \tag{3}$$

where $F = S_0 e^{(r-\delta)T}$ is the risk-neutral forward price.

Let us define a new measure \tilde{Q} such that :

$$\left. \frac{d\tilde{Q}}{dQ} \right|_{F_T} = e^{-\frac{\sigma^2}{2} T + \sigma W_T} \tag{4}$$

Then, according to Girsanov theorem, we have :

$$e^{rT} C(S, T, K, r, \sigma, \delta; A_1, \dots, A_m) = F \tilde{Q}(A_1, \dots, A_m, X_T > k) - K Q(A_1, \dots, A_m, X_T > k) \tag{5}$$

with : $X_T = \ln(S_T / S_0)$, $k = \ln(K / S_0)$

Similarly, we have the following expression for a path-dependent put option :

$$e^{rT} P(S, T, K, r, \sigma, \delta; A_1, \dots, A_m) = K Q(A_1, \dots, A_m, X_T < k) - F \tilde{Q}(A_1, \dots, A_m, X_T < k)$$

Hence, it suffices to calculate probabilities under the Q -measure : a simple change of drift from $\mu = r - \delta - \sigma^2 / 2$ to $\tilde{\mu} = r - \delta + \sigma^2 / 2$ will provide the corresponding probabilities under the \tilde{Q} -measure.

In general, the greater the number of path dependent conditions, the higher the dimension of the solution will be.

Low-dimensional problems can be solved in terms of univariate or bivariate standard Gaussian distribution functions, $N[a]$ and $N_2[a, b; \rho]$ respectively. This is the case for standard barrier and lookback options, as well as for partial or outside barrier options. If we want to find closed form formulae for more complex path-dependent options, we need to use higher-dimensional Gaussian distributions.

In general, if $X = (X_1, \dots, X_n)$ is a vector of n joint standardized Gaussian random variables with a symmetric, positive-definite $(n \times n)$ matrix of variances and covariances Σ , then the density of X is given by :

$$f(x_1, \dots, x_n) = \frac{e^{-\frac{X^T \Sigma^{-1} X}{2}}}{(2\pi)^{n/2} \sqrt{\text{Det}(\Sigma)}} \quad (6)$$

where X^T is the transpose of X , $\text{Det}(\Sigma)$ is the determinant of Σ and Σ^{-1} is the inverse of Σ .

For example, if we denote by ρ_{12}, ρ_{23} and ρ_{13} the correlation coefficients between three standardized Gaussian random variables X_1, X_2 and X_3 , it is easily shown, by applying formula (6), that the joint density of X_1, X_2 and X_3 is given by :

$$f(a, b, c; \rho_{12}, \rho_{23}, \rho_{13}) = \frac{e^{-\frac{1}{2}(\lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2 + 2\lambda_4 ab + 2\lambda_5 bc + 2\lambda_6 ac)}}{(2\pi)^{3/2} \sqrt{\text{Det}(\Sigma)}} \quad (7)$$

with :

$$\lambda_1 = \frac{1 - \rho_{23}^2}{\text{Det}(\Sigma)}; \lambda_2 = \frac{1 - \rho_{13}^2}{\text{Det}(\Sigma)}; \lambda_3 = \frac{1 - \rho_{12}^2}{\text{Det}(\Sigma)}; \lambda_4 = \frac{\rho_{13}\rho_{23} - \rho_{12}}{\text{Det}(\Sigma)}; \lambda_5 = \frac{\rho_{13}\rho_{12} - \rho_{23}}{\text{Det}(\Sigma)}; \lambda_6 = \frac{\rho_{12}\rho_{23} - \rho_{13}}{\text{Det}(\Sigma)} \quad (8)$$

and

$$\text{Det}(\Sigma) = 1 - \rho_{12}^2 - \rho_{23}^2 - \rho_{13}^2 + 2\rho_{12}\rho_{23}\rho_{13} \quad (9)$$

However, using these general multinormal expressions becomes analytically cumbersome and computationally inefficient as the dimension of the integral rises. Actually, simplified expressions can be used when dealing with the finite-dimensional distributions of geometric Brownian motion GBM. Let $Q(\cdot)$ refer to the equivalent martingale measure, let $X_t = \ln(S_t / S_0)$,

and let T_0, T_1, T_2, T_3 be three different dates during the option life $[T_0, T_3]$, such that :

$$T_0 < T_1 < T_2 < T_3 .$$

By using the Markov property of Brownian motion, as well as the independence of Brownian increments, one can write :

$$Q(X_{T_1} < a, X_{T_2} < b, X_{T_3} < c) = \int_{-\infty}^a \int_{-\infty}^b e^{\frac{\left(\frac{x-\mu T_1}{\sigma\sqrt{T_1}}\right)^2 - 2\rho_{12}\left(\frac{x-\mu T_1}{\sigma\sqrt{T_1}}\right)\left(\frac{y-\mu T_2}{\sigma\sqrt{T_2}}\right) + \left(\frac{y-\mu T_2}{\sigma\sqrt{T_2}}\right)^2}{2(1-\rho_{12}^2)}} \frac{1}{2\pi\sigma^2\sqrt{T_1 T_2(1-\rho_{12}^2)}} N\left[\frac{c-\mu(T_3-T_2)-y}{\sigma\sqrt{T_3-T_2}}\right] dy dx \quad (10)$$

where $\rho_{12} = \sqrt{T_1/T_2}$ and $N[\cdot]$ is the univariate standard Gaussian distribution.

Manipulating this triple integral and standardizing it yields :

$$Q(X_{T_1} < a, X_{T_2} < b, X_{T_3} < c) = \int_{-\infty}^{\frac{a-\mu T_1}{\sigma\sqrt{T_1}}} \int_{-\infty}^{\frac{b-\mu T_2}{\sigma\sqrt{T_2}}} \int_{-\infty}^{\frac{c-\mu T_3}{\sigma\sqrt{T_3}}} \frac{e^{-\frac{x^2}{2} - \frac{(y-\rho_{12}x)^2}{2(1-\rho_{12}^2)} - \frac{(z-\rho_{23}y)^2}{2(1-\rho_{23}^2)}}}{(2\pi)^{3/2} \sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)}} dz dy dx \quad (11)$$

which can be written in a more compact form as :

$$N_3\left[\frac{a-\mu T_1}{\sigma\sqrt{T_1}}, \frac{b-\mu T_2}{\sigma\sqrt{T_2}}, \frac{c-\mu T_3}{\sigma\sqrt{T_3}}; \rho_{12}, \rho_{23}\right] \quad (12)$$

where $\rho_{23} = \sqrt{T_2/T_3}$

Similarly, if $T_4 > T_3$, we have :

$$N_4\left[\frac{a-\mu T_1}{\sigma\sqrt{T_1}}, \frac{b-\mu T_2}{\sigma\sqrt{T_2}}, \frac{c-\mu T_3}{\sigma\sqrt{T_3}}, \frac{d-\mu T_4}{\sigma\sqrt{T_4}}; \rho_{12}, \rho_{23}, \rho_{34}\right] = Q(X_{T_1} < a, X_{T_2} < b, X_{T_3} < c, X_{T_4} < d) = \int_{-\infty}^{\frac{a-\mu T_1}{\sigma\sqrt{T_1}}} \int_{-\infty}^{\frac{b-\mu T_2}{\sigma\sqrt{T_2}}} \int_{-\infty}^{\frac{c-\mu T_3}{\sigma\sqrt{T_3}}} \int_{-\infty}^{\frac{d-\mu T_4}{\sigma\sqrt{T_4}}} \frac{e^{-\frac{w^2}{2} - \frac{(x-\rho_{12}w)^2}{2(1-\rho_{12}^2)} - \frac{(y-\rho_{23}x)^2}{2(1-\rho_{23}^2)} - \frac{(z-\rho_{34}y)^2}{2(1-\rho_{34}^2)}}}{(2\pi)^2 \sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)(1-\rho_{34}^2)}} dz dy dx dw \quad (13)$$

with : $\rho_{34} = \sqrt{T_3/T_4}$

More generally, it is easily shown that given $\{T_0, T_1, T_2, \dots, T_d\}$ with $T_0 < T_1 < T_2 < \dots < T_d$, and

$d \in \mathbb{N}$, the density of the d correlated Gaussian random variables $X_{T_i} = \ln(S_{T_i}/S_0)$ is given by :

$$\begin{aligned}
& P(X_{T_1} \in dx_1, X_{T_2} \in dx_2, \dots, X_{T_d} \in dx_d) \\
&= \exp \left\{ \frac{1}{2} \left((\arccos[\rho_{1,2}])' \right)^2 (x_1^2 - 2\rho_{1,2}x_1x_2 + x_2^2) \right\} (2\pi)^{-\frac{d}{2}} \prod_{k=2}^{d-1} \left(\arcsin[\rho_{k,k+1}] \right)' \exp \left\{ \frac{1}{2} \left((\arccos[\rho_{k,k+1}])' \right)^2 (x_{k+1} - \rho_{k,k+1}x_k)^2 \right\}
\end{aligned} \tag{14}$$

where $\rho_{a,b} = \sqrt{T_a/T_b}$, $T_a < T_b$ and the notation $(\cdot)'$ refers to the first-order derivative of (\cdot)

Alternatively, the finite-dimensional distribution of order d of arithmetic Brownian motion with drift μ and volatility σ , is given by :

$$\begin{aligned}
& Q(X_{T_1} < x_1, X_{T_2} < x_2, \dots, X_{T_d} < x_d) \\
&= N_d \left[\frac{x_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{x_2 - \mu T_2}{\sigma \sqrt{T_2}}, \dots, \frac{x_d - \mu T_d}{\sigma \sqrt{T_d}}; \rho_{12}, \rho_{23}, \dots, \rho_{d-1,d} \right] \\
&= \int_{-\infty}^{\frac{x_1 - \mu T_1}{\sigma \sqrt{T_1}}} \int_{-\infty}^{\frac{x_2 - \mu T_2}{\sigma \sqrt{T_2}}} \dots \int_{-\infty}^{\frac{x_d - \mu T_d}{\sigma \sqrt{T_d}}} \frac{e^{-\frac{y_1^2}{2} - \sum_{k=1}^{d-1} \frac{(y_{k+1} - \rho_{k,k+1}y_k)^2}{2(1-\rho_{k,k+1}^2)}}}{(2\pi)^{d/2} \sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)\dots(1-\rho_{d-1,d}^2)}} dy_d dy_{d-1} \dots dy_1
\end{aligned} \tag{15}$$

We will now show, in sections 2 and 3 of this paper, how to use these finite-dimensional distributions to value step barrier options.

2. Standard step-up and step-down barrier options

The analytical valuation of a d -dimensional standard step-up barrier option involves the calculation of the distribution of d increasing maxima and terminal value of a geometric Brownian motion. Likewise, valuing a d -dimensional standard step-down barrier option involves the calculation of the distribution of d decreasing minima and terminal value of a geometric Brownian motion.

Let us focus on an up-and-out put option with four different barriers over four intervals of the option life : $[T_0, T_1], [T_1, T_2], [T_2, T_3], [T_3, T_4]$, where T_0 is the beginning of the option life and T_4 is the expiry. We name our four barriers : H_1, H_2, H_3, H_4 . Since we are dealing with a standard step-up barrier option, we have : $H_1 < H_2 < H_3 < H_4$ (but we shall see in section 3 that we could choose another combination).

To find a formula for the value of this option, we need to know the joint probability, under the equivalent martingale measure, that the maximum value of the underlying asset during $[T_0, T_1]$ will not hit H_1 and that the maximum value during $[T_1, T_2]$ will not hit H_2 and that the maximum value during $[T_2, T_3]$ will not hit H_3 and that the maximum value during $[T_3, T_4]$ will not hit H_4 and that the value of the underlying asset at expiry S_4 will be lower than the strike price K . That is, we need to know the value of :

$$Q(M_0^1 < h_1, M_1^2 < h_2, M_2^3 < h_3, M_3^4 < h_4, X_4 < k) \quad (16)$$

where $X_T = \ln(S_T / S_0)$, $k = \ln(K / S_0)$, $h_i = \ln(H_i / S_0)$ and $M_a^b = \sup_{t \in [T_a, T_b]} (X_t)$

Graph 1, in appendix 1, gives an illustration of this problem.

The formula for : $Q(M_0^1 < h_1, X_{T_1} < k)$ is a classical result that is assumed to be known (see, e.g., Karatzas & Shreve [1991]).

Using the Markov property of brownian motion and the independence of brownian increments, the second maximum can be dealt with in the following way :

$$Q(M_0^1 < h_1, M_1^2 < h_2, X_{T_2} < k) = E^Q \left[\mathbf{1}_{\{X_{T_1} < h_1, M_0^1 < h_1\}} E^Q \left[\mathbf{1}_{\{X_{T_2} < k, M_1^2 < h_2\}} \middle| X_{T_1} \right] \right] = \int_{-\infty}^{h_1} \Psi(u) \Phi(u) du \quad (17)$$

where :

$$\Psi(u) = \frac{e^{-\frac{1}{2} \left(\frac{u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2}}{\sigma \sqrt{2\pi T_1}} - e^{\frac{2\mu}{\sigma^2} h_1} \frac{e^{-\frac{1}{2} \left(\frac{u - 2h_1 - \mu T_1}{\sigma \sqrt{T_1}} \right)^2}}{\sigma \sqrt{2\pi T_1}} \quad (18)$$

$$\Phi(u) = N \left[\frac{k - u - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right] - e^{\frac{2\mu}{\sigma^2} (h_2 - u)} N \left[\frac{k - 2h_2 + u - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right]$$

Using identities (A.1), (A.2) and (B.1) in appendix 3, the following solution can be obtained :

$$\begin{aligned}
& N_2 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{k - \mu T_2}{\sigma \sqrt{T_2}}; \rho_{12} \right] - e^{\frac{2\mu}{\sigma^2} h_2} N_2 \left[\frac{h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{k - 2h_2 - \mu T_2}{\sigma \sqrt{T_2}}; -\rho_{12} \right] \\
& - e^{\frac{2\mu}{\sigma^2} h_1} N_2 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{k - 2h_1 - \mu T_2}{\sigma \sqrt{T_2}}; \rho_{12} \right] + e^{\frac{2\mu}{\sigma^2} (h_2 - h_1)} N_2 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{k - 2h_2 + 2h_1 - \mu T_2}{\sigma \sqrt{T_2}}; -\rho_{12} \right]
\end{aligned} \tag{19}$$

where $\rho_{12} = \sqrt{T_1/T_2}$ as in section 2.

The next step is to find: $Q(M_0^1 < h_1, M_1^2 < h_2, M_2^3 < h_3, X_{T_3} < k)$. This probability can be formulated as the following conditional expectation :

$$E^Q \left[\mathbf{1}_{\{X_{T_2} < h_2, M_0^1 < h_1, M_1^2 < h_2\}} E^Q \left[\mathbf{1}_{\{X_{T_3} < k, M_2^3 < h_3\}} \middle| X_2 \right] \right] \tag{20}$$

Building on our acquired knowledge of $Q(M_0^1 < h_1, M_1^2 < h_2, X_{T_2} < k)$, the following integral form of the problem can be written down :

$$\begin{aligned}
& E^Q \left[\mathbf{1}_{\{X_{T_2} < h_2, M_0^1 < h_1, M_1^2 < h_2\}} E^Q \left[\mathbf{1}_{\{X_{T_3} < k, M_2^3 < h_3\}} \middle| X_2 \right] \right] = \\
& \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \Psi^1(u; v) \Phi(v) \, dv du - e^{\frac{2\mu}{\sigma^2} h_2} \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \Psi^2(u; v) \Phi(v) \, dv du \\
& - e^{\frac{2\mu}{\sigma^2} h_1} \int_{-\infty}^{-h_1} \int_{-\infty}^{h_2} \Psi^3(u; v) \Phi(v) \, dv du + e^{\frac{2\mu}{\sigma^2} (h_2 - h_1)} \int_{-\infty}^{-h_1} \int_{-\infty}^{h_2} \Psi^4(u; v) \Phi(v) \, dv du
\end{aligned} \tag{21}$$

where :

$$\Phi(v) = N \left[\frac{k - v - \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right] - e^{\frac{2\mu}{\sigma^2} (h_3 - v)} N \left[\frac{k - 2h_3 + v - \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right] \tag{22}$$

$$\Psi^1(u; v) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v - u - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right]}}{\sigma^2 2\pi \sqrt{T_1(T_2 - T_1)}}; \quad \Psi^2(u; v) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u + \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v + u - 2h_2 - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right]}}{\sigma^2 2\pi \sqrt{T_1(T_2 - T_1)}} \tag{23}$$

$$\Psi^3(u; v) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v - u - 2h_1 - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right]}}{\sigma^2 2\pi \sqrt{T_1(T_2 - T_1)}}; \quad \Psi^4(u; v) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u + \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v + u - 2h_2 + 2h_1 - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right]}}{\sigma^2 2\pi \sqrt{T_1(T_2 - T_1)}}$$

Using identities (A.3) and (B.3) in appendix 3, one can obtain the following explicit expression composed of eight terms involving a special form of the trivariate gaussian cumulative distribution :

$$\begin{aligned}
Q(M_0^1 < h_1, M_1^2 < h_2, M_2^3 < h_3, X_{T_3} < k) = & \\
N_3 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - \mu T_3}{\sigma \sqrt{T_3}}; \rho_{12}, \rho_{23} \right] - e^{\frac{2\mu}{\sigma^2} h_2} N_3 \left[\frac{h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - 2h_2 - \mu T_3}{\sigma \sqrt{T_3}}; -\rho_{12}, \rho_{23} \right] & \quad (24) \\
- e^{\frac{2\mu}{\sigma^2} h_1} N_3 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - 2h_1 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}; \rho_{12}, \rho_{23} \right] & \\
+ e^{\frac{2\mu}{\sigma^2} (h_2 - h_1)} N_3 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{2h_1 - h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - 2h_2 + 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}; -\rho_{12}, \rho_{23} \right] & \\
- e^{\frac{2\mu}{\sigma^2} h_3} N_3 \left[\frac{h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - 2h_3 - \mu T_3}{\sigma \sqrt{T_3}}; \rho_{12}, -\rho_{23} \right] & \\
+ e^{\frac{2\mu}{\sigma^2} (h_3 - h_2)} N_3 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - 2h_3 + 2h_2 - \mu T_3}{\sigma \sqrt{T_3}}; -\rho_{12}, -\rho_{23} \right] & \\
+ e^{\frac{2\mu}{\sigma^2} (h_3 - h_1)} N_3 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - 2h_1 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - 2h_3 + 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}; \rho_{12}, -\rho_{23} \right] & \\
- e^{\frac{2\mu}{\sigma^2} (h_3 + h_1 - h_2)} N_3 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + 2h_1 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{k - 2h_3 + 2h_2 - 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}; -\rho_{12}, -\rho_{23} \right] &
\end{aligned}$$

where it is recalled that :

$$N_3[a, b, c; \rho_{12}, \rho_{23}] = \int_{-\infty}^a \int_{-\infty}^b \int_{-\infty}^c \frac{e^{-\frac{x^2}{2} - \frac{(y - \rho_{12}x)^2}{2(1 - \rho_{12}^2)} - \frac{(z - \rho_{23}y)^2}{2(1 - \rho_{23}^2)}}}{(2\pi)^{3/2} \sqrt{(1 - \rho_{12}^2)(1 - \rho_{23}^2)}} dz dy dx \quad (25)$$

as was shown in section 1. Knowing this probability would be enough to value a three-dimensional standard step up-and-out put option. But, to value a four-dimensional standard step up-and-out put option, we need to add the condition that : $M_3^4 < h_4$, as well as the condition that : $X_{T_4} < k$. This is done by solving :

$$E^Q \left[\mathbf{1}_{\{X_{T_3} < h_3, M_0^1 < h_1, M_1^2 < h_2, M_2^3 < h_3\}} E^Q \left[\mathbf{1}_{\{X_{T_4} < k, M_3^4 < h_4\}} \middle| X_{T_3} \right] \right] =$$

$$\int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \int_{-\infty}^{h_3} \Psi^1(u; v; w) \Phi(w) dw dv du - e^{\frac{2\mu}{\sigma^2} h_2} \int_{-\infty}^{h_1} \int_{-\infty}^{-h_2} \int_{-\infty}^{h_3} \Psi^2(u; v; w) \Phi(w) dw dv du \quad (26)$$

$$- e^{\frac{2\mu}{\sigma^2} h_1} \int_{-\infty}^{h_1} \int_{-\infty}^{-h_2-2h_1} \int_{-\infty}^{h_3} \Psi^3(u; v; w) \Phi(w) dw dv du + e^{\frac{2\mu}{\sigma^2} (h_2-h_1)} \int_{-\infty}^{-h_1} \int_{-\infty}^{2h_1-h_2} \int_{-\infty}^{h_3} \Psi^4(u; v; w) \Phi(w) dw dv du$$

$$- e^{\frac{2\mu}{\sigma^2} h_3} \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \int_{-\infty}^{h_3} \Psi^5(u; v; w) \Phi(w) dw dv du + e^{\frac{2\mu}{\sigma^2} (h_3-h_2)} \int_{-\infty}^{h_1} \int_{-\infty}^{-h_2} \int_{-\infty}^{h_3} \Psi^6(u; v; w) \Phi(w) dw dv du$$

$$+ e^{\frac{2\mu}{\sigma^2} (h_3-h_1)} \int_{-\infty}^{-h_1} \int_{-\infty}^{h_2-2h_1} \int_{-\infty}^{h_3} \Psi^7(u; v; w) \Phi(w) dw dv du - e^{\frac{2\mu}{\sigma^2} (h_3+h_1-h_2)} \int_{-\infty}^{-h_1} \int_{-\infty}^{2h_1-h_2} \int_{-\infty}^{h_3} \Psi^8(u; v; w) \Phi(w) dw dv du$$

with :

$$\Phi(w) = N \left[\frac{k-w-\mu(T_4-T_3)}{\sigma\sqrt{T_4-T_3}} \right] - e^{\frac{2\mu}{\sigma^2} (h_4-w)} N \left[\frac{k-2h_4+w-\mu(T_4-T_3)}{\sigma\sqrt{T_4-T_3}} \right] \quad (27)$$

and :

$$\Psi^1(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u-\mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v-u-\mu(T_2-T_1)}{\sigma\sqrt{T_2-T_1}} \right)^2 + \left(\frac{w-v-\mu(T_3-T_2)}{\sigma\sqrt{T_3-T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2-T_1)(T_3-T_2)}} \quad (28)$$

$$\Psi^2(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u+\mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v+u-\mu(T_2-T_1)}{\sigma\sqrt{T_2-T_1}} \right)^2 + \left(\frac{w-v-2h_2-\mu(T_3-T_2)}{\sigma\sqrt{T_3-T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2-T_1)(T_3-T_2)}}$$

$$\Psi^3(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u-\mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v-u-\mu(T_2-T_1)}{\sigma\sqrt{T_2-T_1}} \right)^2 + \left(\frac{w-v-2h_1-\mu(T_3-T_2)}{\sigma\sqrt{T_3-T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2-T_1)(T_3-T_2)}}$$

$$\Psi^4(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u+\mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v+u-\mu(T_2-T_1)}{\sigma\sqrt{T_2-T_1}} \right)^2 + \left(\frac{w-v-2h_2+2h_1-\mu(T_3-T_2)}{\sigma\sqrt{T_3-T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2-T_1)(T_3-T_2)}}$$

$$\Psi^5(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u+\mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v-u+\mu(T_2-T_1)}{\sigma\sqrt{T_2-T_1}} \right)^2 + \left(\frac{w+v-2h_3-\mu(T_3-T_2)}{\sigma\sqrt{T_3-T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2-T_1)(T_3-T_2)}}$$

$$\Psi^6(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v + u + \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w + v - 2h_3 + 2h_2 - \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2 - T_1)(T_3 - T_2)}}$$

$$\Psi^7(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u + \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v - u + \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w + v - 2h_3 + 2h_1 - \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2 - T_1)(T_3 - T_2)}}$$

$$\Psi^8(u; v; w) = \frac{e^{-\frac{1}{2} \left[\left(\frac{u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v + u + \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w + v - 2h_3 + 2h_2 - 2h_1 - \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right]}}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2 - T_1)(T_3 - T_2)}}$$

Using identities (A.4) and (B.5) in appendix 3, we obtain an explicit expression composed of sixteen terms involving the quadrivariate standard gaussian distribution :

$$\begin{aligned} & Q(M_0^1 < h_1, M_1^2 < h_2, M_2^3 < h_3, M_3^4 < h_4, X_{T_4} < k) \quad \text{with } h_1 < h_2 < h_3 < h_4 \\ &= N_4 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{k - \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, \rho_{23}, \rho_{34} \right] \\ & - e^{\frac{2\mu}{\sigma^2} h_2} N_4 \left[\frac{h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - 2h_2 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{k - 2h_2 - \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, \rho_{23}, \rho_{34} \right] \\ & - e^{\frac{2\mu}{\sigma^2} h_1} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - 2h_1 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{k - 2h_1 - \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, \rho_{23}, \rho_{34} \right] \\ & + e^{\frac{2\mu}{\sigma^2} (h_2 - h_1)} N_4 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{2h_1 - h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - 2h_2 + 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{k - 2h_2 + 2h_1 - \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, \rho_{23}, \rho_{34} \right] \\ & - e^{\frac{2\mu}{\sigma^2} h_3} N_4 \left[\frac{h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{k - 2h_3 - \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, -\rho_{23}, \rho_{34} \right] \\ & + e^{\frac{2\mu}{\sigma^2} (h_3 - h_2)} N_4 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + 2h_2 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{k - 2h_3 + 2h_2 - \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, -\rho_{23}, \rho_{34} \right] \\ & + e^{\frac{2\mu}{\sigma^2} (h_3 - h_1)} N_4 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - 2h_1 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{k - 2h_3 + 2h_1 - \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, -\rho_{23}, \rho_{34} \right] \end{aligned} \tag{29}$$

$$\begin{aligned}
& -e^{\frac{2\mu}{\sigma^2}(h_3+h_4-h_2)} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{-h_2 + 2h_1 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_2 - 2h_1 - \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_3 + 2h_2 - 2h_1 - \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}, -\rho_{23}, \rho_{34} \right] \\
& -e^{\frac{2\mu}{\sigma^2}h_4} N_4 \left[\frac{h_1 + \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_2 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{h_3 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 - \mu T_4}{\sigma\sqrt{T_4}}; \rho_{12}, \rho_{23}, -\rho_{34} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_4-h_2)} N_4 \left[\frac{h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{-h_2 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{h_3 - 2h_2 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 + 2h_2 - \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}, \rho_{23}, -\rho_{34} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_4-h_1)} N_4 \left[\frac{-h_1 + \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_2 - 2h_1 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{h_3 - 2h_1 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 + 2h_1 - \mu T_4}{\sigma\sqrt{T_4}}; \rho_{12}, \rho_{23}, -\rho_{34} \right] \\
& -e^{\frac{2\mu}{\sigma^2}(h_4+h_1-h_2)} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{2h_1 - h_2 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{h_3 - 2h_2 + 2h_1 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 + 2h_2 - 2h_1 - \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}, \rho_{23}, -\rho_{34} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_4-h_3)} N_4 \left[\frac{h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_2 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 + 2h_3 - \mu T_4}{\sigma\sqrt{T_4}}; \rho_{12}, -\rho_{23}, -\rho_{34} \right] \\
& -e^{\frac{2\mu}{\sigma^2}(h_4-h_3+h_2)} N_4 \left[\frac{h_1 + \mu T_1}{\sigma\sqrt{T_1}}, \frac{-h_2 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_2 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 + 2h_3 - 2h_2 - \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}, -\rho_{23}, -\rho_{34} \right] \\
& -e^{\frac{2\mu}{\sigma^2}(h_4-h_3+h_1)} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_2 - 2h_1 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_1 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 + 2h_3 - 2h_1 - \mu T_4}{\sigma\sqrt{T_4}}; \rho_{12}, -\rho_{23}, -\rho_{34} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_4-h_3+h_2-h_1)} N_4 \left[\frac{-h_1 + \mu T_1}{\sigma\sqrt{T_1}}, \frac{-h_2 + 2h_1 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_2 - 2h_1 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{k - 2h_4 + 2h_3 - 2h_2 + 2h_1 - \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}, -\rho_{23}, -\rho_{34} \right]
\end{aligned}$$

where we recall that :

$$N_4[a, b, c, d; \rho_{12}, \rho_{23}, \rho_{34}] = \int_{-\infty}^a \int_{-\infty}^b \int_{-\infty}^c \int_{-\infty}^d \frac{e^{-\frac{w^2}{2} \frac{(x-\rho_{12}w)^2}{2(1-\rho_{12}^2)} - \frac{(y-\rho_{23}x)^2}{2(1-\rho_{23}^2)} - \frac{(z-\rho_{34}y)^2}{2(1-\rho_{34}^2)}}}{(2\pi)^2 \sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)(1-\rho_{34}^2)}} dz dy dx dw \quad (30)$$

The same method can be applied to value all other standard step-up or step-down barrier options. As a second example, we provide in appendix 2 the formula for a down-and-out call option with four decreasing down-barriers during the option life.

3. Partial and outside step barrier options

As mentioned earlier, the sequence of monitored barriers needs not be monotonically increasing or decreasing. Besides, monitoring needs not start at the beginning of the option life nor does it have to end at expiry. For example, we may want to value a forward-start early-ending down-and-out call option with monitoring of three different barriers H_1, H_2, H_3 over three intervals of the option life: $[T_1, T_2], [T_2, T_3], [T_3, T_4]$ with $T_0 < T_1 < T_2 < T_3 < T_4 < T_5$, where T_5 is the option expiry. This time we choose: $H_1 > H_2, H_2 < H_3, H_3 < H_1$, instead of having: $H_1 > H_2 > H_3$, as illustrated by graph 3 in appendix 1.

To value this option, we need to calculate :

$$Q(m_1^2 > h_1, m_2^3 > h_2, m_3^4 > h_3, X_{T_5} > k) \text{ with } h_1 > h_2, h_2 < h_3, h_3 < h_1$$

which is equal to :

$$Q(m_1^2 > h_1, m_2^3 > h_2, X_{T_3} > h_3, m_3^4 > h_3, X_{T_4} > h_3, X_{T_5} > k) \quad (31)$$

Note that we must impose the condition that : $X_{T_3} > h_3$, in this case.

To find this probability, we first solve :

$$E^Q \left[\mathbf{1}_{\{X_{T_1} > h_1\}} E^Q \left[\mathbf{1}_{\{m_1^2 > h_1, X_{T_2} > k\}} \middle| X_{T_1} \right] \right], \text{ then :} \quad (32)$$

$$E^Q \left[\mathbf{1}_{\{X_{T_2} > h_1, m_1^2 > h_1\}} E^Q \left[\mathbf{1}_{\{m_2^3 > h_1, X_{T_3} > k\}} \middle| X_{T_2} \right] \right], \text{ then :} \quad (33)$$

$$E^Q \left[\mathbf{1}_{\{X_{T_3} > h_3, m_1^2 > h_1, m_2^3 > h_2\}} E^Q \left[\mathbf{1}_{\{m_3^4 > h_3, X_{T_4} > k\}} \middle| X_{T_3} \right] \right], \text{ and eventually :} \quad (34)$$

$$E^Q \left[\mathbf{1}_{\{X_{T_4} > h_3, m_1^2 > h_1, m_2^3 > h_2, X_{T_3} > h_3, m_3^4 > h_3\}} E^Q \left[\mathbf{1}_{\{X_{T_5} > k\}} \middle| X_{T_4} \right] \right] \quad (35)$$

The result is a sum of eight terms involving the quintivariate standard gaussian distribution :

$$Q(m_1^2 > h_1, m_2^3 > h_2, X_{T_3} > h_3, m_3^4 > h_3, X_{T_4} > h_3, X_{T_5} > k) = N_5 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_1 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-h_3 + \mu T_4}{\sigma \sqrt{T_4}}, \frac{-k + \mu T_5}{\sigma \sqrt{T_5}}; \rho_{12}, \rho_{23}, \rho_{34}, \rho_{45} \right] \quad (36)$$

$$\begin{aligned}
& -e^{\frac{2\mu}{\sigma^2}h_1} N_5 \left[\frac{-h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_1 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_1 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{-h_3 + 2h_1 + \mu T_4}{\sigma\sqrt{T_4}}, \frac{-k + 2h_1 + \mu T_5}{\sigma\sqrt{T_5}}; -\rho_{12}, \rho_{23}, \rho_{34}, \rho_{45} \right] \\
& -e^{\frac{2\mu}{\sigma^2}h_2} N_5 \left[\frac{-h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{-h_1 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_2 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{-h_3 + 2h_2 + \mu T_4}{\sigma\sqrt{T_4}}, \frac{-k + 2h_2 + \mu T_5}{\sigma\sqrt{T_5}}; \rho_{12}, -\rho_{23}, \rho_{34}, \rho_{45} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_2-h_1)} N_5 \left[\frac{-h_1 + \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_1 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_2 - 2h_1 + \mu T_3}{\sigma\sqrt{T_3}}, \frac{-h_3 + 2h_2 - 2h_1 + \mu T_4}{\sigma\sqrt{T_4}}, \frac{-k + 2h_2 - 2h_1 + \mu T_5}{\sigma\sqrt{T_5}}; \right. \\
& \quad \left. -\rho_{12}, -\rho_{23}, \rho_{34}, \rho_{45} \right] \\
& -e^{\frac{2\mu}{\sigma^2}h_3} N_5 \left[\frac{-h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{-h_1 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 - \mu T_3}{\sigma\sqrt{T_3}}, \frac{h_3 + \mu T_4}{\sigma\sqrt{T_4}}, \frac{-k + 2h_3 + \mu T_5}{\sigma\sqrt{T_5}}; \rho_{12}, \rho_{23}, -\rho_{34}, \rho_{45} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_3-h_1)} N_5 \left[\frac{-h_1 + \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_1 - \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_1 - \mu T_3}{\sigma\sqrt{T_3}}, \frac{h_3 - 2h_1 + \mu T_4}{\sigma\sqrt{T_4}}, \frac{-k + 2h_3 - 2h_1 + \mu T_5}{\sigma\sqrt{T_5}}; -\rho_{12}, \rho_{23}, -\rho_{34}, \rho_{45} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_3-h_2)} N_5 \left[\frac{-h_1 + \mu T_1}{\sigma\sqrt{T_1}}, \frac{-h_1 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_2 - \mu T_3}{\sigma\sqrt{T_3}}, \frac{h_3 - 2h_2 + \mu T_4}{\sigma\sqrt{T_4}}, \frac{-k + 2h_3 - 2h_2 + \mu T_5}{\sigma\sqrt{T_5}}; \rho_{12}, -\rho_{23}, -\rho_{34}, \rho_{45} \right] \\
& -e^{\frac{2\mu}{\sigma^2}(h_3-h_2+h_1)} N_5 \left[\frac{-h_1 - \mu T_1}{\sigma\sqrt{T_1}}, \frac{h_1 + \mu T_2}{\sigma\sqrt{T_2}}, \frac{-h_3 + 2h_2 - 2h_1 - \mu T_3}{\sigma\sqrt{T_3}}, \frac{h_3 - 2h_2 + 2h_1 + \mu T_4}{\sigma\sqrt{T_4}}, \frac{-k + 2h_3 - 2h_2 + 2h_1 + \mu T_5}{\sigma\sqrt{T_5}}; \right. \\
& \quad \left. -\rho_{12}, -\rho_{23}, -\rho_{34}, \rho_{45} \right]
\end{aligned}$$

where :

$$N_5 [u_1, u_2, u_3, u_4, u_5; \rho_{12}, \rho_{23}, \rho_{34}, \rho_{45}] = \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} \int_{-\infty}^{u_3} \int_{-\infty}^{u_4} \int_{-\infty}^{u_5} \frac{e^{-\frac{u^2}{2} - \frac{(v-\rho_{12}u)^2}{2(1-\rho_{12}^2)} - \frac{(w-\rho_{23}v)^2}{2(1-\rho_{23}^2)} - \frac{(x-\rho_{34}w)^2}{2(1-\rho_{34}^2)} - \frac{(y-\rho_{45}x)^2}{2(1-\rho_{45}^2)}}}{(2\pi)^{5/2} \sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)(1-\rho_{34}^2)(1-\rho_{45}^2)}} dy dx dw dv du \quad (37)$$

We may also require that the monitored barriers be outside barriers. This can be easily done in our analytical framework.

Let us take the example of an up-and-out put option written on an asset S with strike K , expiry T_3 and three outside barriers H_1, H_2, H_3 (with $H_1 < H_2 < H_3$) monitored with respect to another asset V . Both S and V follow geometric brownian motion with risk-neutral drift μ_s and μ_v respectively, and volatility σ_s and σ_v respectively. S and V have correlation coefficient θ .

To value this option, we need to find :

$$\mathcal{Q}\left(M_0^{1(V)} < h_1, M_1^{2(V)} < h_2, M_2^{3(V)} < h_3, X_{T_3} < k\right) \quad (38)$$

with :

$$M_a^{b(V)} = \sup_{t \in [T_a, T_b]} \{\ln(V_t / V_0)\}, \quad h_i = \ln(H_i / V_0), \quad k = \ln(K / S_0) \quad \text{and} \quad X_{T_3} = \ln(S_{T_3} / S_0) \quad (39)$$

In terms of conditional expectation, this probability can be expressed as :

$$E^{\mathcal{Q}} \left[\mathbf{1}_{\left\{M_0^{1(V)} < h_1, M_1^{2(V)} < h_2, M_2^{3(V)} < h_3, Y_{T_3} < h_3\right\}} E^{\mathcal{Q}} \left[\mathbf{1}_{\{X_{T_3} < k\}} \middle| Y_{T_3} \right] \right] \quad (40)$$

$$\text{with : } Y_{T_3} = \ln(V_{T_3} / V_0)$$

We know how to calculate

$$E^{\mathcal{Q}} \left[\mathbf{1}_{\left\{M_0^{1(V)} < h_1, M_1^{2(V)} < h_2, M_2^{3(V)} < h_3, Y_{T_3} < h_3\right\}} \right] \quad (41)$$

Now, $E^{\mathcal{Q}} \left[\mathbf{1}_{\{X_{T_3} < k\}} \middle| Y_{T_3} \right]$ is the expectation, under the equivalent martingale measure, of the

logarithm of a geometric Brownian motion S_{T_3} conditional on the logarithm of another

correlated geometric brownian motion V_{T_3} . The risk-neutral joint density of X_{T_3} and Y_{T_3} is :

$$f_{(X_{T_3}, Y_{T_3})}(a, b) = \frac{e^{-\frac{1}{2(1-\theta^2)} \left[\left(\frac{a-\mu_V T_3}{\sigma_V \sqrt{T_3}} \right)^2 - 2\theta \left(\frac{a-\mu_V T_3}{\sigma_V \sqrt{T_3}} \right) \left(\frac{b-\mu_S T_3}{\sigma_S \sqrt{T_3}} \right) + \left(\frac{b-\mu_S T_3}{\sigma_S \sqrt{T_3}} \right)^2 \right]}}{2\pi\sigma_S\sigma_V T_3 \sqrt{1-\theta^2}} \quad (42)$$

Hence, the density of X_{T_3} conditional on Y_{T_3} is :

$$f_{(X_{T_3}|Y_{T_3})}(b|a) = \frac{e^{-\frac{\left(b-\mu_S T_3 - \theta \frac{\sigma_S}{\sigma_V} (a-\mu_V T_3) \right)^2}{2(1-\theta^2)\sigma_S^2 T_3}}}{\sigma_S \sqrt{2\pi(1-\theta^2)} T_3} \quad (43)$$

$$\text{i.e., } X_{T_3} | Y_{T_3} \sim \mathcal{N} \left(\mu_S T_3 + \theta \frac{\sigma_S}{\sigma_V} (Y_{T_3} - \mu_V T_3); \sigma_S^2 T_3 (1-\theta^2) \right) \quad (44)$$

where \mathcal{N} refers to the univariate Gaussian distribution with mean $\mu_S T_3 + \theta \frac{\sigma_S}{\sigma_V} (Y_3 - \mu_V T_3)$ and

variance $\sigma_S^2 T_3 (1 - \theta^2)$. Then, performing the necessary calculations, one can obtain :

$$\begin{aligned}
& E^Q \left[\mathbf{1}_{\left\{ M_0^{(V)} < h_1, M_1^{2(V)} < h_2, M_2^{3(V)} < h_3, Y_{T_3} < k \right\}} E^Q \left[\mathbf{1}_{\{X_{T_3} < k\}} \middle| Y_{T_3} \right] \right] \\
&= N_4 \left[\frac{h_1 - \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{h_2 - \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{h_3 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k - \mu_S T_3}{\sigma_S \sqrt{T_3}}; \rho_{12}, \rho_{23}, \theta \right] \tag{45} \\
&- e^{\frac{2\mu_V h_2}{\sigma_V^2}} N_4 \left[\frac{h_1 + \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{-h_2 - \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{h_3 - 2h_2 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k - 2h_2 \theta \frac{\sigma_S}{\sigma_V} - \mu_S T_3}{\sigma_S \sqrt{T_3}}; -\rho_{12}, \rho_{23}, \theta \right] \\
&- e^{\frac{2\mu_V h_1}{\sigma_V^2}} N_4 \left[\frac{-h_1 - \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{h_2 - 2h_1 - \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{h_3 - 2h_1 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k - 2h_1 \theta \frac{\sigma_S}{\sigma_V} - \mu_S T_3}{\sigma_S \sqrt{T_3}}; \rho_{12}, \rho_{23}, \theta \right] \\
&+ e^{\frac{2\mu_V (h_2 - h_1)}{\sigma_V^2}} N_4 \left[\frac{-h_1 + \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{2h_1 - h_2 - \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{h_3 - 2h_2 + 2h_1 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k + 2(h_1 - h_2) \theta \frac{\sigma_S}{\sigma_V} - \mu_S T_3}{\sigma_S \sqrt{T_3}}; -\rho_{12}, \rho_{23}, \theta \right] \\
&- e^{\frac{2\mu_V h_3}{\sigma_V^2}} N_4 \left[\frac{h_1 + \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{h_2 + \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{-h_3 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k - 2h_3 \theta \frac{\sigma_S}{\sigma_V} - \mu_S T_3}{\sigma_S \sqrt{T_3}}; \rho_{12}, -\rho_{23}, \theta \right] \\
&+ e^{\frac{2\mu_V (h_3 - h_2)}{\sigma_V^2}} N_4 \left[\frac{h_1 - \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{-h_2 + \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{-h_3 + 2h_2 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k + 2(h_2 - h_3) \theta \frac{\sigma_S}{\sigma_V} - \mu_S T_3}{\sigma_S \sqrt{T_3}}; -\rho_{12}, -\rho_{23}, \theta \right] \\
&+ e^{\frac{2\mu_V (h_3 - h_1)}{\sigma_V^2}} N_4 \left[\frac{-h_1 + \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{h_2 - 2h_1 + \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{-h_3 + 2h_1 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k + 2(h_1 - h_3) \theta \frac{\sigma_S}{\sigma_V} \mu_S T_3}{\sigma_S \sqrt{T_3}}; \rho_{12}, -\rho_{23}, \theta \right] \\
&- e^{\frac{2\mu_V (h_3 + h_1 - h_2)}{\sigma_V^2}} N_4 \left[\frac{-h_1 - \mu_V T_1}{\sigma_V \sqrt{T_1}}, \frac{-h_2 + 2h_1 + \mu_V T_2}{\sigma_V \sqrt{T_2}}, \frac{-h_3 + 2h_2 - 2h_1 - \mu_V T_3}{\sigma_V \sqrt{T_3}}, \frac{k + 2(h_2 - h_3 - h_1) \theta \frac{\sigma_S}{\sigma_V} \mu_S T_3}{\sigma_S \sqrt{T_3}}; -\rho_{12}, -\rho_{23}, \theta \right]
\end{aligned}$$

In the case of outside barriers, we have to resort to a new convolution of normal distribution functions, slightly different from the one given in section 1. This is because we are now faced with two different kinds of correlation coefficients : the first one $(\pm\rho_{12}, \pm\rho_{23})$, which is the one we have dealt with so far, is purely path-dependent ; the second one (θ) measures a correlation of returns between two assets. As a result, we obtain the following minor modification :

$$N_4[a, b, c, d; \rho_{12}, \rho_{23}, \theta] = \int_{-\infty}^a \int_{-\infty}^b \int_{-\infty}^c \int_{-\infty}^d e^{-\frac{w^2}{2} - \frac{(x-\rho_{12}w)^2}{2(1-\rho_{12}^2)} - \frac{(y-\rho_{23}x)^2}{2(1-\rho_{23}^2)} - \frac{(z-\theta\sqrt{T_3}y)^2}{2(1-\theta^2)}} \frac{dzdydx dw}{(2\pi)^2 \sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)(1-\theta^2)}} \quad (46)$$

Note too that the final option valuation formula implies a new change of measure. As shown in section 1, the option value is :

$$e^{-rT_3} K \mathcal{Q} \left(M_0^{1(V)} < h_1, M_1^{2(V)} < h_2, M_2^{3(V)} < h_3, X_{T_3} < k \right) - S_0 e^{-\delta_S T_3} \tilde{\mathcal{Q}} \left(M_0^{1(V)} < h_1, M_1^{2(V)} < h_2, M_2^{3(V)} < h_3, X_{T_3} < k \right) \quad (47)$$

As before, we have : $\tilde{\mu}_S = \mu_S + \sigma_S^2$. However, application of Girsanov theorem shows that :

$$\tilde{\mu}_V = \mu_V + \theta \sigma_S \sigma_V .$$

4. Numerical results

The formulae given in the previous sections would be of little use if their implementation did not provide fast and accurate option values. This raises the question of how to perform the numerical integration implied by the computation of the special forms of multivariate Gaussian distributions involved. As the dimension of the integrals is quite moderate, one can turn to low discrepancy sequences.

The fundamentals of quasi random integration are very well covered, for instance, in Niederreiter (1992). It has been shown by Paskov (1996) that the Sobol algorithm (Sobol, 1967) performs particularly well when applied to financial problems. Faure sequences have

also been proved to be a judicious choice (Faure, 1982 ; Boyle et al., 1996). There is evidence that Halton sequences (Halton, 1960) are slightly less reliable than Sobol or Faure sequences in high dimensions. However, they have the attraction of simplicity, and since we are dealing here with five-dimensional integrals at most, they can be used without significant loss of precision, as will be verified in a moment.

To put our analytical approach to the test, we are indeed going to compare the Halton quasi random integration of our formulae with Monte Carlo simulation results. We choose to implement conditional Monte Carlo because, in our experience, it consistently outperforms alternative Monte Carlo variance reduction techniques, such as matching moments, stratification or importance sampling. Good references on conditional Monte Carlo are El Babsiri and Noël (1998) or Andersen (1998). In particular, El Babsiri and Noël (1998) show how generating extrema over a certain period by their own law, conditional on the simulated terminal value of the underlying asset over the given period, substantially increases the speed and the accuracy of Monte Carlo simulation, and their method will be followed here. To further reduce variance, we combine conditional Monte Carlo with antithetic variates, and we call this « advanced Monte Carlo », as opposed to « plain Monte Carlo » in which the asset price path is discretized in time without applying any variance reduction technique and the extrema are monitored four times per business day (which is inefficient and biased high).

We start with the implementation of the formula for a four-dimensional step down-and-out call option given in appendix 2. The dimension, here, is quite moderate, but the fact that we have 32 terms to compute (16 terms under the Q -measure and 16 terms under the \tilde{Q} -measure) makes the issue of efficiency not so trivial. Our Monte Carlo estimator, after n simulations, will be given by :

$$E^Q \left[(S_{T_4} - K)^+ \mathbf{1}_{\{m_0 > H_1\}} \mathbf{1}_{\{m_1 > H_2\}} \mathbf{1}_{\{m_2 > H_3\}} \mathbf{1}_{\{m_3 > H_4\}} \middle| S_{T_1}, S_{T_2}, S_{T_3}, S_{T_4} \right]$$

$$= e^{-rT_4} \frac{1}{n} \sum_{i=1}^n \left[\left(S_{T_4}^{(i)} - K \right)^+ \prod_{j=1}^4 \left(1 - e^{-\frac{2 \ln(H_j / S_{T_{j-1}}^{(i)}) \left(\ln(S_{T_j}^{(i)} / S_{T_{j-1}}^{(i)}) - \ln(H_j / S_{T_{j-1}}^{(i)}) \right)}{\sigma^2 T_j}} \right) \mathbf{1}_{\left\{ S_{T_1}^{(i)} > H_1, S_{T_2}^{(i)} > H_2, S_{T_3}^{(i)} > H_3, S_{T_4}^{(i)} > H_4 \right\}} \right] \quad (48)$$

where $S_t^{(i)}$ denotes the value of S_t randomly drawn at simulation n° i

The results are provided in Table 1. One can observe the quality of convergence of quasi random integration of analytical formulae, using only a small sample of Halton points, which makes it extremely fast to compute, even on an ordinary personal computer. In contrast, convergence is much slower with Monte Carlo simulation.

Due to its intrinsic uncertainty, the accuracy of Monte Carlo simulation should obviously be gauged with respect to analytical values, provided that the process of quasi random integration does not alter the exact theoretical solutions. One way to estimate the possible distortion created by quasi random integration is to use it to compute option values that can be computed otherwise with great precision, so that we have a benchmark at our disposal.

This the case for one-dimensional options, as we can rightfully trust our numerical estimates of the univariate gaussian distribution. However, our test of quasi random integration will be biased if we simply perform one-dimensional quasi-random integration, because then we shall not feel entitled to extrapolate our findings to the behaviour of quasi random integration in dimension four. A possible solution is to compute four-dimensional formulae which exactly converge to one-dimensional formulae. For example, a step down-and-out call option with $H_1=90$, $H_2=90$, $H_3=90$ and $H_4=90$, should have exactly the same value as that of a standard down-and-out call option with $H=90$. Table 2 shows the results of this simple test. They confirm the robustness of quasi random integration. Almost instantaneously, we obtain a price whose divergence from the calibrating exact price is less than 0.003%. In contrast, using advanced Monte Carlo, the immediate price we obtain exhibits an awful divergence of 11.9% from the calibrating exact price. After performing two million simulations, that divergence has shrunk to 0.008%, but the procedure is extremely time-consuming and cannot be applied in real trading time.

In Table 3, we turn to the implementation of the formula for a four-dimensional step up-and-out put option given in section 2. Again, we test the precision of quasi random integration by using it compute the value of the same step up-and-out put option with all four barriers fixed at 120, which should be strictly equal to a standard up-and-out put option with a knock-out barrier at 120. This time, the quasi instantaneous analytical price we obtain diverges from the exact calibrating price from 0.002% (Table 4).

In Table 5, we implement the formula for a forward-start early-ending three-step down-and-out call option given in section 3. As one more dimension is involved, we increase the size of our samples of Halton quasi random points. As a result, computational time is slightly longer, but it remains extremely efficient compared with Monte Carlo simulation. Of course, computational time could be cut simply by using more hardware resources.

Eventually, in Table 6, we implement the formula for an outside three-step up-and-out put option given in section 3.

Table 1 : four-dimensional standard step down-and-out call option

Number of simulations / Halton points	Volatility = 18%	Volatility = 25%	Volatility = 32%	Computational Time (PIII 1 GHz)	Standard error (for simulation)
QRA 50,000	11.42	11.42	11.26	00 : 18'	
QRA 10,000	11.37	11.42	11.29	00 : 04'	
AMC 2,000,000	11.41	11.40	11.26	03 : 07'	0.0067
AMC 200,000	11.19	11.56	11.06	00 : 16'	0.0604
PMC 100,000	11.76	12.01	11.92	18 : 23'	0.1242

Inputs : $S = 100$; $K = 100$; $r = 0.06$; $\delta = 0.02$; $T_1 = 0.6$; $T_2 = 1.2$; $T_3 = 1.8$; $T_4 = 2.4$
 $H_1 = 90$; $H_2 = 84$; $H_3 = 80$; $H_4 = 76$; QRA = Quasi Random Analytical (Halton) ;
 AMC = Advanced Monte Carlo simulation ; PMC = Plain Monte Carlo simulation

Table 2 : four-dimensional step down-and-out call option theoretically equal to a standard down-and-out call option

	Exact value	QRA 10,000	QRA 50,000	AMC 10,000	AMC 50,000	AMC 2,000,000
4-D step DOC	10.46	10.49	10.44	9.35	9.40	10.38

Inputs : $S = 100$; $K = 100$; $r = 0.06$; $\delta = 0.02$; $T_1 = 0.6$; $T_2 = 1.2$; $T_3 = 1.8$; $T_4 = 2.4$
 $H_1 = 90$; $H_2 = 90$; $H_3 = 90$; $H_4 = 90$; volatility = 32%

Table 3 : four-dimensional standard step up-and-out put option

Number of simulations / Halton points	Volatility = 18%	Volatility = 25%	Volatility = 32%	Computational Time (PIII 1 GHz)	Standard error (for simulation)
QRA 50,000	5.60	7.70	9.14	00 : 18'	
QRA 10,000	5.59	7.71	9.17	00 : 04'	
AMC 2,000,000	5.64	7.74	9.14	03 : 09'	0.0044
AMC 200,000	5.82	7.58	9.26	00 : 16'	0.0592
PMC 100,000	6.02	7.08	9.52	18 : 28'	0.1308

Inputs : $S = 100$; $K = 100$; $r = 0.06$; $\delta = 0.02$; $T_1 = 0.6$; $T_2 = 1.2$; $T_3 = 1.8$; $T_4 = 2.4$
 $H_1 = 120$; $H_2 = 122$; $H_3 = 125$; $H_4 = 128$

Table 4 : four-dimensional step up-and-out put option theoretically equal to a standard up-and-out put option

	Exact value	QRA 10,000	QRA 50,000	AMC 10,000	AMC 50,000	AMC 2,000,000
4-D step UOP	8.92	8.94	8.89	9.27	8.98	8.94

Inputs : $S = 100$; $K = 100$; $r = 0.06$; $\delta = 0.02$; $T_1 = 0.6$; $T_2 = 1.2$; $T_3 = 1.8$; $T_4 = 2.4$
 $H_1 = 120$; $H_2 = 120$; $H_3 = 120$; $H_4 = 120$; volatility = 32%

Table 5 : forward-start early-ending three-step down-and-out call option

Number of simulations / Halton points	Volatility = 18%	Volatility = 25%	Volatility = 32%	Computational Time (PIII 1 GHz)	Standard error (for simulation)
QRA 200,000	10.45	11.67	12.95	01 : 06'	
PMC 1,000,000	10.50	11.78	13.09	58 : 33'	0.014
QRA 100,000	10.40	11.62	12.91	00 : 42'	

Inputs : $S = 100$; $K = 100$; $r = 0.06$; $\delta = 0.02$; $T_1 = 0.3$; $T_2 = 0.6$; $T_3 = 0.9$; $T_4 = 1.2$
 $T_5 = 2$; $H_1 = 92$; $H_2 = 85$; $H_3 = 88$

Table 6 : outside three-step up-and-out put option

Number of simulations / Halton points	Correlation = - 0.5	Correlation = 0	Correlation = 0.5	Computational Time (PIII 1 GHz)	Standard error (for simulation)
QRA 10,000	1.01	1.52	1.91	00 : 03'	
QRA 100,000	1.01	1.51	1.91	00 : 30'	
AMC 2,000,000	1.06	1.51	1.93	01 : 54'	0.002
AMC 200,000	1.07	1.54	1.95	00 : 16'	0.008

Inputs : $S = 15$; $K = 18$; $V = 100$; $H_1 = 125$; $H_2 = 130$; $H_3 = 135$
 $r = 0.06$; $\sigma_S = 0.27$; $\delta_S = 0.01$; $\sigma_V = 0.32$; $\delta_V = 0.02$; $T_1 = 0.5$; $T_2 = 0.8$; $T_3 = 1.5$

Conclusion

This paper has shown ways to obtain explicit formulae for a variety of barrier options with complex payoffs involving joint distributions of several extrema. Combined with quasi random integration, analytical formulae provide extremely fast and accurate option values, compared with Monte Carlo simulation. They can also provide exact hedge ratios by mere differentiation. Probably most important of all, analytical solutions tell us how the various parameters affect the solution and bring out the interrelationships among them.

More sophisticated stochastic processes may be required to model the motion of the underlying asset, making it impossible to find explicit solutions when joint extrema are involved. Still, analytical formulae remain valuable as accurate benchmarks to test more general numerical schemes.

In the cases that have been covered, only one barrier, whether down-and-out or up-and-out, or no barrier at all, is monitored at any one time. One may need to value partial double barrier options with periods when two barriers, one down-and-out and the other up-and-out, are simultaneously monitored. Classical examples are the so-called « hot dog » or « corridor » options with double barriers that change as step functions of time. Exact values can be found for these contingent claims too. However we would not recommend the use of analytical pricing in this case. Indeed, such formulae involve products of infinite series, so that the number of terms increases very quickly, which makes it cumbersome to calculate analytically and inefficient to compute numerically.

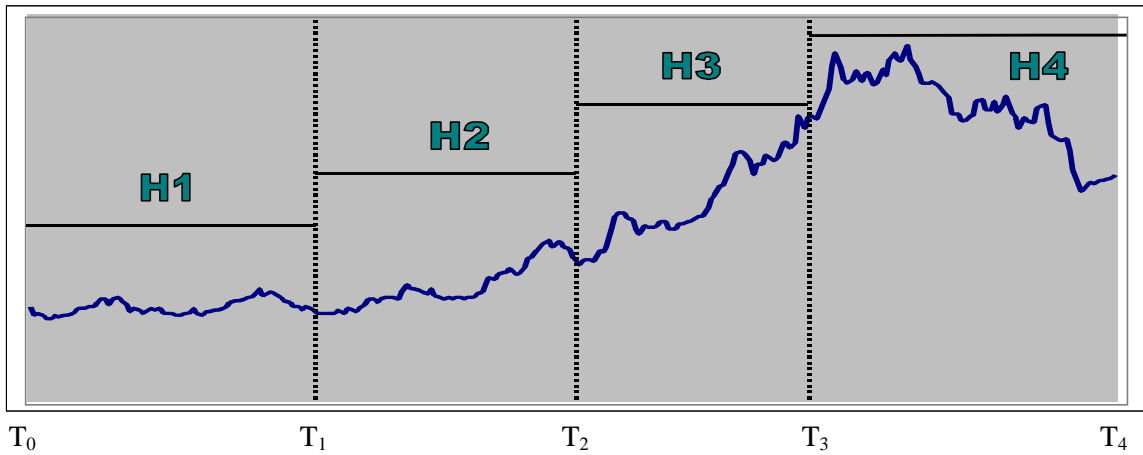
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APPENDIX 1 : GRAPHS

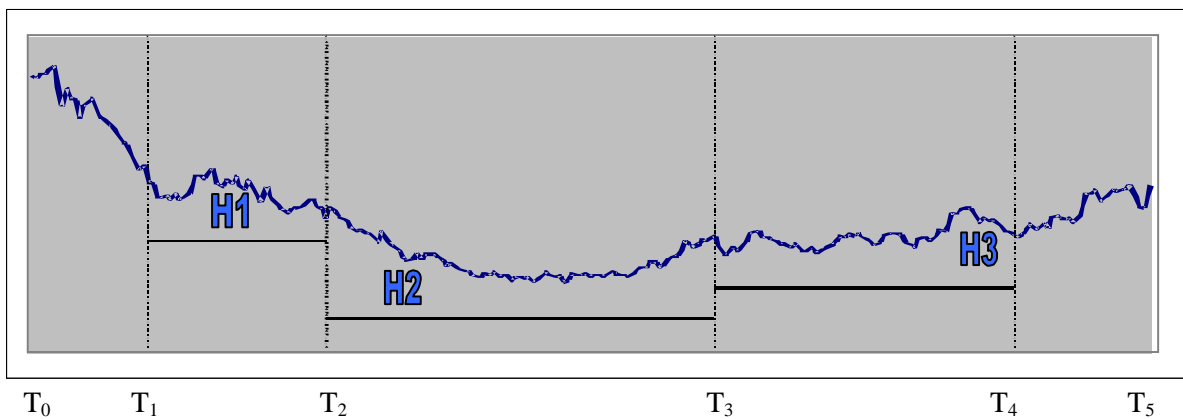
Graph 1 : four-dimensional standard step up-and-out option



Graph 2 : four-dimensional standard step down-and-out option



Graph 3 : forward-start early-ending three-step down-and-out option



**APPENDIX 2 : FORMULA FOR THE PRICE OF A FOUR-DIMENSIONAL
STANDARD STEP DOWN-AND-OUT CALL OPTION**

At expiry T_4 , a four-dimensional European standard step down-and-out call option pays off : $(S_{T_4} - K)^+$, provided that the barrier H_1 is not crossed before T_1 and that the barrier H_2 is not crossed between T_1 and T_2 and that the barrier H_3 is not crossed between T_2 and T_3 and that the barrier H_4 is not crossed between T_3 and the option expiry T_4 , with : $H_1 < S_0, H_2 < S_0, H_3 < S_0, H_4 < S_0$, and : $H_1 > H_2 > H_3 > H_4$.

As shown in section 1, finding a formula for the price of a four-dimensional European standard step down-and-out call option involves computing the following joint cumulative distribution :

$$Q(m_0^1 > h_1, m_1^2 > h_2, m_2^3 > h_3, m_3^4 > h_4, X_{T_4} > k) \quad (49)$$

where:

$$m_a^b = \inf_{t \in [T_a, T_b]} \{\ln(S_t / S_0)\}$$

$$h_i = \ln(H_i / S_0)$$

$$k = \ln(K / S_0)$$

$$X_{T_4} = \ln(S_{T_4} / S_0)$$

By successively solving :

$$E^Q \left[\mathbf{1}_{\{X_{T_1} > h_1, m_0^1 > h_1\}} E^Q \left[\mathbf{1}_{\{X_{T_2} > k, m_1^2 > h_2\}} \middle| X_{T_1} \right] \right] \quad (50)$$

$$E^Q \left[\mathbf{1}_{\{X_{T_2} > h_2, m_0^1 > h_1, m_1^2 > h_2\}} E^Q \left[\mathbf{1}_{\{X_{T_3} > k, m_2^3 > h_3\}} \middle| X_{T_2} \right] \right] \quad (51)$$

$$E^Q \left[\mathbf{1}_{\{X_{T_3} > h_3, m_0^1 > h_1, m_1^2 > h_2, m_2^3 > h_3\}} E^Q \left[\mathbf{1}_{\{X_{T_4} > k, m_3^4 > h_4\}} \middle| X_{T_3} \right] \right] \quad (52)$$

by means of the method explained in section 2 of this paper, using identities (A.5), (A.6) and (B.2) in Appendix 3 at stage (i), identities (A.7) and (B.4) in Appendix 3 at stage (ii), identities

(A.8) and (B.6) in Appendix 3 at stage (iii), one can obtain a closed form expression made up of sixteen terms involving the quadrivariate standard gaussian distribution :

$$\begin{aligned}
& Q(m_0^1 > H_1, m_1^2 > H_2, m_2^3 > H_3, m_3^4 > H_4, X_{T_4} > K) = \\
& N_4 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, \rho_{23}, \rho_{34} \right] \\
& - e^{\frac{2\mu}{\sigma^2} h_4} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_4 + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, \rho_{23}, -\rho_{34} \right] \\
& - e^{\frac{2\mu}{\sigma^2} h_3} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_3 + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, -\rho_{23}, \rho_{34} \right] \\
& + e^{\frac{2\mu}{\sigma^2} (h_4 - h_3)} N_4 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_4 - 2h_3 + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, -\rho_{23}, -\rho_{34} \right] \\
& - e^{\frac{2\mu}{\sigma^2} h_2} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + 2h_2 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_2 + \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, \rho_{23}, \rho_{34} \right] \\
& + e^{\frac{2\mu}{\sigma^2} (h_4 - h_2)} N_4 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + 2h_2 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_4 - 2h_2 + \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, \rho_{23}, -\rho_{34} \right] \\
& + e^{\frac{2\mu}{\sigma^2} (h_3 - h_2)} N_4 \left[\frac{-h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - 2h_2 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_3 - 2h_2 + \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, -\rho_{23}, \rho_{34} \right] \\
& - e^{\frac{2\mu}{\sigma^2} (h_4 - h_3 + h_2)} N_4 \left[\frac{-h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - 2h_2 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_4 - 2h_3 + 2h_2 + \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, -\rho_{23}, -\rho_{34} \right] \\
& - e^{\frac{2\mu}{\sigma^2} h_1} N_4 \left[\frac{h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + 2h_1 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + 2h_1 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_1 + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, \rho_{23}, \rho_{34} \right] \\
& + e^{\frac{2\mu}{\sigma^2} (h_4 - h_1)} N_4 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + 2h_1 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_4 - 2h_1 + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, \rho_{23}, -\rho_{34} \right] \\
& + e^{\frac{2\mu}{\sigma^2} (h_3 - h_1)} N_4 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + 2h_1 - \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - 2h_1 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_3 - 2h_1 + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, -\rho_{23}, \rho_{34} \right] \\
& - e^{\frac{2\mu}{\sigma^2} (h_4 - h_3 + h_1)} N_4 \left[\frac{h_1 + \mu T_1}{\sigma \sqrt{T_1}}, \frac{-h_2 + 2h_1 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{h_3 - 2h_1 - \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_4 - 2h_3 + 2h_1 + \mu T_4}{\sigma \sqrt{T_4}}; \rho_{12}, -\rho_{23}, -\rho_{34} \right] \\
& + e^{\frac{2\mu}{\sigma^2} (h_2 - h_1)} N_4 \left[\frac{h_1 - \mu T_1}{\sigma \sqrt{T_1}}, \frac{h_2 - 2h_1 + \mu T_2}{\sigma \sqrt{T_2}}, \frac{-h_3 + 2h_2 - 2h_1 + \mu T_3}{\sigma \sqrt{T_3}}, \frac{-k + 2h_2 - 2h_1 + \mu T_4}{\sigma \sqrt{T_4}}; -\rho_{12}, \rho_{23}, \rho_{34} \right]
\end{aligned} \tag{53}$$

$$\begin{aligned}
& -e^{\frac{2\mu}{\sigma^2}(h_1-h_2+h_3)} N_4 \left[\frac{h_1 + \mu T_1}{\sigma\sqrt{T_1}}; \frac{h_2 - 2h_1 - \mu T_2}{\sigma\sqrt{T_2}}; \frac{-h_3 + 2h_2 - 2h_1 - \mu T_3}{\sigma\sqrt{T_3}}; \frac{-k + 2h_4 - 2h_2 + 2h_1 + \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}; \rho_{23}; -\rho_{34} \right] \\
& -e^{\frac{2\mu}{\sigma^2}(h_3-h_2+h_1)} N_4 \left[\frac{h_1 + \mu T_1}{\sigma\sqrt{T_1}}; \frac{h_2 - 2h_1 - \mu T_2}{\sigma\sqrt{T_2}}; \frac{h_3 - 2h_2 + 2h_1 + \mu T_3}{\sigma\sqrt{T_3}}; \frac{-k + 2h_3 - 2h_2 + 2h_1 + \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}, -\rho_{23}, \rho_{34} \right] \\
& +e^{\frac{2\mu}{\sigma^2}(h_1-h_3+h_2-h_1)} N_4 \left[\frac{h_1 - \mu T_1}{\sigma\sqrt{T_1}}; \frac{h_2 - 2h_1 + \mu T_2}{\sigma\sqrt{T_2}}; \frac{h_3 - 2h_2 + 2h_1 - \mu T_3}{\sigma\sqrt{T_3}}; \frac{-k + 2h_4 - 2h_3 + 2h_2 - 2h_1 + \mu T_4}{\sigma\sqrt{T_4}}; -\rho_{12}, -\rho_{23}, -\rho_{34} \right]
\end{aligned}$$

where the function $N_4[a; b; c; d; \rho_{12}; \rho_{23}; \rho_{34}]$ has been defined in Section 1

APPENDIX 3 : USEFUL IDENTITIES

In the following, u, v, w, x are variables of integration, $\lambda, \alpha, \beta, \gamma$ are arbitrary integers indexing barrier level, and $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ are any real numbers. There is a one-to-one correspondence between each operator \pm or \mp appearing on the left-hand side of every identity and each symmetrical operator appearing on the right-hand side of the same identity (symmetrical meaning here in the same order of appearance). To make sure that these notations are clear, proposition A.3) is expanded at the end of this list into the two identities it is equivalent to.

These straightforward identities, which can easily be verified, are helpful to simplify multiple integrals that are frequently encountered in multidimensional barrier option pricing. They are readily extended to higher dimensions.

$$A.1) \quad -\frac{1}{2} \left(\frac{u - \mu T_1}{\sigma\sqrt{T_1}} \right)^2 - \frac{2\mu}{\sigma^2} u = -\frac{1}{2} \left(\frac{u + \mu T_1}{\sigma\sqrt{T_1}} \right)^2$$

$$A.2) \quad -\frac{1}{2} \left(\frac{u \pm 2h_\alpha - \mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \frac{2\mu}{\sigma^2} (h_\lambda \mp h_\alpha - u) = -\frac{1}{2} \left(\frac{u \pm 2h_\alpha + \mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \frac{2\mu}{\sigma^2} (h_\lambda \pm h_\alpha)$$

$$\begin{aligned}
\text{A.3)} \quad & -\frac{1}{2} \left\{ \left(\frac{u \pm \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v \pm u \pm 2h_\beta \pm 2h_\alpha - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \mp h_\beta \mp h_\alpha - v) \\
& = -\frac{1}{2} \left\{ \left(\frac{u \mp \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v \pm u \pm 2h_\beta \pm 2h_\alpha + \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \pm h_\beta \pm h_\alpha)
\end{aligned}$$

$$\begin{aligned}
\text{A.4)} \quad & -\frac{1}{2} \left\{ \left(\frac{u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v \pm u \pm \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w \pm v \pm 2h_\alpha \pm 2h_\beta \pm 2h_\gamma - \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \mp h_\beta \mp h_\alpha \mp h_\gamma - w) \\
& = -\frac{1}{2} \left\{ \left(\frac{u + \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v \pm u \mp \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w \pm v \pm 2h_\alpha \pm 2h_\beta \pm 2h_\gamma + \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \pm h_\beta \pm h_\alpha \pm h_\gamma)
\end{aligned}$$

$$\text{A.5)} \quad -\frac{1}{2} \left(\frac{-u + \mu T_1}{\sigma \sqrt{T_1}} \right)^2 - \frac{2\mu}{\sigma^2} u = -\frac{1}{2} \left(\frac{-u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2$$

$$\text{A.6)} \quad -\frac{1}{2} \left(\frac{-u \pm 2h_\alpha + \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \frac{2\mu}{\sigma^2} (h_\lambda \pm h_\alpha - u) = -\frac{1}{2} \left(\frac{-u \pm 2h_\alpha - \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \frac{2\mu}{\sigma^2} (h_\lambda \mp h_\alpha)$$

$$\begin{aligned}
\text{A.7)} \quad & -\frac{1}{2} \left\{ \left(\frac{u \pm \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{-v \mp u \pm 2h_\beta \pm 2h_\alpha + \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \pm h_\beta \pm h_\alpha - v) \\
& = -\frac{1}{2} \left\{ \left(\frac{u \mp \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{-v \mp u \pm 2h_\beta \pm 2h_\alpha - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \mp h_\beta \mp h_\alpha)
\end{aligned}$$

$$\begin{aligned}
\text{A.8)} \quad & -\frac{1}{2} \left\{ \left(\frac{u + \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v - u + \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{-w - v \pm 2h_\alpha \pm 2h_\beta \pm 2h_\gamma + \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \pm h_\beta \pm h_\alpha \pm h_\gamma - w) \\
& = -\frac{1}{2} \left\{ \left(\frac{u - \mu T_1}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v - u - \mu(T_2 - T_1)}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{-w - v \pm 2h_\alpha \pm 2h_\beta \pm 2h_\gamma - \mu(T_3 - T_2)}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda \mp h_\beta \mp h_\alpha \mp h_\gamma)
\end{aligned}$$

$$\text{B.1)} \quad \int_{-\infty}^{\Lambda_1} \int_{-\infty}^{\Lambda_2} e^{-\frac{1}{2} \left\{ \left(\frac{u}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v \mp u}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right\}} \frac{dv du}{\sigma^2 2\pi \sqrt{T_1(T_2 - T_1)}} = N_2 \left[\frac{\Lambda_1}{\sigma \sqrt{T_1}}, \frac{\Lambda_2}{\sigma \sqrt{T_2}}; \pm \rho_{12} \right]$$

$$\text{B.2)} \quad \int_{-\infty}^{\Lambda_1} \int_{-\infty}^{\Lambda_2} e^{-\frac{1}{2} \left\{ \left(\frac{-u}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v \mp u}{\sigma \sqrt{T_2 - T_1}} \right)^2 \right\}} \frac{dv du}{\sigma^2 2\pi \sqrt{T_1(T_2 - T_1)}} = N_2 \left[\frac{\Lambda_1}{\sigma \sqrt{T_1}}, \frac{\Lambda_2}{\sigma \sqrt{T_2}}; \mp \rho_{12} \right]$$

$$\text{B.3)} \quad \int_{-\infty}^{\Lambda_1} \int_{-\infty}^{\Lambda_2} \int_{-\infty}^{\Lambda_3} e^{-\frac{1}{2} \left\{ \left(\frac{u}{\sigma \sqrt{T_1}} \right)^2 + \left(\frac{v \pm u}{\sigma \sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w \pm v}{\sigma \sqrt{T_3 - T_2}} \right)^2 \right\}} \frac{dw dv du}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2 - T_1)(T_3 - T_2)}} = N_3 \left[\frac{\Lambda_1}{\sigma \sqrt{T_1}}, \frac{\Lambda_2}{\sigma \sqrt{T_2}}, \frac{\Lambda_3}{\sigma \sqrt{T_3}}; \mp \rho_{12}, \mp \rho_{23} \right]$$

$$B.4) \int_{-\infty}^{\Lambda_1} \int_{-\infty}^{\Lambda_2} \int_{-\infty}^{\Lambda_3} e^{-\frac{1}{2} \left[\left(\frac{u}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{-v \pm u}{\sigma\sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w \pm v}{\sigma\sqrt{T_3 - T_2}} \right)^2 \right]} \frac{dw dv du}{\sigma^3 (2\pi)^{3/2} \sqrt{T_1(T_2 - T_1)(T_3 - T_2)}} = N_3 \left[\frac{\Lambda_1}{\sigma\sqrt{T_1}}, \frac{\Lambda_2}{\sigma\sqrt{T_2}}, \frac{\Lambda_3}{\sigma\sqrt{T_3}}; \mp \rho_{12}, \pm \rho_{23} \right]$$

$$B.5) \int_{-\infty}^{\Lambda_1} \int_{-\infty}^{\Lambda_2} \int_{-\infty}^{\Lambda_3} \int_{-\infty}^{\Lambda_4} e^{-\frac{1}{2} \left[\left(\frac{u}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v \pm u}{\sigma\sqrt{T_2 - T_1}} \right)^2 + \left(\frac{w \pm v}{\sigma\sqrt{T_3 - T_2}} \right)^2 + \left(\frac{x \pm w}{\sigma\sqrt{T_4 - T_3}} \right)^2 \right]} \frac{dx dw dv du}{\sigma^4 (2\pi)^2 \sqrt{T_1(T_2 - T_1)(T_3 - T_2)(T_4 - T_3)}} \\ = N_4 \left[\frac{\Lambda_1}{\sigma\sqrt{T_1}}, \frac{\Lambda_2}{\sigma\sqrt{T_2}}, \frac{\Lambda_3}{\sigma\sqrt{T_3}}, \frac{\Lambda_4}{\sigma\sqrt{T_4}}; \mp \rho_{12}, \mp \rho_{23}, \mp \rho_{34} \right]$$

$$B.6) \int_{-\infty}^{\Lambda_1} \int_{-\infty}^{\Lambda_2} \int_{-\infty}^{\Lambda_3} \int_{-\infty}^{\Lambda_4} e^{-\frac{1}{2} \left[\left(\frac{u}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v \pm u}{\sigma\sqrt{T_2 - T_1}} \right)^2 + \left(\frac{-w \pm v}{\sigma\sqrt{T_3 - T_2}} \right)^2 + \left(\frac{x \pm w}{\sigma\sqrt{T_4 - T_3}} \right)^2 \right]} \frac{dx dw dv du}{\sigma^4 (2\pi)^2 \sqrt{T_1(T_2 - T_1)(T_3 - T_2)(T_4 - T_3)}} \\ = N_4 \left[\frac{\Lambda_1}{\sigma\sqrt{T_1}}, \frac{\Lambda_2}{\sigma\sqrt{T_2}}, \frac{\Lambda_3}{\sigma\sqrt{T_3}}, \frac{\Lambda_4}{\sigma\sqrt{T_4}}; \mp \rho_{12}, \mp \rho_{23}, \pm \rho_{34} \right]$$

For example, proposition A.3) is expanded into :

$$(i) -\frac{1}{2} \left\{ \left(\frac{u + \mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v + u + 2h_\beta + 2h_\alpha - \mu(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda - h_\beta - h_\alpha - v) \\ = -\frac{1}{2} \left\{ \left(\frac{u - \mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v + u + 2h_\beta + 2h_\alpha + \mu(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda + h_\beta + h_\alpha)$$

$$(ii) -\frac{1}{2} \left\{ \left(\frac{u - \mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v - u - 2h_\beta - 2h_\alpha - \mu(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda + h_\beta + h_\alpha - v) \\ = -\frac{1}{2} \left\{ \left(\frac{u + \mu T_1}{\sigma\sqrt{T_1}} \right)^2 + \left(\frac{v - u - 2h_\beta - 2h_\alpha + \mu(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}} \right)^2 \right\} + \frac{2\mu}{\sigma^2} (h_\lambda - h_\beta - h_\alpha)$$