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THE BRAUER GROUP AND INDECOMPOSABLE (2, 1)-CYCLES

BRUNO KAHN

ABSTRACT. We show that the torsion in the group of indecomposable (2, 1)-cycles on a smooth projective variety over an algebraically closed field is isomorphic to a twist of its Brauer group, away from the characteristic. In particular, this group is infinite as soon as $b_2 - \rho > 0$. We derive a new insight into Roïtman's theorem on torsion 0-cycles over a surface.

INTRODUCTION

Let X be a smooth projective variety over an algebraically closed field k . The group

$$C(X) = H^1(X, \mathcal{K}_2) \simeq CH^2(X, 1) \simeq H^3(X, \mathbf{Z}(2))$$

has been widely studied. Its most interesting part is the *indecomposable quotient*

$$H_{\text{ind}}^1(X, \mathcal{K}_2) \simeq CH_{\text{ind}}^2(X, 1) \simeq H_{\text{ind}}^3(X, \mathbf{Z}(2))$$

defined as the cokernel of the natural homomorphism

$$(1) \quad \text{Pic}(X) \otimes k^* \xrightarrow{\theta} C(X).$$

It vanishes for $\dim X \leq 1$.

Let $\text{Br}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$ be the Brauer group of X : it sits in an exact sequence

$$(2) \quad 0 \rightarrow \text{NS}(X) \otimes \mathbf{Q}/\mathbf{Z} \rightarrow H_{\text{ét}}^2(X, \mathbf{Q}/\mathbf{Z}(1)) \rightarrow \text{Br}(X) \rightarrow 0$$

(see [7, p. 629, (5.8.4)] for the p -primary part in characteristic p).

Theorem 1. *Let p be the exponential characteristic of k . There are natural isomorphisms*

$$\begin{aligned} \beta' : \text{Br}(X)\{p'\}(1) &\xrightarrow{\sim} H_{\text{ind}}^3(X, \mathbf{Z}(2))\{p'\} \\ \beta_p : H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)) &\xrightarrow{\sim} H_{\text{ind}}^3(X, \mathbf{Z}(2))\{p\} \end{aligned}$$

where $\{p\}$ (resp. $\{p'\}$) denotes p -primary torsion (resp. prime-to- p torsion.)

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Here we write $A(1)$ for $\varinjlim_{(n,p)=1} nA \otimes \mu_n$ for a prime-to- p torsion abelian group A , and

$$H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)) = \varinjlim_s H_{\text{ét}}^1(X, \nu_s(2))$$

where $\nu_s(2)$ is the s -th sheaf of logarithmic Hodge-Witt differentials of weight 2 [7, 10, 6].

Theorem 1 gives an interpretation of the Brauer group (away from p)¹ in terms of algebraic cycles. In view of (2), it also implies:

Corollary 1. *If $b_2 - \rho > 0$, $H_{\text{ind}}^3(X, \mathbf{Z}(2))$ is infinite. In characteristic zero, if $p_g > 0$ then $H_{\text{ind}}^3(X, \mathbf{Z}(2))$ is infinite. \square*

To my knowledge, this is the first general result on indecomposable (2,1)-cycles. It relates to the following open question:

Question 1 (See also Remark 1). Is there a surface X such that $b_2 - \rho > 0$ but $H_{\text{ind}}^3(X, \mathbf{Z}(2)) \otimes \mathbf{Q} = 0$?

Many examples of complex surfaces X for which $H_{\text{ind}}^3(X, \mathbf{Z}(2))$ is not torsion have been given: we refer for example to [3] and the references therein. In all of them, one shows that a version of the Beilinson regulator with values in a quotient of Deligne cohomology takes non torsion values on this group. On the other hand, there are examples of complex surfaces X with $p_g > 0$ for which the regulator vanishes rationally [13, Th. 1.6], but there seems to be no such X for which one can decide whether $H_{\text{ind}}^3(X, \mathbf{Z}(2)) \otimes \mathbf{Q} = 0$.

Question 1 evokes Mumford's nonrepresentability theorem for the Albanese kernel $T(X)$ in the Chow group $CH_0(X)$ under the given hypothesis. It is of course much harder, but not unrelated. The link comes through the *transcendental part of the Chow motive of X* , introduced and studied in [8]. If we denote this motive by $t_2(X)$ as in loc. cit., we have

$$T(X)_{\mathbf{Q}} = \text{Hom}_{\mathbf{Q}}(t_2(X), \mathbb{L}^2) = H^4(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}}$$

¹The group $H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2))$ is very different from $\text{Br}(X)\{p\}$: suppose that k is the algebraic closure of a finite field \mathbf{F}_q over which X is defined. In [10, Rk 5.6], Milne proves

$$\det(1 - \gamma t \mid H^i(X, \mathbf{Q}_p(n))) = \prod_{v(a_{ij})=v(q^n)} (1 - (q^n/a_{ij})t)$$

where γ is the ‘‘arithmetic’’ Frobenius of X over \mathbf{F}_q and the a_{ij} are the eigenvalues of the ‘‘geometric’’ Frobenius acting on the crystalline cohomology $H^i(X/W) \otimes \mathbf{Q}_p$ (or, equivalently, on l -adic cohomology for $l \neq p$ by Katz-Messing). We get $V_p(\text{Br}(X)\{p\})$ for $i = 2, n = 1$ and $V_p(H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)))$ for $i = 2, n = 2$.

[8, Prop. 7.2.3]. Here, all groups are taken in the category $\mathbf{Ab} \otimes \mathbf{Q}$ of abelian groups modulo groups of finite exponent and $\mathrm{Hom}_{\mathbf{Q}}$ denotes the refined Hom group on the category $\mathcal{M}_{\mathrm{rat}}^{\mathrm{eff}}(k, \mathbf{Q})$ of effective Chow motives with \mathbf{Q} coefficients (see Section 2 for all this), while \mathbb{L} is the Lefschetz motive; to justify the last term, note that Chow correspondences act on motivic cohomology, so that motivic cohomology of a Chow motive makes sense. We show:

Theorem 2 (see Proposition 3). *If X is a surface, we have an isomorphism in $\mathbf{Ab} \otimes \mathbf{Q}$:*

$$H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))_{\mathbf{Q}} \simeq H^3(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}}.$$

Corollary 2. *In Theorem 2, assume that k has infinite transcendence degree over its prime subfield. If $T(X) = 0$, then $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))$ is finite.*

Proof. Under the hypothesis on k , $T(X) = 0 \iff t_2(X) = 0$ [8, Cor. 7.4.9 b)]. Thus, $T(X) = 0 \Rightarrow H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))_{\mathbf{Q}} = 0$ by Theorem 2. This means that $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))$ has finite exponent, hence is finite by Theorem 1 and the known structure of $\mathrm{Br}(X)$. \square

Remark 1. 1) For $l \neq p$, $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))\{l\}$ finite $\iff b_2 - \rho = 0$ by Theorem 1. Under Bloch's conjecture, this implies $t_2(X) = 0$ [8, Cor. 7.6.11], hence $T(X) = 0$ and (by Theorem 2) $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))$ finite. This provides conjectural converses to Corollaries 1 (for a surface) and 2. 2) The quotient of $H_{\mathrm{ind}}^3(X, \mathbf{Z}(2))_{\mathrm{tors}}$ by its maximal divisible subgroup is dual to $\mathrm{NS}(X)_{\mathrm{tors}}$, at least away from p : we leave this to the interested reader.

We also apply Theorem 2 to give in Section 4 a new proof of Roïtman's theorem that $T(X)$ is uniquely divisible, up to a group of finite exponent.

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1. PROOF OF THEOREM 1

This proof is an elaboration of the arguments of Colliot-Thélène and Raskind in [4], completed by Gros-Suwa [6, Ch. IV] for $l = \mathrm{char} k$. We use motivic cohomology as it smoothens the exposition and is more inspirational, but stress that these ideas go back to Bloch [2] and [4].

We refer to [9, §2] for an exposition of ordinary and étale motivic cohomology and the facts used below, especially to [9, Th. 2.6] for the comparison with étale cohomology of twisted roots of unity and logarithmic Hodge-Witt sheaves.

Multiplication by l^s on étale motivic cohomology yields “Bockstein” exact sequences

$$0 \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}(n))/l^s \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}/l^s(n)) \rightarrow {}_{l^s}H_{\text{ét}}^{i+1}(X, \mathbf{Z}(n)) \rightarrow 0$$

for any prime l , $s \geq 1$, $n \geq 0$ and $i \in \mathbf{Z}$. Since $\varprojlim^1 H_{\text{ét}}^i(X, \mathbf{Z}(n))/l^s = 0$, one gets in the limit exact sequences:

$$(3) \quad 0 \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}(n)) \xrightarrow{a} H_{\text{ét}}^i(X, \hat{\mathbf{Z}}(n)) \xrightarrow{b} \hat{T}(H_{\text{ét}}^{i+1}(X, \mathbf{Z}(n))) \rightarrow 0$$

where $\hat{T}(-) = \text{Hom}(\mathbf{Q}/\mathbf{Z}, -)$ denotes the total Tate module. This first yields:

Proposition 1. *For $i \neq 2n$, $\text{Ker } b$ is finite in (3) and $H_{\text{ét}}^i(X, \mathbf{Z}(n))$ is an extension of a finite group by a divisible group.*

Proof. This is the argument of [4, 1.8 and 2.2]. Let us summarise it: $H_{\text{ét}}^i(X, \mathbf{Z}(n))$ is “of weight 0” and $H_{\text{ét}}^i(X, \hat{\mathbf{Z}}(n))$ is “of weight $i - 2n$ ” by Deligne’s proof of the Weil conjectures. It follows that a has finite image in every l -component, hence has finite image by Gabber’s theorem [5]. One derives the structure of $H_{\text{ét}}^i(X, \mathbf{Z}(n))$ from this. \square

Consider now the case $n = 2$. Recall that $H^i(X, \mathbf{Z}(2)) \xrightarrow{\sim} H_{\text{ét}}^i(X, \mathbf{Z}(2))$ for $i \leq 3$ from the Merkurjev-Suslin theorem (cf. [9, (2-6)]). For $l \neq p$, let

$$\begin{aligned} H_{\text{ind}}^2(X, \mu_{l^n}^{\otimes 2}) &= \text{Coker}(\text{Pic}(X) \otimes \mu_{l^n} \rightarrow H_{\text{ét}}^2(X, \mu_{l^n}^{\otimes 2})) \\ H_{\text{ind}}^2(X, \mathbf{Z}_l(2)) &= \text{Coker}(\text{Pic}(X) \otimes \mathbf{Z}_l(1) \rightarrow H_{\text{ét}}^2(X, \mathbf{Z}_l(2))). \end{aligned}$$

Lemma 1. *There is a canonical isomorphism $H_{\text{ind}}^2(X, \mathbf{Z}_l(2)) \simeq T_l(\text{Br}(X))(1)$. In particular, this group is torsion-free.*

Proof. Straightforward from the Kummer exact sequence. \square

We have a commutative diagram

$$(4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}(X) \otimes \mu_{l^s} & \longrightarrow & H_{\text{ét}}^2(X, \mu_{l^s}^{\otimes 2}) & \longrightarrow & H_{\text{ind}}^2(X, \mu_{l^s}^{\otimes 2}) \longrightarrow 0 \\ & & \downarrow \wr & & \downarrow \alpha_s & & \\ 0 & \longrightarrow & {}_{l^s}(\text{Pic}(X) \otimes k^*) & \longrightarrow & {}_{l^s}H^3(X, \mathbf{Z}(2)) & \longrightarrow & {}_{l^s}H_{\text{ind}}^3(X, \mathbf{Z}(2)) \longrightarrow 0 \end{array}$$

where the upper row is exact and the lower row is a complex. This diagram is equivalent to the one in [4, 2.8], but the proof of its commutativity is easier, as a consequence of the compatibility of Bockstein

boundaries with cup-product in hypercohomology. This yields maps:

$$(5) \quad H_{\text{ind}}^2(X, \mu_{l^s}^{\otimes 2}) \xrightarrow{\beta_s} {}_l H_{\text{ind}}^3(X, \mathbf{Z}(2))$$

an inverse limit commutative diagram

$$(6) \quad \begin{array}{ccccccc} 0 \rightarrow \text{NS}(X) \otimes \mathbf{Z}_l(1) & \longrightarrow & H_{\text{ét}}^2(X, \mathbf{Z}_l(2)) & \longrightarrow & H_{\text{ind}}^2(X, \mathbf{Z}_l(2)) & \longrightarrow & 0 \\ & & \wr \downarrow & & \hat{\alpha} \downarrow & & \hat{\beta} \downarrow \\ 0 \rightarrow T_l(\text{Pic}(X) \otimes k^*) & \longrightarrow & T_l(H^3(X, \mathbf{Z}(2))) & \longrightarrow & T_l(H_{\text{ind}}^3(X, \mathbf{Z}(2))) & \longrightarrow & 0. \end{array}$$

(note that $\text{Pic}(X) \otimes \mu_{l^s} \xrightarrow{\sim} \text{NS}(X) \otimes \mu_{l^s}$) and a direct limit commutative diagram

$$(7) \quad \begin{array}{ccccccc} 0 \rightarrow \text{Pic}(X) \otimes \mu_{l^\infty} & \longrightarrow & H^2(X, \mathbf{Q}_l/\mathbf{Z}_l(2)) & \longrightarrow & \text{Br}(X)\{l\}(1) & \longrightarrow & 0 \\ & & \wr \downarrow & & \alpha_l \downarrow & & \beta_l \downarrow \\ 0 \rightarrow (\text{Pic}(X) \otimes k^*)\{l\} & \longrightarrow & H^3(X, \mathbf{Z}(2))\{l\} & \longrightarrow & H_{\text{ind}}^3(X, \mathbf{Z}(2))\{l\} & \longrightarrow & 0 \end{array}$$

where β_l defines the map β' in Theorem 1.

We shall use the following fact, which is proven in [4, 2.7] (and could be reproven here with motivic cohomology in the same fashion):

Lemma 2. *In (1), $N := \text{Ker } \theta$ has no l -torsion.*

Proposition 2. *β_s is surjective in (5) and $\hat{\beta}$ is bijective in (6); N is uniquely divisible; the lower row of (7) is exact and β_l is bijective.*

Proof. Since $\text{Pic}(X) \otimes k^*$ is l -divisible, Lemma 2 yields exact sequences

$$(8) \quad 0 \rightarrow {}_l A \rightarrow {}_l A \rightarrow N/l^s \rightarrow 0$$

$$(9) \quad 0 \rightarrow {}_l A \rightarrow {}_l H^3(X, \mathbf{Z}(2)) \rightarrow {}_l H_{\text{ind}}^3(X, \mathbf{Z}(2)) \rightarrow 0$$

where $A = \text{Im } \theta$, and (9) implies the surjectivity of β_s , hence of $\hat{\beta}$ since the groups $H_{\text{ind}}^2(X, \mu_{l^s}^{\otimes 2})$ are finite. Since α_s is surjective in (4), we also get that all groups in (8) and (9) are finite. Now the upper row of (6) is exact; in its lower row, the homology at $T_l(H^3(X, \mathbf{Z}(2)))$ is isomorphic to \widehat{N}_l thanks to (8) and (9). A diagram chase then yields an exact sequence

$$H^2(X, \mathbf{Z}(2))_l^\wedge \simeq \text{Ker } \hat{\alpha} \rightarrow \text{Ker } \hat{\beta} \rightarrow \widehat{N}_l \rightarrow 0$$

(see Proposition 1). Thus $\text{Ker } \hat{\beta}$ is of weight 0, hence finite, hence 0 by Lemma 1. This also shows the divisibility of N , which thanks to (8) and (9) implies the exactness of the lower row of (4), hence of (7). Now α_l is surjective, and also injective since $\text{Ker } \alpha_l \simeq H^2(X, \mathbf{Z}(2)) \otimes \mathbf{Q}_l/\mathbf{Z}_l$ is 0 by Proposition 1. Hence β_l is bijective. \square

The case of p -torsion is similar and easier: by the same argument as above,

$$H^2(X, \mathbf{Q}_p/\mathbf{Z}_p(2)) \xrightarrow{\sim} H^3(X, \mathbf{Z}(2))\{p\}$$

and that $H^3(X, \mathbf{Z}(2))\{p\} \xrightarrow{\sim} H_{\text{ind}}^3(X, \mathbf{Z}(2))\{p\}$ since k^* is uniquely p -divisible, hence also $\text{Pic}(X) \otimes k^*$. This concludes the proof of Theorem 1.

2. REFINED Hom GROUPS

Let \mathcal{A} be an additive category; write $\mathcal{A} \otimes \mathbf{Q}$ for the category with the same objects as \mathcal{A} and Hom groups tensored with \mathbf{Q} , and $\mathcal{A} \boxtimes \mathbf{Q}$ for the pseudo-abelian envelope of $\mathcal{A} \otimes \mathbf{Q}$. If \mathcal{A} is abelian, then $\mathcal{A} \otimes \mathbf{Q} = \mathcal{A} \boxtimes \mathbf{Q}$ is still abelian and is the localisation of \mathcal{A} by the Serre subcategory $\mathcal{A}_{\text{tors}}$ of objects A such that $n1_A = 0$ for some integer $n > 0$ (e.g. [1, Prop. B.3.1]).

For $\mathcal{A} = \mathbf{Ab}$, the category of abelian groups, one has a natural functor “tensoring objects with \mathbf{Q} ”

$$\mathbf{Ab} \otimes \mathbf{Q} \rightarrow \mathbf{Vec}_{\mathbf{Q}}$$

to \mathbf{Q} -vector spaces, but this functor is an equivalence of categories only on the full subcategory of $\mathbf{Ab} \otimes \mathbf{Q}$ given by finitely generated abelian groups. For clarity, we shall write

$$(10) \quad A_{\mathbf{Q}}, \quad A \otimes \mathbf{Q}$$

for the image of an abelian group $A \in \mathbf{Ab}$ respectively in $\mathbf{Ab} \otimes \mathbf{Q}$ and $\mathbf{Vec}_{\mathbf{Q}}$.

Let F be an additive functor (covariant or contravariant) from \mathcal{A} to \mathbf{Ab} , the category of abelian groups: it then induces a functor

$$F_{\mathbf{Q}} : \mathcal{A} \boxtimes \mathbf{Q} \rightarrow \mathbf{Ab} \otimes \mathbf{Q}.$$

In particular, we get a bifunctor

$$\text{Hom}_{\mathbf{Q}} : (\mathcal{A} \boxtimes \mathbf{Q})^{\text{op}} \times \mathcal{A} \boxtimes \mathbf{Q} \rightarrow \mathbf{Ab} \otimes \mathbf{Q}$$

which refines the bifunctor Hom of $\mathcal{A} \boxtimes \mathbf{Q}$.

We shall apply this to $\mathcal{A} = \mathcal{M}_{\text{rat}}^{\text{eff}}(k)$, the category of effective Chow motives with integral coefficients: the category $\mathcal{A} \boxtimes \mathbf{Q}$ is then equivalent to the category $\mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$ of Chow motives with rational coefficients.

3. CHOW-KÜNNETH DECOMPOSITION OF \mathcal{K}_2 -COHOMOLOGY

In this section, X is a connected surface. Its Chow motive $h(X) \in \mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$ then enjoys a refined Chow-Künneth decomposition

$$(11) \quad h(X) = h_0(X) \oplus h_1(X) \oplus h_2^{\text{alg}}(X) \oplus t_2(X) \oplus h_3(X) \oplus h_4(X)$$

[8, Prop. 7.2.1 and 7.2.3]. The projectors defining this decomposition act on the groups $H^i(X, \mathbf{Z}(2))_{\mathbf{Q}}$; we propose to compute the corresponding direct summands $H^i(M, \mathbf{Z}(2))_{\mathbf{Q}}$. To be more concrete, we shall express this in terms of the \mathcal{K}_2 -cohomology of X .

We keep the notation

$$H_{\text{ind}}^1(X, \mathcal{K}_2) = \text{Coker}(\text{Pic}(X) \otimes k^* \rightarrow H^1(X, \mathcal{K}_2))$$

to which we adjoin

$$H_{\text{ind}}^0(X, \mathcal{K}_2) = \text{Coker}(K_2(k) \rightarrow H^0(X, \mathcal{K}_2)).$$

To relate with the notation in Section 1, recall that $H^2(k, \mathbf{Z}(2)) = K_2(k)$ and $H^2(X, \mathbf{Z}(2)) = H^0(X, \mathcal{K}_2)$.

We shall also need a smooth connected hyperplane section C of X , appearing in the construction of (11) [11, 12], and its own Chow-Künneth decomposition attached to the choice of a rational point:

$$(12) \quad h(C) = h_0(C) \oplus h_1(C) \oplus h_2(C).$$

The projectors defining (12) have integral coefficients, while those defining (11) only have rational coefficients in general.

The following proposition extends the computations of [8, 7.2.1 and 7.2.3] to weight 2 motivic cohomology.

Proposition 3. *a) We have the following table for $H^i(M, \mathbf{Z}(2))$:*

$M =$	$h_0(C)$	$h_1(C)$	$h_2(C)$
$i = 2$	$K_2(k)$	$H_{\text{ind}}^0(C, \mathcal{K}_2)$	0
$i = 3$	0	$V(C)$	k^*
$i > 3$	0	0	0

where $V(C) = \text{Ker}(H^1(C, \mathcal{K}_2) \xrightarrow{N} k^*)$ is Bloch's group.

b) We have the following table for $H^i(M, \mathbf{Z}(2))$, where all groups are taken in $\mathbf{Ab} \otimes \mathbf{Q}$ (see Section 2):

$M =$	$h_0(X)$	$h_1(X)$	$h_2^{\text{alg}}(X)$	$t_2(X)$	$h_3(X)$	$h_4(X)$
$i = 2$	$K_2(k)$	A	0	B	0	0
$i = 3$	0	$\text{Pic}^0(X)k^*$	$\text{NS}(X) \otimes k^*$	$H_{\text{ind}}^1(X, \mathcal{K}_2)$	0	0
$i = 4$	0	0	0	$T(X)$	$\text{Alb}(X)$	\mathbf{Z}
$i > 4$	0	0	0	0	0	0

where

$$\text{Pic}^0(X)k^* = \text{Im}(\text{Pic}^0(X) \otimes k^* \rightarrow H^1(X, \mathcal{K}_2))$$

$$A = \text{Im}(H_{\text{ind}}^0(X, \mathcal{K}_2) \rightarrow H_{\text{ind}}^0(C, \mathcal{K}_2))$$

$$B = \text{Ker}(H_{\text{ind}}^0(X, \mathcal{K}_2) \rightarrow H_{\text{ind}}^0(C, \mathcal{K}_2)).$$

Proof. We proceed by exclusion as in the proof of [8, Th. 7.8.4]. Let us start with a). We use the notation (10) of Section 2.

- For $i > 3$, $H^i(M, \mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^i(C, \mathbf{Z}(2))_{\mathbf{Q}} = 0$.
- One has $h_2(C) = \mathbb{L}$, hence

$$H^i(h_2(C), \mathbf{Z}(2))_{\mathbf{Q}} = H^{i-2}(k, \mathbf{Z}(1))_{\mathbf{Q}} = \begin{cases} k_{\mathbf{Q}}^* & \text{if } i = 3 \\ 0 & \text{else.} \end{cases}$$

- One has

$$H^i(h_0(C), \mathbf{Z}(2))_{\mathbf{Q}} = H^i(k, \mathbf{Z}(2))_{\mathbf{Q}} = \begin{cases} K_2(k)_{\mathbf{Q}} & \text{if } i = 2 \\ 0 & \text{if } i > 2. \end{cases}$$

- The case of $M = h_1(C)$ follows from the two previous ones by exclusion.

Let us come to b).

- For $i > 4$, $H^i(M, \mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^i(X, \mathbf{Z}(2))_{\mathbf{Q}} = 0$.
- One has $h_4(X) = \mathbb{L}^2$, hence

$$H^i(h_4(X), \mathbf{Z}(2))_{\mathbf{Q}} = H^{i-4}(k, \mathbf{Z})_{\mathbf{Q}} = \begin{cases} \mathbf{Z}_{\mathbf{Q}} & \text{if } i = 4 \\ 0 & \text{else.} \end{cases}$$

- One has $h_3(X) = h_1(X)(1)$, hence

$$H^i(h_3(X), \mathbf{Z}(2))_{\mathbf{Q}} = H^{i-2}(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}}.$$

As $h_1(X)$ is a direct summand of $h_1(C)$, $H^{i-2}(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}}$ is a direct summand of $H^{i-2}(C, \mathbf{Z}(1))_{\mathbf{Q}}$. This group is 0 for $i \neq 3, 4$. For $i = 3$, one has $H^1(C, \mathbf{Z}(1))_{\mathbf{Q}} = H^1(h_0(C), \mathbf{Z}(1))_{\mathbf{Q}}$, hence

$$H^1(h_1(C), \mathbf{Z}(1))_{\mathbf{Q}} = H^1(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}} = 0.$$

For $i = 4$, $H^2(h_1(X), \mathbf{Z}(1))_{\mathbf{Q}} = \text{Alb}(X)_{\mathbf{Q}}$ (cf. Murre [11]).

- One has $h_2^{\text{alg}}(X) = \text{NS}(X)(1)$, hence

$$\begin{aligned} H^i(h_2^{\text{alg}}(X), \mathbf{Z}(2))_{\mathbf{Q}} &= (H^{i-2}(k, \mathbf{Z}(1)) \otimes \text{NS}(X))_{\mathbf{Q}} \\ &= \begin{cases} (\text{NS}(X) \otimes k^*)_{\mathbf{Q}} & \text{if } i = 3 \\ 0 & \text{else.} \end{cases} \end{aligned}$$

- One has

$$H^i(h_0(X), \mathbf{Z}(2))_{\mathbf{Q}} = H^i(k, \mathbf{Z}(2))_{\mathbf{Q}} = \begin{cases} K_2(k)_{\mathbf{Q}} & \text{if } i = 2 \\ 0 & \text{if } i > 2. \end{cases}$$

- As $h^1(X)$ is a direct summand of $h^1(C)$, $H^i(h^1(X), \mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^i(C, \mathbf{Z}(2))_{\mathbf{Q}}$: this group is therefore 0 $i > 3$. This completes row $i = 4$ by exclusion.
- The action of refined Chow-Künneth projectors respects the homomorphism $(\text{Pic}(X) \otimes k^*)_{\mathbf{Q}} \rightarrow H^3(X, \mathbf{Z}(2))_{\mathbf{Q}}$. As the action of π_2^{tr} (defining $t_2(X)$) is 0 on $\text{Pic}(X)_{\mathbf{Q}}$, we get $H^3(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \simeq H_{\text{ind}}^1(X, \mathcal{K}_2)_{\mathbf{Q}}$, which completes row $i = 3$ by exclusion.
- The construction of π_2^{tr} [8, proof of 2.3] shows that the composition

$$h(C) \xrightarrow{i^*} h(X) \rightarrow t_2(X)$$

is 0. Hence the composition

$$H^i(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \rightarrow H^i(X, \mathbf{Z}(2))_{\mathbf{Q}} \xrightarrow{i^*} H^i(C, \mathbf{Z}(2))_{\mathbf{Q}}$$

is 0 for all i . Applying this for $i = 2$, we see that $H^2(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \subseteq B_{\mathbf{Q}}$. On the other hand, $H^2(h_1(X), \mathbf{Z}(2))_{\mathbf{Q}}$ is a direct summand of $H^2(h_1(C), \mathbf{Z}(2))_{\mathbf{Q}}$, hence injects in $A_{\mathbf{Q}}$. By exclusion, we have $H^2(t_2(X), \mathbf{Z}(2))_{\mathbf{Q}} \oplus H^2(h_1(X), \mathbf{Z}(2))_{\mathbf{Q}} \simeq H_{\text{ind}}^0(X, \mathbf{Z}(2))_{\mathbf{Q}}$, hence row $i = 2$.

□

Remark 2. Let us clarify the “reasoning by exclusion” that has been used repeatedly in this proof. Let F be a functor from smooth projective varieties to $\mathbf{Ab} \otimes \mathbf{Q}$, provided with an action of Chow correspondences. Then F automatically extends to $\mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$, and we wish to compute the effect of a Chow-Künneth decomposition of $h(X)$ on $F(X)$. The reasoning above is as follows in its simplest form:

Suppose that we have a motivic decomposition $h(X) = M \oplus M'$, hence a decomposition $F(X) = F(M) \oplus F(M')$. Suppose that we know an exact sequence

$$0 \rightarrow A \rightarrow F(X) \rightarrow B \rightarrow 0$$

and an isomorphism $F(M) \simeq A$. Then $F(M') \simeq B$.

Of course this reasoning is incorrect as it stands; to justify it, one should check that if π is the projector with image M yielding the decomposition of $h(X)$, then $F(\pi)$ does have image A . This can be checked in all cases of the above proof, but such a verification would be tedious, double the length of the proof and probably make it unreadable. I hope the reader will not disagree with this expository choice.

4. GENERALISATION

In this section, we take the gist of the previous arguments. For convenience we pass from effective Chow motives $\mathcal{M}_{\text{rat}}^{\text{eff}}(k, \mathbf{Q})$ to all Chow motives $\mathcal{M}_{\text{rat}}(k, \mathbf{Q})$. Since étale motivic cohomology has an action of

Chow correspondences and verifies the projective bundle formula, it yields well-defined contravariant functors

$$H_{\text{ét}}^i : \mathcal{M}_{\text{rat}}(k, \mathbf{Q}) \rightarrow \mathbf{Ab} \otimes \mathbf{Q}$$

such that $H_{\text{ét}}^i(X, \mathbf{Z}(n))_{\mathbf{Q}} = H_{\text{ét}}^{i-2n}(h(X)(-n))$ for any smooth projective k -variety X and $i, n \in \mathbf{Z}$. We also have (contravariant) realisation functors

$$H_l^i : \mathcal{M}_{\text{rat}}(k, \mathbf{Q}) \rightarrow \mathcal{C}_l \otimes \mathbf{Q}$$

extending l -adic cohomology for $l \neq \text{char } k$, where \mathcal{C}_l denotes the category of $l\mathbf{Z}$ -adic inverse systems of abelian groups [SGA 5, V.3.1.1]. For $l = \text{char } k$ we use logarithmic Hodge-Witt cohomology as in Theorem 1 [10, §2], [6].

Definition 1. Let $M \in \mathcal{M}_{\text{rat}}(k, \mathbf{Q})$. If $i \in \mathbf{Z}$, we say that M is *pure of weight i* if $H_l^j(M) = 0$ for all $j \neq i$ and all primes l .

For example, if $h(X) = \bigoplus_{i=0}^{2d} h_i(X)$ is a Chow-Künneth decomposition of the motive $h(X)$ of a d -dimensional smooth projective variety X , then $h_i(X)$ is pure of weight i . If $d = 2$, the motive $t_2(X)(-2)$ is pure of weight -2 as a direct summand of $h_2(X)(-2)$.

Theorem 3. *Let M be pure of weight i . Then $H_{\text{ét}}^j(M)$ is uniquely divisible for $j \neq i, i + 1$. If moreover $i \neq 0$, then $H_{\text{ét}}^i(M)$ is uniquely divisible and $H_{\text{ét}}^{i+1}(M)\{l\} \simeq H_l^i(M) \otimes \mathbf{Q}/\mathbf{Z}$.*

(An object $A \in \mathbf{Ab} \otimes \mathbf{Q}$ is *uniquely divisible* if multiplication by n is an automorphism of A for any integer $n \neq 0$.)

Proof. As in Section 1, we have Bockstein exact sequences in $\mathcal{C}_l \otimes \mathbf{Q}$

$$0 \rightarrow H_{\text{ét}}^j(M)/l^* \xrightarrow{a} H_l^j(M) \rightarrow {}_{l^*}H_{\text{ét}}^{j+1}(M) \rightarrow 0$$

which yields the first statement. For the second one, the weight argument of [4] yields $\text{Im } a = 0$. \square

Let X be a surface. Applying Theorem 3 to $M = t_2(X)(-2)$ as above, we get that $H_{\text{ét}}^i(t_2(X), \mathbf{Z}(2))$ is uniquely divisible for $i \neq 3$ and

$$H_{\text{ét}}^3(t_2(X), \mathbf{Z}(2))\{l\} \simeq H_{\text{tr}}^3(X, \mathbf{Z}_l(2)) \otimes \mathbf{Q}/\mathbf{Z} \simeq \text{Br}(X)\{l\}$$

in $\mathbf{Ab} \otimes \mathbf{Q}$, recovering a slightly weaker version of Theorem 1 in view of Proposition 3. For $i = 4$, the exact sequence [9, (2-7)]

$$0 \rightarrow CH^2(X) \rightarrow H_{\text{ét}}^4(X, \mathbf{Z}(2)) \rightarrow H^0(X, \mathcal{H}_{\text{ét}}^3(\mathbf{Q}/\mathbf{Z}(2))) \rightarrow 0$$

shows that $CH^2(X) \xrightarrow{\sim} H_{\text{ét}}^4(X, \mathbf{Z}(2))$ since $\dim X = 2$, whence

$$T(X) = H^4(t_2(X), \mathbf{Z}(2)) \xrightarrow{\sim} H_{\text{ét}}^4(t_2(X), \mathbf{Z}(2))$$

yielding a new proof of Roitman's theorem up to small torsion.

Remark 3. This argument is not integral because the projector π_2^{tr} defining $t_2(X)$ is not an integral correspondence. It is however l -integral for any l prime to a denominator N of π_2^{tr} . This N is essentially controlled by the degree of the Weil isogeny

$$\text{Pic}_{X/k}^0 \rightarrow \text{Pic}_{C/k}^0 = \text{Alb}(C) \rightarrow \text{Alb}(X)$$

where C is the ample curve involved in the construction of π_2^{tr} . If one could show that various C 's can be chosen so that the corresponding degrees have gcd equal to 1, one would deduce a full proof of Roïtman's theorem from the above.

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