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AMALGAMATED PRODUCTS OF C*-BUNDLES

ÉTIENNE BLANCHARD

ABSTRACT. We describe which classical amalgamated products of continuous C*-bundles are continuous C*-bundles and we analyse the involved extension problems for continuous C*-bundles.

INTRODUCTION

Different (fibrewise) amalgamated products of continuous C*-bundles have been studied over the last years ([1], [8], [6], [4]), one of the main questions being to know when these amalgamated products are still continuous C*-bundles.

In order to gather these different approaches in a joint survey, we first recall a few definitions from the theory of deformations of C*-algebras and we fix several notations which will be used in the sequel.

Then we present a few possible extension properties for continuous C*-bundles. More precisely, given a compact Hausdorff space X which is perfect, i.e. without any isolated point, we first recall in §2 that there is no general C(X)-linear version of the Hahn-Banach extension theorem for continuous C(X)-algebra. But we describe in §3 a Tietze extension property for continuous C(X)-algebras which will enable us to characterize in the following sections:

- when the canonical fiberwise amalgamated tensor products of a given continuous C(X)-algebra A with any other continuous C(X)-algebra B is a continuous C(X)-algebra ([6] Theorem 1.1 and Theorem 1.2),
- when the canonical fiberwise amalgamated free products of a given continuous C(X)-algebra A with any other continuous C(X)-algebra B is a continuous C(X)-algebra ([4] Theorem 3.7 and Corollary 4.8]).

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1. C(X)-ALGEBRAS

We recall first a few definitions from deformation theory for C*-algebras and we fix the notations which will be used in the sequel.

Let X be a compact Hausdorff space and C(X) the C*-algebra of continuous functions on X with values in the complex field C.

Definition 1.1. A C(X)-algebra is a C*-algebra A endowed with a unital *-homomorphism from C(X) to the centre of the multiplier C*-algebra $\mathcal{M}(A)$ of A.

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Given a closed subset \(Y \subset X\), we denote by \(C_0(X \setminus Y)\) the closed ideal of continuous functions on \(X\) that vanish of \(Y\). If \(A\) is a \(C(X)\)-algebra, then the subset \(C_0(X \setminus Y)\). \(A\) is a closed ideal in \(A\) (by Cohen factorisation Theorem) and we denote by \(\pi_Y\) the quotient map \(A \to A/C_0(X \setminus Y)\).

If the closed subset \(Y\) is reduced to a point \(x\) and the element \(a\) belongs to the \(C(X)\)-algebra \(A\), we usually write \(\pi_x, A_x\) and \(a_x\) for \(\pi^X_{(x)}, \pi^X_{(1)}(A)\) and \(\pi^X_{(x)}(a)\).

Note that the function

\[(1.1) \quad x \mapsto \|a_x\| = \inf\{\|1 - f + f(x)\|a\| : f \in C(X)\}\]

is always upper semi-continuous by construction. And the \(C(X)\)-algebra \(A\) is said to be continuous (or to be a continuous \(C^*\)-bundle over \(X\)) if the function \(x \mapsto \|a_x\|\) is actually continuous for all \(a\) in \(A\).

**Definition 1.2.** A continuous field of states on a unital \(C(X)\)-algebra \(A\) is a unital positive \(C(X)\)-linear map \(\varphi : A \to C(X)\).

**Remark 1.3.** A (unital) separable \(C(X)\)-algebra \(A\) is continuous if and only if (iff) there exists a continuous field of states \(\varphi : A \to C(X)\) such that for all \(x \in X\), the induced state \(\varphi_x : a_x \in A_x \mapsto \varphi(a)(x)\) is faithful on \(A_x\) \((2)\).

2. **Hahn-Banach extension properties**

Given a compact Hausdorff space \(X\), a continuous unital \(C(X)\)-algebra \(A\), a unital \(C(X)\)-subalgebra \(B \subset A\) and a continuous field of states \(\phi : B \to C(X)\), there does not exist in general a \(C(X)\)-linear positive unital map \(\varphi : A \to C(X)\) extending \(\phi\), i.e. a continuous field of states \(\varphi\) on \(A\) making the following diagram commutative:

\[
\begin{array}{ccc}
B & \xrightarrow{\phi} & C(X) \\
\cap & \| & \\
A & \xrightarrow{\varphi} & C(X)
\end{array}
\]

The problem happens as soon as the interior of \(X\) is non empty. Indeed, consider:

- the compact space \(X := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}^*\}\),
- the unital continuous \(C(X)\)-algebra \(A := C(X) \oplus C(X)\) and
- the \(C(X)\)-subalgebra \(B := C(X).1_A + C_0(X \setminus \{0\}) \oplus C_0(X \setminus \{0\})\) \(\subset A\)

And let \(\phi : B \to C(X)\) be the continuous field of states on \(B\) fixed by the formulae

\[
\phi( (b_1, b_2) ) (\frac{1}{n}) = \begin{cases} 
\frac{b_1(\frac{1}{n})}{n} & \text{if } n \text{ is odd} \\
\frac{b_2(\frac{1}{n})}{n} & \text{otherwise}
\end{cases}
\]

for \((b_1, b_2) \in C_0(X \setminus \{0\}) \oplus C_0(Y \setminus \{0\})\)

Then, there cannot be any continuous field of states \(\varphi : A \to C(X)\) such that \(\varphi(b) = \phi(b)\) for all \(b \in B\). Indeed, if \(a = 1 \oplus 0 \in A\), one has that:
(1) \( \varphi(a)\left(\frac{1}{n}\right) = 1 \) if \( n \) is odd and
(2) \( \varphi(a)\left(\frac{1}{n}\right) = 0 \) otherwise.

Hence, the function \( x \mapsto \varphi(a)(x) \) cannot be continuous at \( x = 0 \).

On the other hand, if \( Z \) is a second countable compact Hausdorff space \( Z \) and \( X \subset Z \) is a non empty closed subspace, then any continuous field of states \( \phi : \pi_X^Z(A) \to C(X) \) on the restriction \( \pi_X^Z(A) \) can be extended to a continuous field of states \( \phi : A \to C(Z) \) by Michael continuous selection theorem (see e.g. [2, Proposition 3.13]), i.e. such that the following diagramme commutes:

\[
\begin{array}{ccc}
\pi_X^Z(A) & \overset{\phi}{\longrightarrow} & C(X) \\
\downarrow & & \downarrow \\
A & \overset{\varphi}{\longrightarrow} & C(Z)
\end{array}
\]

3. Tietze extension properties

Given a second countable compact Hausdorff space \( X \) and a closed non empty subspace \( Y \subset X \), we describe in this section when a continuous \( C(Y) \)-algebra \( A \) can be extended to \( X \), i.e. when there exists a continuous \( C(X) \)-algebra isomorphic \( \pi_Y^X(D) \cong A \).

If the \( \text{C}^* \)-algebra \( A \) is an exact separable \( \text{C}^* \)-algebra, then there exists a unital embedding of the \( C(Y) \)-algebra \( A \) into the trivial \( C(Y) \)-algebra \( C(Y; \mathcal{O}_2) \cong C(Y) \otimes \mathcal{O}_2 \), where \( \mathcal{O}_2 \) is the unital Cuntz \( \text{C}^* \)-algebra generated by two isometries \( s_1, s_2 \) satisfying the relation \( 1_{\mathcal{O}_2} = s_1(s_1)^* + s_2(s_2)^* \) \([3]\). Hence, the continuous \( C(X) \)-algebra \( D := \{ f \in C(X, \mathcal{O}_2) ; \pi_Y^X(f) \in A \} \) answers the question in that case.

But there are continuous \( C(Y) \)-algebras which are not exact \( \text{C}^* \)-algebras. Thus, in order to study extensions in the general case, let us define in \( X \times Y \times [0,1] \):
- the open subspace \( U = \{(x,y,t) \in X \times Y \times [0,1] ; 0 < t \} \)
- the closed subspace \( Z = \{(x,y,t) \in X \times Y \times [0,1] ; 0 \leq t.d(x,Y) \leq 2d(x,Y) - d(x,y) \} \).

And let \( \bar{d} \) be the metric on \( Z \) given by \( \bar{d}((x,y,t),(x',y',t')) = d(x,x') + d(y,y') + |t-t'| \).

**Proposition 3.1.** \([6]\) The coordinate map \( p_1 : (x,y,t) \mapsto x \) gives a structure of \( C(X) \)-algebra on \( C(Z) \) and the ideal \( C_0(U \cap Z) \) is a continuous \( C(X) \)-algebra such that \( C_0(U \cap Z) |_Y \cong C_0(Y \times (0,1]) \), i.e. the map \( (x,y,t) \in U \cap Z \mapsto x \in X \) is open.

**Proof.** Given a function \( f \in C_0(U \cap Z) \), let us prove the continuity of the function

\[ x \in X \mapsto \|\pi_x^Y(f)\| = \sup\{|f(z)| ; z \in p_1^{-1}\{x\}\} \]

This map is already upper semi-continuous (u. s. c.) by construction. Hence, it only remains to show that for any point \( x_0 \in X \) and any constant \( \varepsilon > 0 \), one has \( \|\pi_x^Y(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon \) for all points \( x \) in a neighbourhood of \( x_0 \) in \( X \).

The uniform continuity of the function \( f \) implies that there exists \( \delta > 0 \) such that \( |f(z) - f(z')| < \varepsilon \) for all \( z, z' \) in \( Z \) with \( \bar{d}(z,z') < \delta \). Now three cases can appear:
1) If \( x_0 \in Y \) and \( x \in Y \) satisfies \( d(x_0, x) < \delta/2 \), then \( |f(x, x, t) - f(x_0, x_0, t)| < \varepsilon \) for all \( t \in [0, 1] \). And so \( \|\pi^X_x(f)\| > \|\pi^X_{x_0}(f)\| - \varepsilon \).

2) If \( x_0 \in Y \) and \( x \in X \setminus Y \) satisfies \( d(x_0, x) < \delta/4 \), then for all \( y \in Y \), the relation \( d(x, y) \leq 2d(x, Y) \) implies that \( d(y, x_0) \leq d(y, x) + d(x, x_0) \leq 2d(x, Y) + d(x, x_0) \leq \frac{3}{2} \delta \) and so \( |f(x, y, t) - f(x_0, x_0, t)| < \varepsilon \) for all \( t \in [0, 2 - \frac{d(x,y)}{d(x,Y)}] \). Whence the inequality \( \|\pi^X_x(f)\| > \|\pi^X_{x_0}(f)\| - \varepsilon \).

3) If \( x_0 \not\in Y \) and the triple \((x_0, y_0, t_0) \in U \cap Z \) satisfies \( |f(x_0, y_0, t_0)| = \|\pi^X_{x_0}(f)\| \neq 0 \), then \( d(x_0, y_0) < 2d(x_0, Y) \). Thus, there exists by continuity a constant \( \alpha(x_0) \in [0, \delta/2] \) such that all \( x \in X \) in the ball of radius \( \alpha(x_0) \) around \( x_0 \) satisfy:

\[
\begin{align*}
a) & \ d(x, Y) > 0, \quad b) \ d(x, y) < 2d(x, Y), \quad c) \ t_0 < 2 - \frac{d(x,y)}{d(x,Y)} + \delta/2. \\
\end{align*}
\]

And so \( \|\pi^X_x(f)\| \geq \left| f(x, y_0, \inf\{t_0, 2 - \frac{d(x,y)}{d(x,Y)}\}) \right| > \|\pi^X_{x_0}(f)\| - \varepsilon. \) \( \square \)

**Remark 3.2.** S. Wassermann pointed out that if \( Y = \{0, 1\} \subset X = [0, 1] \), then \( Z = \{(x, 0, t) \in [0, 1] \times \{0\} \times [0, 1]; t \leq \frac{2-3x}{1-x} \} \cup \{(x, 1, t) \in [0, 1] \times \{1\} \times [0, 1]; t \leq \frac{3x-1}{x} \} \). Hence, the \( C(X) \)-algebra \( C(Z) \) is not continuous at \( x = \frac{1}{3} \) and \( x = \frac{2}{3} \).

But the essential ideal \( C_0(Z \cap \{0, 1\} \times \{0, 1\} \times \{0, 1\}) \) is a continuous \( C(X) \)-algebra.

The following Corollary will be essential in the proof of Proposition [4.2]

**Corollary 3.3.** Let \( X \) be a second countable compact space, \( Y \subset X \) a non zero closed subset and \( A \) a continuous \( C(Y) \)-algebra.

\[
\begin{align*}
a) & \ B := C(X) \otimes A \otimes C([0, 1]) \text{ is a continuous } C(X \times Y \times [0, 1]) \text{-algebra.} \\
b) & \ D := [C_0(U).B]_Y = C_0(U).B/C_0(U \setminus U \cap Z).B \text{ is a continuous } C(X) \text{-algebra.} \\
c) & \text{There is an isomorphism of } C(Y) \text{-algebras } D_Y \cong A \otimes C_0([0, 1]). \\
\end{align*}
\]

**Proof.** a) holds because the \( C^* \)-algebras \( C(X) \) and \( C_0((0, 1]) \) are nuclear.

b) Let \( b \in D \). Then for all \( x \in X \), we have

\[
\|\pi^X_x(b)\| = \|b + C_0(X \setminus \{x\}).D\| = \sup\{\|\pi^X_z(b)\|; z \in p^{-1}_1(\{x\})\},
\]

whence the continuity of the map \( x \mapsto \|\pi^X_x(b)\| \) by a) and Proposition [3.1]

c) One has \( D_Y \cong [C_0(U).B]_Y \cong A \otimes C_0([0, 1]) \) by Proposition [3.1] \( \square \)

4. Amalgamated tensor products of continuous \( C(X) \)-algebras

Given a fixed compact Hausdorff space \( X \), we study in this section the continuity properties of the different tensor products amalgamated over \( C(X) \) of two given continuous \( C(X) \)-algebras \( A \) and \( B \).

More precisely, let \( A \otimes B \) denote the algebraic tensor product (over \( \mathbb{C} \)) of \( A \) and \( B \), let \( \mathcal{I}_X(A, B) \) be the ideal in \( A \otimes B \) generated by the differences \( a \otimes b - a \otimes fb \) \((a \in A, b \in B, f \in C(X))\) and let \( A \otimes^\mathcal{I}_X B \) denote the quotient of \( A \otimes B \) by \( \mathcal{I}_X(A, B) \).

If \( C_\Delta(X \times X) \subset C(X \times X) \) is the ideal of continuous function of \( X \times X \) which are
zero on the diagonal and \( A \overset{m}{\otimes} B \) (resp. \( A \overset{M}{\otimes} B \)) is the minimal (resp. maximal) tensor product over \( \mathbb{C} \) of the two continuous \( C(X) \)-algebras \( A \) and \( B \), then the quotient \( A \overset{m}{\otimes} B := A \overset{m}{\otimes} B / C_\Delta(X \times X)A \overset{m}{\otimes} B \) (resp. \( A \overset{M}{\otimes} B := A \overset{M}{\otimes} B / C_\Delta(X \times X)A \overset{M}{\otimes} B \)) is the minimal (resp. maximal) completion of the algebraic amalgamated tensor product \( A \odot B \). Further, the \(*\)-algebra \( A \odot B \) embeds in the \( C(X) \)-algebra \( A \overset{m}{\otimes} B \) \( \square \)). and we have

\[
\forall x \in X, \quad (A \overset{m}{\otimes} B)_x \cong A_x \otimes B_x \quad \text{and} \quad (A \overset{M}{\otimes} B)_x \cong A_x^M \otimes B_x .
\]

Let us also recall a characterisation of exactness given by Kirchberg and Wassermann.

**Proposition 4.1.** \([8, \text{Theorem 4.5}]\) Let \( Y = \mathbb{N} \cup \{\infty\} \) be the one point compactification of \( \mathbb{N} \) and let \( D \) be a \( C^* \)-algebra. Then the following assertions are equivalent.

i) The \( C^* \)-algebra \( A \) is exact.

ii) For all continuous \( C(Y) \)-algebra \( B \), the minimal tensor product \( A \overset{m}{\otimes} B \) is a continuous \( C(Y) \)-algebra with fibres \( A \overset{m}{\otimes} B_y \) \( (y \in Y) \).

It induces the following results for fibrewise tensor products of continuous \( C(X) \)-algebras.

**Proposition 4.2.** \([6, \square]\) Let \( X \) be a second countable compact Hausdorff space and \( A \) a separable unital continuous \( C(X) \)-algebra.

If the topological space \( X \) is perfect (i.e. without isolated point), then the following assertions \( \alpha_e \) and \( \beta_e \) (resp. \( \alpha_n \) and \( \beta_n \)) are equivalent.

\( \alpha_e \) The \( C^* \)-algebra \( A \) is exact.

\( \beta_e \) For all continuous \( C(X) \)-algebra \( B \), the amalgamated tensor product \( A \overset{m}{\otimes} B \) is a continuous \( C(X) \)-algebra with fibres \( A \overset{m}{\otimes} B_x \) \( (x \in X) \).

\( \alpha_n \) The \( C^* \)-algebra \( A \) is nuclear.

\( \beta_n \) For all continuous \( C(X) \)-algebra \( B \), the amalgamated tensor product \( A \overset{M}{\otimes} B \) is a continuous \( C(X) \)-algebra with fibres \( A \overset{M}{\otimes} B_x \) \( (x \in X) \).

**Proof.** \( \alpha_n \Rightarrow \beta_n \) If the \( C^* \)-algebra \( A \) is exact, then the spatial tensor product \( A \overset{m}{\otimes} D \) is a continuous \( C(X \times X) \)-algebra with fibres \( A_x \overset{m}{\otimes} D_{x'} \) \( (x, x' \in X) \) \( \square \)). Hence, its restriction to the diagonal is as desired.

\( \beta_n \Rightarrow \alpha_e \) Suppose conversely that the \( C(X) \)-algebra \( A \) satisfies \( \beta_e \). And let us prove step by step that the \( C^* \)-algebra \( A \) is exact.

**Step a)** All the fibres \( A_x \) are exact \( (x \in X) \). Indeed, given a point \( x \) in \( X \), take a sequence of points \( x_n \) in \( X \) converging to \( x \) such that there is a topological isomorphism
Let \( X := \{ x_n : n \in \mathbb{N} \} \cup \{ x \} \cong \mathbb{N} \cup \{ \infty \} \). Then, for any separable continuous \( C(Y) \)-algebra \( B \), there is a continuous \( C(X) \)-algebra \( B \) such that \( B = B \otimes C_0((0, 1]) \) (Corollary 3.3).

Now, the continuity of the \( C(X) \)-algebra \( B \otimes \) \( A \) given by \( \beta_e \) implies that of its restriction \( (B \otimes A)|_Y \cong (C_0((0, 1]) \otimes B) \otimes A|_Y, \) whence that of the \( C(Y) \)-algebra \( B \otimes A|_Y \) since there is an isometric \( C(Y) \)-linear embedding \( B \hookrightarrow B|_Y \). And this implies the exactness of the \( C^* \)-algebra \( A \) by Proposition 4.1.

**Step b)** If \( B \) is a \( C^\ast \)-algebra and \( B \) is the constant \( C(X) \)-algebra \( C(X; B) \), then for all \( x \in X \), we have the exact sequence
\[
0 \to C_X(X)A \otimes B \to (A \otimes B)_x = A \otimes B \to A \otimes B \to 0.
\]

**Step c)** If \( B \) is a \( C(X) \)-algebra, then for all \( x \in X \), we have the sequence of epimorphisms \( (A \otimes B)_x \to (A \otimes B)_x \to A \otimes B_x \)

**Step d)** Now, let \( B \) be a \( C^\ast \)-algebra, \( K \triangleleft B \) a closed two sided ideal in \( B \) and take an element \( d \in \ker\{A \otimes B \to A \otimes B/K\} \). Then for all \( x \in X \), we have
\[
d_x \in \ker\{A \otimes B_x \to (A \otimes B/K)_x\} = \ker\{A \otimes B_x \to A \otimes B/K\} \quad \text{by b)}
\]
\[
= A_x \otimes K \quad \text{by a)}
\]
\[
= (A \otimes K)_x \quad \text{by c)}
\]

Thus, \( d \in A \otimes K \). And so, the \( C^\ast \)-algebra \( A \) is exact.

\( \alpha_n \Rightarrow \beta_n \) has a similar proof to that of \( \alpha_e \Rightarrow \beta_e \).

\( \beta_n \Rightarrow \alpha_n \) If a \( C^\ast \)-algebra \( A \) satisfies \( \beta_n \), then all the fibres \( A_x (x \in X) \) are nuclear by [8, Theorem 3.2] and so the \( C^\ast \)-algebra \( A \) itself is nuclear (see e.g. [2, Proposition 3.23]).

**Remark 4.3.** These characterisations do not hold anymore if the compact space \( X \) is not perfect. Indeed, if the space \( X \) is reduced to a point, then both the amalgamated tensor products \( A \otimes B \) and \( A \otimes B \) are constant, hence continuous.

Proposition 4.2 implies the following characterisation of exact continuous \( C(X) \)-algebras in the framework of \( C(X) \)-algebras.

**Corollary 4.4.** Let \( X \) be a perfect compact metric space and \( A \) be a separable continuous \( C(X) \)-algebra. Then the following are equivalent

1. The \( C^\ast \)-algebra \( A \) is exact.
2. For all exact sequence of continuous \( C(X) \)-algebras \( 0 \to J \to B \to D \to 0 \), the sequence \( 0 \to A \otimes C(X) J \to A \otimes C(X) B \to A \otimes C(X) D \to 0 \) is exact.
Proof. (2)⇒(1) If the unital continuous $C(X)$-algebra $A$ satisfies (2) and the sequence $0 \to J_0 \to B_0 \to D_0 \to 0$ is an exact sequence of $C^*$-algebras, then the sequence $0 \to J := C(X) \otimes J_0 \to B := C(X) \otimes B_0 \to D := C(X) \otimes D_0 \to 0$ is an exact sequence of $C(X)$-algebras. And the condition (2) implies the exactness of the sequence

$$0 \to A \otimes_{C(X)} J = A \otimes_{C(X)} m J_0 \to A \otimes_{C(X)} B = A \otimes_{C(X)} m B_0 \to A \otimes_{C(X)} D = A \otimes_{C(X)} m D_0 \to 0,$$

whence the exactness of $A$.

(1)⇒(2) If the $C(X)$-algebra $A$ is an exact $C^*$-algebra and $0 \to J \to B \to D \to 0$ is an exact sequence of $C(X)$-algebras, then the two first lines of the following diagram are exact by assumption (1)

$$
\begin{array}{c}
C_\Delta(X \times X) A \otimes_{C(X)} J & \to & C_\Delta(X \times X) A \otimes_{C(X)} B & \to & C_\Delta(X \times X) A \otimes_{C(X)} D \\
\downarrow & & \downarrow & & \downarrow \\
A \otimes_{C(X)} J & \to & A \otimes_{C(X)} B & \to & A \otimes_{C(X)} D \\
\downarrow & & \downarrow & & \downarrow \\
A \otimes_{C(X)} J & \to & A \otimes_{C(X)} B & \to & A \otimes_{C(X)} D \\
\end{array}
$$

Besides, all the columns of the diagram are exact by definition, whence the exactness of the last line by a diagram chasing. \hfill \square

5. Amalgamated free products of continuous $C(X)$-algebras

In this section, we describe the continuity properties of different free products amalgamated over $C(X)$ of two given unital continuous $C(X)$-algebras $A$ and $B$.

**Proposition 5.1.** ([4]) Let $X$ be a second countable perfect compact Hausdorff space and $A$ a separable unital continuous $C(X)$-algebra.

Then the following assertions are equivalent.

1. $\alpha_e$) The $C^*$-algebra $A$ is exact.

2. $\gamma_e$) For all separable unital continuous $C(X)$-algebra $B$ and all continuous fields of faithful states $\phi : A \to C(X)$, $\psi : B \to C(X)$, the reduced amalgamated free product $(C, \phi \ast \psi) = (A, \phi) \ast_{C(X)} (B, \psi)$ is a continuous $C(X)$-algebra with fibres $(C_x, \phi_x \ast \psi_x) = (A_x, \phi_x) \ast (B_x, \psi_x)$.

**Proof.** $\gamma_e) \Rightarrow \alpha_e$) Let $B$ be a unital separable continuous $C(X)$-algebra and let $\psi$ be a continuous field of faithful states $\psi$ on $B$. Set $D = A \otimes_{C(X)} B$ and let $E$ be the Hilbert $D$-bimodule $E = L^2(D, \phi \otimes \psi) \otimes_{C(X)} D$.

Then, the following assertions are equivalent ([4 Lemma 4.5]).

1. $D$ is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \otimes_{m} B_x$ ($x \in X$).

2. The Pimsner $C^*$-algebra $\mathcal{T}_D(E \oplus D)$ of the full Hilbert $D$-bimodule $E \oplus D$ is a continuous $C(X)$-algebra with fibres $\mathcal{T}_{D_x}(E_x \oplus D_x)$ ($x \in X$).
But there is a $C(X)$-linear isomorphism $\mathcal{T}_D(E \oplus D) \cong C \times \mathbb{N}$. And so, these two assertions are equivalent to the continuity of the reduced amalgamated free product $(C, \phi \ast \psi) = (A, \phi) \ast_{C(X)} (B, \psi)$ since the group $\mathbb{Z}$ is amenable ([2 Corollaire 5.10]).

$\alpha_x \Rightarrow \gamma_x$) Conversely, if $A$ is an exact C*-algebra and $B$ is a unital separable continuous $C(X)$, the amalgamated tensor product $D = A \underset{C(X)}{\overline{\otimes}} B$ is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \otimes_{C(X)} B_x$ for $x \in X$ ([1]). Hence, the reduced amalgamated free product $(C, \phi \ast \psi) = (A, \phi) \ast_{C(X)} (B, \psi)$ is a continuous $C(X)$-algebra ([1 Theorem 4.1]).

\[ \square \]

Remark 5.2. There is no similar result for full amalgamated free product. Indeed, the full amalgamated free product $A \underset{C(X)}{\ast} B$ of two unital continuous $C(X)$-algebras $A$ and $B$ is always a continuous $C(X)$-algebra with fibres $A_x \ast B_x$ ($x \in X$) ([1] Theorem 3.7]).

Sketch of proof. The algebraic amalgamated free product $A \underset{C(X)}{\mathcal{O}} B$ is a dense $C(X)$-submodule of the amalgamated Haagerup tensor product $A \underset{C(X)}{\mathcal{H}} B$, which itself is contained in the full amalgamated free product $A \underset{C(X)}{\ast} B$ ([2]). And for all $d \in A \underset{C(X)}{\mathcal{O}} B$, one has

\[
\|d_x\|_{A_x \otimes_{C(X)} B_x} = \inf \left\{ \| \sum_i a_i a_i^* \| \cdot \sum_i b_i^* b_i \|^{\frac{1}{2}} : d_x = \sum_i a_i \otimes b_i \right\} 
\]

\[
= \sup \left\{ \left\langle \xi, \sum_i \pi(a_i) \sigma(b_i) \eta \right\rangle : \xi, \eta \text{ unit vectors in the Hilbert space } \ell^2(\mathbb{N}) \right\}
\]

Hence, the map $x \mapsto \|d_x\|$ is both upper and lower semi-continuous if $d \in A \underset{C(X)}{\mathcal{O}} B$.

The proof for the continuity of the map $x \mapsto \|d_x\|$ for elements $d$ in the algebraic amalgamated free product $A \underset{C(X)}{\mathcal{O}} B$ is similar ([1]).

References


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