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# THE BRAUER GROUP AND INDECOMPOSABLE (2, 1)-CYCLES

BRUNO KAHN

ABSTRACT. We show that the torsion in the group of indecomposable (2, 1)-cycles on a smooth projective variety over an algebraically closed field is isomorphic to a twist of its Brauer group, away from the characteristic. In particular, this group is infinite as soon as  $b_2 - \rho > 0$ . We derive a new insight into Roïtman's theorem on torsion 0-cycles over a surface.

## INTRODUCTION

Let  $X$  be a smooth projective variety over an algebraically closed field  $k$ . The group

$$C(X) = H^1(X, \mathcal{K}_2) \simeq CH^2(X, 1) \simeq H^3(X, \mathbf{Z}(2))$$

has been widely studied. Its most interesting part is the *indecomposable quotient*

$$H_{\text{ind}}^1(X, \mathcal{K}_2) \simeq CH_{\text{ind}}^2(X, 1) \simeq H_{\text{ind}}^3(X, \mathbf{Z}(2))$$

defined as the cokernel of the natural homomorphism

$$(1) \quad \text{Pic}(X) \otimes k^* \xrightarrow{\theta} C(X).$$

It vanishes for  $\dim X \leq 1$ .

Let  $\text{Br}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$  be the Brauer group of  $X$ : it sits in an exact sequence

$$(2) \quad 0 \rightarrow \text{NS}(X) \otimes \mathbf{Q}/\mathbf{Z} \rightarrow H_{\text{ét}}^2(X, \mathbf{Q}/\mathbf{Z}(1)) \rightarrow \text{Br}(X) \rightarrow 0$$

(see [7, p. 629, (5.8.4)] for the  $p$ -primary part in characteristic  $p$ ).

**Theorem 1.** *Let  $p$  be the exponential characteristic of  $k$ . There are natural isomorphisms*

$$\begin{aligned} \beta' : \text{Br}(X)\{p'\}(1) &\xrightarrow{\sim} H_{\text{ind}}^3(X, \mathbf{Z}(2))\{p'\} \\ \beta_p : H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)) &\xrightarrow{\sim} H_{\text{ind}}^3(X, \mathbf{Z}(2))\{p\} \end{aligned}$$

where  $\{p\}$  (resp.  $\{p'\}$ ) denotes  $p$ -primary torsion (resp. prime-to- $p$  torsion.)

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Here we write  $A(1)$  for  $\varinjlim_{(n,p)=1} nA \otimes \mu_n$  for a prime-to- $p$  torsion abelian group  $A$ , and

$$H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)) = \varinjlim_s H_{\text{ét}}^1(X, \nu_s(2))$$

where  $\nu_s(2)$  is the  $s$ -th sheaf of logarithmic Hodge-Witt differentials of weight 2 [7, 10, 6].

Theorem 1 gives an interpretation of the Brauer group (away from  $p$ )<sup>1</sup> in terms of algebraic cycles. In view of (2), it also implies:

**Corollary 1.** *If  $b_2 - \rho > 0$ ,  $H_{\text{ind}}^3(X, \mathbf{Z}(2))$  is infinite. In characteristic zero, if  $p_g > 0$  then  $H_{\text{ind}}^3(X, \mathbf{Z}(2))$  is infinite.  $\square$*

To my knowledge, this is the first general result on indecomposable (2,1)-cycles. It relates to the following open question:

*Question 1* (See also Remark 1). Is there a surface  $X$  such that  $b_2 - \rho > 0$  but  $H_{\text{ind}}^3(X, \mathbf{Z}(2)) \otimes \mathbf{Q} = 0$ ?

Many examples of complex surfaces  $X$  for which  $H_{\text{ind}}^3(X, \mathbf{Z}(2))$  is not torsion have been given: we refer for example to [3] and the references therein. In all of them, one shows that a version of the Beilinson regulator with values in a quotient of Deligne cohomology takes non torsion values on this group. On the other hand, there are examples of complex surfaces  $X$  with  $p_g > 0$  for which the regulator vanishes rationally [13, Th. 1.6], but there seems to be no such  $X$  for which one can decide whether  $H_{\text{ind}}^3(X, \mathbf{Z}(2)) \otimes \mathbf{Q} = 0$ .

Question 1 evokes Mumford's nonrepresentability theorem for the Albanese kernel  $T(X)$  in the Chow group  $CH_0(X)$  under the given hypothesis. It is of course much harder, but not unrelated. The link comes through the *transcendental part of the Chow motive of  $X$* , introduced and studied in [8]. If we denote this motive by  $t_2(X)$  as in loc. cit., we have

$$T(X)_{\mathbf{Q}} = \text{Hom}_{\mathbf{Q}}(t_2(X), \mathbb{L}^2) = H^4(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}}$$

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<sup>1</sup>The group  $H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2))$  is very different from  $\text{Br}(X)\{p\}$ : suppose that  $k$  is the algebraic closure of a finite field  $\mathbf{F}_q$  over which  $X$  is defined. In [10, Rk 5.6], Milne proves

$$\det(1 - \gamma t \mid H^i(X, \mathbf{Q}_p(n))) = \prod_{v(a_{ij})=v(q^n)} (1 - (q^n/a_{ij})t)$$

where  $\gamma$  is the ‘‘arithmetic’’ Frobenius of  $X$  over  $\mathbf{F}_q$  and the  $a_{ij}$  are the eigenvalues of the ‘‘geometric’’ Frobenius acting on the crystalline cohomology  $H^i(X/W) \otimes \mathbf{Q}_p$  (or, equivalently, on  $l$ -adic cohomology for  $l \neq p$  by Katz-Messing). We get  $V_p(\text{Br}(X)\{p\})$  for  $i = 2, n = 1$  and  $V_p(H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)))$  for  $i = 2, n = 2$ .

[8, Prop. 7.2.3]. Here, all groups are taken in the category  $\mathbf{Ab} \otimes \mathbf{Q}$  of abelian groups modulo groups of finite exponent and  $\mathrm{Hom}_{\mathbf{Q}}$  denotes the refined Hom group on the category  $\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(k, \mathbf{Q})$  of effective Chow motives with  $\mathbf{Q}$  coefficients (see Section 2 for all this), while  $\mathbb{L}$  is the Lefschetz motive; to justify the last term, note that Chow correspondences act on motivic cohomology, so that motivic cohomology of a Chow motive makes sense. We show:

**Theorem 2** (see Proposition 3). *If  $X$  is a surface, we have an isomorphism in  $\mathbf{Ab} \otimes \mathbf{Q}$ :*

$$H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))_{\mathbf{Q}} \simeq H^3(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}}.$$

**Corollary 2** ([4, Prop. 2.15]). *In Theorem 2, assume that  $k$  has infinite transcendence degree over its prime subfield. If  $T(X) = 0$ , then  $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))$  is finite.*

*Proof.* Under the hypothesis on  $k$ ,  $T(X) = 0 \iff t_2(X) = 0$  [8, Cor. 7.4.9 b)]. Thus,  $T(X) = 0 \Rightarrow H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))_{\mathbf{Q}} = 0$  by Theorem 2. This means that  $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))$  has finite exponent, hence is finite by Theorem 1 and the known structure of  $\mathrm{Br}(X)$ .  $\square$

*Remark 1.* 1) For  $l \neq p$ ,  $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))\{l\}$  finite  $\iff b_2 - \rho = 0$  by Theorem 1. Under Bloch's conjecture, this implies  $t_2(X) = 0$  [8, Cor. 7.6.11], hence  $T(X) = 0$  and (by Theorem 2)  $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))$  finite. This provides conjectural converses to Corollaries 1 (for a surface) and 2.  
2) The quotient of  $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))_{\mathrm{tors}}$  by its maximal divisible subgroup is dual to  $\mathrm{NS}(X)_{\mathrm{tors}}$ , at least away from  $p$ : we leave this to the interested reader.

We also apply Theorem 2 to give in Section 4 a new proof of Roïtman's theorem that  $T(X)$  is uniquely divisible, up to a group of finite exponent.

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## 1. PROOF OF THEOREM 1

This proof is an elaboration of the arguments of Colliot-Thélène and Raskind in [4], completed by Gros-Suwa [6, Ch. IV] for  $l = \mathrm{char} k$ . We use motivic cohomology as it smoothens the exposition and is more inspirational, but stress that these ideas go back to Bloch [2] and [4].

We refer to [9, §2] for an exposition of ordinary and étale motivic cohomology and the facts used below, especially to [9, Th. 2.6] for the comparison with étale cohomology of twisted roots of unity and logarithmic Hodge-Witt sheaves.

Multiplication by  $l^s$  on étale motivic cohomology yields “Bockstein” exact sequences

$$0 \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}(n))/l^s \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}/l^s(n)) \rightarrow {}_{l^s}H_{\text{ét}}^{i+1}(X, \mathbf{Z}(n)) \rightarrow 0$$

for any prime  $l$ ,  $s \geq 1$ ,  $n \geq 0$  and  $i \in \mathbf{Z}$ . Since  $\varprojlim^1 H_{\text{ét}}^i(X, \mathbf{Z}(n))/l^s = 0$ , one gets in the limit exact sequences:

$$(3) \quad 0 \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}(n)) \xrightarrow{a} H_{\text{ét}}^i(X, \hat{\mathbf{Z}}(n)) \xrightarrow{b} \hat{T}(H_{\text{ét}}^{i+1}(X, \mathbf{Z}(n))) \rightarrow 0$$

where  $\hat{T}(-) = \text{Hom}(\mathbf{Q}/\mathbf{Z}, -)$  denotes the total Tate module. This first yields:

**Proposition 1.** *For  $i \neq 2n$ ,  $\text{Ker } b$  is finite in (3) and  $H_{\text{ét}}^i(X, \mathbf{Z}(n))$  is an extension of a finite group by a divisible group.*

*Proof.* This is the argument of [4, 1.8 and 2.2]. Let us summarise it:  $H_{\text{ét}}^i(X, \mathbf{Z}(n))$  is “of weight 0” and  $H_{\text{ét}}^i(X, \hat{\mathbf{Z}}(n))$  is “of weight  $i - 2n$ ” by Deligne’s proof of the Weil conjectures. It follows that  $a$  has finite image in every  $l$ -component, hence has finite image by Gabber’s theorem [5]. One derives the structure of  $H_{\text{ét}}^i(X, \mathbf{Z}(n))$  from this.  $\square$

Consider now the case  $n = 2$ . Recall that  $H^i(X, \mathbf{Z}(2)) \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbf{Z}(2))$  for  $i \leq 3$  from the Merkurjev-Suslin theorem (cf. [9, (2-6)]). For  $l \neq p$ , let

$$\begin{aligned} H_{\text{ind}}^2(X, \mu_{l^n}^{\otimes 2}) &= \text{Coker}(\text{Pic}(X) \otimes \mu_{l^n} \rightarrow H_{\text{ét}}^2(X, \mu_{l^n}^{\otimes 2})) \\ H_{\text{ind}}^2(X, \mathbf{Z}_l(2)) &= \text{Coker}(\text{Pic}(X) \otimes \mathbf{Z}_l(1) \rightarrow H_{\text{ét}}^2(X, \mathbf{Z}_l(2))). \end{aligned}$$

**Lemma 1.** *There is a canonical isomorphism  $H_{\text{ind}}^2(X, \mathbf{Z}_l(2)) \simeq T_l(\text{Br}(X))(1)$ . In particular, this group is torsion-free.*

*Proof.* Straightforward from the Kummer exact sequence.  $\square$

We have a commutative diagram

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}(X) \otimes \mu_{l^s} & \longrightarrow & H_{\text{ét}}^2(X, \mu_{l^s}^{\otimes 2}) & \longrightarrow & H_{\text{ind}}^2(X, \mu_{l^s}^{\otimes 2}) \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \alpha_s & & \\ 0 & \longrightarrow & {}_{l^s}(\text{Pic}(X) \otimes k^*) & \longrightarrow & {}_{l^s}H^3(X, \mathbf{Z}(2)) & \longrightarrow & {}_{l^s}H_{\text{ind}}^3(X, \mathbf{Z}(2)) \longrightarrow 0 \end{array}$$

where the upper row is exact and the lower row is a complex. This diagram is equivalent to the one in [4, 2.8], but the proof of its commutativity is easier, as a consequence of the compatibility of Bockstein

boundaries with cup-product in hypercohomology. This yields maps:

$$(5) \quad H_{\text{ind}}^2(X, \mu_{l^s}^{\otimes 2}) \xrightarrow{\beta_s} {}_l H_{\text{ind}}^3(X, \mathbf{Z}(2))$$

an inverse limit commutative diagram

$$(6) \quad \begin{array}{ccccccc} 0 \rightarrow & \text{NS}(X) \otimes \mathbf{Z}_l(1) & \longrightarrow & H_{\text{ét}}^2(X, \mathbf{Z}_l(2)) & \longrightarrow & H_{\text{ind}}^2(X, \mathbf{Z}_l(2)) & \rightarrow 0 \\ & \wr \downarrow & & \hat{\alpha} \downarrow & & \hat{\beta} \downarrow & \\ 0 \rightarrow & T_l(\text{Pic}(X) \otimes k^*) & \longrightarrow & T_l(H^3(X, \mathbf{Z}(2))) & \longrightarrow & T_l(H_{\text{ind}}^3(X, \mathbf{Z}(2))) & \rightarrow 0. \end{array}$$

(note that  $\text{Pic}(X) \otimes \mu_{l^s} \xrightarrow{\sim} \text{NS}(X) \otimes \mu_{l^s}$ ) and a direct limit commutative diagram

$$(7) \quad \begin{array}{ccccccc} 0 \rightarrow & \text{Pic}(X) \otimes \mu_{l^\infty} & \longrightarrow & H^2(X, \mathbf{Q}_l/\mathbf{Z}_l(2)) & \longrightarrow & \text{Br}(X)\{l\}(1) & \rightarrow 0 \\ & \wr \downarrow & & \alpha_l \downarrow & & \beta_l \downarrow & \\ 0 \rightarrow & (\text{Pic}(X) \otimes k^*)\{l\} & \longrightarrow & H^3(X, \mathbf{Z}(2))\{l\} & \longrightarrow & H_{\text{ind}}^3(X, \mathbf{Z}(2))\{l\} & \rightarrow 0 \end{array}$$

where  $\beta_l$  defines the map  $\beta'$  in Theorem 1.

We shall use the following fact, which is proven in [4, 2.7] (and could be reproven here with motivic cohomology in the same fashion):

**Lemma 2.** *In (1),  $N := \text{Ker } \theta$  has no  $l$ -torsion.*

**Proposition 2** (cf. [4, Rk. 2.13]).  *$\beta_s$  is surjective in (5) and  $\hat{\beta}$  is bijective in (6);  $N$  is uniquely divisible; the lower row of (7) is exact and  $\beta_l$  is bijective.*

*Proof.* Since  $\text{Pic}(X) \otimes k^*$  is  $l$ -divisible, Lemma 2 yields exact sequences

$$(8) \quad 0 \rightarrow {}_l(\text{Pic}(X) \otimes k^*) \rightarrow {}_l A \rightarrow N/l^s \rightarrow 0$$

$$(9) \quad 0 \rightarrow {}_l A \rightarrow {}_l H^3(X, \mathbf{Z}(2)) \rightarrow {}_l H_{\text{ind}}^3(X, \mathbf{Z}(2)) \rightarrow 0$$

where  $A = \text{Im } \theta$ , and (9) implies the surjectivity of  $\beta_s$ , hence of  $\hat{\beta}$  since the groups  $H_{\text{ind}}^2(X, \mu_{l^s}^{\otimes 2})$  are finite. Since  $\alpha_s$  is surjective in (4), we also get that all groups in (8) and (9) are finite. Now the upper row of (6) is exact; in its lower row, the homology at  $T_l(H^3(X, \mathbf{Z}(2)))$  is isomorphic to  $\hat{N}_l$  thanks to (8) and (9). A diagram chase then yields an exact sequence

$$H^2(X, \mathbf{Z}(2))_l^\wedge \simeq \text{Ker } \hat{\alpha} \rightarrow \text{Ker } \hat{\beta} \rightarrow \hat{N}_l \rightarrow 0$$

(see Proposition 1). Thus  $\text{Ker } \hat{\beta}$  is of weight 0, hence finite, hence 0 by Lemma 1. This also shows the divisibility of  $N$ , which thanks to (8) and (9) implies the exactness of the lower row of (4), hence of (7). Now  $\alpha_l$  is surjective, and also injective since  $\text{Ker } \alpha_l \simeq H^2(X, \mathbf{Z}(2)) \otimes \mathbf{Q}_l/\mathbf{Z}_l$  is 0 by Proposition 1. Hence  $\beta_l$  is bijective.  $\square$

The case of  $p$ -torsion is similar and easier: by the same argument as above,

$$H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)) \xrightarrow{\sim} H^3(X, \mathbf{Z}(2))\{p\}$$

and that  $H^3(X, \mathbf{Z}(2))\{p\} \xrightarrow{\sim} H_{\text{ind}}^3(X, \mathbf{Z}(2))\{p\}$  since  $k^*$  is uniquely  $p$ -divisible, hence also  $\text{Pic}(X) \otimes k^*$ . This concludes the proof of Theorem 1.

## 2. REFINED Hom GROUPS

Let  $\mathcal{A}$  be an additive category; write  $\mathcal{A} \otimes \mathbf{Q}$  for the category with the same objects as  $\mathcal{A}$  and Hom groups tensored with  $\mathbf{Q}$ , and  $\mathcal{A} \boxtimes \mathbf{Q}$  for the pseudo-abelian envelope of  $\mathcal{A} \otimes \mathbf{Q}$ . If  $\mathcal{A}$  is abelian, then  $\mathcal{A} \otimes \mathbf{Q} = \mathcal{A} \boxtimes \mathbf{Q}$  is still abelian and is the localisation of  $\mathcal{A}$  by the Serre subcategory  $\mathcal{A}_{\text{tors}}$  of objects  $A$  such that  $n1_A = 0$  for some integer  $n > 0$  (e.g. [1, Prop. B.3.1]).

For  $\mathcal{A} = \mathbf{Ab}$ , the category of abelian groups, one has a natural functor “tensoring objects with  $\mathbf{Q}$ ”

$$\mathbf{Ab} \otimes \mathbf{Q} \rightarrow \mathbf{Vec}_{\mathbf{Q}}$$

to  $\mathbf{Q}$ -vector spaces, but this functor is an equivalence of categories only on the full subcategory of  $\mathbf{Ab} \otimes \mathbf{Q}$  given by finitely generated abelian groups. For clarity, we shall write

$$(10) \quad A_{\mathbf{Q}}, \quad A \otimes \mathbf{Q}$$

for the image of an abelian group  $A \in \mathbf{Ab}$  respectively in  $\mathbf{Ab} \otimes \mathbf{Q}$  and  $\mathbf{Vec}_{\mathbf{Q}}$ .

Let  $F$  be an additive functor (covariant or contravariant) from  $\mathcal{A}$  to  $\mathbf{Ab}$ , the category of abelian groups: it then induces a functor

$$F_{\mathbf{Q}} : \mathcal{A} \boxtimes \mathbf{Q} \rightarrow \mathbf{Ab} \otimes \mathbf{Q}.$$

In particular, we get a bifunctor

$$\text{Hom}_{\mathbf{Q}} : (\mathcal{A} \boxtimes \mathbf{Q})^{\text{op}} \times \mathcal{A} \boxtimes \mathbf{Q} \rightarrow \mathbf{Ab} \otimes \mathbf{Q}$$

which refines the bifunctor  $\text{Hom}$  of  $\mathcal{A} \boxtimes \mathbf{Q}$ .

We shall apply this to  $\mathcal{A} = \mathcal{M}_{\text{rat}}^{\text{eff}}(k)$ , the category of effective Chow motives with integral coefficients: the category  $\mathcal{A} \boxtimes \mathbf{Q}$  is then equivalent to the category  $\mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$  of Chow motives with rational coefficients.

## 3. CHOW-KÜNNETH DECOMPOSITION OF $\mathcal{K}_2$ -COHOMOLOGY

In this section,  $X$  is a connected surface. Its Chow motive  $h(X) \in \mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$  then enjoys a refined Chow-Künneth decomposition

$$(11) \quad h(X) = h_0(X) \oplus h_1(X) \oplus h_2^{\text{alg}}(X) \oplus t_2(X) \oplus h_3(X) \oplus h_4(X)$$

[8, Prop. 7.2.1 and 7.2.3]. The projectors defining this decomposition act on the groups  $H^i(X, \mathbf{Z}(2))_{\mathbf{Q}}$ ; we propose to compute the corresponding direct summands  $H^i(M, \mathbf{Z}(2))_{\mathbf{Q}}$ . To be more concrete, we shall express this in terms of the  $\mathcal{K}_2$ -cohomology of  $X$ .

We keep the notation

$$H_{\text{ind}}^1(X, \mathcal{K}_2) = \text{Coker}(\text{Pic}(X) \otimes k^* \rightarrow H^1(X, \mathcal{K}_2))$$

to which we adjoin

$$H_{\text{ind}}^0(X, \mathcal{K}_2) = \text{Coker}(K_2(k) \rightarrow H^0(X, \mathcal{K}_2)).$$

To relate with the notation in Section 1, recall that  $H^2(k, \mathbf{Z}(2)) = K_2(k)$  and  $H^2(X, \mathbf{Z}(2)) = H^0(X, \mathcal{K}_2)$ .

We shall also need a smooth connected hyperplane section  $C$  of  $X$ , appearing in the construction of (11) [11, 12], and its own Chow-Künneth decomposition attached to the choice of a rational point:

$$(12) \quad h(C) = h_0(C) \oplus h_1(C) \oplus h_2(C).$$

The projectors defining (12) have integral coefficients, while those defining (11) only have rational coefficients in general.

The following proposition extends the computations of [8, 7.2.1 and 7.2.3] to weight 2 motivic cohomology.

**Proposition 3.** *a) We have the following table for  $H^i(M, \mathbf{Z}(2))$ :*

$M =$	$h_0(C)$	$h_1(C)$	$h_2(C)$
$i = 2$	$K_2(k)$	$H_{\text{ind}}^0(C, \mathcal{K}_2)$	0
$i = 3$	0	$V(C)$	$k^*$
$i > 3$	0	0	0

where  $V(C) = \text{Ker}(H^1(C, \mathcal{K}_2) \xrightarrow{N} k^*)$  is Bloch's group.

*b) We have the following table for  $H^i(M, \mathbf{Z}(2))$ , where all groups are taken in  $\mathbf{Ab} \otimes \mathbf{Q}$  (see Section 2):*

$M =$	$h_0(X)$	$h_1(X)$	$h_2^{\text{alg}}(X)$	$t_2(X)$	$h_3(X)$	$h_4(X)$
$i = 2$	$K_2(k)$	$A$	0	$B$	0	0
$i = 3$	0	$\text{Pic}^0(X)k^*$	$\text{NS}(X) \otimes k^*$	$H_{\text{ind}}^1(X, \mathcal{K}_2)$	0	0
$i = 4$	0	0	0	$T(X)$	$\text{Alb}(X)$	$\mathbf{Z}$
$i > 4$	0	0	0	0	0	0

where

$$\text{Pic}^0(X)k^* = \text{Im}(\text{Pic}^0(X) \otimes k^* \rightarrow H^1(X, \mathcal{K}_2))$$

$$A = \text{Im}(H_{\text{ind}}^0(X, \mathcal{K}_2) \rightarrow H_{\text{ind}}^0(C, \mathcal{K}_2))$$

$$B = \text{Ker}(H_{\text{ind}}^0(X, \mathcal{K}_2) \rightarrow H_{\text{ind}}^0(C, \mathcal{K}_2)).$$



*Proof.* We proceed by exclusion as in the proof of [8, Th. 7.8.4]. Let us start with a). We use the notation (10) of Section 2.

- For  $i > 3$ ,  $H^i(M, \mathbf{Z}(2))_{\mathbf{Q}}$  is a direct summand of  $H^i(C, \mathbf{Z}(2))_{\mathbf{Q}} = 0$ .
- One has  $h_2(C) = \mathbb{L}$ , hence

$$H^i(h_2(C), \mathbf{Z}(2))_{\mathbf{Q}} = H^{i-2}(k, \mathbf{Z}(1))_{\mathbf{Q}} = \begin{cases} k_{\mathbf{Q}}^* & \text{if } i = 3 \\ 0 & \text{else.} \end{cases}$$

- One has

$$H^i(h_0(C), \mathbf{Z}(2))_{\mathbf{Q}} = H^i(k, \mathbf{Z}(2))_{\mathbf{Q}} = \begin{cases} K_2(k)_{\mathbf{Q}} & \text{if } i = 2 \\ 0 & \text{if } i > 2. \end{cases}$$

- The case of  $M = h_1(C)$  follows from the two previous ones by exclusion.

Let us come to b).

- For  $i > 4$ ,  $H^i(M, \mathbf{Z}(2))_{\mathbf{Q}}$  is a direct summand of  $H^i(X, \mathbf{Z}(2))_{\mathbf{Q}} = 0$ .
- One has  $h_4(X) = \mathbb{L}^2$ , hence

$$H^i(h_4(X), \mathbf{Z}(2))_{\mathbf{Q}} = H^{i-4}(k, \mathbf{Z})_{\mathbf{Q}} = \begin{cases} \mathbf{Z}_{\mathbf{Q}} & \text{if } i = 4 \\ 0 & \text{else.} \end{cases}$$

- One has  $h_3(X) = h_1(X)(1)$ , hence

$$H^i(h_3(X), \mathbf{Z}(2))_{\mathbf{Q}} = H^{i-2}(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}}.$$

As  $h_1(X)$  is a direct summand of  $h_1(C)$ ,  $H^{i-2}(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}}$  is a direct summand of  $H^{i-2}(C, \mathbf{Z}(1))_{\mathbf{Q}}$ . This group is 0 for  $i \neq 3, 4$ . For  $i = 3$ , one has  $H^1(C, \mathbf{Z}(1))_{\mathbf{Q}} = H^1(h_0(C), \mathbf{Z}(1))_{\mathbf{Q}}$ , hence

$$H^1(h_1(C), \mathbf{Z}(1))_{\mathbf{Q}} = H^1(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}} = 0.$$

For  $i = 4$ ,  $H^2(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}} = \text{Alb}(X)_{\mathbf{Q}}$  (cf. Murre [11]).

- One has  $h_2^{\text{alg}}(X) = \text{NS}(X)(1)$ , hence

$$\begin{aligned} H^i(h_2^{\text{alg}}(X), \mathbf{Z}(2))_{\mathbf{Q}} &= (H^{i-2}(k, \mathbf{Z}(1)) \otimes \text{NS}(X))_{\mathbf{Q}} \\ &= \begin{cases} (\text{NS}(X) \otimes k^*)_{\mathbf{Q}} & \text{if } i = 3 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

- One has

$$H^i(h_0(X), \mathbf{Z}(2))_{\mathbf{Q}} = H^i(k, \mathbf{Z}(2))_{\mathbf{Q}} = \begin{cases} K_2(k)_{\mathbf{Q}} & \text{if } i = 2 \\ 0 & \text{if } i > 2. \end{cases}$$

- As  $h^1(X)$  is a direct summand of  $h^1(C)$ ,  $H^i(h^1(X), \mathbf{Z}(2))_{\mathbf{Q}}$  is a direct summand of  $H^i(C, \mathbf{Z}(2))_{\mathbf{Q}}$ : this group is therefore 0  $i > 3$ . This completes row  $i = 4$  by exclusion.
- The action of refined Chow-Künneth projectors respects the homomorphism  $(\text{Pic}(X) \otimes k^*)_{\mathbf{Q}} \rightarrow H^3(X, \mathbf{Z}(2))_{\mathbf{Q}}$ . As the action of  $\pi_2^{\text{tr}}$  (defining  $t_2(X)$ ) is 0 on  $\text{Pic}(X)_{\mathbf{Q}}$ , we get  $H^3(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \simeq H_{\text{ind}}^1(X, \mathcal{K}_2)_{\mathbf{Q}}$ , which completes row  $i = 3$  by exclusion.
- The construction of  $\pi_2^{\text{tr}}$  [8, proof of 2.3] shows that the composition

$$h(C) \xrightarrow{i^*} h(X) \rightarrow t_2(X)$$

is 0. Hence the composition

$$H^i(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \rightarrow H^i(X, \mathbf{Z}(2))_{\mathbf{Q}} \xrightarrow{i^*} H^i(C, \mathbf{Z}(2))_{\mathbf{Q}}$$

is 0 for all  $i$ . Applying this for  $i = 2$ , we see that  $H^2(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \subseteq B_{\mathbf{Q}}$ . On the other hand,  $H^2(h_1(X), \mathbf{Z}(2))_{\mathbf{Q}}$  is a direct summand of  $H^2(h_1(C), \mathbf{Z}(2))_{\mathbf{Q}}$ , hence injects in  $A_{\mathbf{Q}}$ . By exclusion, we have  $H^2(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \oplus H^2(h_1(X), \mathbf{Z}(2))_{\mathbf{Q}} \simeq H_{\text{ind}}^0(X, \mathbf{Z}(2))_{\mathbf{Q}}$ , hence row  $i = 2$ .

□

*Remark 2.* Let us clarify the “reasoning by exclusion” that has been used repeatedly in this proof. Let  $F$  be a functor from smooth projective varieties to  $\mathbf{Ab} \otimes \mathbf{Q}$ , provided with an action of Chow correspondences. Then  $F$  automatically extends to  $\mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$ , and we wish to compute the effect of a Chow-Künneth decomposition of  $h(X)$  on  $F(X)$ . The reasoning above is as follows in its simplest form:

Suppose that we have a motivic decomposition  $h(X) = M \oplus M'$ , hence a decomposition  $F(X) = F(M) \oplus F(M')$ . Suppose that we know an exact sequence

$$0 \rightarrow A \rightarrow F(X) \rightarrow B \rightarrow 0$$

and an isomorphism  $F(M) \simeq A$ . Then  $F(M') \simeq B$ .

Of course this reasoning is incorrect as it stands; to justify it, one should check that if  $\pi$  is the projector with image  $M$  yielding the decomposition of  $h(X)$ , then  $F(\pi)$  does have image  $A$ . This can be checked in all cases of the above proof, but such a verification would be tedious, double the length of the proof and probably make it unreadable. I hope the reader will not disagree with this expository choice.

#### 4. GENERALISATION

In this section, we take the gist of the previous arguments. For convenience we pass from effective Chow motives  $\mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$  to all Chow motives  $\mathcal{M}_{\text{rat}}(k, \mathbf{Q})$ . Since étale motivic cohomology has an action of

Chow correspondences and verifies the projective bundle formula, it yields well-defined contravariant functors

$$H_{\text{ét}}^i : \mathcal{M}_{\text{rat}}(k, \mathbf{Q}) \rightarrow \mathbf{Ab} \otimes \mathbf{Q}$$

such that  $H_{\text{ét}}^i(X, \mathbf{Z}(n))_{\mathbf{Q}} = H_{\text{ét}}^{i-2n}(h(X)(-n))$  for any smooth projective  $k$ -variety  $X$  and  $i, n \in \mathbf{Z}$ . We also have (contravariant) realisation functors

$$H_l^i : \mathcal{M}_{\text{rat}}(k, \mathbf{Q}) \rightarrow \mathcal{C}_l \otimes \mathbf{Q}$$

extending  $l$ -adic cohomology for  $l \neq \text{char } k$ , where  $\mathcal{C}_l$  denotes the category of  $l\mathbf{Z}$ -adic inverse systems of abelian groups [SGA 5, V.3.1.1]. For  $l = \text{char } k$  we use logarithmic Hodge-Witt cohomology as in Theorem 1 [10, §2], [6].

**Definition 1.** Let  $M \in \mathcal{M}_{\text{rat}}(k, \mathbf{Q})$ . If  $i \in \mathbf{Z}$ , we say that  $M$  is *pure of weight  $i$*  if  $H_l^j(M) = 0$  for all  $j \neq i$  and all primes  $l$ .

For example, if  $h(X) = \bigoplus_{i=0}^{2d} h_i(X)$  is a Chow-Künneth decomposition of the motive  $h(X)$  of a  $d$ -dimensional smooth projective variety  $X$ , then  $h_i(X)$  is pure of weight  $i$ . If  $d = 2$ , the motive  $t_2(X)(-2)$  is pure of weight  $-2$  as a direct summand of  $h_2(X)(-2)$ .

**Theorem 3.** *Let  $M$  be pure of weight  $i$ . Then  $H_{\text{ét}}^j(M)$  is uniquely divisible for  $j \neq i, i + 1$ . If moreover  $i \neq 0$ , then  $H_{\text{ét}}^i(M)$  is uniquely divisible and  $H_{\text{ét}}^{i+1}(M)\{l\} \simeq H_l^i(M) \otimes \mathbf{Q}/\mathbf{Z}$ .*

(An object  $A \in \mathbf{Ab} \otimes \mathbf{Q}$  is *uniquely divisible* if multiplication by  $n$  is an automorphism of  $A$  for any integer  $n \neq 0$ .)

*Proof.* As in Section 1, we have Bockstein exact sequences in  $\mathcal{C}_l \otimes \mathbf{Q}$

$$0 \rightarrow H_{\text{ét}}^j(M)/l^* \xrightarrow{a} H_l^j(M) \rightarrow {}_{l^*}H_{\text{ét}}^{j+1}(M) \rightarrow 0$$

which yields the first statement. For the second one, the weight argument of [4] yields  $\text{Im } a = 0$ .  $\square$

Let  $X$  be a surface. Applying Theorem 3 to  $M = t_2(X)(-2)$  as above, we get that  $H_{\text{ét}}^i(t_2(X), \mathbf{Z}(2))$  is uniquely divisible for  $i \neq 3$  and

$$H_{\text{ét}}^3(t_2(X), \mathbf{Z}(2))\{l\} \simeq H_{\text{tr}}^3(X, \mathbf{Z}_l(2)) \otimes \mathbf{Q}/\mathbf{Z} \simeq \text{Br}(X)\{l\}$$

in  $\mathbf{Ab} \otimes \mathbf{Q}$ , recovering a slightly weaker version of Theorem 1 in view of Proposition 3. For  $i = 4$ , the exact sequence [9, (2-7)]

$$0 \rightarrow CH^2(X) \rightarrow H_{\text{ét}}^4(X, \mathbf{Z}(2)) \rightarrow H^0(X, \mathcal{H}_{\text{ét}}^3(\mathbf{Q}/\mathbf{Z}(2))) \rightarrow 0$$

shows that  $CH^2(X) \xrightarrow{\sim} H_{\text{ét}}^4(X, \mathbf{Z}(2))$  since  $\dim X = 2$ , whence

$$T(X) = H^4(t_2(X), \mathbf{Z}(2)) \xrightarrow{\sim} H_{\text{ét}}^4(t_2(X), \mathbf{Z}(2))$$

yielding a new proof of Roitman's theorem up to small torsion.

*Remark 3.* This argument is not integral because the projector  $\pi_2^{\text{tr}}$  defining  $t_2(X)$  is not an integral correspondence. It is however  $l$ -integral for any  $l$  prime to a denominator  $N$  of  $\pi_2^{\text{tr}}$ . This  $N$  is essentially controlled by the degree of the Weil isogeny

$$\text{Pic}_{X/k}^0 \rightarrow \text{Pic}_{C/k}^0 = \text{Alb}(C) \rightarrow \text{Alb}(X)$$

where  $C$  is the ample curve involved in the construction of  $\pi_2^{\text{tr}}$ . If one could show that various  $C$ 's can be chosen so that the corresponding degrees have gcd equal to 1, one would deduce a full proof of Roïtman's theorem from the above.

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