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# Simulation of strong nonlinear waves with vectorial lattice Boltzmann schemes

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**Abstract.** We show that a hyperbolic system with a mathematical entropy can be discretized with vectorial lattice Boltzmann schemes using the methodology of kinetic representation of the dual entropy. We test this approach for the shallow water equations in one and two spatial dimensions. We obtain interesting results for a shock tube, reflection of a shock wave and non-stationary two-dimensional propagation. This contribution shows the ability of vectorial lattice Boltzmann schemes to simulate strong nonlinear waves in non-stationary situations.

**Keywords:** Hyperbolic conservation laws, entropy, shock wave, shallow water equations.

**PACS numbers:** 02.70.Ns, 05.20.Dd, 47.10.+g, 47.11.+j.

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## Introduction

The computation of discrete shock waves with lattice Boltzmann approaches began with viscous Burgers approximations in the framework of lattice-gas automata (see Boghosian and Levermore[2] and Elton *et al.*[10]). With the lattice Boltzmann methods described *e.g.* by Lallemand and Luo[18], the first tentative results were proposed by d’Humières[14] and Alexander *et al.*[1] among others. A D1Q2 entropic scheme for the one-dimensional viscous Burgers equation has been developed by Boghosian *et al.*[3]. The extension to gas-dynamic equations, and in particular to shock tube problems, is studied in the works of Philippi *et al.*[20], Nie, Shan and Chen[19], Karlin and Asinari[15], and Chikatamarla and Karlin[6].

In this contribution, we test the ability of lattice Boltzmann schemes to approach weak entropy solutions of hyperbolic equations. It is well known that a first-order hyperbolic equation exhibits shock waves. In order to enforce uniqueness, the notion of mathematical entropy has been proposed by Godunov[13] and Friedrichs and Lax[12]. A mathematical entropy is a strictly convex function of the conserved variables satisfying *ad hoc* differential constraints to ensure a complementary conservation law for regular solutions (see, *e.g.*, our book with Després[8]). The gradient of the entropy defines the so-called “entropy variables.” The Legendre-Fenchel-Moreau duality for convex functions allows us to define the dual of the entropy, which is a convex function of the entropy variables.

We start from the mathematical framework developed by Bouchut[5], making the link between the finite-volume method and kinetic models in the framework of the BGK approximation. The key notion is the representation of the dual entropy with the help of convex functions associated with the discrete velocities of the lattice. If we suppose that a single distribution of particles is present, our previous contribution[9] shows that Burgers equation can be simulated in this way. We have also shown that the approach can be extended to the nonlinear wave equation but is not compatible with the system of shallow water equations.

In Section 1, we develop vectorial lattice Boltzmann schemes with a kinetic representation of the dual entropy. This framework is applied in Section 2 for the approximation of one-dimensional shallow water equations, and in Section 3 for the two-dimensional case. Stationary and non-stationary two-dimensional simulations are presented in Section 4.

### 1) Dual entropy vectorial lattice Boltzmann schemes

In order to treat complex physics with particle-like methods, a classical idea is to multiply the number of particle distributions, as proposed by Khobalatte and Perthame[17], Shan and Chen[21], Bouchut[4], Dellar[7], and Wang *et al.*[22]. We follow here the idea of a dual entropy decomposition with vectorial particle distributions, as proposed by Bouchut[5]. We consider a hyperbolic system composed of  $N$  conservation laws with space described by points in  $x \in \mathbb{R}^d$ . The unknowns are the conserved variables  $W \in \mathbb{R}^N$  (*i.e.*  $W^k \in \mathbb{R}$ ). The nonlinear physical fluxes :  $F_\alpha(W) \in \mathbb{R}^N$  (with  $1 \leq \alpha \leq d$ ) are given regular

functions. The system is of first-order:

$$(1) \quad \partial_t W^k + \sum_{\alpha=1}^d \partial_\alpha F_\alpha^k(W) = 0, \quad 1 \leq k \leq N.$$

We suppose that a mathematical entropy  $\eta(W)$  is given, with associated entropy fluxes  $\zeta_\alpha(W)$  for  $0 \leq \alpha \leq d$ :

$$d\zeta_\alpha(W) \equiv d\eta(W) \bullet dF_\alpha(W).$$

The entropy variables  $\varphi_k \equiv \frac{\partial \eta(W)}{\partial W^k}$  are defined as the jacobian of the entropy:

$$d\eta(W) \equiv \sum_{k=1}^N \varphi_k dW^k.$$

The dual entropy  $\eta^*(\varphi)$  and the so-called ‘‘dual entropy fluxes’’  $\zeta_\alpha^*(\varphi)$  satisfy

$$(2) \quad \eta^*(\varphi) = \varphi \bullet W - \eta(W), \quad \zeta_\alpha^*(\varphi) \equiv \varphi \bullet F_\alpha(W) - \zeta_\alpha(W).$$

They can be differentiated without difficulty (see *e.g.*[8]):

$$d\eta^*(\varphi) \equiv \sum_k d\varphi_k W^k, \quad d\zeta_\alpha^*(\varphi) \equiv \sum_k d\varphi_k F_\alpha^k(W).$$

• With Bouchut[5], we introduce  $N$  particle distributions  $f_j^k$  (for  $1 \leq k \leq N$ ) and  $q$  velocities ( $0 \leq j \leq q-1$ ). The conserved moments  $W^k$  are simply the first discrete integrals of these distributions:

$$(3) \quad W^k = \sum_{j=0}^{q-1} f_j^k, \quad 1 \leq k \leq N.$$

We suppose that the particle distributions  $f_j^k$  are solutions of the Boltzmann equations with discrete velocities:

$$\partial_t f_j^k + v_j^\alpha \partial_\alpha f_j^k = Q_j^k, \quad 0 \leq j \leq q-1, \quad 1 \leq k \leq N$$

We suppose  $\sum_j Q_j^k = 0$  in order to enforce the conservation laws (1). The non-equilibrium fluxes take the natural form  $\Phi_\alpha^k \equiv \sum_j v_j^\alpha f_j^k$  and we have a system of  $N$  conservation laws:

$$\partial_t W^k + \sum_\alpha \partial_\alpha \Phi_\alpha^k = 0, \quad 1 \leq k \leq N.$$

In the following, we use the term ‘‘Perthame-Bouchut hypothesis’’ to refer to the fact that the dual mathematical entropy  $\eta^*(\varphi)$  can be decomposed into  $q$  scalar potentials,  $h_j^*$ . The potentials  $h_j^*$  are supposed to be regular convex functions of the entropy variables  $\varphi$ , and satisfy the two identities

$$(4) \quad \sum_{j=0}^{q-1} h_j^*(\varphi) \equiv \eta^*(\varphi), \quad \sum_{j=0}^{q-1} v_j^\alpha h_j^*(\varphi) \equiv \zeta_\alpha^*(\varphi), \quad \forall \varphi.$$

The equilibrium fluxes  $(f^{\text{eq}})_j^k$  are easy to derive from the potentials  $h_j^*$ :

$$(f^{\text{eq}})_j^k = \frac{\partial h_j^*}{\partial \varphi_k}, \quad \sum_{j=0}^{q-1} (f^{\text{eq}})_j^k = W^k, \quad 1 \leq k \leq N.$$

- We introduce the Legendre dual of the convex potentials  $h_j^*$ :

$$h_j(f_j, f_j^2, \dots, f_j^N) \equiv \sup_{\varphi} \left( \left[ \sum_{k=1}^N \varphi_k f_j^k \right] - h_j^*(\varphi) \right), \quad 0 \leq j \leq q-1.$$

We observe that each function  $h_j(\bullet)$  is a convex function of  $N$  variables. The so-called “microscopic entropy”  $H(f)$  can now be defined according to

$$H(f) \equiv \sum_{j=0}^{q-1} h_j(f_j^1, f_j^2, \dots, f_j^N).$$

This is a convex function in the domain where the  $h_j$ 's are convex.

- We can establish a “H-theorem” for the continuous dynamics relative to time and space in a way similar to the maximal entropy approach developed by Karlin and his co-workers[16]. Under a BGK-type hypothesis

$$Q_j^k \equiv \frac{1}{\tau} \left( (f^{\text{eq}})_j^k - f_j^k \right)$$

we have

$$\partial_t H(f) + \sum_{\alpha} \partial_{\alpha} \left( \sum_j v_j^{\alpha} h_j(f_j^1, f_j^2, \dots, f_j^N) \right) \leq 0.$$

To establish this result, we derive the microscopic entropy relative to time:

$$\frac{\partial H}{\partial t} = \sum_{jk} \frac{\partial h_j}{\partial f_j^k} \frac{\partial f_j^k}{\partial t} = \sum_{jk} \frac{\partial h_j}{\partial f_j^k} Q_j^k - \sum_{jk} \frac{\partial h_j}{\partial f_j^k} v_j^{\alpha} \partial_{\alpha} f_j^k = \sum_{jk} \frac{\partial h_j}{\partial f_j^k} Q_j^k - \partial_{\alpha} \left( \sum_{j=0}^{q-1} v_j^{\alpha} h_j \right).$$

Then

$$\begin{aligned} \frac{\partial H}{\partial t} + \partial_{\alpha} \left( \sum_j v_j^{\alpha} h_j \right) &= \frac{1}{\tau} \sum_j \frac{\partial h_j}{\partial f_j^k} (f_j^k) \left[ (f^{\text{eq}})_j^k - f_j^k \right] \\ &\leq \frac{1}{\tau} \sum_{jk} \frac{\partial h_j}{\partial f_j^k} (f_j^{\text{eq}}) \left[ (f^{\text{eq}})_j^k - f_j^k \right] \text{ by convexity of the potentials } h_j. \end{aligned}$$

This last expression is equal to  $\frac{1}{\tau} \sum_k \varphi_k \left[ (f^{\text{eq}})_j^k - f_j^k \right]$  due to Legendre duality:  

$$\frac{\partial h_j}{\partial f_j^k} (f^{\text{eq}}) = \varphi_k.$$

In consequence,

$$\frac{\partial H}{\partial t} + \partial_{\alpha} \left( \sum_j v_j^{\alpha} h_j \right) \leq \sum_k \varphi_k \sum_j \left[ (f^{\text{eq}})_j^k - f_j^k \right] = 0$$

by construction of the values  $f^{\text{eq}}$  in equilibrium. The H-theorem is thereby proven.  $\square$

## 2) “D1Q3Q2” lattice Boltzmann scheme for shallow water

We apply the previous ideas to the shallow-water equations in one spatial dimension

$$\partial_t \rho + \partial_x q = 0, \quad \partial_t q + \partial_x \left( \frac{q^2}{\rho} + \frac{p_0}{\rho_0^{\gamma}} \rho^{\gamma} \right) = 0,$$

Velocity  $u$ , pressure  $p$  and sound velocity  $c > 0$  are given by the expressions:

$$u \equiv \frac{q}{\rho}, \quad p \equiv \frac{p_0}{\rho_0^{\gamma}} \rho^{\gamma}, \quad c^2 \equiv \frac{\gamma p}{\rho} = \gamma \frac{p_0}{\rho_0^{\gamma}} \rho^{\gamma-1}.$$

The entropy  $\eta$  and the entropy flux  $\zeta$  can be determined explicitly without difficulty (see *e.g.*[9]):

$$\eta = \frac{1}{2} \rho u^2 + \frac{p}{\gamma - 1}, \quad \zeta = \eta u + p u$$

Then the entropy variables  $\varphi = (\theta \equiv \partial_\rho \eta, \beta \equiv \partial_q \eta)$  can be related to the usual ones:

$$\theta = \frac{c^2}{\gamma - 1} - \frac{u^2}{2}, \quad \beta = u.$$

Thanks to (2), the dual entropy  $\eta^*$  and the dual entropy flux  $\zeta^*$  can be worked out explicitly:  $\eta^* = p$  and  $\zeta^* = p u$ . We observe that

$$\eta^* = K \left( \theta + \frac{\beta^2}{2} \right)^2 = p, \quad \zeta^* = K \beta \left( \theta + \frac{\beta^2}{2} \right)^2 = p u, \quad \text{with } K = k \left( \frac{\gamma - 1}{\gamma k} \right)^{\frac{\gamma}{\gamma - 1}}.$$

• We model this system with a kinetic approach and a D1Q3 stencil. We have to find the particle components of the entropy variables, *id est* the (still unknown) convex functions  $h_j^*$  satisfying the Perthame-Bouchut hypothesis (4), that now can be written in the form:

$$(5) \quad h_+^*(\theta, \beta) + h_0^*(\theta, \beta) + h_-^*(\theta, \beta) = p, \quad \lambda h_+^*(\theta, \beta) - \lambda h_-^*(\theta, \beta) = p u,$$

where  $\lambda \equiv \frac{\Delta x}{\Delta t}$  is the numerical velocity of the mesh. We use a simple quadratic function as in our previous contribution[9]. We suggest that when  $\gamma = 2$ :

$$(6) \quad h_0^* = h_0^*(\theta) = \frac{a}{2} K \theta^2,$$

with the introduction of a parameter  $a$  that has to be made precise for real numerical computations. With this choice (6), the resolution of the system (5) with unknowns  $h_\pm^*$  is straightforward, resulting in

$$(7) \quad h_\pm^*(\theta, \beta) = \frac{K}{2} \left( \theta + \frac{\beta^2}{2} \right)^2 \left( 1 \pm \frac{\beta}{\lambda} \right) - \frac{a K}{4} \theta^2.$$

• From the previous potentials, (6) and (7), it is possible to derive the entire distribution at equilibrium. Observe first that with a vectorial lattice Boltzmann scheme, it is necessary to use two families,  $f$  and  $g$ , of particle distributions, one for mass conservation and the other for momentum conservation. We have in this case

$$f_j^{\text{eq}} = \frac{\partial h_j^*}{\partial \theta}, \quad g_j^{\text{eq}} = \frac{\partial h_j^*}{\partial \beta}.$$

With (6), the function  $h_0^*$  is independent of  $\beta$ . Then  $g_0 = \frac{\partial h_0^*}{\partial \beta}$  is unnecessary for the computation. With a very basic D1Q3 stencil, we define a ‘‘D1Q3Q2’’ lattice Boltzmann scheme. The equilibrium distribution is obtained by differentiation of the relations (6) and (7):

$$\begin{cases} f_0^{\text{eq}} &= \frac{\partial h_0^*}{\partial \theta} &= a K \theta = a \frac{\rho_0}{2 c_0^2} \left( c^2 - \frac{u^2}{2} \right) = \frac{a}{2} \left( \rho - \frac{\rho_0 u^2}{2 c_0^2} \right) \\ f_\pm^{\text{eq}} &= \frac{\partial h_\pm^*}{\partial \theta} &= \frac{\rho}{2} \left( 1 \pm \frac{u}{\lambda} \right) - \frac{a}{4} \left( \rho - \frac{\rho_0 u^2}{2 c_0^2} \right) \\ g_\pm^{\text{eq}} &= \frac{\partial h_\pm^*}{\partial \beta} &= \frac{\rho u}{2} \pm \frac{\rho}{2} \left( \frac{u^2}{\lambda} + \frac{c^2}{2 \lambda} \right). \end{cases}$$

From these equilibria, we implement the lattice Boltzmann method within the multiple-relaxation-time (MRT) framework. The conserved moments follow the general paradigm introduced in (3):

$$\rho = f_0 + f_+ + f_-, \quad q = g_+ + g_-.$$

The non-conserved moments are chosen in the usual way:

$$J_\rho = \lambda(f_+ - f_-), \quad \varepsilon_\rho = \lambda^2(f_+ + f_- - 2f_0), \quad J_q = \lambda(g_+ - g_-).$$

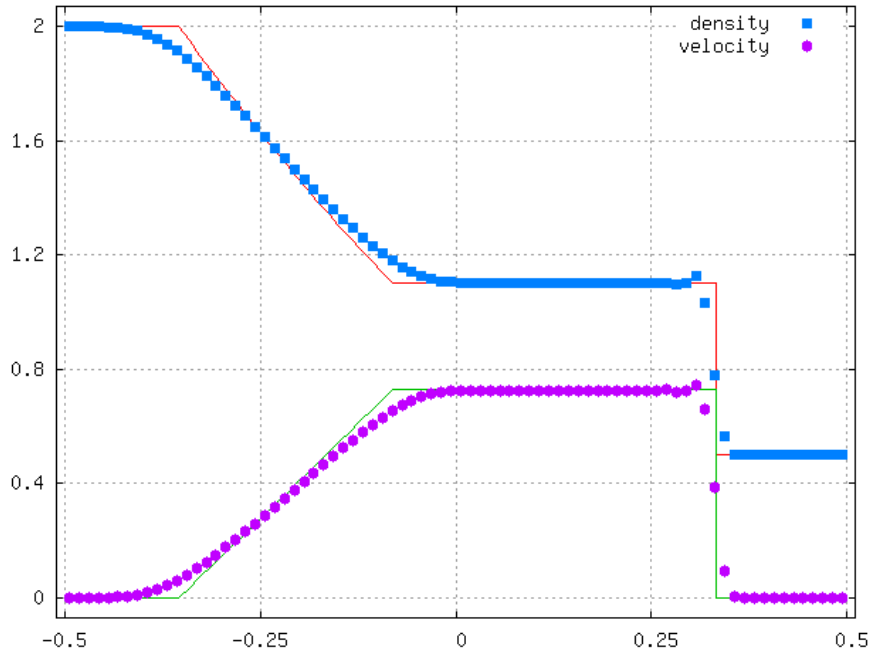
The relaxation step of the scheme is particularly simple when all the relaxation parameters are equal to a constant value  $\tau$  as proposed in the BGK hypothesis. When a general MRT scheme is used, we follow the rule[18] of the moments  $m_\ell^*$  after relaxation:

$$(8) \quad m_\ell^* = m_\ell + s_\ell (m_\ell^{\text{eq}} - m_\ell).$$

- We have tested the previous ideas for a Riemann problem for a shock tube. We have chosen the following numerical data and parameters:

$$\gamma = 2, \quad \frac{\rho_\ell}{\rho_0} = 2, \quad \frac{\rho_r}{\rho_0} = 0.5, \quad q_\ell = q_r = 0, \quad \frac{\lambda}{c_0} = 8, \quad a = 0.15, \quad s_j \equiv 1.8.$$

The numerical results are displayed in Fig. 1. The rarefaction wave (on the left) and the shock wave (on the right) are correctly captured.



**Figure 1.** Riemann problem for shallow water equations. Density (blue, top) and velocity (pink, bottom) fields were computed with the D1Q3Q2 lattice Boltzmann scheme with 80 mesh points and compared to the exact solution.

### 3) “D2Q5Q4Q4” vectorial lattice Boltzmann scheme

We study now the two-dimensional shallow-water equations

$$(9) \quad \begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_y(\rho v) & = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \frac{p_0}{\rho_0^2} \rho^2) + \partial_y(\rho u v) & = 0 \\ \partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(\rho v^2 + \frac{p_0}{\rho_0^2} \rho^2) & = 0. \end{cases}$$

We have three conservation laws in two spatial dimensions. We extend the previous D1Q3Q2 vectorial lattice Boltzmann scheme into a D2Q5Q4Q4 scheme. The D2Q5 stencil is associated with the following velocities:

$$(10) \quad v_0 = (0, 0), \quad v_1 = (\lambda, 0), \quad v_2 = (0, \lambda), \quad v_3 = (-\lambda, 0), \quad v_4 = (0, -\lambda).$$

We now have three particle distributions:  $f \in \text{D2Q5}$ ,  $g_x \in \text{D2Q4}$  and  $g_y \in \text{D2Q4}$ . The natural question is to find an intrinsic method to determine the equilibrium values  $f_j^{\text{eq}}$  for  $0 \leq j \leq 4$  and  $(g_{xj}^{\text{eq}}, g_{yj}^{\text{eq}})$  for  $1 \leq j \leq 4$ . As in the one-dimensional case, a key point is to be able to explicitly determine the dual entropy. In this two-dimensional case, the entropy variables  $\varphi \in \mathbb{R}^3$  can be written as

$$\begin{aligned} \varphi &= (\theta, u, v), \quad \theta = \frac{\partial \eta}{\partial \rho} = \frac{c^2}{\gamma - 1} - \frac{u^2 + v^2}{2} \\ \eta^*(\theta, u, v) &\equiv p \equiv \frac{\rho_0}{2c_0^2} \left( \theta + \frac{1}{2}(u^2 + v^2) \right)^2. \end{aligned}$$

In order to determine the equilibrium distributions, we search for convex functions  $h_j^*(\theta, u, v)$  for  $0 \leq j \leq 4$ , such that the first set of Perthame-Bouchut conditions (4) are satisfied:

$$(11) \quad \sum_{j=0}^4 h_j^*(\theta, u, v) \equiv \eta^*(\theta, u, v).$$

Then

$$f_j^{\text{eq}} = \frac{\partial h_j^*}{\partial \theta}, \quad g_{xj}^{\text{eq}} = \frac{\partial h_j^*}{\partial u}, \quad g_{yj}^{\text{eq}} = \frac{\partial h_j^*}{\partial v}.$$

We also have to take into account the dual entropy fluxes  $\zeta_\alpha$  in order to correctly represent the first-order terms of the model, (1) or (9) in our case. With the second set of Perthame-Bouchut conditions (4), we have:

$$(12) \quad \sum_{j=0}^4 v_j^1 h_j^*(\theta, u, v) \equiv \eta^* u, \quad \sum_{j=0}^4 v_j^2 h_j^*(\theta, u, v) \equiv \eta^* v.$$

For the D2Q5 stencil, the conditions of (11) (12) take the form

$$(13) \quad h_0^* + h_1^* + h_2^* + h_3^* + h_4^* \equiv p, \quad \lambda (h_1^* - h_3^*) \equiv p u, \quad \lambda (h_2^* - h_4^*) \equiv p v.$$

We mimic for shallow water in two spatial dimensions what we have done for the one-dimensional case (6), and we suggest here to set as previously

$$h_0^*(\theta) = \frac{a}{2} K \theta^2.$$

Because this function  $h_0^*$  does not depend explicitly on the variables  $u$  and  $v$ , we are not defining a D1Q5Q5Q5 scheme, but rather simply a D1Q5Q4Q4 vectorial lattice Boltzmann



scheme. The positive parameter  $a$  still has to be fixed. Nevertheless, we still have many degrees of freedom. We suggest moreover to break into two parts the first relation of (13):

$$(14) \quad h_1^* + h_3^* = \frac{1}{2}(p - h_0^*), \quad h_2^* + h_4^* = \frac{1}{2}(p - h_0^*).$$

We have now a set of five independent equations (6), (13) and (14) with 5 unknowns  $h_j^*$ . The end of the algebraic determination of the system (6), (13) and (14) is then completely elementary.

- When the potentials  $h_j^*$  are known, the computation of the equilibrium values is straightforward. With the  $5 + 4 + 4 = 13$  particle distributions, we can construct 13 moments for the D2Q5Q4Q4 lattice Boltzmann scheme. We suggest the following five moments associated with the distribution  $f_j$ :

$$\begin{cases} \rho = f_0 + f_1 + f_2 + f_3 + f_4, & J_{x,\rho} = \lambda(f_1 - f_3), & J_{y,\rho} = \lambda(f_2 - f_4), \\ \varepsilon_\rho = f_1 + f_2 + f_3 + f_4 - 4f_0, & XX_\rho = f_1 - f_2 + f_3 - f_4. \end{cases}$$

For the eight moments relative to the distributions  $g_{xj}$  and  $g_{yj}$ , we have chosen

$$\begin{cases} q_x = g_{x1} + g_{x2} + g_{x3} + g_{x4}, & f_{xx} = \lambda(g_{x1} - g_{x3}), \\ f_{xy} = \lambda(g_{x2} - g_{x4}), & XX_u = g_{x1} - g_{x2} + g_{x3} - g_{x4} \end{cases}$$

and

$$\begin{cases} q_y = g_{y1} + g_{y2} + g_{y3} + g_{y4}, & f_{yx} = \lambda(g_{y1} - g_{y3}), \\ f_{yy} = \lambda(g_{y2} - g_{y4}), & XX_v = g_{y1} - g_{y2} + g_{y3} - g_{y4}. \end{cases}$$

- The value at equilibrium of the previous moments can be determined, taking into account that the three moments  $\rho$ ,  $q_x$  and  $q_y$  are at equilibrium. We have:

$$\begin{cases} J_{x,\rho}^{\text{eq}} = \rho u = q_x, & J_{y,\rho}^{\text{eq}} = \rho v = q_y, \\ \varepsilon_\rho^{\text{eq}} = \left(1 - \frac{5a}{2}\right)\rho + \frac{5}{4} \frac{\rho_0(u^2 + v^2)}{c_0^2}, & XX_\rho^{\text{eq}} = 0 \end{cases}$$

We have also

$$\begin{cases} f_{xx}^{\text{eq}} = \rho u^2 + p, & f_{xy}^{\text{eq}} = \rho u v, & XX_u^{\text{eq}} = 0 \\ f_{yx}^{\text{eq}} = \rho u v, & f_{yy}^{\text{eq}} = \lambda(g_{y2} - g_{y4}), & XX_v^{\text{eq}} = 0. \end{cases}$$

The MRT algorithm can be implemented without difficulty. It is just necessary to write a relation of the type (8) for the 10 moments that are not at equilibrium. Our present choice is the BGK variant of the scheme, with all parameters  $s_\ell$  set equal. The boundary conditions of wall constraint, supersonic inflow or supersonic outflow are treated with an easy adaptation of the usual methods of bounce-back and “anti-bounce-back”.

#### 4) First test cases

We propose two bi-dimensional test cases for the shallow-water equations: A stationary shock reflection and a classical non-stationary forward-facing step, first proposed by Emery [11] for gas dynamics. The first test case is a the reflection of an incident shock wave of angle  $-\pi/4$  issued from a “left” state into a new shock of angle  $\text{atan}(4/3)$  due

to the physical nature of the “top” state (in green on the left picture of Fig. 2) and the “right” state (in indigo). The exact solution is determined through the use of the Rankine-Hugoniot relations. We have chosen

$$\begin{cases} \rho_\ell = 1, & u_\ell = 1.59497132403753, & v_\ell = 0, \\ \rho_t = 1.17150636388320, & u_t = 1.47822089880855, & v_t = -0.116750425228984, \\ \rho_r = 1.38196199044604, & u_r = 1.33228286727232, & v_r = 0. \end{cases}$$

The stationary result of the vectorial lattice Boltzmann scheme for this first test case can be compared with the pure finite-volume approach with the Godunov[13] scheme, solving a discontinuity at each interface at each time step. We have used three meshes of  $35 \times 20$ ,  $70 \times 40$  and  $140 \times 80$  grid points. The contours of constant density are presented in Fig. 2. The numerical results are similar.



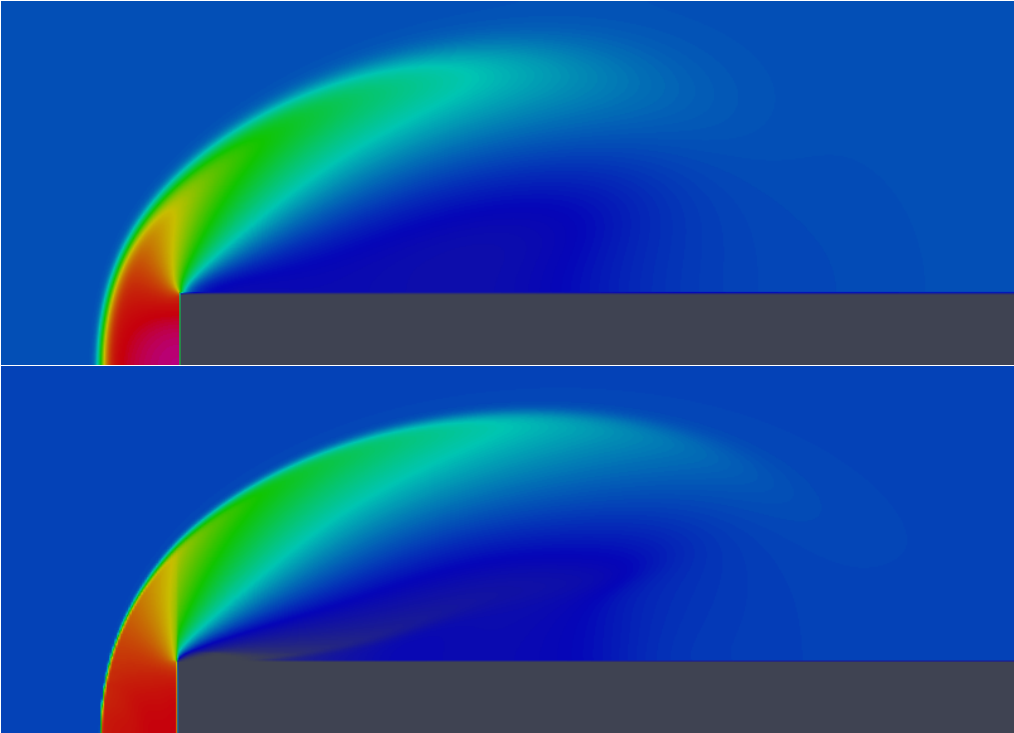
**Figure 2.** Shock reflection, mesh  $140 \times 80$ . Exact solution (left), Lattice Boltzmann scheme D2Q5Q4Q4 (middle) and Godunov scheme (right).

- The second test case (Emery[11]) is purely non-stationary. At time zero a small step is created inside a flow at Froude number equal to 3. A strong shock wave separates from the wall and various nonlinear waves are generated which mutually interact. Our present experiment (Figs. 3 and 4) shows the ability of a vectorial lattice Boltzmann scheme to approach such a flow. We have refined the mesh, using three families of meshes:  $120 \times 40$ ,  $240 \times 80$  and  $480 \times 120$ . We have used

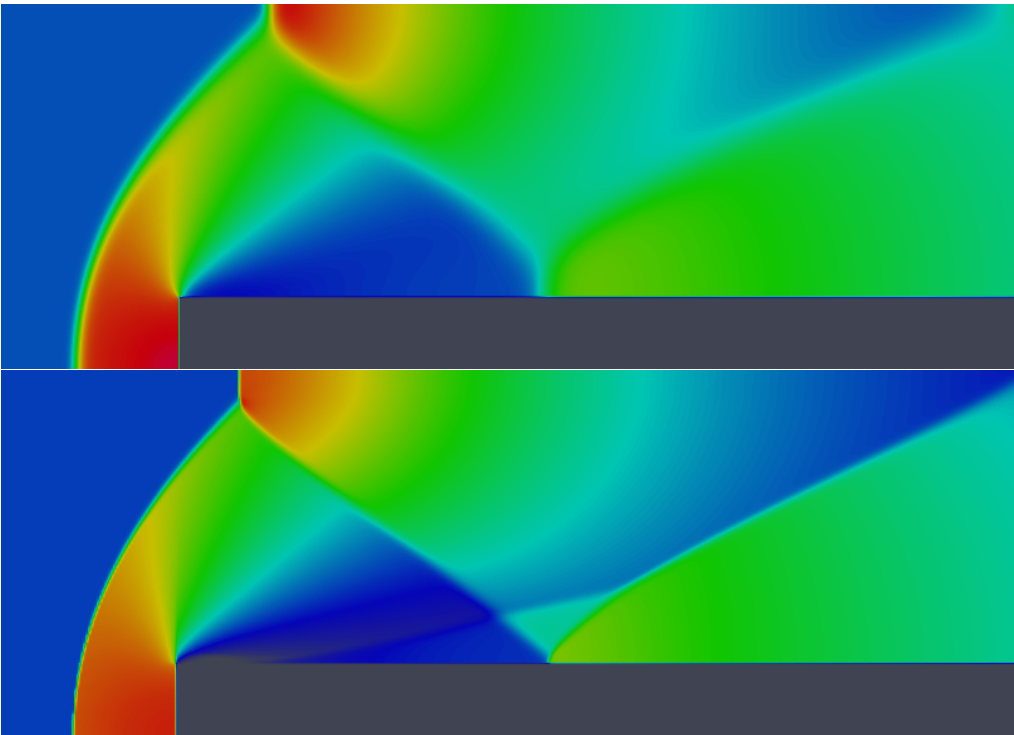
$$\lambda = 80, \quad a = 0.05, \quad s_j = 1.8 \forall j$$

to achieve experimental stability. The time step is very small (due to the high value of  $\lambda = \frac{\Delta x}{\Delta t}$ ), and in consequence the computation is relatively slow.

- We present our results for the finer mesh, at a dimensionalized time equal to  $1/2$  (Fig. 3) and 4 (Fig. 4). The results show the ability of the vectorial scheme based on the decomposition of the dual entropy to capture such flows. Nevertheless, the Godunov scheme, well known to be only order one, gives better non-stationary results compared to the new approach.



**Figure 3.** Emery test case for the shallow-water equations, mesh  $480 \times 120$ ,  $t = 1/2$ , density profile, D2Q5Q4Q4 vectorial lattice Boltzmann scheme (top) and Godunov scheme (bottom).



**Figure 4.** Emery test case for the shallow water equations, mesh  $480 \times 120$ ,  $t = 4$ , density profile, D2Q5Q4Q4 vectorial lattice Boltzmann scheme (top) and Godunov scheme (bottom).

## Conclusion

We have extended the methodology of kinetic decomposition of the dual entropy previously studied for one-dimensional problems into a general framework of vectorial lattice Boltzmann schemes for systems of conservation laws in several spatial dimensions, in the spirit of Bouchut[5]. The key point is to decompose the dual entropy of the system into convex potentials satisfying the Perthame-Bouchut hypothesis. Our first choices show that the system of shallow-water equations can be solved numerically without major difficulty. Nevertheless, our first numerical experiments show that the resulting scheme contains high numerical viscosity. Future work is necessary to reduce this effect.

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