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Abstract. We analyze the relationship between two compactifications of the moduli space of maps from curves to a toric fano varieties: the Kontsevich moduli space of stable maps and the Ciocan-Fontanine–Kim moduli space of stable quasi-maps. We exhibit the moduli space of stable maps as a union of two (reducible) components: the moduli space of relevant maps and the moduli space of irrelevant maps. We equip these spaces with virtual classes such that their sum equals the virtual class of the moduli space of stable maps. The moduli space of relevant maps is birational to the space of qusi-maps and the enumerative invariants defined as virtual intersection numbers on the space of relevant maps are equal to the quasi-maps invariants. On the other hand, the virtual intersection numbers on the space of stable irrelevant maps are zero. This shows that Gromov–Witten invariants agree with quasi-map invariants for toric Fano varieties.

Contents

1. Introduction

In recent years many birational models of moduli spaces of stable maps have been constructed for particular targets. In this paper we consider the Kontsevich moduli spaces of stable maps and the moduli spaces of stable quasi-maps to toric Fano varieties of Ciocan–Fontanine and Kim. These are spaces which come equipped with virtual classes in the sense of [1], [11]. The purpose of this paper is to understand the relation between the two virtual fundamental classes: we will show that enumerative invariants defined as virtual intersection numbers in the two moduli spaces coincide. We do this by analyzing the geometry of the two types of moduli spaces. For this, we write the moduli space of stable maps as a union of two

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moduli spaces: the space of *relevant maps* and the space of *irrelevant maps*. The relation between invariants is then proved in the following steps:

Step 1: we endow the space of relevant maps and the space of irrelevant maps with virtual classes such that the virtual class of the moduli space of stable maps is the sum of the virtual class of the space of relevant maps and the virtual class of the space of irrelevant maps;

Step 2: we show that the invariants of the spaces of relevant maps and quasi-maps agree;

Step 3: we show that the invariants of the spaces of irrelevant maps are zero.

This is achieved in Sections 2-3. The main ingredient is the construction of Ciocan–Fontanine and Kim which exhibits the moduli space of stable maps and stable quasi-maps as the zero locus of a section of a bundle on a smooth space. This allows us to define virtual classes of unions of components of the moduli space of stable maps as in Step 1 above and to use properties of Segre classes of cones in [8] to prove Steps 2 and 3.

In the last section we discuss a possible alternative approach to Steps 2 and 3 which does not rely on the Ciocan-Fontanine–Kim construction. For this we construct a birational morphism from the moduli space of relevant maps to the space of quasi-maps and a morphism from the space of irrelevant maps to a space of smaller virtual dimension. To these morphisms we can apply the virtual push-forward theorem in [15]. The missing part is Step 1, for which a more general version of the virtual push forward theorem is needed.

Relation to other works. As explained above, the moduli space of stable maps consists of a union of components which is virtually birational to the moduli space of stable quasimaps and a union of components which do not contribute to the invariants. The main difficulty is to define and relate virtual classes of components (or of some other space which dominates a union of components) of the moduli space of stable maps. The phenomenon of relating virtual classes has already appeared in several contexts such as the relationship between Gromov-Witten and Gopakumar-Vafa invariants, the Li-Zinger formula for elliptic Gromov-Witten invariants of a hypersurface in a projective space, and the relation between Gromov-Witten invariants and quasi-map invariants. More precisely, Pandharipande [18] computes contributions to Gromov–Witten invariants of Calabi–Yau 3-folds coming from components of the moduli space of stable maps which consist of multiple covers. He also shows the vanishing of contributions of covers of elliptic curves conjectured by Gopakumar and Vafa. Li–Zinger, Vakil–Zinger, Zinger [12, 13, 20, 21] and Chang–Li [2] define virtual classes of (a desingularization) of the main component of the moduli space of stable genus one maps to a hypersurface in a projective space and prove that Gromov–Witten invariants are equal to invariants of the main components plus genus zero Gromov–Witten invariants. Both analytic methods of Zinger and the algebraic ones of the more recent work of Chang and Li are rather technical. In [16] a relation between virtual classes of stable maps and quasi-maps to Grassmannians is given. For Grassmannians one has a rational application from an open dense set inside the moduli space of stable maps to the moduli space of stable quotients. Once we resolve the indeterminacy locus the problem of relating virtual classes can be easily solved by applying the general virtual push-forward theorem in [15].

In the case of Gromov–Witten invariants and quasi-map invariants one can employ a completely different method based on virtual localization. This has been done in [5, 6, 17, 19]. Although very powerful, virtual localization is rather mysterious and does not explain the geometric picture. In this paper we continue the approach in [16] with the difference that now we only have a morphism from a union of components. Our proof heavily relies on the

very specific properties of moduli spaces of maps and quasi-maps to toric varieties (or more generally to GIT quotients).
It would be highly desirable to have a general machinery of defining virtual classes (or obstruction theories) of components of the moduli space of stable maps and a more general push forward theorem in order to address similar questions. We will address this in future work.

Notation and conventions. We take the ground field to be $\mathbb{C}$.
Unless otherwise specified we will try to respect the following convention: we will use normal fonts for Deligne-Mumford stacks (e.g. $\overline{M}_{g,n}(X,\beta)$, $\overline{Q}_{g,n}(X,\beta)$, etc.), Artin stacks for which we know that they are not Deligne-Mumford stacks will be generally denoted by gothic letters (e.g. $\mathfrak{M}_{g,n}$, $\mathfrak{Pic}_{g,n}$, etc).
By a commutative diagram of stacks we mean a 2-commutative diagram of stacks and by a cartesian diagram of stacks we mean a 2-cartesian diagram of stacks.

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2. Stable quasi-maps to toric varieties

2.1. **Stable maps and stable quasimaps.** We begin by recalling the definition of stable quasimaps to a toric variety [3]. We use the following notation throughout the paper.

Notation 2.1. Let X denote a smooth projective toric variety defined by a fan Σ. Fix a $\mathbb{Z}$-basis $\mathcal{L}_1, \ldots, \mathcal{L}_r$ for $\mathrm{Pic}(X)$.

Definition 2.2. Fix an integer g and integers d_ρ for all $\rho \in \Sigma(1)$. A *stable quasimap to X* consists of the data

$$(C, p_1, ..., p_n, \{L_\rho\}_{\rho\in\Sigma(1)}, \{u_\rho\}_{\rho\in\Sigma(1)}, \{\phi_m\}_{m\in M})$$

where

(1) C is a nodal curve of arithmetic genus g
(2) $p_1, \ldots, p_n$ are distinct marked smooth points on C
(3) L_ρ is a line bundle on C of degree d_ρ
(4) u_ρ is a global section of L_ρ
(5) ϕ_m is an isomorphism from $\otimes_{\rho\in\Sigma(1)} L_\rho^{\langle\rho,m\rangle}$ to $\mathcal{O}_C$

such that

- (nondegeneracy) there is a finite, possibly empty, set of non-singular points $B \subset C$, disjoint from $\{p_1, \ldots, p_n\}$, such that for every $y \in C\backslash B$, there exists a maximal cone $\sigma \in \Sigma$ with $u_\rho(y) \neq 0$ for all rays ρ of Σ such that $\rho \notin \sigma$
- (stability) $\omega_C(\sum p_i) \otimes L^\epsilon$ is ample for all $\epsilon > 0$, where $L = \otimes L_\rho$.

The *degree* of this stable quasimap is the element $\beta \in H_2(X;\mathbb{Z})$ defined by $\beta \cdot D_\rho = d_\rho$ for all $\rho \in \Sigma(1)$. Here $D_\rho \subset X$ is the toric divisor corresponding to the ray ρ of Σ.

Ciocan-Fontanine–Kim (ibid.) have defined what it means for two stable quasimaps to be isomorphic, and what is meant by a family of quasimaps.

Notation 2.3. We denote the moduli space of stable quasimaps [3] to X by $\overline{Q}_{g,n}(X,\beta)$. We denote the moduli space of stable maps [9, 10] to X by $\overline{M}_{g,n}(X,\beta)$.

2.2. The main constructions.

Notation 2.4. Let $\mathfrak{M}_{g,n}$ denote the stack of prestable curves of genus g with n marked points, and let $\mathfrak{Pic}_{g,n} \to \mathfrak{M}_{g,n}$ denote the relative Picard stack. Let $\mathfrak{Pic}^r_{g,n}$ denote the r-fold fiber product of $\mathfrak{Pic}_{g,n}$ over $\mathfrak{M}_{g,n}$.

Notation 2.5. Consider the universal curve $\pi\colon \mathcal{C} \to \overline{M}_{g,n}(X,\beta)$. Let $\rho \in \Sigma(1)$. We denote by L_ρ the line bundle over $\mathcal{C}$ given by the pullback of $\mathcal{O}(D_\rho)$ along the canonical map $\mathcal{C} \to X$.

Construction 2.6. Let $\pi\colon \mathcal{C} \to \overline{M}_{g,n}(X,\beta)$ be the universal curve and $\nu\colon \overline{M}_{g,n}(X,\beta) \to \mathfrak{Pic}^r_{g,n}$ be the forgetful morphism. Then $R^\bullet\pi_* L_\rho$ is a perfect relative obstruction theory for ν. The same holds if we replace $\overline{M}_{g,n}(X,\beta)$ by $\overline{Q}_{g,n}(X,\beta)$.

Definition 2.7. Let $(a_{i\rho})$ denote the matrix entries of the canonical map $\mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X)$ with respect to the canonical basis for $\mathbb{Z}^{\Sigma(1)}$ and our chosen basis $\mathcal{L}_1,\ldots,\mathcal{L}_r$ for $\mathrm{Pic}(X)$. Let $\pi\colon \mathcal{C} \to \mathfrak{Pic}^r_{g,n}$ be the universal curve and let $\mathcal{Q}_1,\ldots,\mathcal{Q}_r$ denote the canonical line bundles on $\mathcal{C}$. We define line bundles $\mathcal{L}_\rho \to \mathcal{C}$, $\rho \in \Sigma(1)$, by:

$$\mathcal{L}_\rho := \otimes_{i=1}^r \mathcal{Q}_i^{a_{i\rho}}$$

Construction 2.8. We present a construction of Ciocan-Fontanine–Kim [3]. Let $\pi\colon \mathcal{C} \to \mathfrak{Pic}^r_{g,n}$ be the universal curve and consider the open substack of $\mathfrak{Pic}^r_{g,n}$ obtained by requiring that

$$A = \omega_C(\sum p_i) \otimes L^\epsilon$$

is π-relatively ample for some ϵ. By abuse of notation we will continue to denote by π : $\mathcal{C} \to \mathfrak{Pic}^r_{g,n}$ the universal curve and by $\mathcal{L}_\rho$ the universal line bundles. Note that π is now a projective morphism. Let us fix $\mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}(1)$ a global section. Fix m depending on g,n,β sufficiently large as in Theorem 3.2.5 in [4]. Let $\mathfrak{Pic}^r_{g,n,\beta}$ be the substack of $\mathfrak{Pic}^r_{g,n}$ such that

- the degree of L_ρ is d_ρ
- $H^1(C, L_\rho(m)) = 0$

Let $\mathfrak{X}$ be the total space of the bundle $\oplus_\rho \pi_*\mathcal{L}_\rho(m)$ over $\mathfrak{Pic}^r_{g,n,\beta}$. Note that $\mathfrak{X}$ is an Artin stack which parametrizes

$$(C, p_1, ..., p_n, L_\rho, v_\rho \in H^0(L_\rho(m)))$$

Let V be the open substack of $\mathfrak{X}$ such that

(1) (generic nondegeneracy) u_ρ sends all but possibly finitely many points of C to $Z(\Sigma)$
(2) (stable map stability) $\omega_C(\sum p_i) \otimes L^\epsilon$ is ample for all $\epsilon > 2$

Let $\mathcal{N}_\rho$ be the vector bundle on $\mathcal{C} \to \mathfrak{Pic}^r_{g,n\beta}$ determined by the exact sequence

$$0 \to \mathcal{L}_\rho \to \mathcal{L}_\rho(m) \to \mathcal{N}_\rho \to 0 \tag{1}$$

We denote by $r : V \to \mathfrak{Pic}^r_{g,n,\beta}$ the projection. We consider the vector bundle

$$\mathcal{N} = r^*\pi_* \oplus \mathcal{N}_\rho$$

on V which comes equipped with a tautological section v induced by the morphisms $\mathcal{L}_\rho \to \mathcal{L}_\rho(m)$. Let M be the zero locus of v. Then we have that $\overline{M}_{0,n}(X,\beta)$ is the open substack of M obtained by imposing the strong nondegeneracy condition. If $f : \overline{M}_{0,n}(X,\beta) \to V$ is the embedding we have obtained that the complex

$$[f^*T_V \to f^*\mathcal{N}]$$

concentrated in $[0,1]$ is a dual obstruction theory for $\overline{M}_{0,n}(X,\beta)$ relative to $\mathfrak{Pic}^r_{g,n\beta}$. It can be seen that this obstruction theory is the same as the previous one.

In a completely analogous fashion we can construct a smooth stack $\hat{V}$ and a vector bundle $\hat{\mathcal{N}}$ equipped with a section $\hat{v}$ such that $\overline{Q}_{0,n}(X,\beta)$ is the open substack of $Z(\hat{v})$ by imposing the strong nondegeneracy condition. The only difference is that in the above construction we replace the stable map stability with the stable quotient stability condition.

Construction 2.9. Let A be a relatively very ample line bundle for $\mathcal{C} \to \mathfrak{Pic}^r_{g,n}$. Let D_α be a divisor consisting of rational tails and let $p : \mathcal{C} \to \hat{\mathcal{C}}$ be the contraction of rational tails. Then $\hat{A} = p_*A(m_\alpha D_\alpha)$ is relatively very ample for $\hat{\mathcal{C}} \to \mathfrak{Pic}^r_{g,n}$. Then we have that the line bundles A and $\hat{A}$ determine stacks V and $\hat{V}$ which are birational. More precisely we have a morphism $q : V \to \hat{V}$ defined on fibers in the following way. Let C be a curve with rational tail free part $\hat{C}$ and rational tails $C^0_1, ...C^0_s$ attached to $\hat{C}$ in x_i. Then

$$q(C, p_1, ..., p_n, G_\rho, u_\rho) = (\hat{C}, p_1, ..., p_n, \hat{G}_\rho, \hat{u}_\rho) \tag{2}$$

where $\hat{G}_\rho = G_\rho|_{\hat{C}}(\sum_i b_i x_i)$ and b_i denotes the degree of G_ρ on c^0_i.

The image of the moduli space of stable maps in $\hat{V}$. We have the composition

$$\begin{tikzcd} \overline{M}_{0,n}(X,\beta) \arrow[r, "f"] \arrow[dr] & V \arrow[d, "q"] \\ & \hat{V} \end{tikzcd} \tag{3}$$

Let us determine the image of $\overline{M}_{0,n}(X,\beta)$ in $\hat{V}$. We will see that this image strictly contains $\overline{Q}_{0,n}(X,\beta)$ and in particular q does not define a map from $\overline{M}_{0,n}(X,\beta)$ to $\overline{Q}_{0,n}(X,\beta)$ but only from a substack of $\overline{M}_{0,n}(X,\beta)$.
Let (C, L_ρ, u_ρ) be a stable map to X. Suppose L_ρ has negative degree $(-a)$ on some rational tail and the image in V has positive degree b on the rational tail. Then its image in V is a stable quotient $(\hat{C}, \hat{L}_\rho, \hat{u}_\rho)$ with $\hat{u}_\rho = x^b u_\rho|_{\hat{C}}$. Note that by construction $\hat{A}$ has a zero of order $a+b$ at x. This shows that $q(C, L_\rho, u_\rho)$ is not in a zero locus of a section of $\hat{N}$. More precisely, if Z_A denotes the zero locus of the fixed section of A, the image of $\overline{M}_{0,n}(X,\beta)$ in V is the subset of V

$$\{(C, L'_\rho, u_\rho) | L'_\rho \text{ has degree } d+m, u_\rho \in H^0(C, L'_\rho), u_\rho(Z_A) = 0\}.$$

If $Z_{\hat{A}}$ denotes the zero locus of the fixed section of $\hat{A}$, then $Z_{\hat{A}}$ comes with points x_j of multiplicity m_α which correspond the contraction of the divisor D_α. Then the image of $q(\overline{M}_{0,n}(X,\beta))$ in $\hat{V}$ is the subset of $\hat{V}$

$$\bigcup_{i \le m_\alpha} \{(\hat{C}, \hat{L}'_\rho, \hat{u}_\rho) | \hat{L}'_\rho \text{ has degree } d+m, \hat{u}_\rho \in H^0(\hat{C}, \hat{L}'_\rho),$$
$$\hat{u}_\rho \text{ has a zero of order i at } x_j \text{ and simple zeroes for the other } x \in Z_{\hat{A}}\}$$

Construction 2.10. This is very similar to Construction 2.8 and it will be the key ingredient in the proof of the main theorem. We construct an additional space W with embeddings $\overline{M}_{0,n}(X,\beta) \overset{i}{\hookrightarrow} W \overset{j}{\hookrightarrow} V$ such that $j \circ i = f$ and i and j have perfect obstruction theories.

For this consider the morphism which contracts rational tails defined in [16]

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "p"] \arrow[ddr, "\pi"'] & \\ & \hat{\mathcal{C}} \arrow[d, "\hat{\pi}"] \\ & \mathfrak{Pic}^r_{g,n,\beta} \end{tikzcd} \tag{4}$$

As in Construction 2.8, by considering an open subset of $\mathfrak{Pic}^r_{g,n,\beta}$ we may assume that π and subsequently $\hat{\pi}$ are projective. Let $\mathcal{O}_{\hat{\mathcal{C}}}(\hat{1})$ be a $\hat{\pi}$-ample line bundle and $\mathcal{O}_{\mathcal{C}}(1)$ be a π-ample line bundle. We have that $p^*\mathcal{O}_{\hat{\mathcal{C}}}(\hat{1}) \otimes \mathcal{O}_{\mathcal{C}}(1)$ is π-ample. As before, we fix $m_1, m_2 \in \mathbb{Z}$ sufficiently large and we consider $\mathfrak{Pic}^r_{g,n,\beta}$ the open substack of $\mathfrak{Pic}^r_{g,n}$ such that

(1) the degree of L_ρ is d_ρ
(2) $H^1(\hat{C}, L_\rho|_{\hat{C}}(m_1\hat{1})) = 0$
(3) $H^1(C, L_\rho(m_2 1)) = 0$

where $L_\rho|_{\hat{C}}(m_1\hat{1})$ denotes the line bundle $L_\rho|_{\hat{C}} \otimes \mathcal{O}_{\hat{\mathcal{C}}}(\hat{1})^{\otimes m_1}$ and $L_\rho(m_2 1)$ denotes the line bundle $L_\rho \otimes \mathcal{O}_{\mathcal{C}}(1)^{\otimes m_2}$. Let us fix $\hat{s} : \mathcal{O}_{\mathcal{C}} \to p^*\mathcal{O}_{\hat{\mathcal{C}}}(\hat{1}m_1)$ a global section of $p^*\mathcal{O}(\hat{1}m_1)$, $s : \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}}(1m_2)$ a global section of $\mathcal{O}_{\mathcal{C}}(1m_2)$ and $\hat{s} \otimes s$ a global section of $p^*\mathcal{O}_{\hat{\mathcal{C}}}(\hat{1}) \otimes \mathcal{O}_{\mathcal{C}}(1)$. As before, let $\mathfrak{Y}$ be the total space of the sheaf (not bundle!) $\oplus_\rho \pi_* L_\rho(m\hat{1})$ on $\mathfrak{Pic}^r_{g,n}$. Note that $\mathfrak{Y}$ is an Artin stack which parametrizes

$$(C, p_1, ..., p_n, L_\rho, v_\rho \in H^0(L_\rho(m\hat{1})))$$

Let W be the open substack of $\mathfrak{Y}$ such that stable map stability and generic nondegeneracy hold. Note that W is a Deligne-Mumford stack.
Let $\mathcal{N}^i_\rho$ be the vector bundle on $\mathcal{C} \to \mathfrak{Pic}^r_{g,n,\beta}$ determined by the exact sequence

$$0 \to \mathcal{L}_\rho \to L_\rho(m_1\hat{1}) \to \mathcal{N}^i_\rho \to 0 \tag{5}$$

We denote by $r_W : W \to \mathfrak{Pic}^r_{g,n,\beta}$ the projection. We consider the vector bundle

$$\mathcal{N}^i = r^*\pi_* \oplus \mathcal{N}^i_\rho$$

on W which comes equipped with a tautological section v^i induced by the morphisms $\mathcal{L}_\rho \to \mathcal{L}_\rho(m_1\hat{1})$. Let M be the zero locus of v^i. Then we have that $\overline{M}_{0,n}(X, \beta)$ is the open substack of M obtained by imposing the nondegeneracy condition. Denote by $i : \overline{M}_{0,n}(X, \beta) \to W$ the embedding.
Let us now embed W in V. Let $\mathcal{N}^j_\rho$ be the vector bundle on $\mathcal{C} \to \mathfrak{Pic}^r_{g,n}$ determined by the exact sequence

$$0 \to \mathcal{L}_\rho(m\hat{1}) \to \mathcal{L}_\rho(m_1\hat{1} \otimes m_2 1) \to \mathcal{N}^j_\rho \to 0 \tag{6}$$

If $r_V : V \to \mathfrak{Pic}^r_{g,n,\beta}$ is the projection, let us consider the vector bundle

$$\mathcal{N}^j = r_V^*\pi_* \oplus N^j_\rho$$

on V which comes equipped with a tautological section v^j. Then W is the zero locus of a section of v^j and we denote the embedding of W in V by j.
To conclude, by the above constructions we have a sequence of embeddings

$$\overline{M}_{0,n}(X, \beta) \overset{i}{\hookrightarrow} W \overset{j}{\hookrightarrow} V \tag{7}$$

such that $j \circ j = f$. Moreover, if in notations of Construction 2.9 we take $A = p^*\mathcal{O}_{\hat{\mathcal{C}}}(\hat{1}) \otimes \mathcal{O}_{\mathcal{C}}(1)$ and $\mathcal{N}_\rho$ is the vector bundle determined by the exact sequence

$$0 \to \mathcal{L}_\rho \to \mathcal{L}_\rho(m_1\hat{1} \otimes m_2 1) \to \mathcal{N}_\rho \to 0 \tag{8}$$

we have that $\mathcal{N} = \oplus r_V^* \mathcal{N}_\rho$ contains the normal cone C_f.

Construction 2.11. Let us factorize the map $\overline{Q}_{0,n}(X,\beta) \hookrightarrow \hat{V}$ through a suitable $\hat{W}$. In notation as above let $\hat{W}$ be the substack of $\mathfrak{Y}$ defined by imposing generic nondegeneracy and quasi-map stability. Note that by our condition on $\mathcal{O}_{\hat{\mathcal{C}}}(\hat{1})$ we have that $\hat{W}$ is smooth.

Definition 2.12. By Construction 2.10 we can define the virtual class of W by

$$[W]^{\text{virt}} = j^![V].$$

Lemma 2.13. *We have*

$$[\overline{M}_{0,n}(X,\beta)]^{\text{virt}} = (i^![W]^{\text{virt}}) \cap \overline{M}_{0,n}(X,\beta)$$
$$[\overline{Q}_{0,n}(X,\beta)]^{\text{virt}} = (i^![\hat{W}]) \cap \overline{Q}_{0,n}(X,\beta)$$

Proof. Let us prove the first equality. It is enough to show that $[M]^{\text{virt}} = (i^![W]^{\text{virt}})$. Putting the definitions together we get a commutative diagram of vector bundles on $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L}_\rho \arrow[r] \arrow[d] & L_\rho(m_1\hat{1}) \arrow[r] \arrow[d] & \mathcal{N}_\rho^i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L}_\rho \arrow[r] & \mathcal{L}_\rho(m_1\hat{1} \otimes m_2 1) \arrow[r] \arrow[d] & \mathcal{N}_\rho \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{N}_\rho^j \arrow[r] & \mathcal{N}_\rho^j &
\end{tikzcd} \tag{9}$$

The snake lemma implies that the following sequence is exact

$$0 \to \mathcal{N}_\rho^i \to \mathcal{N}_\rho \to \mathcal{N}_\rho^j \to 0. \tag{10}$$

Let C be a fiber of $\pi : \mathcal{C} \to \mathfrak{Pic}_{g,n}^r$ in some point and $N_\rho^i = \mathcal{N}_\rho^i|_C$. Note that since N_ρ^i is supported on some points of C we have that $H^1(C, N_\rho^i) = 0$. The same argument holds for the other two vector bundles. This shows that pushing forward the above sequence along π we get an exact sequence of vector bundles on V

$$0 \to \mathcal{N}^i \to \mathcal{N} \to \mathcal{N}^j \to 0. \tag{11}$$

This shows that $[\overline{M}_{0,n}(X,\beta)]^{\text{virt}} = i^!(j^![V])$ and by the definition of $[W]^{\text{virt}}$ we get the conclusion.
The proof of the second equality copies verbatim. ∎

2.3. A dimension count.

Lemma 2.14. *Let D_α, with $\alpha = (a_1, ..., a_k)$, with $a_i > 0$ be the divisor in $\overline{M}_{0,n}(X,\beta)$ such that the degrees of $L_1, ..., L_k$ on the generic rational tail are equal to $-a_1, ..., -a_k$. Suppose that the degrees of $L_1 \otimes A, ..., L_k \otimes A$ on the same rational tail are $b_1, ..., b_k$. Let $a = \sum_{i=1}^k a_i$ and $b = \sum_{i=1}^k b_i$. Then*

$$q_* D_\alpha^j \cdot [V] = 0$$

for $j \le a + b$.

Proof. We have that $\beta = \beta^0 + \beta^1$ where β^0 is the class of the fixed rational tail and β^1 the class of the rest of the curve. We have that

$$\begin{aligned} K_X \cdot \beta^0 &= \sum_{i=1}^{l} \deg(L_i|C^0) \\ &= \sum_{i=1}^{k} -a_i + \sum_{i=k+1}^{l} \deg(L_i|C^0). \end{aligned}$$

The Fano condition implies that $K_X \cdot \beta^0 > 0$ and by the above $\sum_{i=1}^{k} a_i < \sum_{i=k+1}^{l} \deg(L_i|C^0)$. On the other hand $\sum_{i=1}^{l} \deg(L_i|C^0) \otimes A = \sum_{i=1}^{k} b_i + \sum_{i=k+1}^{l} \deg(L_i|C^0)$. The above implies that

$$\sum_{i=1}^{l} \deg(L_i|C^0) \otimes A > b + a. \tag{12}$$

We have that $p(D^j) \subset q(D)$ and $q(D) \simeq \hat{D}$, where $\hat{D}$ is the space of sections of line bundles L_i' of degrees equal to $K_X \cdot \beta_1 + \deg(A|_{\hat{C}})$. By (12)

$$\dim \hat{D} < \dim(V) - (b + a).$$

From this we obtain that $\dim(D^j) = \dim(V) - j > \dim \hat{D}$ for $j \leq a + b$ and therefore $q_* D^j \cdot [V] = 0$ for all $j \leq a + b$. $\square$

Proposition 2.15. $q_* c(N)[V] = c(\hat{N})[\hat{V}]$

Corollary 2.16. $\prod_i \psi_i^{k_i} [\overline{M}_{0,n}(X, \beta)]^{\text{virt}} = \prod_i \psi_i^{k_i} [\overline{Q}_{0,n}(X, \beta)]^{\text{virt}}$.

3. The geometry of the components of the moduli space of quasi-maps

3.1. **Relevant and irrelevant maps.** To a component of the moduli space of maps we can associate a decorated graph τ_α, such that the general curve C is of topological type τ and to each edge e of the graph we associate $\alpha^e = (a_1^e, ..., a_k^e) \in \mathbb{Z}$ if L_ρ has degree a_ρ on the corresponding irreducible subcurve of C. We will denote such a component by M^{τ_α}. Then we can define the following partial ordering of the components of $\overline{M}_{0,n}(X, \beta)$ (or $\overline{Q}_{0,n}(X, \beta)$).

Definition 3.1. We say that $\tau_{\alpha_1} \leq \tau_{\alpha_2}$ if τ_{α_1} is obtained from τ_{α_2} by replacing a union of edges $e_1, ..., e_t$ in τ_{α_2} with one edge e and $\alpha^e = \sum_{i=1}^{t} \alpha^{e_i}$. In this case we call $M^{\tau_{\alpha_1}}$ smaller than $M^{\tau_{\alpha_1}}$.

Proposition 3.2. *There exists a morphism* $c : \overline{M}_{0,n}^{rel}(X, \beta) \to \overline{Q}_{0,n}(X, \beta)$ *and similarly* $d : W^{rel} \to \hat{W}$

Proof. This essentially follows from the fact that we have a morphism $q : V \to \hat{V}$ and $q(\overline{M}_{0,n}^{rel}(X, \beta))$ is included in $\overline{Q}_{0,n}(X, \beta)$. We use notations as in Construction 2.9. Let us consider family of prestable curves $\mathcal{C} \to B$ with smooth generic fiber and special fiber C_0 isomorphic to $\hat{C} \cup D$ where D is some tree of rational curves and let $f : \mathcal{C} \to X$ be a morphism over B. Then it suffices to show that $q(C_0, L_\rho', u_\rho')$ is such that u_ρ' has a zero of order at least $a + b$ at x. A section of L_ρ determines a Weil divisor W on $\mathcal{C}$ which intersects the generic curve in d points. Then, L_ρ vanishes on D with multiplicity at least a. This shows that u_ρ'

has a zero of order at least $a+b$ at x.
Moreover, we have a commutative diagram

$$\begin{tikzcd}
\overline{M}^{rel}_{0,n}(X,\beta) \arrow[r] \arrow[d, "c"'] & M^{rel} \arrow[r, "i"] \arrow[d] & W^{rel} \arrow[r, "j"] \arrow[d, "d"'] & V \arrow[d, "q"] \\
\overline{Q}_{0,n}(X,\beta) \arrow[r] & Q \arrow[r] & \hat{W} \arrow[r] & \hat{V}
\end{tikzcd} \tag{13}$$

and by definitions it can be easily seen that the square in the middle is cartesian. □

3.2. **Separating components.**

Definition 3.3. Let $[W^{rel}]$ be the fundamental class of the component of W whose general curve is smooth. Define

$$[W^{irrel}]^{\text{virt}} = [W]^{\text{virt}} - [W^{rel}].$$

Remark 3.4. As W^{rel} is of pure dimension equal to the virtual dimension the definition makes sense.

Lemma 3.5. *We have that $d(W^{rel}) = \hat{W}$ and for any cycle T which represents $[W^{irrel}]^{\text{virt}}$ in $A_*(W^{irrel})$ we have that the image $d(T)$ has dimension strictly smaller than k, where k is the dimension of $\hat{W}$.*

Proof. First equality is obvious. By the same dimension count as before the dimension of $q(W^{irrel})$ is too small. □

Definition 3.6. We define

$$[\overline{M}^{rel}_{0,n}(X,\beta)]^{\text{virt}} = \{(s(M/W^{rel})c(\hat{N}^i)\}_m \cap \overline{M}^{rel}_{0,n}(X,\beta).$$

Proposition 3.7. *We have that $c_*[\overline{M}^{rel}_{0,n}(X,\beta)]^{\text{virt}} = [\overline{Q}_{0,n}(X,\beta)]^{\text{virt}}$.*

Proof. By definition $[\overline{M}^{rel}_{0,n}(X,\beta)]^{\text{virt}} = \{(s(M^{rel}/W^{rel})c(\hat{N}^i)\}_m \cap \overline{M}^{rel}_{0,n}(X,\beta)$. Let M^{rel} be thesubstack of$M$ such that the following diagram is cartesian

$$\begin{tikzcd}
M^{rel} \arrow[r] \arrow[d, "g"] & W^{rel} \arrow[d, "d"] \\
Q \arrow[r] & \hat{W}
\end{tikzcd}$$

The morphism d is not proper, so we need to adapt the result in [8]. However, we still have a morphism $G : \mathbb{P}(C_{M^{rel}/W^{rel}}) \to \mathbb{P}(C_{Q/\hat{W}})$ and if ξ is $c_1(\mathcal{O}_{\mathbb{P}(C_{M^{rel}/W^{rel}})}(1))$ and $\hat{\xi}$ is $c_1(\mathcal{O}_{\mathbb{P}(C_{Q/\hat{W}})}(1))$ then $G^*\hat{\xi} = \xi$. Moreover, the restriction of G to $\mathbb{P}(C_{M^{rel}/W^{rel}}) \cap \overline{M}^{rel}_{0,n}(X,\beta)$ is proper. Let $\pi : \mathbb{P}(C_{M^{rel}/W^{rel}}) \to M^{rel}$ and $\hat{\pi} : \mathbb{P}(C_{Q/\hat{W}}) \to M^{rel}$ be the projections to the base. By the fact that intersection products commute with restrictions to open subsets we have

$$c_*(s(M^{rel}/W^{rel}) \cap \overline{M}^{rel}_{0,n}(X,\beta) = c_*\pi_*(\xi^i\mathbb{P}(C_{M^{rel}/W^{rel}})) \cap \overline{M}^{rel}_{0,n}(X,\beta) \tag{14}$$

$$= c_*\pi_*\xi^i(\mathbb{P}(C_{M^{rel}/W^{rel}}) \cap \overline{M}^{rel}_{0,n}(X,\beta)) \tag{15}$$

If we look at the commutative diagram

$$\begin{tikzcd} \mathbb{P}(C_{M^{rel}/W^{rel}}) \cap \overline{M}^{rel}_{0,n}(X,\beta) \arrow[r, "\pi"] \arrow[d, "G"] & \overline{M}^{rel}_{0,n}(X,\beta) \arrow[d, "c"] \\ \mathbb{P}(C_{Q/\hat{W}}) \cap \overline{Q}_{0,n}(X,\beta) \arrow[r, "\hat{\pi}"] & \overline{Q}_{0,n}(X,\beta) \end{tikzcd} \tag{16}$$

we get

$$c_*(s(M^{rel}/W^{rel}) \cap \overline{M}^{rel}_{0,n}(X,\beta) = \hat{\pi}_*\hat{\xi}^i G_*\mathbb{P}(C_{M^{rel}/W^{rel}}). \tag{17}$$

By the proof of Proposition 4.2 in [8] the morphism G is generically bijective which means that its restriction to an open subset is still generically bijective. We have thus obtained that

$$c_*[\overline{M}^{rel}_{0,n}(X,\beta)]^{\mathrm{virt}} = \{(s(\overline{Q}_{0,n}(X,\beta)/\hat{W})c(\hat{N}^i)\}_m.$$

□

Proposition 3.8. *Gromov-Witten invariants are equal to quasi-map invariants for toric Fano varieties.*

Proof. We have that

$$[\overline{M}_{0,n}(X,\beta)]^{\mathrm{virt}} = \{s(M/[W]^{\mathrm{virt}})c(N^i)\}_m \cap \overline{M}_{0,n}(X,\beta).$$

Denote by W^i the components of W and by M^i the stack $W^i \cap M$. (The point here is that any M^i different from $\overline{M}^{rel}_{0,n}(X,\beta)$ is not included in W^{rel}!) By Lemma 4.2 in [8] we have

$$s(M/[W]^{\mathrm{virt}}) = \sum_i m_i s(M^i/[W^i]^{\mathrm{virt}}) \tag{18}$$

where m_i is the multiplicity of $[W^i]^{\mathrm{virt}}i$ in W. This shows that

$$s(M/W) = s(M^{rel}/W^{rel}) + \sum_i m_i s(M^{irrel}/[W^{irrel}]^{\mathrm{virt}}) \tag{19}$$

and by this we get

$$[M]^{\mathrm{virt}} = \left\{ \left(s(M^{rel}/W^{rel}) + \sum_i m_i s(M^{irrel}/[W^{irrel})]^{\mathrm{virt}} \right) c(N^i) \right\}_m \tag{20}$$

By Proposition 3.7 we have that

$$c_*(\{s(M^{rel}/W^{rel}c(N^i)\} \cap \overline{M}_{0,n}(X,\beta)) = [\overline{Q}_{0,n}(X,\beta)]^{\mathrm{virt}} \tag{21}$$

By abuse of notation we denote by q the restriction of q to $\overline{M}_{0,n}(X,\beta)$. Unlike $q : V \to \hat{V}$ the restriction to $\overline{M}_{0,n}(X,\beta)$ is proper. With this convention let us now prove that

$$q_*\left(\{s(M^{irrel}/W^{irrel})c(N^i)\}_m \cap \overline{M}_{0,n}(X,\beta)\right) = 0 \tag{22}$$

Intersection with chern classes commutes with flat pull-backs and in particular with restricting to open subsets. This shows that

$$\{(s(M^{irrel}/[W^{irrel}]^{\mathrm{virt}}) \cap \overline{M}^{irrel}_{0,n}(X,\beta))c(N^i)\}_m = \tag{23}$$

$$q_*\left(\{s(M^{irrel}/[W^{irrel}]^{\mathrm{virt}}) \cap \overline{M}_{0,n}(X,\beta)\}_{m+j} c_j(N^i)\right). \tag{24}$$

We now use projection formula and the fact that $N^i = q^*\hat{N}^i$, which means that we are left to prove

$$q_*\left(\{s(M^{irrel}/[W^{irrel}]^{\mathrm{virt}})\}_{m+j} \cap \overline{M}_{0,n}(X,\beta)\right) = 0 \tag{25}$$

for all $j \geq 0$. This follows from lemma 3.5 and the argument in the above proposition. By 20, 21, 22 and the projection formula we get

$$\prod ev_i^*\gamma_i\psi_i^{k_i} \cdot [\overline{M}_{0,n}(X,\beta)]^{\mathrm{virt}} = \prod ev_i^*\gamma_i\psi_i^{k_i} \cdot [\overline{Q}_{0,n}(X,\beta)]^{\mathrm{virt}} \tag{26}$$

□

4. Further discussion

Obstruction theories of components.

Lemma 4.1. *Let $\bar{M}^{\geq\tau_{\alpha_0}} = \cup_{\tau_{\alpha_i}\geq\tau_{\alpha_0}}\overline{M}^{\tau_{\alpha_i}}$ be the union of components of $\overline{Q}_{0,n}(X,\beta)$ greater than a fixed τ_{α_0}. Similarly, let $\overline{M}^{\leq\tau_{\alpha_0}} = \cup_{\tau_{\alpha_i}\leq\tau_{\alpha_0}}\overline{M}^{\tau_{\alpha_i}}$ be the union of components of $\overline{Q}_{0,n}(X,\beta)$ lower than a fixed τ_{α_0}. Then we have that*

(1) *the obstruction theory in any point of $\bar{M}^{\geq\tau_{\alpha_0}}$ is equal to $\mathcal{R}^\bullet\pi_*f^*T_X$.*
(2) *the obstruction theory in any point of $\bar{M}^{\leq\tau_{\alpha_0}}$ which is not in $\overline{M}^{\geq\tau_{\alpha_0}} \cap \overline{M}^{\leq\tau_{\alpha_0}}$ is equal to $\mathcal{R}^\bullet\pi_*f^*T_X$.*

Proof. Clear. □

Definition 4.2. Let $\overline{M}^{rel}_{0,n}(X,\beta)$ be the the union of components of $\overline{M}_{0,n}(X,\beta)$ such that all L_ρ are nonnegative on rational tails. Let $\overline{M}^{irrel}_{0,n}(X,\beta)$ be the closure of the complement of $\overline{M}^{rel}_{0,n}(X,\beta)$ in $\overline{M}_{0,n}(X,\beta)$.

Corollary 4.3. *The stack $\overline{M}^{irrel}_{0,n}(X,\beta)$ has a perfect obstruction theory*

Proposition 4.4. *GW invariants of $\overline{M}^{irrel}_{0,n}(X,\beta)$ are zero.*

Proof. We have that

$$\overline{M}^{irrel}_{0,n}(X,\beta) = \bigcup_{\beta_1+\beta_2=\beta} \overline{M}_{0,n+1}(X,\beta_1)\times_X \overline{M}_{0,1}(X,\beta_2)$$

and for at least one ρ we have that $c_1(L_\rho)\cdot\beta_2 < 0$. Let $p_1 : \overline{M}^{irrel}_{0,n}(X,\beta) \to \bigcup_{\beta_1}\overline{M}_{0,n}(X,\beta_1)$ be the projection on the first factor. If X is Fano of index at least two we have that the virtual dimension of $\overline{M}_{0,n}(X,\beta_1)$ is strictly lower than the virtual dimension of $\overline{M}_{0,n}(X,\beta)$. Let us show that $p_{1*}[\overline{M}_{0,n}(X,\beta)]^{\mathrm{virt}} = 0$. The proof ifollows by the virtual push forward theorem. Consider the following commutative diagram

$$\begin{tikzcd} \overline{M}_{0,n+1}(X,\beta_1)\times_X \overline{M}_{0,1}(X,\beta_2) \arrow[r, "p_1"] \arrow[d, "\epsilon"'] & \overline{M}_{0,n}(X,\beta_1) \arrow[d] \\ \mathfrak{M}_{g,n} \arrow[r, "p"] & \mathfrak{M}_{g,n} \end{tikzcd} \tag{27}$$

where p is the contraction of rational tails. We have a normalization sequence

$$0 \to f^*T_X \to f_1^*T_X \oplus f_2^*T_X \to \mathbb{C}^k \to 0$$

Taking cohomology we see that we have a surjective morphism

$$H^1(C, f^*T_X) \to H^1(C_1, f_1^*T_X) \oplus H^1(C_2, f_2^*T_X) \to 0 \tag{28}$$

Composing with the projection on the first factor we get a surjective morphism $H^1(C, f^*T_X) \to H^1(C_1, f_1^*T_X)$ and with this we are under the hypothesis of the virtual push-forward theorem. The Fano condition implies that the rank of the vector bundle with fiber $H^1(C, f^*T_X)$ is strictly larger than the rank of the vector bundle with fiber $H^1(C_1, f_1^*T_X)$. This shows that $p_{1*}[\overline{M}^{irrel}_{0,n}(X,\beta)]^{\mathrm{virt}} = 0$. By projection formula we obtain that

$$\begin{aligned} p_{1*}ev_i^*\gamma \cdot [\overline{M}^{irrel}_{0,n}(X,\beta)]^{\mathrm{virt}} &= ev_i^*\gamma \cdot p_{1*}[\overline{M}_{0,n}(X,\beta)]^{\mathrm{virt}} \\ &= 0 \end{aligned}$$

□

Proposition 4.5. *Let $M^{rel}(X,\beta)$ be the subset of $\overline{M}^{rel}_{0,n}(X,\beta)$ such that for all L_ρ which have (overall) negative degree $(-a)$ on the rational tails the sections u_ρ vanish on all irreducible components of the rational tails with multiplicity $-a$. Then the obstruction theory of $M^{rel}(X,\beta)$ relative to $\mathfrak{Pic}^r_{g,n}$ is $\mathcal{R}^\bullet(\hat{L}_\rho)$.*

Proof. Construct a (compact) moduli space M' which parametrizes $(C \to \hat{C}, L_\rho, \hat{L}, u'_\rho)$. The natural morphism $\overline{M}^{rel}_{0,n}(X,\beta) \to M'$ is an isomorphism around a point as in our hypothesis. □

Proposition 4.6. $q_*[M^{rel}_{0,n}(X,\beta)]^{\mathrm{virt}} = [Q_{0,n}(X,\beta)]^{\mathrm{virt}}$

Proof. By the push forward theorem. □

Remark 4.7. Propositions 4.4 and 4.6 are not enough to conclude that $q_*[\overline{M}^{rel}_{0,n}(X,\beta)]^{\mathrm{virt}} = [\overline{Q}_{0,n}(X,\beta)]^{\mathrm{virt}}$.

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