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Stabilization of persistently excited linear systems

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Notation. In this chapter, $\mathcal{M}_{d,m}(\mathbb{R})$ denotes the set of $d \times m$ matrices with real coefficients. When $m = d$, this set is denoted simply by $\mathcal{M}_d(\mathbb{R})$. As usual, we identify column matrices in $\mathcal{M}_{d,1}(\mathbb{R})$ with vectors in \mathbb{R}^d . The identity matrix in $\mathcal{M}_d(\mathbb{R})$ is denoted by Id_d and $0_{d \times m} \in \mathcal{M}_{d,m}(\mathbb{R})$ denotes the matrix whose entries are all zero, the dimensions d and $d \times m$ being possibly omitted if they are implicit. The notation $\|x\|$ indicates the Euclidean norm of a vector $x \in \mathbb{R}^d$ or the norm on a Hilbert space \mathbb{H} ; we sometimes write $\|x\|_{\mathbb{H}}$ in this latter case. Associated operator norms are also denoted by $\|\cdot\|$. The symbol $|a|$ is reserved for the absolute value of a real or complex number a . The real and imaginary parts of a complex number z are denoted by $\Re(z)$ and $\Im(z)$ respectively, and, when the argument of \Re or \Im is a set, we understand it as the set of real or imaginary parts of the elements of the original set. The Lebesgue measure in the real line is denoted by m .

1 Introduction

Consider the linear control system

$$\dot{x} = Ax + \alpha(t)Bu, \quad x \in \mathbb{R}^d, u \in \mathbb{R}^m, \alpha \in \mathcal{G} \quad (1.1)$$

where x is the state variable, u is a control input, A and B are matrices of appropriate dimensions and α is a scalar measurable signal belonging to a certain class $\mathcal{G} \subset L^\infty(\mathbb{R}_+, [0, 1])$. This is a modification of the linear time-invariant control system

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^d, u \in \mathbb{R}^m, \quad (1.2)$$

where the signal α determines when the input u is active or not. In the case where α takes its values on $\{0, 1\}$, (1.1) switches between the uncontrolled system $\dot{x} = Ax$ and the controlled one $\dot{x} = Ax + Bu$.

We wish to stabilize System (1.1) by means of a linear state feedback $u = -Kx$, for a certain class \mathcal{G} of functions α , that is, we wish to find $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that, for every $\alpha \in \mathcal{G}$, the system

$$\dot{x} = (A - \alpha(t)BK)x$$

is asymptotically stable (and, possibly, uniformly with respect to $\alpha \in \mathcal{G}$). As in the stabilization problem for System (1.2), we may wish to stabilize systems whose uncontrolled dynamics $\dot{x} = Ax$ are unstable, and one must thus impose on α , by an appropriate choice of class \mathcal{G} , conditions guaranteeing that the state feedback will have a *sufficient amount of action* on the system. A condition normally used for this purpose (as in [9, 12, 13, 18, 21, 25]), which arises naturally in adaptive control problems, is that of *persistent excitation* (PE): given constants $T \geq \mu > 0$, $\alpha \in L^\infty(\mathbb{R}_+, [0, 1])$ is said to be a PE signal (with constants T, μ) if, for every $t \in \mathbb{R}_+$,

$$\int_t^{t+T} \alpha(s)ds \geq \mu.$$

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The interest of System (1.1) is not purely theoretical, as such a signal α may model different phenomena, such as failure in the transmission from the controller to the plant, leading to instants of time at which the control is switched off; time-varying parameters affecting the control efficiency, leading to the effective application of a rescaled control $\alpha(t)u(t)$; allocation of control resources, activating the control only up to a certain fraction of its designed value, or only on certain time intervals; among other possible phenomena. This kind of system is also related to problems stemming from identification and adaptive control (see, e.g., [1–3, 8]). In such type of problems, one is lead to consider the stability of linear systems of the kind $\dot{x} = -P(t)x$, $x \in \mathbb{R}^d$, where the matrix $P(\cdot)$ is symmetric non-negative definite. If P is also bounded and has bounded derivative, a *necessary and sufficient* condition for the global exponential stability of $\dot{x} = -P(t)x$, given in [25], is that P is also persistently exciting, in the sense that there exist $T \geq \mu > 0$ such that

$$\int_t^{t+T} \xi^T P(s) \xi ds \geq \mu,$$

for all unitary vectors $\xi \in \mathbb{R}^d$ and all $t \geq 0$.

Still in the context of identification and adaptive control, the condition of persistence of excitation is useful when analyzing the convergence of certain identification methods for linear systems, where the identification error satisfies an equation of the form $\dot{x}(t) = -u(t)u^T(t)x(t)$ [1, 3, 8, 27]. In this case, it can be shown that, under some regularity hypothesis on u , exponential stability of this system is equivalent to finding positive constants μ_1 , μ_2 , and T such that

$$\mu_1 \text{Id} \leq \int_t^{t+T} u(s)u^T(s)ds \leq \mu_2 \text{Id}.$$

A question of practical importance in this case is to estimate the rate of exponential convergence to zero (see [1, 8, 27]) and to compare the different estimates (see [3]).

Nonlinear generalizations of (1.1) also appear in practical situations, such as the control of spacecrafts with magnetic actuators [21, 22], where the control system is

$$\dot{\omega} = S(\omega)\omega + g(t)u$$

with $\omega \in \mathbb{R}^3$ the state variable, u the input, $S(\omega) \in \mathcal{M}_3(\mathbb{R})$ a matrix depending on ω and $g(t)$ a time-varying matrix with $\text{rank}(g(t)) < 3$ for all time t and satisfying a persistent excitation condition. Further examples of systems similar to (1.1) where the persistent excitation condition appears are given in [21].

Let us briefly recall the main results concerning stabilization of linear control systems of the form (1.2), which are presented in most classical control textbooks, such as [7, 26]. The linear control system (1.2) is *stabilizable at the origin* by means of a linear state feedback $u = -Kx$ if

$$\dot{x} = (A - BK)x$$

is asymptotically stable, which is the case if and only if the matrix $A - BK$ is Hurwitz. Such a stabilizing feedback exists if and only if there exists a system of coordinates in which

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

A_3 is Hurwitz and (A_1, B_1) is controllable. System (1.2) is stabilizable with an *arbitrary rate of convergence* if, for every $\lambda > 0$, there exists a state feedback $u = -Kx$ and $C > 0$ such that every solution $x(t)$ of

$$\dot{x} = (A - BK)x$$

satisfies

$$\|x(t)\| \leq Ce^{-\lambda t} \|x(0)\|.$$

This is well-known to be true if and only if (A, B) is controllable.

The goal of this chapter is to present recent developments on the stabilization of persistently excited linear systems. Section 2 deals with finite-dimensional systems and gives two main results on stabilization, concerning neutrally stable systems and systems whose eigenvalues have all non-positive real part. We also present a result stating the existence of persistently excited systems for which the pair (A, b) is controllable but that cannot be stabilized by means of a linear state feedback. The question of whether stabilization at an arbitrary rate may take place is also discussed, showing that this problem actually depends on the ratio μ/T . In Section 3, we present some results for infinite-dimensional systems, generalizing some results from Section 2 to the case of systems defined by a linear operator A which generates a strongly continuous contraction semigroup, with applications to Schrödinger's equation and the wave equation. Section 4 finally discusses some problems that remain open, giving some preliminary results in certain cases.

2 Finite-dimensional systems

We shall consider hereafter the linear control system

$$\dot{x} = Ax + \alpha(t)Bu, \quad x \in \mathbb{R}^d, u \in \mathbb{R}^m, \alpha \in \mathcal{G}(T, \mu) \quad (2.1)$$

where $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$, and $\mathcal{G}(T, \mu) \subset L^\infty(\mathbb{R}_+, [0, 1])$ is the class of (T, μ) -signals defined below.

Definition 2.1. Let T, μ be two positive constants with $T \geq \mu$. We say that a measurable function $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$ is a (T, μ) -signal if, for every $t \in \mathbb{R}_+$, one has

$$\int_t^{t+T} \alpha(s)ds \geq \mu. \quad (2.2)$$

The set of (T, μ) -signals is denoted by $\mathcal{G}(T, \mu)$. We say that a measurable function $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$ is a *persistently exciting signal* (or simply *PE signal*) if it is a (T, μ) -signal for certain positive constants T and μ with $T \geq \mu$.

Notice that, for any (T, μ) -signal α , existence and uniqueness of the solutions of (2.1) are guaranteed by Carathéodory's Theorem (see, for instance, [17]). System (2.1) with $\alpha \in \mathcal{G}(T, \mu)$ is called a *persistently excited system* (*PE system* for short). The main problem we are interested in is the question of *uniform* stabilization of System (2.1) by a linear state feedback of the form $u = -Kx$ with $K \in \mathcal{M}_{m,d}(\mathbb{R})$, which makes System (2.1) take the form

$$\dot{x} = (A - \alpha(t)BK)x. \quad (2.3)$$

The problem is thus the choice of K such that the origin of the linear system (2.3) is globally uniformly asymptotically stable. With this in mind, we can introduce the following notion of stabilizer.

Definition 2.2. Let T and μ be positive constants with $T \geq \mu$. We say that $K \in \mathcal{M}_{m,d}(\mathbb{R})$ is a (T, μ) -stabilizer for System (2.1) if System (2.3) is globally exponentially stable, uniformly with respect to $\alpha \in \mathcal{G}(T, \mu)$.

Remark 2.3. Thanks to Fenichel's Uniformity Lemma (see for instance [14, Lemma 5.2.7]), the above definition can be restated equivalently in the following weaker form: $K \in \mathcal{M}_{m,d}(\mathbb{R})$ is a (T, μ) -stabilizer for System (2.1) if, for every $\alpha \in \mathcal{G}(T, \mu)$, System (2.3) is globally asymptotically stable.

We note that a stabilizer K may depend on the parameters of the system, that is, on A, B, T , and μ , but we do not let K depend on the signal $\alpha \in \mathcal{G}(T, \mu)$. We can now turn to the study of stabilization of PE systems.

2.1 The neutrally stable case

We consider System (2.1) with A neutrally stable, that is, every eigenvalue of A has non-positive real part, and those with real part zero have trivial Jordan blocks. This case has been treated in [2, 9], and [9] presents the following stabilization result.

Theorem 2.4. *Assume that the pair (A, B) is stabilizable and that the matrix A is neutrally stable. Then there exists a matrix $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that, for every $T \geq \mu > 0$, K is a (T, μ) -stabilizer for (2.1).*

Note that, in this case, K does not depend on T or μ .

The first step of the proof is the reduction to the case where (A, B) is controllable and A is skew symmetric. Indeed, since the non-controllable part of the linear system $\dot{x} = Ax + Bu$ is already stable, it is sufficient to consider only the controllable part of (A, B) , and we may thus suppose this pair controllable. Up to a linear change of variables, A and B can be written as

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_3 \end{pmatrix},$$

where A_1 is Hurwitz and all the eigenvalues of A_3 have zero real part; this fact, together with the neutral stability of A_3 , shows that the latter is similar to a skew-symmetric matrix, and we can thus suppose, up to another change of variables, that A_3 is skew-symmetric. The controllability assumption also shows that (A_3, B_3) is controllable, and, if Theorem 2.4 is proved in this case, giving a certain K_3 which is a (T, μ) -stabilizer for (A_3, B_3) for every $T \geq \mu > 0$, then

$$K = \begin{pmatrix} 0 \\ K_3 \end{pmatrix}$$

is a (T, μ) -stabilizer for (A, B) , for every $T \geq \mu > 0$, which proves the desired reduction. Now Theorem 2.4 follows from the following.

Proposition 2.5. *Suppose that the pair (A, B) is controllable and that the matrix A is skew-symmetric. Then $K = B^T \in \mathcal{M}_{m,d}(\mathbb{R})$ is a (T, μ) -stabilizer for (2.1), for every $T \geq \mu > 0$.*

The choice of K in Proposition 2.5 leads to the system

$$\dot{x} = (A - \alpha(t)BB^T)x,$$

for which one may prove that $V(x) = \|x\|^2$ is a Lyapunov function. This last step may be done by computing $\dot{V} = -2\alpha(t)\|B^T x\|^2$ and using a Lasalle-type argument to conclude; for the details of the proof, we refer to [9].

2.2 Spectra with non-positive real part

Theorem 2.4 deals only with control systems whose uncontrolled dynamics $\dot{x} = Ax$ are stable (even though possibly not asymptotically). It is also interesting to consider the stabilizability of systems whose uncontrolled dynamics are not necessarily stable. This has been studied in [13] for the case of a single scalar input $u \in \mathbb{R}$,

$$\dot{x} = Ax + \alpha(t)bu, \quad x \in \mathbb{R}^d, u \in \mathbb{R}, \alpha \in \mathcal{G}(T, \mu), \quad (2.4)$$

where the following result was proved.

Theorem 2.6. *Let (A, b) be a controllable pair and assume that the eigenvalues of A have non-positive real part. Then, for every T, μ with $T \geq \mu > 0$, there exists a (T, μ) -stabilizer for (2.4).*

Note that the uncontrolled system $\dot{x} = Ax$ may have trajectories $x(t)$ such that $\|x(t)\| \xrightarrow[t \rightarrow +\infty]{} +\infty$. Differently from the case of Theorem 2.4, the choice of K now depends on T and μ .

The proof of Theorem 2.6 relies on a compactness argument and a time-contraction procedure, transforming the integral PE constraint (2.2) in a pointwise one. The limit system obtained with the time-contraction procedure can be shown to be stable via a Lyapunov function, and an approximation theorem makes it possible to conclude the stability of a time-contracted system from the stability of the limit system.

Let us detail more precisely the strategy of the proof of Theorem 2.6 proposed in [13]. First, the theorem is proved for the case of the d -integrator, that is, we consider $A = J_d$ with

$$J_d = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

the d -dimensional Jordan block, and we take $b = (0 \ \dots \ 0 \ 1)^T$, so that (J_d, b) is controllable. Then System (2.4) under the feedback law $u = -Kx$ with $K = (k_1 \ \dots \ k_d)$ can be written as

$$\begin{cases} \dot{x}_j = x_{j+1}, & j = 1, \dots, d-1, \\ \dot{x}_d = -\alpha(t)(k_1 x_1 + \dots + k_d x_d) \end{cases} \quad (2.5)$$

In this case, the following lemma holds.

Lemma 2.7. *Let $v > 0$. Then K is a (T, μ) -stabilizer for (2.5) if and only if $(v^d k_1 \ \dots \ v k_d)$ is a $(T/v, \mu/v)$ -stabilizer for (2.5).*

Proof. For $v > 0$, define $D_{d,v} = \text{diag}(v^{d-1}, \dots, v, 1)$. It is easy to verify the relations $vD_{d,v}^{-1}J_d D_{d,v} = J_d$ and $D_{d,v}b = b$, and so a direct computation shows that $x_v(t) = D_{d,v}^{-1}x(vt)$ satisfies

$$\dot{x}_v = J_d x_v - \alpha_v v b K D_{d,v} x_v,$$

where $\alpha_v(t) = \alpha(vt)$. This is the same system as (2.5), but with a switching signal $\alpha_v \in \mathcal{G}(T/v, \mu/v)$ and subject to a linear state feedback given by $vKD_{d,v} = (v^d k_1 \ \dots \ v k_d)$, from where we get the desired result. \blacksquare

Based on the lemma above the strategy is now as follows: instead of looking for a (T, μ) -stabilizer for (2.5), we look for a $(T/v, \mu/v)$ -stabilizer, for $v > 0$ large enough. It was established in [13, Lemma 2.5] that, if $(v_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers with $v_n \rightarrow +\infty$ as $n \rightarrow +\infty$ and $\alpha_n \in \mathcal{G}(T/v_n, \mu/v_n)$ converges weakly- \star in $L^\infty(\mathbb{R}_+, [0, 1])$ to a certain α_\star , then $\alpha_\star(t) \geq \mu/T$ for almost every $t \in \mathbb{R}_+$. We are thus led to consider the system

$$\begin{cases} \dot{x}_j = x_{j+1}, & j = 1, \dots, d-1, \\ \dot{x}_d = -\alpha_\star(t)(k_1 x_1 + \dots + k_d x_d), \end{cases} \quad \alpha_\star \in L^\infty(\mathbb{R}_+, [\mu/T, 1]), \quad (2.6)$$

which is a “limit system” of (2.5) in the following sense [9, Proposition 21]: if $(v_n)_{n \in \mathbb{N}}$ is a sequence of positive real numbers with $v_n \rightarrow +\infty$ as $n \rightarrow +\infty$, $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of functions with $\alpha_n \in \mathcal{G}(T/v_n, \mu/v_n)$ and $\alpha_n \rightharpoonup \alpha_*$ weakly- \star in $L^\infty(\mathbb{R}_+, [0, 1])$, and $(x_{0,n})_{n \in \mathbb{N}}$ is a sequence of unitary vectors in \mathbb{R}^d converging to $x_{0,*}$, then, noting by x_n the solution of (2.5) with initial condition $x_{0,n}$ and subject to the switching signal α_n and x_* the solution of (2.6) with initial condition $x_{0,*}$ and subject to the switching signal α_* , we have that $x_n(t) \rightarrow x_*(t)$ as $n \rightarrow +\infty$, uniformly on compact time intervals.

The stabilizability of (2.6) by a certain feedback K can be established through a common quadratic Lyapunov function, obtained by means of a uniform observability result from [16, Lemma 6.2.1]. The convergence mentioned above allows us to choose the same K for (2.5) with $\alpha \in \mathcal{G}(T/v, \mu/v)$ for a certain $v > 0$ large enough, which proves the desired result for J_d .

The general case can be reduced to the case where all the eigenvalues of A have real part zero. In this case, a change of coordinates in System (2.4) allows us to write A in its Jordan canonical form. We note, in terms of the Kronecker product,

$$J_n^C = J_n \otimes \text{Id}_2, \quad A^{(n)} = \text{Id}_n \otimes A_0, \quad A_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and then System (2.4) becomes

$$\begin{cases} \dot{x}_0 = J_{r_0} x_0 + \alpha b^0 u, \\ \dot{x}_j = (\omega_j A^{(r_j)} + J_{r_j}^C) x_j + \alpha b^j u, \quad j = 1, \dots, h, \end{cases} \quad (2.7)$$

where $\{\pm i\omega_j \mid j = 1, \dots, h\} = \sigma(A) \setminus \{0\}$, all the ω_j are positive and pairwise distinct, r_j is the multiplicity of the eigenvalue $i\omega_j$, $r_0 = 0$ if 0 is not an eigenvalue of A , otherwise r_0 is the multiplicity of 0 as an eigenvalue of A , and $x_0 \in \mathbb{R}^{r_0}$, $x_j \in \mathbb{R}^{2r_j}$, $j = 1, \dots, h$. Here, b^0 and b^j are the vectors of \mathbb{R}^{r_0} and \mathbb{R}^{2r_j} , respectively, with all the components zero except for the last one, which is equal one. Now, the idea follows the case of the d -integrator. We define the feedback law $u = -Kx$ with $K = (K_0 \ K_1 \ \dots \ K_h)$, $K_0 \in \mathcal{M}_{1,r_0}(\mathbb{R})$, $K_j \in \mathcal{M}_{1,2r_j}(\mathbb{R})$, $j = 1, \dots, h$, and we make the change of time-space variables given by

$$\begin{aligned} y_0(t) &= D_{r_0,v}^{-1} x_0(vt), \\ y_j(t) &= (D_{r_j,v}^C)^{-1} e^{-vt\omega_j A^{(r_j)}} x_j(vt), \quad j = 1, \dots, h, \end{aligned}$$

with $D_{r_j,v}^C = D_{r_j,v} \otimes \text{Id}_2$. The system satisfied by the new variables is

$$\begin{cases} \dot{y}_0 = J_{r_0} y_0 - \alpha_v b^0 \left[K_{0,v} y_0 + \sum_{\ell=1}^h K_{\ell,v} e^{vt\omega_\ell A^{(r_\ell)}} y_\ell \right], \\ \dot{y}_j = J_{r_j}^C y_j - \alpha_v e^{-vt\omega_j A^{(r_j)}} b^j \left[K_{0,v} y_0 + \sum_{\ell=1}^h K_{\ell,v} e^{vt\omega_\ell A^{(r_\ell)}} y_\ell \right], \quad j = 1, \dots, h, \end{cases} \quad (2.8)$$

with $\alpha_v(t) = \alpha(vt)$, $K_{0,v} = vK_0 D_{r_0,v}$, $K_{\ell,v} = vK_\ell D_{r_\ell,v}^C$ for $\ell = 1, \dots, h$. As in the case of the d -integrator, $K = (K_0 \ K_1 \ \dots \ K_h)$ is a (T, μ) -stabilizer for (2.7) if and only if $K_v = (K_{0,v} \ K_{1,v} \ \dots \ K_{h,v})$ is a $(T/v, \mu/v)$ -stabilizer for (2.8), and so it suffices to exhibit a $(T/v, \mu/v)$ -stabilizer for (2.8) for a certain $v > 0$.

We look for a $(T/v, \mu/v)$ -stabilizer of (2.8) under the form $K_v = (K_{0,v} \ K_{1,v} \ \dots \ K_{h,v})$ with

$$K_{j,v} = \mathcal{K}_j \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & k_1^j & 0 & k_2^j & \dots & 0 & k_{r_j}^j \end{pmatrix}, \quad \mathcal{K}_j = \begin{pmatrix} k_1^j & \dots & k_{r_j}^j \end{pmatrix} \in \mathcal{M}_{1,r_j}(\mathbb{R})$$

for $j = 1, \dots, h$ and $K_{0,v} = \mathcal{K}_0$. We write $b_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$, so that $K_{j,v} = \mathcal{K}_j \otimes b_0^T$. We have that $K_{j,v} e^{vt\omega_j A^{(r_j)}} = \mathcal{K}_j \otimes b_0^T e^{vt\omega_j A_0}$. Noting $\tilde{b}^j \in \mathbb{R}^{r_j}$ the vector with all coordinates equal to zero except the last one that is equal

to one, we have $b^j = \tilde{b}^j \otimes b_0$, and thus $e^{-vt\omega_j A^{(r_j)}} b^j = \tilde{b}^j \otimes e^{-vt\omega_j A_0} b_0$. We finally write, for $j, \ell \in \{1, \dots, h\}$,

$$\begin{aligned} C_{00}^{(v)}(t) &= \alpha_v(t), & C_{0j}^{(v)}(t) &= \alpha_v(t) b_0^T e^{vt\omega_j A_0}, \\ C_{j0}^{(v)}(t) &= \alpha_v(t) e^{-vt\omega_j A_0} b_0, & C_{j\ell}^{(v)}(t) &= \alpha_v(t) e^{-vt\omega_j A_0} b_0 b_0^T e^{vt\omega_j A_0}, \end{aligned}$$

and thus System (2.8) can be written as

$$\begin{cases} \dot{y}_0 = J_{r_0} y_0 - \left[(b^0 \mathcal{K}_0 \otimes C_{00}^{(v)}) y_0 + \sum_{\ell=1}^h (b^0 \mathcal{K}_\ell \otimes C_{0\ell}^{(v)}) y_\ell \right], \\ \dot{y}_j = J_{r_j}^c y_j - \left[(\tilde{b}^j \mathcal{K}_0 \otimes C_{j0}^{(v)}) y_0 + \sum_{\ell=1}^h (\tilde{b}^j \mathcal{K}_\ell \otimes C_{j\ell}^{(v)}) y_\ell \right], \quad j = 1, \dots, h. \end{cases}$$

We can arrange the $C_{j\ell}^{(v)}$ in a matrix as

$$C^{(v)}(t) = \left(C_{j\ell}^{(v)}(t) \right)_{0 \leq j, \ell \leq h}.$$

We are now in a situation similar to the case of the d -integrator, but where the scalar switching signal α is replaced by the matrix $C^{(v)}$. As before, we can also define a limit system for this case, which is stabilizable by a similar argument, and, the convergence result used before being still valid in this context, we conclude in the same manner than for the d -integrator that System (2.8) admits a $(T/v, \mu/v)$ -stabilizer for a certain $v > 0$ large enough.

The proof of Theorem 2.6 depends deeply on the fact that all the eigenvalues of A have non-positive real part. Actually, it is not true that any controllable system of the form (2.4) admits a (T, μ) -stabilizer when the ratio μ/T is small, as it was shown in [13] for the case of dimension $d = 2$.

Theorem 2.8. *There exists $\rho_* \in (0, 1)$ such that, for every controllable pair $(A, b) \in \mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2$, every $T > 0$ and every $\rho \in (0, \rho_*)$, if $\lambda > 0$ is large enough, then $(A + \lambda Id_2, b)$ does not admit a $(T, \rho T)$ -stabilizer.*

Theorem 2.6 has been proved only for the single-input case of System (2.4), but the general multi-input case of System (2.1) can be retrieved from Theorem 2.6 by induction on the number of inputs m .

Theorem 2.9. *Let $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ such that (A, B) is a controllable pair and assume that the eigenvalues of A have non-positive real part. Then, for every T, μ with $T \geq \mu > 0$, there exists a (T, μ) -stabilizer for (2.1).*

Proof. We prove our result by induction on m . Theorem 2.6 proves the case $m = 1$. Now, suppose the theorem has been proved for $m - 1$, that is, for every $d \in \mathbb{N}^*$, for every $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m-1}(\mathbb{R})$ such that (A, B) is a controllable pair and the eigenvalues of A have non-positive real part, and for every T, μ with $T \geq \mu > 0$, there exists a (T, μ) -stabilizer for (2.1).

Take $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ such that (A, B) is a controllable pair and the eigenvalues of A have non-positive real part and fix $T \geq \mu > 0$. Note by $b \in \mathbb{R}^d$ the first column of B ; we may suppose, without loss of generality, that $b \neq 0$, for otherwise the first input does not influence the system and it may thus be excluded, reducing the system to the case with $m - 1$ inputs. We consider the pair (A, b) , which may not be controllable, but can be decomposed according to Kalman decomposition: there exists an invertible $P \in \mathcal{M}_d(\mathbb{R})$ such that

$$PAP^{-1} = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix}, \quad Pb = \begin{pmatrix} b_1 \\ 0 \end{pmatrix},$$

with $A_1 \in \mathcal{M}_{d_1}(\mathbb{R})$, $b_1 \in \mathbb{R}^{d_1}$, all the other matrices have appropriate dimensions, and (A_1, b_1) is controllable (see, for instance, [26, Theorem 13.1]). Now, up to the change of variables $z = Px$, System (2.1) can be written as

$$\dot{z} = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix} z + \alpha(t) \begin{pmatrix} b_1 & B_{12} \\ 0 & B_2 \end{pmatrix} u. \quad (2.9)$$

By the controllability of (A, B) and (A_1, b_1) , it follows that (A_2, B_2) is also controllable. Now $B_2 \in \mathcal{M}_{d-d_1, m-1}(\mathbb{R})$, and so, by the induction hypothesis, (A_2, B_2) admits a (T, μ) -stabilizer $K_2 \in \mathcal{M}_{m-1, d-d_1}(\mathbb{R})$. Theorem 2.6 gives a (T, μ) -stabilizer $K_1 \in \mathcal{M}_{1, d_1}(\mathbb{R})$ for (A_1, b_1) . We affirm that $K \in \mathcal{M}_{m, d}(\mathbb{R})$ given by

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

is a (T, μ) -stabilizer for (A, B) . Indeed, with this feedback, System (2.9) becomes

$$\dot{z} = \begin{pmatrix} A_1 - \alpha(t)b_1K_1 & A_{12} - \alpha(t)B_{12}K_2 \\ 0 & A_2 - \alpha(t)B_2K_2 \end{pmatrix} z.$$

Noting $z = \begin{pmatrix} z_1 & z_2 \end{pmatrix}^T$ with $z_1 \in \mathbb{R}^{d_1}$ and $z_2 \in \mathbb{R}^{d-d_1}$, we can thus write

$$\begin{cases} \dot{z}_1 = (A_1 - \alpha(t)b_1K_1)z_1 + (A_{12} - \alpha(t)B_{12}K_2)z_2, \\ \dot{z}_2 = (A_2 - \alpha(t)B_2K_2)z_2. \end{cases} \quad (2.10)$$

Denote by $\Phi_1(t, s)$ and $\Phi_2(t, s)$ the flows associated respectively with $A_1 - \alpha(t)b_1K_1$ and $A_2 - \alpha(t)B_2K_2$; by construction of K_1 and K_2 we can find $C > 0$ and $\gamma > 0$, both independent of $\alpha \in \mathcal{G}(T, \mu)$, such that

$$\|\Phi_j(t, s)\| \leq Ce^{-\gamma(t-s)}, \quad \text{for } j = 1, 2 \text{ and for all } t \geq s \geq 0.$$

We can write the solution of (2.10) in terms of the initial condition $\begin{pmatrix} z_{0,1} & z_{0,2} \end{pmatrix}^T$ using the variation-of-constants formula as

$$\begin{cases} z_1(t) = \Phi_1(t, 0)z_{0,1} + \int_0^t \Phi_1(t, s)(A_{12} - \alpha(s)B_{12}K_2)z_2(s)ds, \\ z_2(t) = \Phi_2(t, 0)z_{0,2}. \end{cases}$$

It is thus easy to see that

$$\begin{cases} \|z_1(t)\| \leq Ce^{-\gamma t} \|z_{0,1}\| + C'te^{-\gamma t} \|z_{0,2}\|, \\ \|z_2(t)\| \leq Ce^{-\gamma t} \|z_{0,2}\|, \end{cases}$$

with $C' = C^2(\|A_{12}\| + \|B_{12}K_2\|)$, and so K is a (T, μ) -stabilizer for (2.9), as we wanted to prove. The theorem is thus established by induction. \blacksquare

2.3 Arbitrary rate of convergence

For a control system $\dot{x} = Ax + Bu$ with (A, B) controllable, it is always possible to find a state feedback $u = -Kx$ such that the eigenvalues of the matrix $A - BK$ corresponding to the closed-loop system $\dot{x} = (A - BK)x$ are given by certain prescribed values $\lambda_1, \dots, \lambda_d$. This allows us to choose K such that $\dot{x} = (A - BK)x$ is exponentially stable with a certain prescribed exponential decay rate λ . In [13], the generalization of this property to System (2.4) is studied, and it is shown that the problem of stabilizing (2.4) with an arbitrary rate of exponential convergence gives rise to a bifurcation phenomenon depending on the ratio μ/T . This problem is formulated in terms of the maximal rates of convergence and divergence, defined below.

Definition 2.10. Let $(A, b) \in \mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$ be a controllable pair. For $T \geq \mu > 0$, $\alpha \in \mathcal{G}(T, \mu)$, $x_0 \in \mathbb{R}^d$, and $K \in \mathcal{M}_{1,d}(\mathbb{R})$, we denote by $x(t; x_0, \alpha, K)$ the solution of $\dot{x} = (A - \alpha(t)bK)x$ with initial condition $x(0; x_0, \alpha, K) = x_0$.

(a) The *maximal and minimal Lyapunov exponents* associated with $\dot{x} = (A - \alpha(t)bK)x$ are, respectively,

$$\lambda^+(\alpha, K) = \sup_{\|x_0\|=1} \limsup_{t \rightarrow +\infty} \frac{\log(\|x(t; x_0, \alpha, K)\|)}{t}, \quad \lambda^-(\alpha, K) = \inf_{\|x_0\|=1} \liminf_{t \rightarrow +\infty} \frac{\log(\|x(t; x_0, \alpha, K)\|)}{t}.$$

(b) The *rates of convergence and divergence* associated with (2.4) are, respectively,

$$\text{rc}(A, b, T, \mu, K) = - \sup_{\alpha \in \mathcal{G}(T, \mu)} \lambda^+(\alpha, K), \quad \text{rd}(A, b, T, \mu, K) = \inf_{\alpha \in \mathcal{G}(T, \mu)} \lambda^-(\alpha, K).$$

(c) The *maximal rates of convergence and divergence* associated with (2.4) are, respectively,

$$\text{RC}(A, T, \mu) = \sup_{K \in \mathcal{M}_{1,d}(\mathbb{R})} \text{rc}(A, b, T, \mu, K), \quad \text{RD}(A, T, \mu) = \sup_{K \in \mathcal{M}_{1,d}(\mathbb{R})} \text{rd}(A, b, T, \mu, K).$$

Notice that

$$\begin{aligned} \text{rc}(A, b, T, \mu, K) &\leq \min_{\alpha \in [\mu/T, 1]} \min\{-\Re(\sigma(A - \alpha bK^T))\}, \\ \text{rd}(A, b, T, \mu, K) &\leq \min_{\alpha \in [\mu/T, 1]} \min\{\Re(\sigma(A - \alpha bK^T))\}, \end{aligned}$$

and, since a linear change of coordinates $x' = Px$ does not affect the Lyapunov exponents,

$$\begin{aligned} \text{rc}(A, b, T, \mu, K) &= \text{rc}(PAP^{-1}, Pb, T, \mu, (P^{-1})^T K), \\ \text{rd}(A, b, T, \mu, K) &= \text{rd}(PAP^{-1}, Pb, T, \mu, (P^{-1})^T K). \end{aligned}$$

In particular, this shows that $\text{RC}(A, T, \mu)$ and $\text{RD}(A, T, \mu)$ do not depend on b . It is also immediate to obtain that, for every $\lambda \in \mathbb{R}$,

$$\begin{aligned} \text{RC}(A + \lambda \text{Id}_d, T, \mu) &= \text{RC}(A, T, \mu) - \lambda, & \text{RD}(A + \lambda \text{Id}_d, T, \mu) &= \text{RD}(A, T, \mu) + \lambda, \\ \text{RC}(A, T, \rho T) &= \text{RC}(A/T, 1, \rho), & \text{RD}(A, T, \rho T) &= \text{RD}(A/T, 1, \rho), \end{aligned}$$

and that both $\text{RC}(A, T, \mu)$ and $\text{RD}(A, T, \mu)$ are monotone with respect to μ .

The property of stabilizing (2.4) can be translated in terms of the maximal rate of convergence as the property of having $\text{RC}(A, T, \mu) = +\infty$. A first result proved in [13] is that, in dimension 2, the maximal rates of convergence and divergence are either both finite or both infinite.

Theorem 2.11. *Suppose $d = 2$ and (A, b) controllable. Then, for System (2.4), we have that $\text{RC}(A, T, \mu) = +\infty$ if and only if $\text{RD}(A, T, \mu) = +\infty$.*

The answer to the question of whether it is possible to stabilize (2.4) at an arbitrary rate of convergence was found to depend on the parameter $\rho = \mu/T$, as stated the two following theorems from [13].

Theorem 2.12. *There exists $\rho^* \in (0, 1)$ (only depending on d) such that for every controllable pair $(A, b) \in \mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$, every $T > 0$ and every $\rho \in (\rho^*, 1]$ one has $\text{RC}(A, T, \rho T) = \text{RD}(A, T, \rho T) = +\infty$.*

Theorem 2.13. *There exists $\rho_* \in (0, 1)$ such that for every controllable pair $(A, b) \in \mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2$, every $T > 0$ and every $\rho \in (0, \rho_*)$ one has $\text{RC}(A, T, \rho T) < +\infty$.*

Theorem 2.12 is proved by means of a perturbative argument, using a Lyapunov function for the system $\dot{x} = (A - bK)x$ and showing that it remains a Lyapunov function for $\dot{x} = (A - \alpha(t)bK)x$ if $\alpha \in \mathcal{G}(T, \mu)$ and μ/T is large enough.

The idea of the proof of Theorem 2.13 is to actually construct, for each gain $K \in \mathcal{M}_{1,2}(\mathbb{R})$, a (T, μ) -signal α which destabilizes the system, that is, for which one can find a solution of $\dot{x} = (A - \alpha(t)bK)x$ which does not tend to zero as $t \rightarrow +\infty$. This construction exploits the overshoot phenomenon that happens when switching between systems $\dot{x} = Ax$ and $\dot{x} = (A - bK)x$, and it is interesting to note that the overshoot prevents stabilization in the case where μ/T is small, but not for μ/T large. The techniques used in this analysis rely deeply on the fact that the system is 2-dimensional, which prevents an immediate generalization of this result to higher dimensions.

We also note that Theorem 2.8 is actually a corollary of Theorem 2.13, since, for a given controllable pair (A, b) , it suffices to take $\lambda > \text{RC}(A, T, \rho T)$ and so $\text{RC}(A + \lambda \text{Id}_2, T, \rho T) < 0$, which means that (A, b) does not admit a $(T, \rho T)$ -stabilizer.

The signal α constructed in the proof of Theorem 2.13 takes its values on $\{0, 1\}$ and is periodic. As K increases in norm, α oscillates faster between 0 and 1, which suggests that, by taking α in a subclass of $\mathcal{G}(T, \mu)$ where the variation of α is controlled, one might be able to obtain a result guaranteeing the arbitrary rate of convergence. This intuition has been proved true in [12], taking the class

$$\mathcal{D}(T, \mu, M) = \{\alpha \in \mathcal{G}(T, \mu) \mid \alpha \text{ is } M\text{-Lipschitz}\}.$$

In this case, we consider the system

$$\dot{x} = Ax + \alpha(t)bu, \quad x \in \mathbb{R}^2, u \in \mathbb{R}, \alpha \in \mathcal{D}(T, \mu, M). \quad (2.11)$$

Theorem 2.14. *Let $(A, b) \in \mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2$ be a controllable pair, $T \geq \mu > 0$, and $M > 0$. Then, for every $\lambda > 0$, there exists $K \in \mathcal{M}_{1,2}(\mathbb{R})$ and $C > 0$ such that, for every $\alpha \in \mathcal{D}(T, \mu, M)$ and every $x_0 \in \mathbb{R}^2$, the solution x of $\dot{x} = (A - \alpha(t)bK)x$ with initial condition x_0 satisfies*

$$\|x(t)\| \leq Ce^{-\lambda t} \|x_0\|.$$

The proof of this theorem relies on the planar dynamics and cannot be directly generalized to higher dimensions. The time is separated into “good” time intervals, where the feedback is sufficiently active in order to stabilize the system, and “bad” time intervals, where the feedback is not enough active and an explosive behavior may occur; this explosive behavior is due not only to the dynamics of A , but it may also come from the dynamics of $A - \bar{\alpha}bK$ when $\bar{\alpha}$ is too small, and a technique of worst-case trajectory, similar to those presented in [4, 6, 24], is used to analyze the maximal rate of explosion on “bad” time intervals and show that it is compensated by the convergence on “good” ones.

Theorems 2.12 and 2.13 show that the question of whether (2.4) can be stabilized at an arbitrary rate of convergence gives rise to a bifurcation phenomenon depending on the parameter $\rho = \mu/T$. Hence it is of interest to study the quantity

$$\rho(A, T) = \inf\{\rho \in (0, 1] \mid \text{RC}(A, T, \rho T) = +\infty\}. \quad (2.12)$$

Theorem 2.12 implies that $\rho(A, T) \leq \rho^*$ for a certain ρ^* only depending on d . Moreover, in the case $d = 2$, Theorem 2.13 establishes a uniform lower bound $\rho(A, T) \geq \rho_* > 0$. Further properties of $\rho(A, T)$ are stated in the following proposition from [13].

Proposition 2.15. (a) $\rho(A, T)$ does not depend on $\text{Tr}(A)$ and $\rho(A, T) = \rho(A/T, 1)$.

(b) $\rho(J_d, T)$ does not depend on T .

(c) $T \mapsto \rho(A, T)$ is locally Lipschitz on $(0, +\infty)$.

(d) $\lim_{T \rightarrow +\infty} \rho(A, T) = \sup_{T > 0} \rho(A, T)$ and $\lim_{T \rightarrow 0^+} \rho(A, T) = \inf_{T > 0} \rho(A, T)$.

3 Infinite-dimensional systems

Systems of the form (2.1) may be generalized to the infinite-dimensional case. In this section, we consider the linear control system

$$\dot{z} = Az + \alpha(t)Bu, \quad z \in H, u \in U, \alpha \in \mathcal{G}(T, \mu) \quad (3.1)$$

where H and U are Hilbert spaces, $A : D(A) \subset H \rightarrow H$ generates a strongly continuous semigroup $\{e^{At} \mid t \geq 0\}$ and $B \in \mathcal{L}(U, H)$ is a bounded linear operator. Given a state feedback $u = -Kz$ with $K \in \mathcal{L}(H, U)$, $\alpha \in \mathcal{G}(T, \mu)$ and $z_0 \in H$, System (3.1) admits a unique *mild* solution $z \in \mathcal{C}(\mathbb{R}_+, H)$ (see, for instance, [5]), i.e., there exists a unique continuous function z defined in \mathbb{R}_+ and satisfying

$$z(t) = e^{At}z_0 - \int_0^t e^{A(t-s)}\alpha(s)BKz(s)ds \quad \text{for every } t \geq 0.$$

In the following example from [18], we exhibit an exactly controllable system defined by a skew-adjoint operator A for which the analogues of Proposition 2.5 does not hold. Thus we do not expect immediate generalizations of the results on Section 2 to hold, and extra analysis will be necessary in the infinite-dimensional case.

Example 3.1. Let us consider the damped wave equation on a string of unitary length with fixed endpoints, whose dynamics are described by

$$\begin{cases} v_{tt}(t, x) = v_{xx}(t, x) - \alpha(t)\zeta(x)^2v_t(t, x), & (t, x) \in (0, \infty) \times (0, 1), \\ v(0, x) = y_0(x), & x \in (0, 1), \\ v_t(0, x) = y_1(x), & x \in (0, 1), \\ v(t, 0) = v(t, 1) = 0, & t \in (0, \infty), \end{cases} \quad (3.2)$$

where $\zeta \in L^\infty(0, 1)$ and $\alpha \in L^\infty(\mathbb{R}_+, [0, 1])$. This can be written under the form (3.1) by setting the real Hilbert spaces H and U to be $H = H_0^1(0, 1) \times L^2(0, 1)$, $U = L^2(0, 1)$, with the usual scalar product in $L^2(0, 1)$ and the scalar product $\langle v, w \rangle_{H_0^1(0, 1)} = \langle v_x, w_x \rangle_{L^2(0, 1)}$ in $H_0^1(0, 1)$, and defining $z = (v_1, v_2)^T$,

$$\begin{aligned} D(A) &= (H^2(0, 1) \cap H_0^1(0, 1)) \times H_0^1(0, 1), \\ A &= \begin{pmatrix} 0 & 1 \\ \frac{d^2}{dx^2} & 0 \end{pmatrix}, \quad \text{i.e.,} \quad A \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ \frac{d^2v_1}{dx^2} \end{pmatrix}, \\ B &= \begin{pmatrix} 0 \\ \zeta \end{pmatrix}, \quad \text{i.e.,} \quad Bu = \begin{pmatrix} 0 \\ \zeta u \end{pmatrix}, \end{aligned}$$

so that $\|B\|_{\mathcal{L}(U, H)} \leq \|\zeta\|_{L^\infty(0, 1)}$, and taking the feedback $u = -B^*z$.

A straightforward computation shows that $D(A^*) \supset D(A)$ and that A^* and $-A$ coincide in $D(A)$; since A is surjective, it follows that $A^* = -A$, so that A is skew-adjoint and, by Stone's theorem (see, for instance, [28, Theorem 3.8.6]), A generates a strongly continuous unitary group $\{e^{At} \mid t \in \mathbb{R}\}$. If ζ is not the zero function in $L^\infty(0, 1)$, we also have the exact controllability of the pair (A, B) in time greater than 2 (see, for instance, [15, Theorem 2.55]).

However, we do not have asymptotic stability of (3.2) for some choices of ζ . Assume that $\zeta = \chi_{(a, b)}$ is the characteristic function of a proper subinterval $(a, b) \subsetneq (0, 1)$, where we may assume, without loss of generality, that $b < 1$. Then there exist $T \geq \mu > 0$, a (T, μ) -signal α , and a corresponding nonzero

periodic solution. This follows from the results in [23] (see also [19]) and can be illustrated by an explicit counterexample. Set $b' = \frac{1+b}{2}$. Take $T = 2$ and $\mu = 1 - b'$. Then

$$\alpha = \sum_{k=0}^{\infty} \chi_{[2k-\mu, 2k+\mu)} \quad (3.3)$$

is a (T, μ) -signal and

$$v(t, x) = \sum_{k=0}^{\infty} (\chi_{[b'+2k, 1+2k]}(x+t) - \chi_{[-1-2k, -b'-2k]}(x-t)) \quad (3.4)$$

is a periodic, nonzero, mild solution of (3.2) corresponding to α . Notice, in particular, that this solution does not converge to zero, even in the weak sense.

Note that (3.4) corresponds to the propagation of a wave with a sufficiently small support, and α is designed in (3.3) so that, when the support of $v(t, \cdot)$ passes through the interval (a, b) , α switches off the actuator, so that the wave is preserved and asymptotic stability is not achieved.

3.1 Exponential stability under persistent excitation

We suppose from now on that A generates a strongly continuous contraction semigroup $\{e^{At} \mid t \geq 0\}$, i.e., $\|e^{At}\| \leq 1$ for every $t \geq 0$. Even though Proposition 2.5 does not generalize well to the infinite-dimensional setting, as seen in Example 3.1, we may obtain asymptotic stability of (3.1) under the feedback law $u = -B^*z$, that is, of

$$\dot{z} = (A - \alpha(t)BB^*)z, \quad (3.5)$$

if we assume some additional hypothesis, as shown in [18]. Stability is studied through the use of a Lyapunov function, namely

$$V(z) = \frac{1}{2} \|z\|_{\mathbb{H}}^2, \quad (3.6)$$

which can be estimated as follows.

Lemma 3.2. *Let $0 \leq a \leq b < \infty$. Then, for any measurable function $\alpha : \mathbb{R}_+ \rightarrow [0, 1]$, any mild solution $z(\cdot)$ of System (3.5) satisfies*

$$V(z(b)) - V(z(a)) \leq - \left(2 + 2(b-a)^2 \|B\|^4\right)^{-1} \int_0^{b-a} \alpha(t+a) \|B^* e^{At} z(a)\|_{\mathbb{U}}^2 dt.$$

For the proof of this lemma, we refer to [18]. As a consequence of this estimate, one obtains a criterion for exponential stability of (3.5).

Theorem 3.3. *Suppose there exist two constants $c, \vartheta > 0$ such that*

$$\int_0^{\vartheta} \alpha(t) \|B^* e^{At} z_0\|_{\mathbb{U}}^2 dt \geq c \|z_0\|_{\mathbb{H}}^2, \quad \text{for all } z_0 \in \mathbb{H} \text{ and all } \alpha \in \mathfrak{G}(T, \mu). \quad (3.7)$$

Then there exist two constants $M \geq 1$ and $\gamma > 0$ such that, for any initial data $z_0 \in \mathbb{H}$ and any $\alpha \in \mathfrak{G}(T, \mu)$, the corresponding solution z of (3.5) satisfies

$$\|z(t)\|_{\mathbb{H}} \leq M e^{-\gamma t} \|z_0\|_{\mathbb{H}}, \quad \text{for all } t \geq 0. \quad (3.8)$$

Proof. Fix $\alpha \in \mathcal{G}(T, \mu)$ and $s \geq 0$, and define V by (3.6). Lemma 3.2 with $a = s$ and $b = s + \vartheta$ then yields

$$V(z(s + \vartheta)) - V(z(s)) \leq -\frac{1}{2(1 + \vartheta^2 \|B\|^4)} \int_0^\vartheta \alpha(t + s) \|B^* e^{At} z(s)\|_{\mathcal{U}}^2 dt,$$

and so (3.7) implies

$$V(z(s + \vartheta)) - V(z(s)) \leq -\frac{c}{1 + \vartheta^2 \|B\|^4} V(z(s)).$$

The desired estimate (3.8) follows from standard arguments. \blacksquare

An application of Theorem 3.3 to the wave equation is given in the following example.

Example 3.4. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider the damped wave equation on Ω ,

$$\begin{cases} v_{tt}(t, x) = \Delta v(t, x) - \alpha(t) \zeta(x)^2 v_t(t, x), & (t, x) \in (0, \infty) \times \Omega, \\ v(0, x) = y_0(x), & x \in \Omega, \\ v_t(0, x) = y_1(x), & x \in \Omega, \\ v(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \end{cases} \quad (3.9)$$

with $\alpha \in \mathcal{G}(T, \mu)$, which is a d -dimensional generalization of Example 3.1. We now suppose that the damping term acts almost everywhere, that is, $\zeta \in L^\infty(\Omega)$ satisfies

$$|\zeta(x)| \geq \zeta_0 > 0 \quad \text{for almost all } x \in \Omega \quad (3.10)$$

and for a certain constant $\zeta_0 > 0$. Taking $\mathbb{H} = H_0^1(\Omega) \times L^2(\Omega)$, $\mathcal{U} = L^2(\Omega)$, with the usual scalar product in $L^2(\Omega)$ and the scalar product $\langle v, w \rangle_{H_0^1(\Omega)} = \langle \nabla v, \nabla w \rangle_{(L^2(\Omega))^d}$ in $H_0^1(\Omega)$, we can write (3.9) under the form (3.1) with

$$\begin{aligned} D(A) &= (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega), \\ A &= \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \zeta \end{pmatrix}, \end{aligned}$$

and with the feedback $u = -B^* z$. Hypothesis (3.10) on ζ makes it now possible to show, as done in [18], that (3.7) is satisfied for this system with $\vartheta = T$, and so the damped wave equation with a persistently exciting intermittent damping acting almost everywhere is exponentially stable.

3.2 Weak stability under persistent excitation

It is interesting to remark that inequality (3.7) generalizes the exact observation inequality (see, for instance, [28, (6.1.1)]): when α is identically equal to 1, (3.7) reduces to an exact observability criterion for the pair (A, B^*) in time ϑ . A generalized form of the approximate observability criterion, which weakens (3.7), also gives rise to an asymptotic stability result. We recall that we assume A to generate a strongly continuous contraction semigroup $\{e^{At} \mid t \geq 0\}$.

Theorem 3.5. *Suppose there exists $\vartheta > 0$ such that, for all $\alpha \in \mathcal{G}(T, \mu)$,*

$$\int_0^\vartheta \alpha(t) \|B^* e^{At} z_0\|_{\mathcal{U}}^2 dt = 0 \quad \Rightarrow \quad z_0 = 0. \quad (3.11)$$

Then each solution z of System (3.5) converges weakly to 0 in \mathbb{H} as $t \rightarrow +\infty$ for any initial data $z_0 \in \mathbb{H}$ and any $\alpha \in \mathcal{G}(T, \mu)$.

The first step of the proof of this theorem is to show that, for each $z_0 \in \mathbf{H}$ and each $\alpha \in \mathcal{G}(T, \mu)$, the weak ω -limit set

$$\omega(z_0, \alpha) = \{z_\infty \in \mathbf{H} \mid \exists \text{ sequence } (s_n)_{n \in \mathbb{N}}, s_n \rightarrow +\infty, \text{ so that the solution } z \text{ of} \\ (3.5) \text{ with initial condition } z_0 \text{ satisfies } z(s_n) \xrightarrow{n \rightarrow +\infty} z_\infty\}$$

is non-empty; this follows from the fact that the norm of a solution decreases along trajectories, and so any trajectory admits a weak limit point. The main part of the proof consists on establishing that

$$z_\infty \in \omega(z_0, \alpha) \quad \Rightarrow \quad \exists \alpha_\infty \in \mathcal{G}(T, \mu) \text{ such that } \int_0^\vartheta \alpha_\infty(t) \|B^* e^{At} z_\infty\|_U^2 dt = 0, \quad (3.12)$$

and thus the assertion of the theorem follows from (3.11). We refer to [18] for the detailed proof of (3.12).

The following example shows an application of Theorem 3.5 to Schrödinger equation.

Example 3.6. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and consider the internally damped Schrödinger equation on Ω ,

$$\begin{cases} iv_t(t, x) = -\Delta v(t, x) - i\alpha(t)\zeta(x)^2 v(t, x), & (t, x) \in (0, \infty) \times \Omega, \\ v(0, x) = y_0(x), & x \in \Omega, \\ v(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \end{cases} \quad (3.13)$$

with $\alpha \in \mathcal{G}(T, \mu)$ and $\zeta \in L^\infty(\Omega)$. Assume that there exist $\zeta_0 > 0$ and a nonempty open set $\omega \subset \Omega$ such that

$$|\zeta(x)| \geq \zeta_0 > 0 \quad \text{for almost all } x \in \omega. \quad (3.14)$$

We write (3.13) under the form (3.1) by setting $\mathbf{H} = \mathbf{U} = L^2(\Omega)$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $A = i\Delta$, $B : z \mapsto \zeta z$ and with the feedback $u = -B^*z$. It is shown in [18] that (3.11) is satisfied for this system with $\vartheta > T - \mu$, and so the solutions of the internally damped Schrödinger equation with a persistently exciting intermittent damping converge weakly to 0.

3.3 Other conditions of excitation

Condition (2.2) for System (2.1) means that, in *every* time interval of length T , α will activate the control u . It is a natural question whether this condition can be relaxed in certain cases, allowing intervals of arbitrary length where no feedback control is active. For instance, it follows directly from the results in [19, Example 2 and Theorem 3.2] (see also [23]) that, for the damped wave equation on a bounded domain $\Omega \subset \mathbb{R}^d$

$$\begin{cases} v_{tt}(t, x) = \Delta v(t, x) - \alpha(t)v_t(t, x), & (t, x) \in (0, \infty) \times \Omega, \\ v(0, x) = y_0(x), & x \in \Omega, \\ v_t(0, x) = y_1(x), & x \in \Omega, \\ v(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \end{cases} \quad (3.15)$$

if $\alpha \in L^\infty(\mathbb{R}_+, \{0, 1\})$ and $\{t \mid \alpha(t) = 1\} = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ with $a_n < b_n < a_{n+1}$, the condition

$$\sum_{n \in \mathbb{N}} (b_n - a_n)^3 = +\infty \quad (3.16)$$

implies that every solution of (3.15) converges to zero. Condition (3.16) allows for α to be zero on arbitrary long time intervals; actually, the exact distribution of the intervals (a_n, b_n) is unimportant, the only importance being their size.

Condition (3.7) in Theorem 3.3 suggests a generalization of the class $\mathcal{G}(T, \mu)$ where convergence to zero may also be true, allowing α to be zero on longer time intervals. We consider the system

$$\dot{z} = Az + \alpha(t)Bu, \quad z \in H, u \in U, \quad (3.17)$$

where A generates a strongly continuous contraction semigroup $\{e^{At} \mid t \geq 0\}$, subject to the feedback $u = -B^*z$, leading to the system

$$\dot{z} = (A - \alpha(t)BB^*)z. \quad (3.18)$$

Definition 3.7. We say that $\alpha \in L^\infty([0, T], [0, 1])$ is of class $\mathcal{K}(A, B, T, c)$ if

$$\int_0^T \alpha(t) \|B^*e^{At}z_0\|_U^2 dt \geq c \|z_0\|_H^2, \quad \text{for all } z_0 \in H. \quad (3.19)$$

With this definition, the following result of stability is presented in [18].

Theorem 3.8. Suppose that there exist constants $\rho, T_0 > 0$ and a continuous function $c : (0, \infty) \rightarrow (0, \infty)$ such that, for all $T \in (0, T_0]$, the following implication holds.

$$\tilde{\alpha} \in L^\infty([0, T], [0, 1]), \int_0^T \tilde{\alpha}(t) dt \geq \rho T \quad \Rightarrow \quad \tilde{\alpha} \in \mathcal{K}(A, B, T, c(T)).$$

Let $((a_n, b_n))_{n \in \mathbb{N}}$ be a sequence of disjoint intervals in \mathbb{R}_+ and $\alpha \in L^\infty(\mathbb{R}_+, [0, 1])$ with $\int_{a_n}^{b_n} \alpha(t) dt \geq \rho(b_n - a_n)$ and $\sum_{n \in \mathbb{N}} c(b_n - a_n) = \infty$. Then any mild solution z of (3.18) satisfies $\|z(t)\|_H \rightarrow 0$ as $t \rightarrow +\infty$.

With this result, asymptotic estimates of $c(T)$ for T small may be used to obtain stability conditions, as we illustrate in the following example from [18].

Example 3.9. As in Example 3.4, we consider again the wave equation

$$\begin{cases} v_{tt}(t, x) = \Delta v(t, x) - \alpha(t)\zeta(x)^2 v_t(t, x), & (t, x) \in (0, \infty) \times \Omega, \\ v(0, x) = y_0(x), & x \in \Omega, \\ v_t(0, x) = y_1(x), & x \in \Omega, \\ v(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \end{cases} \quad (3.20)$$

where Ω is a bounded domain in \mathbb{R}^d and $\zeta \in L^\infty(\Omega)$ satisfies

$$|\zeta(x)| \geq \zeta_0 > 0 \quad \text{for almost all } x \in \Omega$$

for a certain constant $\zeta_0 > 0$. In the same way we did in Example 3.4, this system can be written under the form (3.17) by setting $H = H_0^1(\Omega) \times L^2(\Omega)$ and $U = L^2(\Omega)$ with the same scalar products as before, and with the same operators A and B .

We claim that, for this system, the function $c(T)$ appearing in the statement of Theorem 3.8 is of order T^3 for T small.

Take $\alpha \in L^\infty(\mathbb{R}_+, [0, 1])$ satisfying

$$\int_0^T \alpha(t) dt \geq \rho T$$

for some positive constant ρ . Denote by $(\varphi_n)_{n \in \mathbb{N}}$ the orthonormal basis of $L^2(\Omega)$ made of eigenfunctions of the Laplace–Dirichlet operator $-\Delta$ on Ω and, for each $n \in \mathbb{N}$, denote by $\lambda_n > 0$ the eigenvalue corresponding to φ_n . Let $t \mapsto z(t) = (v(t, \cdot), v_t(t, \cdot))$ be a solution to (3.20) with initial condition

$$\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} \sum_{n \in \mathbb{N}} a_n \varphi_n \\ \sum_{n \in \mathbb{N}} b_n \varphi_n \end{pmatrix}$$

where $(\sqrt{\lambda_n}a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ belong to ℓ^2 , so that $\|z(0)\|_{\mathbb{H}}^2 = \sum_{n \in \mathbb{N}} (\lambda_n a_n^2 + b_n^2)$ and

$$v(t, x) = \sum_{n \in \mathbb{N}} \left[a_n \varphi_n(x) \cos(\sqrt{\lambda_n}t) + \frac{b_n}{\sqrt{\lambda_n}} \varphi_n(x) \sin(\sqrt{\lambda_n}t) \right].$$

Then we have

$$\begin{aligned} \int_0^T \alpha(t) \|B^* z(t)\|^2 dt &\geq \zeta_0^2 \sum_{n \in \mathbb{N}} \int_0^T \alpha(t) \left[-\sqrt{\lambda_n} a_n \sin(\sqrt{\lambda_n}t) + b_n \cos(\sqrt{\lambda_n}t) \right]^2 dt = \\ &= \zeta_0^2 \sum_{n \in \mathbb{N}} \int_0^T \alpha(t) (\lambda_n a_n^2 + b_n^2) \sin^2(\sqrt{\lambda_n}t + \theta_n) dt \end{aligned}$$

for some $\theta_n \in \mathbb{R}$, and we want to show that

$$\zeta_0^2 \int_0^T \alpha(t) \sin^2(\sqrt{\lambda_n}t + \theta_n) dt \geq c(T)$$

with $c(T)$ of order T^3 for T small.

For every $\varepsilon \in (0, 1)$, consider the set

$$A_n^\varepsilon = \{t \in [0, T] \mid |\sin(\sqrt{\lambda_n}t + \theta_n)| > \varepsilon\}.$$

We note that $|\sin(\sqrt{\lambda_n}t + \theta_n)| \leq \sqrt{\lambda_n} |t - t_0|$ for every t_0 such that $\sin(\sqrt{\lambda_n}t_0 + \theta_n) = 0$, and so, using this estimate, we can show that

$$m(A_n^\varepsilon) \geq T \left(1 - \frac{2\varepsilon}{\pi} \right) - 2 \frac{\varepsilon}{\lambda_1},$$

where we recall that m is the Lebesgue measure, and where we assume, without loss of generality, that $\min_{n \in \mathbb{N}} \lambda_n = \lambda_1$. For $\varepsilon = \frac{\rho \lambda_1}{6} T$, we have that $m(A_n^\varepsilon) \geq T \left(1 - \frac{\rho}{3} - \frac{\rho \lambda_1}{3\pi} T \right) \geq T \left(1 - \frac{\rho}{2} \right)$ for T small enough, and so

$$\int_{A_n^\varepsilon} \alpha(t) dt \geq \frac{\rho T}{2}.$$

Thus,

$$\int_0^T \alpha(t) \sin^2(\sqrt{\lambda_n}t + \theta_n) dt \geq \varepsilon^2 \frac{\rho T}{2} = \frac{\rho^3 \lambda_1^2}{72} T^3$$

and we may take, for T small,

$$c(T) = \frac{\zeta_0^2 \rho^3 \lambda_1^2}{72} T^3.$$

In particular, Theorem 3.8 shows that, if $\alpha \in L^\infty(\mathbb{R}_+, [0, 1])$ satisfies

$$\int_{a_n}^{b_n} \alpha(t) dt \geq \rho (b_n - a_n)$$

for a certain $\rho > 0$ and a certain sequence of disjoint intervals $((a_n, b_n))_{n \in \mathbb{N}}$ with

$$\sum_{n \in \mathbb{N}} (b_n - a_n)^3 = +\infty,$$

then every solution of (3.20) tends (strongly) to 0 as $t \rightarrow +\infty$.

In the case of finite-dimensional systems, $H = \mathbb{R}^d$, $U = \mathbb{R}^m$, when A is skew-symmetric and (A, B) is controllable, it is possible (see [18]) to take $c(T) \sim \kappa T^{2r+1}$ for T small in Theorem 3.8, where $\kappa > 0$ is a constant and r is the smallest non-negative integer such that

$$\text{rank} \begin{pmatrix} B & AB & \cdots & A^r B \end{pmatrix} = d.$$

This provides the following stability criterion in the finite-dimensional case.

Proposition 3.10. *Let A be skew-symmetric, (A, B) be controllable and r be as above. Then for every $\rho > 0$ and every $\alpha \in L^\infty(\mathbb{R}_+, [0, 1])$ such that there exist a sequence $((a_n, b_n))_{n \in \mathbb{N}}$ of disjoint intervals in \mathbb{R}_+ with $\int_{a_n}^{b_n} \alpha(t) dt \geq \rho(b_n - a_n)$ and $\sum_{n=1}^\infty (b_n - a_n)^{2r+1} = +\infty$, we have that*

$$\lim_{t \rightarrow +\infty} \|z(t)\| = 0$$

for every solution of $\dot{z} = (A - \alpha(t)BB^T)z$.

4 Further discussion and open problems

The results from the previous sections give rise to several questions concerning persistently excited linear systems that, up to our knowledge, remain open. We now present some of these questions that have drawn our attention.

4.1 Generalization of Theorem 2.11 to higher dimensions

Theorem 2.11 was only established in the 2-dimensional case, and an interesting open problem is to find if it still holds true in dimension greater than two.

In order to prove that $\text{RC}(A, T, \mu) = +\infty$ implies that $\text{RD}(A, T, \mu) = +\infty$, the proof provided in [13] consists on showing that, if $C > 0$ is large enough and $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix} \in \mathcal{M}_{1,2}(\mathbb{R}^d)$ is such that $\text{rc}(A, b, T, \mu, K) > C$, then $\text{rd}(A, b, T, \mu, K_-) > C$ for $K_- = \begin{pmatrix} k_1 & -k_2 \end{pmatrix}$, and, to do so, the solutions of $\dot{x} = (A - \alpha b K_-)x$ are regarded as solutions of $\dot{x} = (A - \alpha b K)x$ going backwards in time. For this to be possible, it is necessary to extend α backwards in time, and the result is actually reduced to find such an extension that satisfies certain properties. The search for such an extension of α is stated in terms of the controllability of the angular part $\omega = x/\|x\|$ of the control system $\dot{x} = (A - \xi b K_-)x$ with respect to the control $\xi \in [0, 1]$, i.e., the controllability of the system

$$\dot{\omega} = (A - \xi b K_-)\omega - \left[\omega^T \frac{(A - \xi b K_-)^T + (A - \xi b K_-)}{2} \omega \right] \omega, \quad \omega \in \mathbb{S}^1 \subset \mathbb{R}^2, \xi \in [0, 1]. \quad (4.1)$$

The techniques used in [13] to prove the controllability of (4.1) with respect to the control ξ rely on the fact that the dynamics take place in the unit circle \mathbb{S}^1 , and cannot be immediately generalized to higher dimensions. This is essentially the reason why the proof of Theorem 2.11 given in [13] only holds in dimension 2, and so a key to the generalization of this theorem is to study the controllability of (4.1) in dimension greater than two.

4.2 Generalizations of Theorem 2.14

Theorem 2.14 shows that stabilization at an arbitrary rate of convergence is possible for System (2.11), where we assume that the switching signal α is Lipschitz continuous with a Lipschitz constant bounded by

a certain $M > 0$. The proof provided in [12], however, relies deeply on the planar structure of the dynamics, and it is an interesting question whether this result still holds true in higher dimensions. As we mentioned before, the proof goes by decomposing the time on “good” and “bad” time intervals, and the main idea is to show that the rate of convergence on “good” time intervals is more important than the possible explosion on “bad” time intervals. The estimates on the rate of explosion on “bad” time intervals are based on techniques of worst-case trajectory, similar to those presented in [4, 6, 24], which rely on the planar structure of the dynamics to get the desired estimates. Obtaining fine estimates on “bad” time intervals in higher dimensions is a much harder problem.

Another possible generalization of Theorem 2.14 is to consider an intermediate class between $\mathcal{G}(T, \mu)$ and $\mathcal{D}(T, \mu, M)$. More precisely, we wish to know if stabilization at an arbitrary rate is still possible if we consider system $\dot{x} = Ax + \alpha(t)Bu$ subject to a persistently exciting signal in a class \mathcal{B} larger than $\mathcal{D}(T, \mu, M)$. By Theorem 2.13, this is not true for the whole class of persistently exciting signals $\mathcal{G}(T, \mu)$ when μ/T is small. As we mentioned before, Theorem 2.13 has been proved by constructing, for each gain $K \in \mathcal{M}_{1,2}(\mathbb{R})$, a signal $\alpha \in \mathcal{G}(T, \mu)$ that destabilizes the system, and such signals α oscillate faster between 0 and 1 as K increases in norm, so a natural class to consider would be the class $\mathcal{BV}(T, \mu, V)$ of (T, μ) -signals of total variation bounded by V on every interval $[t, t + T]$, $t \in \mathbb{R}_+$. That is, we define the class $\mathcal{BV}(T, \mu, V)$ by setting that $\alpha \in \mathcal{BV}(T, \mu, V)$ if $\alpha \in \mathcal{G}(T, \mu)$ and if, for every interval $[t, t + T]$ with $t \geq 0$ and every partition $P = \{t = t_0 < t_1 < \dots < t_n = t + T\}$ of $[t, t + T]$, one has

$$\sum_{i=1}^n |\alpha(t_i) - \alpha(t_{i-1})| \leq V.$$

Even though we do not know whether stabilization at an arbitrary rate can be established for the class $\mathcal{BV}(T, \mu, V)$, it is not hard to prove this result to be true if we restrict ourselves to signals in $\mathcal{BV}(T, \mu, V)$ taking their values only in $\{0, 1\}$. Let us prove this fact. Given $T \geq \mu > 0$ and $V > 0$, we define the class $\mathcal{BV}_d(T, \mu, V)$ by setting that $\alpha \in \mathcal{BV}_d(T, \mu, V)$ if $\alpha \in \mathcal{BV}(T, \mu, V)$ and if α takes its values only on $\{0, 1\}$. We wish to study the control system

$$\dot{x} = Ax + \alpha(t)Bu, \quad x \in \mathbb{R}^d, u \in \mathbb{R}^m, \alpha \in \mathcal{BV}_d(T, \mu, V) \quad (4.2)$$

where $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$ and (A, B) is controllable. We first notice that the bound on the total variation of $\alpha \in \mathcal{BV}_d(T, \mu, V)$ in an interval $[t, t + T]$ is actually a bound on the number of jumps between 0 and 1 that α may have in $[t, t + T]$.

Lemma 4.1. *Let $T \geq \mu > 0$, $V > 0$, and $\alpha \in \mathcal{BV}_d(T, \mu, V)$, and fix $t \geq 0$. Then there exist $N \in \mathbb{N}$ and numbers a_i, b_i , $i = 1, \dots, N$, with*

$$t = a_1 \leq b_1 \leq a_2 \leq b_2 \leq \dots \leq a_N \leq b_N = t + T, \quad (4.3)$$

such that

$$\alpha(s) = \begin{cases} 1 & \text{if } s \in (a_i, b_i) \text{ for a certain } i \in \{1, \dots, N\}, \\ 0 & \text{if } s \in (b_i, a_{i+1}) \text{ for a certain } i \in \{1, \dots, N-1\}. \end{cases} \quad (4.4)$$

Furthermore,

$$N \leq \frac{V}{2} + 2 \quad \text{and} \quad \sum_{i=1}^N (b_i - a_i) \geq \mu.$$

Proof. Consider the set \mathcal{N} of all positive integers $n \in \mathbb{N}^*$ such that we can find a partition $P = \{t = t_0 < t_1 < \dots < t_n = t + T\}$ with $|\alpha(t_i) - \alpha(t_{i-1})| = 1$ for every $i = 1, \dots, n$ and note $M = \sup \mathcal{N}$, with the convention $M = 0$ if $\mathcal{N} = \emptyset$. Since the total variation of α in $[t, t + T]$ is less than V , it follows that

$M \leq V$. The case $M = 0$ happens if and only if α is constant on $[t, t + T]$, and this constant must be equal 1 due to the fact that $\alpha \in \mathcal{G}(T, \mu)$ and $\mu > 0$, and so, in this case, we set $N = 1$, $a_1 = t$, $b_1 = t + T$, and we have the desired result. We suppose from now on that $M \geq 1$.

Take a partition $P = \{t = t_0 < t_1 < \dots < t_M = t + T\}$ with $|\alpha(t_i) - \alpha(t_{i-1})| = 1$ for every $i = 1, \dots, M$ (such a partition exists, since M is finite). Then, for every $i = 1, \dots, M$, there exists $c_i \in [t_{i-1}, t_i]$ such that α is constant on $[t_{i-1}, c_i)$ and on $(c_i, t_i]$. Indeed, take $c_i = \inf\{t \in [t_{i-1}, t_i] \mid \alpha(t) = \alpha(t_i)\}$; so α is constant and equal to $\alpha(t_{i-1})$ on $[t_{i-1}, c_i)$, and, if α were not constant on $(c_i, t_i]$, we would be able to choose τ_1, τ_2 with $c_i \leq \tau_1 < \tau_2 < t_i$ and $\alpha(\tau_1) = \alpha(t_i)$, $|\alpha(t_i) - \alpha(\tau_2)| = 1$, in such a way that $\{t = t_0 < t_1 < \dots < t_{i-1} < \tau_1 < \tau_2 < t_i < \dots < t_M = t + T\}$ would be a partition of $[t, t + T]$ with $M + 3$ elements and for which α would change its value between any two consecutive elements, thus contradicting the fact that $M = \sup \mathcal{N}$. Thus we have the desired property on c_i .

If $\alpha(t) = 0$, we define $a_1 = b_1 = t$ and $a_2 = c_1$, $b_2 = c_2$, $a_3 = c_3$, $b_3 = c_4$, and so on, so that, if M is odd, we end with $a_N = c_M$ and $b_N = t + T$, with $N = \frac{M-1}{2} + 2$ and, if M is even, we end with $b_{N-1} = c_M$ and $a_N = b_N = t + T$, with $N = \frac{M}{2} + 2$. Similarly, if $\alpha(t) = 1$, we define $a_1 = t$, $b_1 = c_1$, $a_2 = c_2$, $b_2 = c_3$, and so on, so that, if M is odd, we end with $b_{N-1} = c_M$, $a_N = b_N = t + T$, with $N = \frac{M-1}{2} + 2$, and, if M is even, we end with $a_N = c_M$ and $b_N = t + T$, with $N = \frac{M}{2} + 1$. In all cases, we have $N \leq \frac{M}{2} + 2 \leq \frac{V}{2} + 2$. Since α is constant on $[t_{i-1}, c_i)$ and $(c_i, t_i]$ for all $i = 1, \dots, M$ the construction of a_i and b_i guarantees that α is constant on (a_i, b_i) for all $i = 1, \dots, N$ and on (b_i, a_{i+1}) for all $i = 1, \dots, N - 1$, and our construction also guarantees that (4.4) holds. Finally, since α is a (T, μ) -signal, we have

$$\sum_{i=1}^N (b_i - a_i) = \int_t^{t+T} \alpha(s) ds \geq \mu.$$

■

We shall also need the following result, which gives an estimate on the overshoot constant of the system $\dot{x} = (A - BK)x$ which is polynomial in the exponential decay rate γ . Its proof can be found, for instance, in [10, 11] (with a better estimation on M and L provided in [20]).

Lemma 4.2. *Let $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ be two matrices such that the pair (A, B) is controllable. Then there exists $M \geq 1$ such that, for any $\gamma \geq 1$, there exists a matrix $K \in \mathcal{M}_{m,d}(\mathbb{R})$ such that*

$$\left\| e^{(A-BK)t} \right\| \leq M\gamma^L e^{-\gamma t}, \quad \text{for all } t \geq 0,$$

with L depending only on d .

Using Lemmas 4.1 and 4.2, we can prove a result on the stabilization at an arbitrary rate for the class $\mathcal{BV}_d(T, \mu, V)$.

Theorem 4.3. *Let $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ be two matrices such that the pair (A, B) is controllable, and let $T \geq \mu > 0$ and $V > 0$. Given $\lambda > 0$, there exist $K \in \mathcal{M}_{m,d}(\mathbb{R})$ and $C > 0$ such that, for every $\alpha \in \mathcal{BV}_d(T, \mu, V)$ and every initial condition $x_0 \in \mathbb{R}^d$, the corresponding solution x of $\dot{x} = (A - \alpha(t)BK)x$ satisfies*

$$\|x(t)\| \leq Ce^{-\lambda t} \|x_0\|, \quad \text{for all } t \geq 0.$$

Proof. We first note that, for every $K \in \mathcal{M}_{m,d}(\mathbb{R})$, every initial condition $x_0 \in \mathbb{R}^d$ and every $\alpha \in \mathcal{BV}_d(T, \mu, V)$, equation $\dot{x} = (A - \alpha(t)BK)x$ can be integrated and, by application of Gronwall's Lemma, we get that, for every $t \geq 0$,

$$\|x(t)\| \leq \|x_0\| e^{(\|A\| + \|BK\|)t}. \quad (4.5)$$

Let M and L be as in Lemma 4.2; up to increasing M , we also have that

$$\|e^{At}\| \leq Me^{at}, \quad \text{for all } t \geq 0$$

for a certain $a \geq 0$. Take $N_0 = \frac{V}{2} + 2$ and, for $\gamma > 0$, define

$$C_\gamma = M^{2N_0} \gamma^{N_0 L} e^{aT} e^{-\gamma\mu}.$$

Then $\lim_{\gamma \rightarrow +\infty} C_\gamma = 0$. Now, take $\lambda > 0$; there exists $\gamma \geq 1$ such that $0 < C_\gamma \leq e^{-\lambda T}$. For this γ , we take $K \in \mathcal{M}_{m,d}(\mathbb{R})$ as in Lemma 4.2, and we take $\alpha \in \mathcal{BV}_d(T, \mu, V)$ and $x_0 \in \mathbb{R}^d$.

Take $t \geq 0$. By Lemma 4.1, there exist $N \leq N_0$ and numbers $a_i, b_i, i = 1, \dots, N$, satisfying (4.3), such that α satisfies (4.4). We thus have

$$x(t+T) = e^{(A-BK)(b_N-a_N)} e^{A(a_N-b_{N-1})} e^{(A-BK)(b_{N-1}-a_{N-1})} e^{A(a_{N-1}-b_{N-2})} \dots e^{A(a_2-b_1)} e^{(A-BK)(b_1-a_1)} x(t).$$

Using the estimates on $\|e^{At}\|$ and $\|e^{(A-BK)t}\|$, we obtain

$$\begin{aligned} \|x(t+T)\| &\leq M^{2N-1} \gamma^{NL} e^{a \sum_{j=1}^{N-1} (a_{j+1}-b_j)} e^{-\gamma \sum_{j=1}^N (b_j-a_j)} \|x(t)\| \leq \\ &\leq M^{2N_0} \gamma^{N_0 L} e^{aT} e^{-\gamma\mu} \|x(t)\| \leq \\ &\leq e^{-\lambda T} \|x(t)\|. \end{aligned}$$

Thus, for any $t \geq 0$, we have

$$\|x(t+T)\| \leq e^{-\lambda T} \|x(t)\|.$$

Now, for $t \geq 0$, writing $t = nT + r$ with $r \in [0, T)$ and $n \in \mathbb{N}$, we obtain

$$\|x(t)\| \leq e^{-\lambda nT} \|x(r)\| = e^{\lambda r} \|x(r)\| e^{-\lambda t},$$

from where we finally get, using (4.5), that

$$\|x(t)\| \leq Ce^{-\lambda t} \|x_0\|$$

with $C = e^{(\lambda + \|A\| + \|BK\|)T}$. ■

Theorem 4.3 thus generalizes Theorem 2.14 to the multi-dimensional case when $\alpha \in \mathcal{BV}_d(T, \mu, V)$, but it is presently not known if this result still holds true when α is in the more general class $\mathcal{BV}(T, \mu, V)$, where α may take its values on the whole interval $[0, 1]$ and not only on $\{0, 1\}$.

4.3 Properties of $\rho(A, T)$

The quantity $\rho(A, T)$ defined in (2.12) is important to study the bifurcation phenomenon that happens when considering stabilizability at an arbitrary rate of System (2.4) with respect to the parameter $\rho = \mu/T$. Even though Proposition 2.15 gives some characterization of $\rho(A, T)$, many questions remain open. For instance, we know that $T \mapsto \rho(J_d, T)$ is constant, but an interesting problem is to find exactly for which matrices $T \mapsto \rho(A, T)$ is constant. If it were the case of $T \mapsto \rho(A, T)$ being constant for all matrices $A \in \mathcal{M}_d(\mathbb{R})$, it would be interesting to know whether this constant depends on A . Otherwise, it would also be interesting to investigate if $T \mapsto \rho(A, T)$ is monotone, and the dependence of $\lim_{T \rightarrow +\infty} \rho(A, T)$ and $\lim_{T \rightarrow 0^+} \rho(A, T)$ on A .

It is also of interest to study the quantity

$$\rho_d = \sup_{\substack{T > 0 \\ A \in \mathcal{M}_d(\mathbb{R})}} \rho(A, T).$$

This definition means that, for every $\rho > \rho_d$, we have $\text{RC}(A, T, \rho T) = +\infty$ for any controllable pair $(A, b) \in \mathcal{M}_d(\mathbb{R}) \times \mathbb{R}^d$ and any $T > 0$. Thanks to Theorem 2.12, we have that $\rho_d < 1$ for every $d \in \mathbb{N}^*$, and an interesting open problem is the study of the behavior of ρ_d as $d \rightarrow +\infty$.

4.4 Stabilizability at an arbitrary rate for systems with several inputs

Section 2.3 deals only with systems with a single input, and it would be interesting to study what happens when we consider the multi-input case, i.e., when

$$\dot{x} = Ax + \alpha(t)Bu, \quad x \in \mathbb{R}^d, u \in \mathbb{R}^m, \alpha \in \mathcal{G}(T, \mu). \quad (4.6)$$

The problem of stabilization at an arbitrary rate of convergence seems to be quite intricate when considering several inputs. The intuition is that adding more (independent) inputs to a system gives us more control on its behavior, and so, even if we do not have stabilizability at an arbitrary rate for a certain system given by a pair of matrices (A, B) , we might obtain such a result by adding more independent inputs (and thus adding more linearly independent columns to B). An interesting question in this sense is to find out the minimal number of linearly independent inputs necessary to stabilize at an arbitrary rate any d -dimensional system, for every $T \geq \mu > 0$. That is, we want to find the minimal p such that any controllable pair (A, B) with $A \in \mathcal{M}_d(\mathbb{R})$, $B \in \mathcal{M}_{d,m}(\mathbb{R})$ and $\text{rank}(B) \geq p$ can be stabilized at an arbitrary rate with respect to the class $\mathcal{G}(T, \mu)$, for any $T \geq \mu > 0$. This property holds for d linearly independent inputs, as shown in the next theorem (which generalizes [9, Proposition 20]), but we do not know if we may have $p < d$.

Theorem 4.4. *Let $A \in \mathcal{M}_d(\mathbb{R})$ and $B \in \mathcal{M}_{d,m}(\mathbb{R})$ be two matrices such that the pair (A, B) is controllable and $\text{rank}(B) = d$, and let $T \geq \mu > 0$. Given $\lambda > 0$, there exists $K \in \mathcal{M}_{m,d}(\mathbb{R})$ and $C > 0$ such that, for every $\alpha \in \mathcal{G}(T, \mu)$ and every initial condition $x_0 \in \mathbb{R}^d$, the corresponding solution x of $\dot{x} = (A - \alpha(t)BK)x$ satisfies*

$$\|x(t)\| \leq Ce^{-\lambda t} \|x_0\|, \quad \text{for all } t \geq 0.$$

Proof. As in the proof of Theorem 4.3, for every $K \in \mathcal{M}_{m,d}(\mathbb{R})$, every initial condition $x_0 \in \mathbb{R}^d$ and every $\alpha \in \mathcal{BV}_d(T, \mu, V)$, an application of Gronwall's Lemma yields the estimate

$$\|x(t)\| \leq \|x_0\| e^{(\|A\| + \|BK\|)t}. \quad (4.7)$$

We take $M \geq 1$ and $a \in \mathbb{R}$ such that $\|e^{At}\| \leq Me^{at}$ for every $t \geq 0$. Since $\text{rank}(B) = d$, up to a change of variables in the input $u = Qv$, we can suppose that $B = (\text{Id}_d \quad 0_{d,m-d})$, and, ignoring the last $m-d$ inputs and considering u as a vector in \mathbb{R}^d , System (4.6) can be written as

$$\dot{x} = Ax + \alpha(t)u.$$

For a given $\lambda > 0$, take $k > 0$ such that $Me^{aT}e^{-k\mu} \leq e^{-\lambda T}$ and consider the state feedback $u = -kx$, which yields the closed-loop system

$$\dot{x} = (A - \alpha(t)k\text{Id}_d)x. \quad (4.8)$$

Take $\alpha \in \mathcal{G}(T, \mu)$ and $x_0 \in \mathbb{R}^d$, and note by x the solution of (4.8) corresponding to α , x_0 and k . Since k is a scalar, the flow associated to (4.8) is $\Phi(t, s) = e^{A(t-s)}e^{-k\int_s^t \alpha(\tau)d\tau}$, and so we have, for every $t \geq 0$,

$$x(t+T) = e^{AT}e^{-k\int_t^{t+T} \alpha(s)ds}x(t).$$

Thus,

$$\|x(t+T)\| \leq Me^{aT}e^{-k\mu} \|x(t)\| \leq e^{-\lambda T} \|x(t)\|.$$

Now, for $t \geq 0$, writing $t = nT + r$ with $r \in [0, T)$ and $n \in \mathbb{N}$, we obtain

$$\|x(t)\| \leq e^{-\lambda nT} \|x(r)\| = e^{\lambda r} \|x(r)\| e^{-\lambda t},$$

from where we finally get, using (4.7), that

$$\|x(t)\| \leq Ce^{-\lambda t} \|x_0\|$$

with $C = e^{(\lambda + \|A\| + \|BK\|)T}$. ■

4.5 Infinite-dimensional systems

Section 3 deals with stabilization results for infinite-dimensional systems, concentrating on operators A that generate strongly continuous contraction semigroups $\{e^{At} \mid t \geq 0\}$, that is, for which $\|e^{At}\| \leq 1$ for all $t \geq 0$. These operators generalize the neutrally stable case presented in Section 2.1, and an interesting question is to investigate whether results from Sections 2.2 and 2.3 can also be generalized to the infinite-dimensional setting. This would mean to consider a strongly continuous semigroup $\{e^{At} \mid t \geq 0\}$ for which $\|e^{At}\|$ cannot be uniformly bounded by a constant for all $t \geq 0$.

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