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AMALGAMATED PRODUCTS OF C*-BUNDLES

ÉTIENNE BLANCHARD

Abstract. We describe which classical amalgamated products of continuous C*-bundles are continuous C*-bundles and we analyse the involved extension problems for continuous C*-bundles.

INTRODUCTION

Different (fibrewise) amalgamated products of continuous C*-bundles have been studied over the last years ([1], [8], [6], [4]), one of the main questions being to know when these amalgamated products are still continuous C*-bundles.

In order to gather these different approaches in a joint survey, we first recall a few definitions from the theory of deformations of C*-algebras and we fix several notations which will be used in the sequel.

Then we present a few possible extension properties for continuous C*-bundles. More precisely, given a compact Hausdorff space $X$ which is perfect, i.e. without any isolated point, we first recall in §2 that there is no general $C(X)$-linear version of the Hahn-Banach extension theorem for continuous $C(X)$-algebra. But we describe in §3 a Tietze extension property for continuous $C(X)$-algebras which will enable us to characterize in the following sections:

- when the canonical fiberwise amalgamated tensor products of a given continuous $C(X)$-algebra $A$ with any other continuous $C(X)$-algebra $B$ is a continuous $C(X)$-algebra ([6] Theorem 1.1 and Theorem 1.2]),
- when the canonical fiberwise amalgamated free products of a given continuous $C(X)$-algebra $A$ with any other continuous $C(X)$-algebra $B$ is a continuous $C(X)$-algebra ([6] Theorem 3.7 and Corollary 4.8]).

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1. $C(X)$-algebras

We recall first a few definitions from deformation theory for C*-algebras and we fix the notations which will be used in the sequel.

Let $X$ be a compact Hausdorff space and $C(X)$ the C*-algebra of continuous functions on $X$ with values in the complex field $\mathbb{C}$.

Definition 1.1. A $C(X)$-algebra is a C*-algebra $A$ endowed with a unital *-homomorphism from $C(X)$ to the centre of the multiplier C*-algebra $\mathcal{M}(A)$ of $A$. 

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Given a closed subset \( Y \subseteq X \), we denote by \( C_0(X \setminus Y) \) the closed ideal of continuous functions on \( X \) that vanish of \( Y \). If \( A \) is a \( C(X) \)-algebra, then the subset \( C_0(X \setminus Y)A \) is a closed ideal in \( A \) (by Cohen factorisation Theorem) and we denote by \( \pi_Y^A \) the quotient map \( A \to A/C_0(X \setminus Y)A \).

If the closed subset \( Y \) is reduced to a point \( x \) and the element \( a \) belongs to the \( C(X) \)-algebra \( A \), we usually write \( \pi_x, A_x \) and \( a_x \) for \( \pi_x^X, \pi_x^X(A) \) and \( \pi_x^X(a) \).

Note that the function
\[
(1.1) \quad x \mapsto \|a_x\| = \inf \{ \|1 - f + f(x)\|a \| : f \in C(X) \}
\]
is always upper semi-continuous by construction. And the \( C(X) \)-algebra \( A \) is said to be continuous (or to be a continuous \( C^* \)-bundle over \( X \)) if the function \( x \mapsto \|a_x\| \) is actually continuous for all \( a \in A \).

**Definition 1.2.** A continuous field of states on a unital \( C(X) \)-algebra \( A \) is a unital positive \( C(X) \)-linear map \( \varphi : A \to C(X) \).

**Remark 1.3.** A (unital) separable \( C(X) \)-algebra \( A \) is continuous if and only if (iff) there exists a continuous field of states \( \varphi : A \to C(X) \) such that for all \( x \in X \), the induced state \( \varphi_x : a_x \in A_x \mapsto \varphi(a)(x) \) is faithful on \( A_x \) (2).

2. **Hahn-Banach extension properties**

Given a compact Hausdorff space \( X \), a continuous unital \( C(X) \)-algebra \( A \), a unital \( C(X) \)-subalgebra \( B \subseteq A \) and a continuous field of states \( \phi : B \to C(X) \), there does not exist in general a \( C(X) \)-linear positive unital map \( \varphi : A \to C(X) \) extending \( \phi \), i.e. a continuous field of states \( \varphi \) on \( A \) making the following diagram commutative:
\[
\begin{array}{ccc}
B & \xrightarrow{\phi} & C(X) \\
\cap & & || \\
A & \xrightarrow{\varphi} & C(X)
\end{array}
\]

The problem happens as soon as the interior of \( X \) is non empty. Indeed, consider:
- the compact space \( X := \{0\} \cup \{ \frac{1}{n} : n \in \mathbb{N}^* \} \),
- the unital continuous \( C(X) \)-algebra \( A := C(X) \oplus C(X) \) and
- the \( C(X) \)-subalgebra \( B := C(X).1_A + \left( C_0(X \setminus \{0\}) \oplus C_0(X \setminus \{0\}) \right) \subseteq A \)

And let \( \phi : B \to C(X) \) be the continuous field of states on \( B \) fixed by the formulae
\[
\phi( (b_1, b_2))(\frac{1}{n}) = \begin{cases} 
\frac{b_1(\frac{1}{n})}{n} & \text{if } n \text{ is odd} \\
\frac{b_2(\frac{1}{n})}{n} & \text{otherwise}
\end{cases} \quad \text{for } (b_1, b_2) \in C_0(X \setminus \{0\}) \oplus C_0(Y \setminus \{0\})
\]
Then, there cannot be any continuous field of states \( \varphi : A \to C(X) \) such that \( \varphi(b) = \phi(b) \) for all \( b \in B \). Indeed, if \( a = 1 \oplus 0 \in A \), one has that:
Proposition 3.1. And let \( \overline{\cdot} \) order to study extensions in the general case, let us define in \( X \)

\[
\| \cdot \|_n \in \mathbb{N}
\]

the closed subspace \( C \), the open subspace \( U \) \( \subset X \)

Hence, the function \( x \mapsto \varphi(a)(x) \) cannot be continuous at \( x = 0 \).

On the other hand, if \( Z \) is a second countable compact Hausdorff space \( Z \) and \( X \subset Z \) is a non empty closed subspace, then any continuous field of states \( \phi : \pi^Z_X(A) \to C(X) \) on the restriction \( \pi^Z_X(A) \) can be extended to a continuous field of states \( \varphi : A \to C(Z) \) by Michael continuous selection theorem (see e.g. \cite[Proposition 3.13]{ref}), i.e. such that the following diagramme commutes:

\[
\begin{array}{ccc}
\pi^Z_X(A) & \xrightarrow{\phi} & C(X) \\
\downarrow & & \downarrow \\
A & \longrightarrow & C(Z)
\end{array}
\]

3. Tietze Extension Properties

Given a second countable compact Hausdorff space \( X \) and a closed non empty subspace \( Y \subset X \), we describe in this section when a continuous \( C(Y) \)-algebra \( A \) can be extended to \( X \), i.e. when there exists a continuous \( C(X) \)-algebra isomorphic \( \pi^X_Y(D) \cong A \).

If the \( C^* \)-algebra \( A \) is an exact separable \( \mathcal{C}^* \)-algebra, then there exists a unital embedding of the \( C(Y) \)-algebra \( A \) into the trivial \( C(Y) \)-algebra \( C(Y; \mathcal{O}_2) \cong C(Y) \otimes \mathcal{O}_2 \), where \( \mathcal{O}_2 \) is the unital Cuntz \( \mathcal{C}^* \)-algebra generated by two isometries \( s_1, s_2 \) satisfying the relation \( 1_{\mathcal{O}_2} = s_1(s_1)^* + s_2(s_2)^* \) (\cite{ref3}). Hence, the continuous \( C(X) \)-algebra \( D := \{ f \in C(X, \mathcal{O}_2) ; \pi^X_Y(f) \in A \} \) answers the question in that case.

But there are continuous \( C(Y) \)-algebras which are not exact \( \mathcal{C}^* \)-algebras. Thus, in order to study extensions in the general case, let us define in \( X \times Y \times [0,1] \):

- the open subspace \( U = \{ (x,y,t) \in X \times Y \times [0,1] ; 0 < t \} \) and
- the closed subspace \( Z = \{ (x,y,t) \in X \times Y \times [0,1] ; 0 \leq t.d(x,Y) \leq 2d(x,Y) - d(x,y) \} \).

And let \( \tilde{d} \) be the metric on \( Z \) given by \( \tilde{d}((x,y,t),(x',y',t')) = d(x,x') + d(y,y') + |t-t'| \).

Proposition 3.1. (\cite{ref5}) The coordinate map \( p_1 : (x,y,t) \mapsto x \) gives a structure of \( C(X) \)-algebra on \( C(Z) \) and the ideal \( C_0(U \cap Z) \) is a continuous \( C(X) \)-algebra such that \( C_0(U \cap Z)_{|Y} \cong C_0(Y \times (0,1]) \), i.e. the map \( (x,y,t) \in U \cap Z \mapsto x \in X \) is open.

Proof. Given a function \( f \) in \( C_0(U \cap Z) \), let us prove the continuity of the function

\[
x \in X \mapsto \| \pi^X_x(f) \| = \sup \{ |f(z)| ; z \in p_1^{-1}(\{x\}) \}
\]

This map is already upper semi-continuous (u. s. c.) by construction. Hence, it only remains to show that for any point \( x_0 \in X \) and any constant \( \varepsilon > 0 \), one has \( \| \pi^X_x(f) \| > \| \pi^X_{x_0}(f) \| - \varepsilon \) for all points \( x \) in a neighbourhood of \( x_0 \) in \( X \).

The uniform continuity of the function \( f \) implies that there exists \( \delta > 0 \) such that \( |f(z) - f(z')| < \varepsilon \) for all \( z, z' \in Z \) with \( \tilde{d}(z,z') < \delta \). Now three cases can appear:
1) If \( x_0 \in Y \) and \( x \in Y \) satisfies \( d(x_0, x) < \delta/2 \), then \( |f(x, x, t) - f(x_0, x_0, t)| < \varepsilon \) for all \( t \in [0, 1] \). And so \( \|\pi_x^Y(f)\| > \|\pi_{x_0}^Y(f)\| - \varepsilon \).

2) If \( x_0 \in Y \) and \( x \in X \setminus Y \) satisfies \( d(x_0, x) < \delta/4 \), then for all \( y \in Y \), the relation \( d(x, y) \leq 2d(x, Y) \) implies that \( d(y, x_0) \leq d(y, x) + d(x, x_0) \leq 2d(x, Y) + d(x, x_0) \leq \frac{3}{2}\delta \) and so \( |f(x, y, t) - f(x_0, x_0, t)| < \varepsilon \) for all \( t \in [0, 2 - \frac{d(x, y)}{d(x, Y)}] \). Whence the inequality \( \|\pi_x^Y(f)\| > \|\pi_{x_0}^Y(f)\| - \varepsilon \).

3) If \( x_0 \not\in Y \) and the triple \((x_0, y_0, t_0) \in U \cap Z \) satisfies \( |f(x_0, y_0, t_0)| = \|\pi_{x_0}^Y(f)\| \neq 0 \), then \( d(x_0, y_0) < 2d(x_0, Y) \). Thus, there exists by continuity a constant \( \alpha(x_0) \in [0, \delta/2] \) such that all \( x \in X \) in the ball of radius \( \alpha(x_0) \) around \( x_0 \) satisfy:

\[
\begin{align*}
\text{a)} & \quad d(x, Y) > 0, \\
\text{b)} & \quad d(x, y) < 2d(x, Y), \\
\text{c)} & \quad t_0 < 2 - \frac{d(x, y)}{d(x, Y)} + \delta/2.
\end{align*}
\]

And so \( \|\pi_x^Y(f)\| \geq |f(x, y, \inf\{t_0, 2 - \frac{d(x, y)}{d(x, Y)}\})| > \|\pi_{x_0}^Y(f)\| - \varepsilon. \)

\[ \square \]

**Remark 3.2.** S. Wassermann pointed out that if \( Y = \{0, 1\} \subset X = [0, 1] \), then \( Z = \{ (x, 0, t) \in [0, 1] \times \{0\} \times [0, 1] : t \leq \frac{2 - 3x}{x - 1} \} \cap \{ (x, 1, t) \in [0, 1] \times \{1\} \times [0, 1] : t \leq \frac{3x - 1}{x} \} \).

Hence, the \( C(X) \)-algebra \( C(Z) \) is not continuous at \( x = \frac{1}{3} \) and \( x = \frac{2}{3} \).

But the essential ideal \( C_0(Z \cap \{0, 1\} \times \{0, 1\} \times (0, 1]) \) is a continuous \( C(X) \)-algebra.

The following Corollary will be essential in the proof of Proposition 4.2.

**Corollary 3.3.** Let \( X \) be a second countable compact space, \( Y \subset X \) a non zero closed subset and \( A \) a continuous \( C(Y) \)-algebra.

\[
\begin{align*}
\text{a)} & \quad B := C(X) \otimes A \otimes C([0, 1]) \text{ is a continuous } C(X \times Y \times [0, 1]) \text{-algebra.} \\
\text{b)} & \quad D := [C_0(U).B]_Y = C_0(U).B / C_0(U \setminus U \cap Z).B \text{ is a continuous } C(X) \text{-algebra.} \\
\text{c)} & \quad \text{There is an isomorphism of } C(Y) \text{-algebras } D_Y \cong A \otimes C_0((0, 1]).
\end{align*}
\]

**Proof.** a) holds because the \( C^* \)-algebras \( C(X) \) and \( C_0((0, 1]) \) are nuclear.

b) Let \( b \in D \). Then for all \( x \in X \), we have

\[
\|\pi_x^Y(b)\| = \|b + C_0(X \setminus \{x\})D\| = \sup\{|\pi_z^Z(b)| : z \in p_1^{-1}(\{x\})\},
\]

whence the continuity of the map \( x \mapsto \|\pi_x^Y(b)\| \) by a) and Proposition 3.1.

c) One has \( D_Y \cong [C_0(U).B]_Y \cong A \otimes C_0((0, 1]) \) by Proposition 3.1.

\[ \square \]

4. **Amalgamated tensor products of continuous \( C(X) \)-algebras**

Given a fixed compact Hausdorff space \( X \), we study in this section the continuity properties of the different tensor products amalgamated over \( C(X) \) of two given continuous \( C(X) \)-algebras \( A \) and \( B \).

More precisely, let \( A \otimes B \) denote the algebraic tensor product (over \( \mathbb{C} \)) of \( A \) and \( B \), let \( \mathcal{I}_X(A, B) \) be the ideal in \( A \otimes B \) generated by the differences \( a \otimes b - a \otimes fb \) (\( a \in A \), \( b \in B, f \in C(X) \)) and let \( A \otimes B \) denote the quotient of \( A \otimes B \) by \( \mathcal{I}_X(A, B) \).

If \( C_\Delta(X \times X) \subset C(X \times X) \) is the ideal of continuous function of \( X \times X \) which are
zero on the diagonal and $A \otimes B$ (resp. $A^M \otimes B$) is the minimal (resp. maximal) tensor product over $C$ of the two continuous $C(X)$-algebras $A$ and $B$, then the quotient $A_{C(X)}^m \otimes B := A \otimes B / C_\Delta(X \times X)A \otimes B$ (resp. $A^M_{C(X)} \otimes B := A^M \otimes B / C_\Delta(X \times X)A^M \otimes B$) is the minimal (resp. maximal) completion of the algebraic amalgamated tensor product $A \bowtie C B$. Further, the $*$-algebra $A^m_{C(X)} \otimes B$ embeds in the $C(X)$-algebra $A^M_{C(X)} \otimes B$ (II) and we have

$$\forall x \in X, \quad (A_{C(X)} \otimes B)_x \cong A_x \otimes B_x \quad \text{and} \quad (A^M_{C(X)} \otimes B)_x \cong A^M_x \otimes B_x.$$  

Let us also recall a characterisation of exactness given by Kirchberg and Wassermann.

**Proposition 4.1.** ([8 Theorem 4.5]) Let $Y = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$ and let $D$ be a $C^*$-algebra. Then the following assertions are equivalent.

i) The $C^*$-algebra $A$ is exact.

ii) For all continuous $C(Y)$-algebra $B$, the minimal tensor product $A_{mB} \otimes B$ is a continuous $C(Y)$-algebra with fibres $A_{mB} \otimes B_y (y \in Y)$.

It induces the following results for fibrewise tensor products of continuous $C(X)$-algebras.

**Proposition 4.2.** ([6], [4]) Let $X$ be a second countable compact Hausdorff space and $A$ a separable unital continuous $C(X)$-algebra.

If the topological space $X$ is perfect (i.e. without isolated point), then the following assertions $\alpha_e$ and $\beta_e$ (resp. $\alpha_n$ and $\beta_n$) are equivalent.

$\alpha_e$) The $C^*$-algebra $A$ is exact.

$\beta_e$) For all continuous $C(X)$-algebra $B$, the amalgamated tensor product $A \bowtie_{C(X)} B$ is a continuous $C(X)$-algebra with fibres $A_{C(X)} \bowtie B_x (x \in X)$.

$\alpha_n$) The $C^*$-algebra $A$ is nuclear.

$\beta_n$) For all continuous $C(X)$-algebra $B$, the amalgamated tensor product $A_{C(X)}^m \bowtie B$ is a continuous $C(X)$-algebra with fibres $A_{C(X)}^m \bowtie B_x (x \in X)$.

**Proof.** $\alpha_e \Rightarrow \beta_e$) If the $C^*$-algebra $A$ is exact, then the spatial tensor product $A_{C(X)}^m \otimes D$ is a continuous $C(X \times X)$-algebra with fibres $A_x \otimes D_x' (x, x' \in X)$ ([8]). Hence, its restriction to the diagonal is as desired.

$\beta_e \Rightarrow \alpha_e$) Suppose conversely that the $C(X)$-algebra $A$ satisfies $\beta_e$. And let us prove step by step that the $C^*$-algebra $A$ is exact.

**Step a)** All the fibres $A_x$ are exact ($x \in X$). Indeed, given a point $x$ in $X$, take a sequence of points $x_n$ in $X$ converging to $x$ such that there is a topological isomorphism
Now, let \( C(Y) \) denote the space of continuous bounded functions on \( Y \). Then, for any separable continuous \( C(Y) \)-algebra \( B \), there is a continuous \( C(X) \)-algebra \( B \) such that \( B_X = B \otimes C_X((0, 1]) \) (Corollary 3.3).

Now, the continuity of the \( C(X) \)-algebra \( B \otimes \) given by \( \beta_e \) implies that of its restriction \( B \otimes A \mid_Y \) since there is an isometric \( C(Y) \)-linear embedding \( B \otimes A \mid_Y \). This implies the exactness of the \( C^* \)-algebra \( A \) by Proposition 4.1.

**Step b)** If \( B \) is a \( C^* \)-algebra and \( B \) is the constant \( C(X) \)-algebra \( C(X; B) \), then for all \( x \in X \), we have the exact sequence

\[
0 \to C_x(X) A \otimes B \to (A \otimes B)_x \to A_x \otimes B \to 0.
\]

**Step c)** If \( B \) is a \( C(X) \)-algebra, then for all \( x \in X \), we have the sequence of epimorphisms \( (A \otimes B)_x \to (A_x \otimes B)_x \to A_x \otimes B_x \).

**Step d)** Now, let \( B \) be a \( C^* \)-algebra, \( K \triangleleft B \) a closed two sided ideal in \( B \) and take an element \( d \in \ker\{A \otimes B \to A \otimes B/K\} \). Then for all \( x \in X \), we have

\[
d_x = \ker\{(A \otimes B)_x \to (A \otimes B)/K_x\} = \ker\{A_x \otimes B \to A_x \otimes B/K\} \quad \text{by \( a \)}
\]

\[
= A_x \otimes K \quad \text{by \( b \)}
\]

Thus, \( d \in A_m \otimes K \). And so, the \( C^* \)-algebra \( A \) is exact.

\( \alpha_n \) has a similar proof to that of \( \beta_e \).

\( \beta_n \) If a \( C^* \)-algebra \( A \) satisfies \( \beta_n \), then all the fibres \( A_x \) \( (x \in X) \) are nuclear by [3, Theorem 3.2] and so the \( C^* \)-algebra \( A \) itself is nuclear (see e.g. [2, Proposition 3.23]).

**Remark 4.3.** These characterisations do not hold anymore if the compact space \( X \) is not perfect. Indeed, if the space \( X \) is reduced to a point, then both the amalgamated tensor products \( A \otimes B \) and \( A \otimes B \) are constant, hence continuous.

Proposition 4.2 implies the following characterisation of exact continuous \( C(X) \)-algebras in the framework of \( C(X) \)-algebras.

**Corollary 4.4.** Let \( X \) be a perfect compact metric space and \( A \) be a separable continuous \( C(X) \)-algebra. Then the following are equivalent

1. \( \text{The } C^* \text{-algebra } A \text{ is exact.} \)
2. \( \text{For all exact sequence of continuous } C(X) \text{-algebras } 0 \to J \to B \to D \to 0, \text{ the sequence } 0 \to A \otimes C(X) \to J \otimes C(X) \to B \otimes C(X) \to D \otimes C(X) \to 0 \text{ is exact.} \)
Proof. (2)⇒(1) If the unital continuous $C(X)$-algebra $A$ satisfies (2) and the sequence $0 \to J_0 \to B_0 \to D_0 \to 0$ is an exact sequence of $C^*$-algebras, then the sequence $0 \to J := C(X) \otimes J_0 \to B := C(X) \otimes B_0 \to D := C(X) \otimes D_0 \to 0$ is an exact sequence of $C(X)$-algebras. And the condition (2) implies the exactness of the sequence

$$0 \to A \overset{m}{\otimes} J = A \otimes J_0 \to A \overset{m}{\otimes} B = A \otimes B_0 \to A \overset{m}{\otimes} D = A \otimes D_0 \to 0.$$ 

whence the exactness of $A$.

(1)⇒(2) If the $C(X)$-algebra $A$ is an exact $C^*$-algebra and $0 \to J \to B \to D \to 0$ is an exact sequence of $C(X)$-algebras, then the two first lines of the following diagram are exact by assumption (1)

$$
\begin{array}{c}
C_\Delta(X \times X)A \overset{m}{\otimes} J \\
\downarrow \\
A \overset{m}{\otimes} J
\end{array} \\
\begin{array}{c}
C_\Delta(X \times X)A \overset{m}{\otimes} B \\
\downarrow \\
A \overset{m}{\otimes} B
\end{array} \\
\begin{array}{c}
C_\Delta(X \times X)A \overset{m}{\otimes} D \\
\downarrow \\
A \overset{m}{\otimes} D
\end{array}
$$

Besides, all the columns of the diagram are exact by definition, whence the exactness of the last line by a diagram chasing. \(\square\)

5. AMALGAMATED FREE PRODUCTS OF CONTINUOUS $C(X)$-ALGEBRAS

In this section, we describe the continuity properties of different free products amalgamated over $C(X)$ of two given unital continuous $C(X)$-algebras $A$ and $B$.

Proposition 5.1. (\cite{4}) Let $X$ be a second countable perfect compact Hausdorff space and $A$ a separable unital continuous $C(X)$-algebra.

Then the following assertions are equivalent.

$\alpha_e$) The $C^*$-algebra $A$ is exact.

$\gamma_e$) For all separable unital continuous $C(X)$-algebra $B$ and all continuous fields of faithful states $\phi : A \to C(X)$, $\psi : B \to C(X)$, the reduced amalgamated free product $(C, \phi \ast \psi) = (A, \phi) \ast (B, \psi)$ is a continuous $C(X)$-algebra with fibres $(C_x, \phi_x \ast \psi_x) = (A_x, \phi_x) \ast (B_x, \psi_x)$.

Proof. $\gamma_e$) ⇒ $\alpha_e$) Let $B$ be a unital separable continuous $C(X)$-algebra and let $\psi$ be a continuous field of faithful states $\psi$ on $B$. Set $D = A \overset{m}{\otimes} B$ and let $E$ be the Hilbert $D$-bimodule $E = L^2(D, \phi \otimes \psi) \otimes C(X)$.

Then, the following assertions are equivalent (\cite{4} Lemma 4.5]).

a) $D$ is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \overset{m}{\otimes} B_x$ ($x \in X$).

b) The Pimsner $C^*$-algebra $T_D(E \oplus D)$ of the full Hilbert $D$-bimodule $E \oplus D$ is a continuous $C(X)$-algebra with fibres $T_{D_x}(E_x \oplus D_x)$ ($x \in X$).
But there is a $C(X)$-linear isomorphism $\mathcal{T}_D(E \oplus D) \cong C \times \mathbb{N}$. And so, these two assertions are equivalent to the continuity of the reduced amalgamated free product $(C, \phi * \psi) = (A, \phi) \ast_C (B, \psi)$ since the group $\mathbb{Z}$ is amenable ([2 Corollaire 5.10]).

$\alpha_x \Rightarrow \gamma_x$) Conversely, if $A$ is an exact $C^*$-algebra and $B$ is a unital separable continuous $C(X)$, the amalgamated tensor product $D = A \underset{C(X)}{\overset{m}{\otimes}} B$ is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \otimes B_x$ for $x \in X$ ([1]). Hence, the reduced amalgamated free product $(C, \phi * \psi) = (A, \phi) \ast_C (B, \psi)$ is a continuous $C(X)$-algebra ([1] Theorem 4.1]).

**Remark 5.2.** There is no similar result for full amalgamated free product. Indeed, the full amalgamated free product $A \overset{f}{\ast}_{C(X)} B$ of two unital continuous $C(X)$-algebras $A$ and $B$ is always a continuous $C(X)$-algebra with fibres $A_x \ast B_x (x \in X)$ ([3] Theorem 3.7]).

**Sketch of proof.** The algebraic amalgamated free product $A \overset{\ell}{\ast}_{C(X)} B$ is a dense $C(X)$-submodule of the amalgamated Haagerup tensor product $A \overset{h}{\otimes}_{C(X)} B$, which itself is contained in the full amalgamated free product $A \overset{f}{\ast}_{C(X)} B$ ([4]). And for all $d \in A \overset{\ell}{\ast}_{C(X)} B$, one has

$$\|d_x\|_{A_x \otimes B_x} = \|d_x\|_{A_x \overset{h}{\otimes} B_x} = \inf \left\{ \| \sum_i a_i a_i^* \|_{\ell^2(\mathbb{N})} \left\| \sum_i b_i^* b_i \right\|_{\ell^2(\mathbb{N})} \ ; \ d_x = \sum_i a_i \otimes b_i \right\}$$

$$= \sup \left\{ \left| \langle \xi, \sum_i \pi(a_i) \sigma(b_i) \eta \rangle \right| ; \xi, \eta \text{ unit vectors in the Hilbert space } \ell^2(\mathbb{N}) \right\}$$

Hence, the map $x \mapsto \|d_x\|$ is both upper and lower semi-continuous if $d \in A \overset{\ell}{\ast}_{C(X)} B$.

The proof for the continuity of the map $x \mapsto \|d_x\|$ for elements $d$ in the algebraic amalgamated free product $A \overset{\ell}{\ast}_{C(X)} B$ is similar ([4]).

**References**


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