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Samy Skander Bahoura

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SUP \times INF INEQUALITIES FOR THE SCALAR CURVATURE EQUATION IN DIMENSIONS 4 AND 5.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the following problem on bounded open set Ω of \mathbb{R}^n :

$$\begin{cases} -\Delta u = Vu^{\frac{n+2}{n-2}} & \text{in } \Omega \subset \mathbb{R}^n, \ n = 4, 5, \\ u > 0 & \text{in } \Omega. \end{cases}$$

We assume that :

$$V \in C^{1,\beta}(\Omega), \ 0 < \beta \leq 1$$

$$0 < a \leq V \leq b < +\infty,$$

$$|\nabla V| \leq A \text{ in } \Omega,$$

$$|\nabla^{1+\beta} V| \leq B \text{ in } \Omega.$$

then, we have a sup \times inf inequality for the solutions of the previous equation, namely:

$$(\sup_K u)^\beta \times \inf_\Omega u \leq c = c(a, b, A, B, \beta, K, \Omega), \text{ for } n = 4,$$

and,

$$(\sup_K u)^{1/3} \times \inf_\Omega u \leq c = c(a, b, A, B, K, \Omega), \text{ for } n = 5, \text{ and } \beta = 1.$$

1. INTRODUCTION AND MAIN RESULT

We work on $\Omega \subset \subset \mathbb{R}^4$ and we consider the following equation:

$$\begin{cases} -\Delta u = Vu^{\frac{n+2}{n-2}} & \text{in } \Omega \subset \mathbb{R}^n, \ n = 4, 5, \\ u > 0 & \text{in } \Omega. \end{cases} \quad (E)$$

with,

$$\begin{cases} V \in C^{1,\beta}(\Omega), \\ 0 < a \leq V \leq b < +\infty \text{ in } \Omega, \\ |\nabla V| \leq A \text{ in } \Omega, \\ |\nabla^{1+\beta} V| \leq B \text{ in } \Omega. \end{cases} \quad (C_\beta)$$

Without loss of generality, we suppose $\Omega = B_1(0)$ the unit ball of \mathbb{R}^n .

The corresponding equation in two dimensions on open set Ω of \mathbb{R}^2 , is:

$$-\Delta u = V(x)e^u, \quad (E')$$

The equation (E') was studied by many authors and we can find very important result about a priori estimates in [8], [9], [12], [16], and [19]. In particular in [9] we have the following interior estimate:

$$\sup_K u \leq c = c(\inf_\Omega V, \|V\|_{L^\infty(\Omega)}, \inf_\Omega u, K, \Omega).$$

And, precisely, in [8], [12], [16], and [19], we have:

$$C \sup_K u + \inf_\Omega u \leq c = c(\inf_\Omega V, \|V\|_{L^\infty(\Omega)}, K, \Omega),$$

and,

$$\sup_K u + \inf_{\Omega} u \leq c = c(\inf_{\Omega} V, \|V\|_{C^\alpha(\Omega)}, K, \Omega).$$

where K is a compact subset of Ω , C is a positive constant which depends on $\frac{\inf_{\Omega} V}{\sup_{\Omega} V}$, and, $\alpha \in (0, 1]$.

For $n \geq 3$ we have the following general equation on a riemannian manifold:

$$-\Delta u + hu = V(x)u^{\frac{n+2}{n-2}}, \quad u > 0. \quad (E_n)$$

Where h, V are two continuous functions. In the case $c_n h = R_g$ the scalar curvature, we call V the prescribed scalar curvature. Here c_n is a universal constant.

The equation (E_n) was studied a lot, when $M = \Omega \subset \mathbb{R}^n$ or $M = \mathbb{S}_n$ see for example, [2-4], [11], [15]. In this case we have a sup \times inf inequality.

In the case $V \equiv 1$ and M compact, the equation (E_n) is Yamabe equation. T.Aubin and R.Schoen proved the existence of solution in this case, see for example [1] and [14] for a complete and detailed summary.

When M is a compact Riemannian manifold, there exist some compactness result for equation (E_n) see [18]. Li and Zhu see [18], proved that the energy is bounded and if we suppose M not diffeomorphic to the three sphere, the solutions are uniformly bounded. To have this result they use the positive mass theorem.

Now, if we suppose M Riemannian manifold (not necessarily compact) and $V \equiv 1$, Li and Zhang [17] proved that the product sup \times inf is bounded. On other hand see [3], [5] and [6] for other Harnack type inequalities, and, see [3] and [7] about some characterisation of the solutions of this equation (E_n) in this case ($V \equiv 1$).

Here we extend a result of [11] on an open set of $\mathbb{R}^n, n = 4, 5$. In fact we consider the prescribed scalar curvature equation on an open set of $\mathbb{R}^n, n = 4, 5$, and, we prove a sup \times inf inequality on compact set of the domain when the derivative of the prescribed scalar curvature is β -holderian, $\beta > 0$.

Our proof is an extension of Chen-Lin result in dimension 4 and 5, see [11], and, the moving-plane method is used to have this estimate. We refer to Gidas-Ni-Nirenberg for the moving-plane method, see [13]. Also, we can see in [10], one of the application of this method.

We have the following result in dimension 4, which is the consequence of the work of Chen-Lin.

Theorem A. For all $a, b, m, A, B > 0$, and for all compact K of Ω , there exists a positive constant $c = c(a, b, A, B, K, \Omega)$ such that:

$$\sup_K u \times \inf_{\Omega} u \leq c,$$

where u is solution of (E) with V, C^2 satisfying (C_β) for $\beta = 1$.

Here, we give an inequality of type sup \times inf for the equation (E) in dimension 4 and with general conditions on the prescribed scalar curvature, exactly we take a $C^{1,\beta}$ condition. In fact we extend the result of Chen-Lin in dimension 4.

Here we prove:

Theorem 1.1. . For all $a, b, A, B > 0, 1 \geq \beta > 0$, and for all compact K of Ω , there exists a positive constant $c = c(a, b, A, B, \beta, K, \Omega)$ such that:

$$(\sup_K u)^\beta \times \inf_{\Omega} u \leq c,$$

where u is solution of (E) with V satisfying (C_β) .

We have the following result in dimension 5, which is the consequence of the work of Chen-Lin.

Theorem B. For all $a, b, m, A, B > 0$, and for all compact K of Ω , there exists a positive constant $c = c(a, b, m, A, B, K, \Omega)$ such that:

$$\sup_K u \leq c, \text{ if } \inf_{\Omega} u \geq m,$$

where u is solution of (E) with V satisfying $(C_{\beta}) = (C_1)$ for $\beta = 1$.

Here, we give an inequality of type $\sup \times \inf$ for the equation (E) in dimension 5 and with general conditions on the prescribed scalar curvature, exactly we take a C^2 condition ($\beta = 1$ in (C_{β})). In fact we extend the result of Chen-Lin in dimension 5.

Here we prove:

Theorem 1.2. . For all $a, b, A, B > 0$, and for all compact K of Ω , there exists a positive constant $c = c(a, b, A, B, K, \Omega)$ such that:

$$(\sup_K u)^{1/3} \times \inf_{\Omega} u \leq c,$$

where u is solution of (E) with V satisfying (C_{β}) for $\beta = 1$.

2. THE METHOD OF MOVING-PLANE.

In this section we will formulate a modified version of the method of moving-plane for use later. Let Ω an open set and Ω^c the complement of Ω . We consider a solution u of the following equation:

$$\begin{cases} \Delta u + f(x, u) = 0, \\ u > 0, \end{cases} \quad (E'')$$

where $f(x, u)$ is nonnegative, Holder continuous in x , C^1 in u , and defined on $\bar{\Omega} \times (0, +\infty)$. Let e be a unit vector in \mathbb{R}^n . For $\lambda < 0$, we let

$T_{\lambda} = \{x \in \mathbb{R}^n, \langle x, e \rangle = \lambda\}$, $\Sigma_{\lambda} = \{x \in \mathbb{R}^n, \langle x, e \rangle > \lambda\}$, and $x^{\lambda} = x + (2\lambda - 2\langle x, e \rangle)e$ to denote the reflexion point of x with respect to T_{λ} , where $\langle \cdot, \cdot \rangle$ is the standard inner product of \mathbb{R}^n . Define:

$$\lambda_1 \equiv \sup\{\lambda < 0, \Omega^c \subset \Sigma_{\lambda}\},$$

$\Sigma'_{\lambda} = \Sigma_{\lambda} - \Omega^c$ for $\lambda \leq \lambda_1$, and $\bar{\Sigma}'_{\lambda}$ the closure of Σ'_{λ} . Let $u^{\lambda}(x) = u(x^{\lambda})$ and $w_{\lambda}(x) = u(x) - u^{\lambda}(x)$ for $x \in \Sigma'_{\lambda}$. Then we have, for any arbitrary function $b_{\lambda}(x)$,

$$\Delta w_{\lambda}(x) + b_{\lambda}(x)w_{\lambda}(x) = Q(x, b_{\lambda}(x)),$$

where,

$$Q(x, b_{\lambda}(x)) = f(x^{\lambda}, u^{\lambda}) - f(x, u) + b_{\lambda}(x)w_{\lambda}(x).$$

The hypothesis $(*)$ is said to be satisfied if there are two families of functions $b_{\lambda}(x)$ and $h^{\lambda}(x)$ defined in Σ'_{λ} , for $\lambda \in (-\infty, \lambda_1)$ such that, the following assertions holds:

$$0 \leq b_{\lambda}(x) \leq c(x)|x|^{-2},$$

where $c(x)$ is independant of λ and tends to zero as $|x|$ tends to $+\infty$,

$$h^{\lambda}(x) \in C^1(\Sigma_{\lambda} \cap \Omega),$$

and satisfies:

$$\begin{cases} \Delta h^{\lambda}(x) \geq Q(x, b_{\lambda}(x)) \text{ in } \Sigma_{\lambda} \cap \Omega \\ h^{\lambda}(x) > 0 \text{ in } \Sigma_{\lambda} \cap \Omega \end{cases}$$

in the distributional sense and,

$h^\lambda(x) = 0$ on T_λ and $h^\lambda(x) = O(|x|^{-t_1})$,
as $|x| \rightarrow +\infty$ for some constant $t_1 > 0$,

$$h^\lambda(x) + \epsilon < w_\lambda(x),$$

in a neighborhood of $\partial\Omega$, where ϵ is a positive constant independent of x .

$$\begin{cases} h^\lambda(x) \text{ and } \nabla_x h^\lambda \text{ are continuous with respect to both variables} \\ x \text{ and } \lambda, \text{ and for any compact set of } \Omega, w_\lambda(x) > h^\lambda(x) \\ \text{holds when } -\lambda \text{ is sufficiently large.} \end{cases}$$

We have the following lemma:

Lemma 2.1. *Let u be a solution of (E'') . Suppose that $u(x) \geq C > 0$ in a neighborhood of $\partial\Omega$ and $u(x) = O(|x|^{-t_2})$ at $+\infty$ for some positive t_2 . Assume there exist $b_\lambda(x)$ and $h^\lambda(x)$ such that the hypothesis (*) is satisfied for $\lambda \leq \lambda_1$. Then $w_\lambda(x) > 0$ in Σ'_λ , and $\langle \nabla u, e \rangle > 0$ on T_λ for $\lambda \in (-\infty, \lambda_1)$.*

For the proof see Chen and Lin, [10].

Remark 2.2. *If we know that $w_\lambda - h^\lambda > 0$ for some $\lambda = \lambda_0 < \lambda_1$ and b_λ and h^λ satisfy the hypothesis (*) for $\lambda_0 \leq \lambda \leq \lambda_1$, then the conclusion of the lemma 2.1 holds.*

3. PROOF OF THE RESULT:

Proof of the theorem 1, $n = 4$:

To prove the theorem, we argue by contradiction and we assume that the $(\sup)^\beta \times \inf$ tends to infinity.

Step 1: blow-up analysis

We want to prove that:

$$\tilde{R}^2 \left(\sup_{B_{\tilde{R}}(0)} u \right)^\beta \times \inf_{B_{3\tilde{R}}(0)} u \leq c = c(a, b, A, B, \beta),$$

If it is not the case, we have:

$$\tilde{R}_i^2 \left(\sup_{B_{\tilde{R}_i}(0)} u_i \right)^\beta \times \inf_{B_{3\tilde{R}_i}(0)} u_i = i^6 \rightarrow +\infty,$$

For positive solutions $u_i > 0$ of the equation (E) and $\tilde{R}_i \rightarrow 0$.
Thus,

$$\frac{1}{i} \tilde{R}_i \left(\sup_{B_{\tilde{R}_i}(0)} u_i \right)^{(1+\beta)/2} \rightarrow +\infty,$$

and,

$$\frac{1}{i} \tilde{R}_i \left[\sup_{B_{\tilde{R}_i}(0)} u_i \right]^{(1+\beta)/2} \rightarrow +\infty,$$

Let a_i such that:

$$u_i(a_i) = \max_{B_{\tilde{R}_i}(0)} u_i,$$

We set,

$$s_i(x) = (\tilde{R}_i - |x - a_i|)^{2/(1+\beta)} u_i(x),$$

we have,

$$s_i(\bar{x}_i) = \max_{B_{\tilde{R}_i}(a_i)} s_i \geq s_i(a_i) = \tilde{R}_i^{2/(1+\beta)} \sup_{B_{\tilde{R}_i}(0)} u_i \rightarrow +\infty,$$

we set,

$$R_i = \frac{1}{2}(\tilde{R}_i - |\bar{x}_i - a_i|),$$

We have, for $|x - \bar{x}_i| \leq \frac{R_i}{i}$,

$$\tilde{R}_i - |x - a_i| \geq \tilde{R}_i - |\bar{x}_i - a_i| - |x - \bar{x}_i| \geq 2R_i - R_i = R_i$$

Thus,

$$\frac{u_i(x)}{u_i(\bar{x}_i)} \leq \beta_i \leq 2^{2/(1+\beta)}.$$

with $\beta_i \rightarrow 1$.

We set,

$$M_i = u_i(\bar{x}_i), \quad v_i^*(y) = \frac{u_i(\bar{x}_i + M_i^{-1}y)}{u_i(\bar{x}_i)}, \quad |y| \leq \frac{1}{i}R_i M_i^{(1+\beta)/2} = 2L_i.$$

And,

$$\frac{1}{i^2}R_i^2 M_i^\beta \times \inf_{B_{3\tilde{R}_i}(0)} u_i \rightarrow +\infty,$$

By the elliptic estimates, v_i^* converge on each compact set of \mathbb{R}^4 to a function $U_0^* > 0$ solution of :

$$\begin{cases} -\Delta U_0^* = V(0)U_0^{*3} \text{ in } \mathbb{R}^4, \\ U_0^*(0) = 1 = \max_{\mathbb{R}^4} U_0^*. \end{cases}$$

For simplicity, we assume that $0 < V(0) = n(n-2) = 8$. By a result of Caffarelli-Gidas-Spruck, see [10], we have:

$$U_0^*(y) = (1 + |y|^2)^{-1}.$$

We set,

$$v_i(y) = v_i^*(y + e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$.

$$U_0(y) = U_0^*(y + e).$$

We want to prove that:

$$\min_{\{0 \leq |y| \leq r\}} v_i^* \leq (1 + \epsilon)U_0^*(r).$$

for $0 \leq r \leq L_i$, with $L_i = \frac{1}{2i}R_i M_i^{(1+\beta)/2}$.

We assume that it is not true, then, there is a sequence of number $r_i \in (0, L_i)$ and $\epsilon > 0$, such that:

$$\min_{\{0 \leq |y| \leq r_i\}} v_i^* \geq (1 + \epsilon)U_0^*(r_i).$$

We have:

$$r_i \rightarrow +\infty.$$

Thus, we have for $r_i \in (0, L_i)$:

$$\min_{\{0 \leq |y| \leq r_i\}} v_i \geq (1 + \epsilon)U_0(r_i).$$

Also, we can find a sequence of number $l_i \rightarrow +\infty$ such that:

$$\|v_i^* - U_0\|_{C^2(B_{l_i}(0))} \rightarrow 0.$$

Thus,

$$\min_{\{0 \leq |y| \leq l_i\}} v_i \geq (1 - \epsilon/2)U_0(l_i).$$

Step 2 : The Kelvin transform and the Moving-plane method

Step 2.1: a linear equation perturbed by a term, and, the auxiliary function

Step 2.1.1: $D_i = |\nabla V_i(x_i)| \rightarrow 0.$

We have the same estimate as in the paper of Chen-Lin. We argue by contradiction. We consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i} R_i M_i^{(1+\beta)/2}.$$

We use the assumption that the sup times inf is not bounded to prove $w_\lambda > h_\lambda$ in $\Sigma_\lambda = \{y, y_1 > \lambda\}$, and on the boundary.

The function v_i has a local maximum near $-e$ and converge to $U_0(y) = U_0^*(y + e)$ on each compact set of \mathbb{R}^5 . U_0 has a maximum at $-e$.

We argue by contradiction and we suppose that:

$$D_i = |\nabla V_i(x_i)| \not\rightarrow 0.$$

Then, without loss of generality we can assume that:

$$\nabla V_i(x_i) \rightarrow e = (1, 0, \dots, 0).$$

Where x_i is :

$$x_i = \bar{x}_i + M_i^{-1}e,$$

with \bar{x}_i is the local maximum in the blow-up analysis.

As in the paper of Chen-Lin, we use the Kelvin transform twice and we set (we take the same notations):

$$I_\delta(y) = \frac{\frac{|y|}{|y|^2} - \delta e}{\left(\left|\frac{|y|}{|y|^2} - \delta e\right|\right)^2},$$

$$v_i^\delta(y) = \frac{v_i(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}},$$

and,

$$V_\delta(y) = V_i(x_i + M_i^{-1}I_\delta(y)).$$

$$U_\delta(y) = \frac{U_0(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}}.$$

Then, U_δ has a local maximum near $e_\delta \rightarrow -e$ when $\delta \rightarrow 0$. The function v_i^δ has a local maximum near $-e$.

We want to prove by the application of the maximum principle and the Hopf lemma that near e_δ we have not a local maximum, which is a contradiction.

We set on $\Sigma'_\lambda = \Sigma_\lambda - \{y, |y - \frac{e_0}{\delta}| \leq \frac{e_0}{r_i}\} \simeq \Sigma_\lambda - \{y, |I_\delta(y)| \geq r_i\}$:

$$h_\lambda(y) = - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta.$$

with,

$$Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^3.$$

And, by the same estimates, we have for $\eta \in A_1 = \{\eta, |\eta| \leq R = \epsilon_0/\delta\}$,

$$V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-1}(\eta_1 - \lambda) + o(1)M_i^{-1}|\eta^\lambda|,$$

and, we have for $\eta \in A_2 = \Sigma_\lambda - A_1$:

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq CM_i^{-1}(|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|),$$

And, we have for some $\lambda_0 \leq -2$ and $C_0 > 0$:

$$w_\lambda(y) = v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n},$$

for $y_1 > \lambda_0$.

Because , by the maximum principle:

$$\begin{aligned} \min_{\{l_i \leq |I_\delta(y)| \leq r_i\}} v_i &= \min\left\{ \min_{\{|I_\delta(y)|=l_i\}}, v_i \min_{\{|I_\delta(y)|=r_i\}} v_i \right\} \geq (1 - \epsilon)U_\delta\left(\frac{\epsilon}{\delta}\right) \\ &\geq (1 + c_1\delta - \epsilon)U_\delta\left(\left(\frac{\epsilon}{\delta}\right)^\lambda\right) \geq (1 + c_1\delta - 2\epsilon)v_i^\delta(y^\lambda), \end{aligned}$$

and for $|I_\delta(y)| \leq l_i$ we use the C^2 convergence of v_i^δ to U_δ .

Thus,

$$w_\lambda(y) > 2\epsilon > 0,$$

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on v_i , we have:

$$0 < h_\lambda(y) = O(1)M_i^{-1}(y_1 - \lambda)(1 + |y|)^{-n} < 2\epsilon < w_\lambda(y).$$

also, we have the same estimate on the boundary, $|I_\delta(\eta)| = r_i$ or $|y - e/\delta| = c_2r_i^{-1}$:

Step 2.1.1: $|\nabla V_i(x_i)|^{1/\beta}[u_i(x_i)] \leq C$

Here, also, we argue by contradiction. We use the same computation as in Chen-Lin paper, we choose the same h_λ , except the fact that here we use the computation with $M_i^{-(1+\beta)}$ in front the regular part of h_λ .

Here also, we consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i}R_iM_i^{(1+\beta)/2}.$$

We argue by contradiction and we suppose that:

$$M_i^\beta D_i \rightarrow +\infty.$$

Then, without loss of generality we can assume that:

$$\frac{\nabla V_i(x_i)}{|\nabla V_i(x_i)|} \rightarrow e = (1, 0, \dots, 0).$$

We use the Kelvin transform twice and around this point and around 0.

$$h_\lambda(y) = \epsilon r_i^{-2} G_\lambda\left(y, \frac{\epsilon}{\delta}\right) - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta.$$

with,

$$Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^3.$$

And, by the same estimates, we have for $\eta \in A_1$

$$V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-1}D_i(\eta_1 - \lambda) + o(1)M_i^{-1}|\eta^\lambda|,$$

and, we have for $\eta \in A_2$, $|I_\delta(\eta)| \leq c_2 M_i D_i^{1/\beta}$,

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq C M_i^{-1} D_i (|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|),$$

and for $M_i D_i^{1/\beta} \leq |I_\delta(\eta)| \leq r_i$,

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq M_i^{-1} D_i |I_\delta(\eta)| + M_i^{-(1+\beta)} |I_\delta(\eta)|^{(1+\beta)},$$

By the same estimates, we have for $|I_\delta(\eta)| \leq r_i$ or $|y - e/\delta| \geq c_3 r_i^{-1}$:

$$h_\lambda(y) \simeq \epsilon r_i^{-2} G_\lambda(y, \frac{e}{\delta}) + c_4 M_i^{-1} D_i \frac{(y_1 - \lambda)}{|y|^n} + o(1) M_i^{-1} D_i \frac{(y_1 - \lambda)}{|y|^n} + o(1) M_i^{-(1+\beta)} G_\lambda(y, \frac{e}{\delta}).$$

with $c_4 > 0$.

And, we have for some $\lambda_0 \leq -2$ and $C_0 > 0$:

$$v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n},$$

for $y_1 > \lambda_0$.

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on v_i , we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y).$$

also, we have the same estimate on the boundary, $|I_\delta(\eta)| = r_i$ or $|y - e/\delta| = c_5 r_i^{-1}$

Step 2.2 conclusion : a linear equation perturbed by a term, and, the auxiliary function

Here also, we use the computations of Chen-Lin, and, we take the same auxiliary function h_λ (which correspond to this step), except the fact that here in front the regular part of this function we have $M_i^{-(1+\beta)}$.

Here also, we consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i} R_i M_i^{(1+\beta)/2}.$$

We set,

$$v_i(z) = v_i^*(z + e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$.

$$U_0(z) = U_0^*(z + e).$$

We have, for $|y| \geq L_i'^{-1}$, $L_i' = \frac{1}{2} R_i M_i$,

$$\bar{v}_i(y) = \frac{1}{|y|^{n-2}} v_i\left(\frac{y}{|y|^2}\right).$$

$$|V_i(\bar{x}_i + M_i^{-1} \frac{y}{|y|^2}) - V_i(\bar{x}_i)| \leq M_i^{-(1+\beta)} (1 + |y|^{-1}).$$

$$x_i = \bar{x}_i + M_i^{-1} e,$$

Then, for simplicity, we can assume that, \bar{v}_i has a local maximum near $e^* = (-1/2, 0, \dots, 0)$.

Also, we have:

$$|V_i(x_i + M_i^{-1} \frac{y}{|y|^2}) - V_i(x_i + M_i^{-1} \frac{y^\lambda}{|y^\lambda|^2})| \leq M_i^{-(1+\beta)} (1 + |y|^{-1}).$$

$$h_\lambda(y) \simeq \epsilon r_i^{-2} G_\lambda(y, 0) - \int_{\Sigma'_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta.$$

where, $\Sigma'_\lambda = \Sigma_\lambda - \{\eta, |\eta| \leq r_i^{-1}\}$, and,

$$Q_\lambda(\eta) = \left(V_i(x_i + M_i^{-1} \frac{y}{|y|^2}) - V_i(x_i + M_i^{-1} \frac{y^\lambda}{|y^\lambda|^2}) \right) (v_i(y^\lambda))^3.$$

we have by the same computations that:

$$\int_{\Sigma'_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta \leq C M_i^{-(1+\beta)} G_\lambda(y, 0) \ll \epsilon r_i^{-2} G_\lambda(y, 0).$$

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on v_i , we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y).$$

also, we have the same estimate on the boundary, $|y| = \frac{1}{r_i}$.

Proof of the theorem 2, $n = 5$:

To prove the theorem, we argue by contradiction and we assume that the $(\sup)^{1/3} \times \inf$ tends to infinity.

Step 1: blow-up analysis

We want to prove that:

$$\tilde{R}^3 \left(\sup_{B_{\tilde{R}}(0)} u \right)^{1/3} \times \inf_{B_{3\tilde{R}}(0)} u \leq c = c(a, b, A, B),$$

If it is not the case, we have:

$$\tilde{R}_i^3 \left(\sup_{B_{\tilde{R}_i}(0)} u_i \right)^{1/3} \times \inf_{B_{3\tilde{R}_i}(0)} u_i = i^6 \rightarrow +\infty,$$

For positive solutions $u_i > 0$ of the equation (E) and $\tilde{R}_i \rightarrow 0$.

Thus,

$$\frac{1}{i} \tilde{R}_i \left(\sup_{B_{\tilde{R}_i}(0)} u_i \right)^{2/3} \rightarrow +\infty,$$

and,

$$\frac{1}{i} \tilde{R}_i \left[\sup_{B_{\tilde{R}_i}(0)} u_i \right]^{4/9} \rightarrow +\infty,$$

Let a_i such that:

$$u_i(a_i) = \max_{B_{\tilde{R}_i}(0)} u_i,$$

We set,

$$s_i(x) = (\tilde{R}_i - |x - a_i|)^{9/4} u_i(x),$$

we have,

$$s_i(\bar{x}_i) = \max_{B_{\tilde{R}_i}(a_i)} s_i \geq s_i(a_i) = \tilde{R}_i^{9/4} \sup_{B_{\tilde{R}_i}(0)} u_i \rightarrow +\infty,$$

we set,

$$R_i = \frac{1}{2} (\tilde{R}_i - |\bar{x}_i - a_i|),$$

We have, for $|x - \bar{x}_i| \leq \frac{R_i}{i}$,

$$\tilde{R}_i - |x - a_i| \geq \tilde{R}_i - |\bar{x}_i - a_i| - |x - \bar{x}_i| \geq 2R_i - R_i = R_i$$

Thus,

$$\frac{u_i(x)}{u_i(\bar{x}_i)} \leq \beta_i \leq 2^{9/4}.$$

with $\beta_i \rightarrow 1$.

We set,

$$M_i = u_i(\bar{x}_i), \quad v_i^*(y) = \frac{u_i(\bar{x}_i + M_i^{-2/3}y)}{u_i(\bar{x}_i)}, \quad |y| \leq \frac{1}{i}R_i M_i^{4/9} = 2L_i.$$

And,

$$\frac{1}{i^3}R_i^3 M_i^{1/3} \times \inf_{B_{3\bar{R}_i}(0)} u_i \rightarrow +\infty,$$

By the elliptic estimates, v_i^* converge on each compact set of \mathbb{R}^5 to a function $U_0^* > 0$ solution of :

$$\begin{cases} -\Delta U_0^* = V(0)U_0^{*7/3} \text{ in } \mathbb{R}^5, \\ U_0^*(0) = 1 = \max_{\mathbb{R}^5} U_0^*. \end{cases}$$

For simplicity, we assume that $0 < V(0) = n(n-2) = 15$. By a result of Caffarelli-Gidas-Spruck, see [10], we have:

$$U_0^*(y) = (1 + |y|^2)^{-3/2}.$$

We set,

$$v_i(y) = v_i^*(y + e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$.

$$U_0(y) = U_0^*(y + e).$$

We want to prove that:

$$\min_{\{0 \leq |y| \leq r\}} v_i^* \leq (1 + \epsilon)U_0^*(r).$$

for $0 \leq r \leq L_i$, with $L_i = \frac{1}{2i}R_i M_i^{4/9}$.

We assume that it is not true, then, there is a sequence of number $r_i \in (0, L_i)$ and $\epsilon > 0$, such that:

$$\min_{\{0 \leq |y| \leq r_i\}} v_i^* \geq (1 + \epsilon)U_0^*(r_i).$$

We have:

$$r_i \rightarrow +\infty.$$

Thus, we have for $r_i \in (0, L_i)$:

$$\min_{\{0 \leq |y| \leq r_i\}} v_i \geq (1 + \epsilon)U_0(r_i).$$

Also, we can find a sequence of number $l_i \rightarrow +\infty$ such that:

$$\|v_i^* - U_0\|_{C^2(B_{l_i}(0))} \rightarrow 0.$$

Thus,

$$\min_{\{0 \leq |y| \leq l_i\}} v_i \geq (1 - \epsilon/2)U_0(l_i).$$

Step 2 : The Kelvin transform and the Moving-plane method

Step 2.1: a linear equation perturbed by a term, and, the auxiliary function

Step 2.1.1: $D_i = |\nabla V_i(x_i)| \rightarrow 0$.

We have the same estimate as in the paper of Chen-Lin. We argue by contradiction. We consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i} R_i M_i^{4/9}.$$

We use the assumption that the sup times inf is not bounded to prove $w_\lambda > h_\lambda$ in $\Sigma_\lambda = \{y, y_1 > \lambda\}$, and on the boundary.

The function v_i has a local maximum near $-e$ and converge to $U_0(y) = U_0^*(y + e)$ on each compact set of \mathbb{R}^5 . U_0 has a maximum at $-e$.

We argue by contradiction and we suppose that:

$$D_i = |\nabla V_i(x_i)| \not\rightarrow 0.$$

Then, without loss of generality we can assume that:

$$\nabla V_i(x_i) \rightarrow e = (1, 0, \dots, 0).$$

Where x_i is :

$$x_i = \bar{x}_i + M_i^{-2/3} e,$$

with \bar{x}_i is the local maximum in the blow-up analysis.

As in the paper of Chen-Lin, we use the Kelvin transform twice and we set (we take the same notations):

$$I_\delta(y) = \frac{\frac{|y|}{|y|^2} - \delta e}{\left(\left|\frac{|y|}{|y|^2} - \delta e\right|\right)^2},$$

$$v_i^\delta(y) = \frac{v_i(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}},$$

and,

$$V_\delta(y) = V_i(x_i + M_i^{-2/3} I_\delta(y)).$$

$$U_\delta(y) = \frac{U_0(I_\delta(y))}{|y|^{n-2}|y - e/\delta|^{n-2}}.$$

Then, U_δ has a local maximum near $e_\delta \rightarrow -e$ when $\delta \rightarrow 0$. The function v_i^δ has a local maximum near $-e$.

We want to prove by the application of the maximum principle and the Hopf lemma that near e_δ we have not a local maximum, which is a contradiction.

We set on $\Sigma'_\lambda = \Sigma_\lambda - \{y, |y - \frac{e}{\delta}| \leq \frac{\epsilon_0}{r_i}\} \simeq \Sigma_\lambda - \{y, |I_\delta(y)| \geq r_i\}$:

$$h_\lambda(y) = - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta.$$

with,

$$Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^{(n+2)/(n-2)}.$$

And, by the same estimates, we have for $\eta \in A_1 = \{\eta, |\eta| \leq R = \epsilon_0/\delta\}$,

$$V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-2/3}(\eta_1 - \lambda) + o(1)M_i^{-2/3}|\eta^\lambda|,$$

and, we have for $\eta \in A_2 = \Sigma_\lambda - A_1$:

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq CM_i^{-2/3}(|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|),$$

And, we have for some $\lambda_0 \leq -2$ and $C_0 > 0$:

$$v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n},$$

for $y_1 > \lambda_0$.

By the same estimates, and by our hypothesis on v_i , we have, for $c_1 > 0$:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y).$$

also, we have the same estimate on the boundary, $|I_\delta(\eta)| = r_i$ or $|y - e/\delta| = c_2 r_i^{-1}$

Step 2.1.1: $|\nabla V_i(x_i)|[u_i(x_i)]^{2/3} \leq C$

Here, also, we argue by contradiction. We use the same computation as in Chen-Lin paper, we take $\alpha = 2$ and we choose the same h_λ , except the fact that here we use the computation with $M_i^{-4/3}$ in front the regular part of h_λ .

Here also, we consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i} R_i M_i^{4/9}.$$

We argue by contradiction and we suppose that:

$$M_i^{2/3} D_i \rightarrow +\infty.$$

Then, without loss of generality we can assume that:

$$\frac{\nabla V_i(x_i)}{|\nabla V_i(x_i)|} \rightarrow e = (1, 0, \dots, 0).$$

We use the Kelvin transform twice and around this point and around 0.

$$h_\lambda(y) = \epsilon r_i^{-3} G_\lambda(y, \frac{e}{\delta}) - \int_{\Sigma_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta.$$

with,

$$Q_\lambda(\eta) = (V_\delta(\eta) - V_\delta(\eta^\lambda))(v_i^\delta(\eta^\lambda))^{(n+2)/(n-2)}.$$

And, by the same estimates, we have for $\eta \in A_1$

$$V_\delta(\eta) - V_\delta(\eta^\lambda) \geq M_i^{-2/3} D_i (\eta - \lambda) + o(1) M_i^{-2/3} |\eta^\lambda|,$$

and, we have for $\eta \in A_2$, $|I_\delta(\eta)| \leq c_2 M_i^{2/3} D_i$,

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq C M_i^{-2/3} D_i (|I_\delta(\eta)| + |I_\delta(\eta^\lambda)|),$$

and for $M_i^{2/3} D_i \leq |I_\delta(\eta)| \leq r_i$,

$$|V_\delta(\eta) - V_\delta(\eta^\lambda)| \leq M_i^{-2/3} D_i |I_\delta(\eta)| + M_i^{-4/3} |I_\delta(\eta)|^2,$$

By the same estimates, we have for $|I_\delta(\eta)| \leq r_i$ or $|y - e/\delta| \geq c_3 r_i^{-1}$:

$$h_\lambda(y) \simeq \epsilon r_i^{-3} G_\lambda(y, \frac{e}{\delta}) + c_4 M_i^{-2/3} D_i \frac{(y_1 - \lambda)}{|y|^n} + o(1) M_i^{-2/3} D_i \frac{(y_1 - \lambda)}{|y|^n} + o(1) M_i^{-4/3} G_\lambda(y, \frac{e}{\delta}).$$

with $c_4 > 0$.

And, we have for some $\lambda_0 \leq -2$ and $C_0 > 0$:

$$v_i^\delta(y) - v_i^\delta(y^{\lambda_0}) \geq C_0 \frac{y_1 - \lambda_0}{(1 + |y|)^n},$$

for $y_1 > \lambda_0$.

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on v_i , we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y).$$

also, we have the same estimate on the boundary, $|I_\delta(\eta)| = r_i$ or $|y - e/\delta| = c_5 r_i^{-1}$:

Step 2.2 conclusion : a linear equation perturbed by a term, and, the auxiliary function

Here also, we use the computations of Chen-Lin, and, we take the same auxiliary function h_λ (which correspond to this step), except the fact that here in front the regular part of this function we have $M_i^{-4/3}$.

Here also, we consider $r_i \in (0, L_i)$ where L_i is the number of the blow-up analysis.

$$L_i = \frac{1}{2i} R_i M_i^{4/9}.$$

We set,

$$v_i(z) = v_i^*(z + e),$$

where v_i^* is the blow-up function. Then, v_i has a local maximum near $-e$.

$$U_0(z) = U_0^*(z + e).$$

We have, for $|y| \geq L_i^{-1}$, $L_i = \frac{1}{2} R_i M_i^{2/3}$,

$$\bar{v}_i(y) = \frac{1}{|y|^{n-2}} v_i \left(\frac{y}{|y|^2} \right).$$

$$|V_i(\bar{x}_i + M_i^{-2/3} \frac{y}{|y|^2}) - V_i(\bar{x}_i)| \leq M_i^{-4/3} (1 + |y|^{-2}).$$

$$x_i = \bar{x}_i + M_i^{-2/3} e,$$

Then, for simplicity, we can assume that, \bar{v}_i has a local maximum near $e^* = (-1/2, 0, \dots, 0)$. Also, we have:

$$|V_i(x_i + M_i^{-2/3} \frac{y}{|y|^2}) - V_i(x_i + M_i^{-2/3} \frac{y^\lambda}{|y^\lambda|^2})| \leq M_i^{-4/3} (1 + |y|^{-2}).$$

$$h_\lambda(y) \simeq \epsilon r_i^{-3} G_\lambda(y, 0) - \int_{\Sigma'_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta.$$

where, $\Sigma'_\lambda = \Sigma_\lambda - \{\eta, |\eta| \leq r_i^{-1}\}$, and,

$$Q_\lambda(\eta) = \left(V_i(x_i + M_i^{-2/3} \frac{y}{|y|^2}) - V_i(x_i + M_i^{-2/3} \frac{y^\lambda}{|y^\lambda|^2}) \right) (v_i(y^\lambda))^{\frac{n+2}{n-2}}.$$

we have by the same computations that:

$$\int_{\Sigma'_\lambda} G_\lambda(y, \eta) Q_\lambda(\eta) d\eta \leq C M_i^{-4/3} G_\lambda(y, 0) \ll \epsilon r_i^{-3} G_\lambda(y, 0).$$

By the same estimates as in Chen-Lin paper (we apply the lemma 2.1 of the second section), and by our hypothesis on v_i , we have:

$$0 < h_\lambda(y) < 2\epsilon < w_\lambda(y).$$

also, we have the same estimate on the boundary, $|\eta| = \frac{1}{r_i}$.

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DEPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ PIERRE ET MARIE CURIE, 2 PLACE JUSSIEU, 75005, PARIS, FRANCE.

E-mail address: samybahoura@yahoo.fr