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# On the continuity of the eigenvalues of a sublaplacian

Amine Aribi, Sorin Dragomir<sup>1</sup>, Ahmad El Soufi<sup>2</sup>

ABSTRACT. We study the behavior of the eigenvalues of a sublaplacian  $\Delta_b$  on a compact strictly pseudoconvex CR manifold  $M$ , as functions on the set  $\mathcal{P}_+$  of positively oriented contact forms on  $M$  by endowing  $\mathcal{P}_+$  with a natural metric topology.

## 1. INTRODUCTION

Let  $M$  be a compact strictly pseudoconvex CR manifold, of CR dimension  $n$ , without boundary. Let  $\mathcal{P}$  be the set of all  $C^\infty$  pseudohermitian structures on  $M$ . Every  $\theta \in \mathcal{P}$  is a contact form on  $M$  i.e.  $\theta \wedge (d\theta)^n$  is a volume form. Let  $\mathcal{P}_\pm$  be the sets of  $\theta \in \mathcal{P}$  such that the Levi form  $G_\theta$  is positive definite (respectively negative definite). For  $\theta \in \mathcal{P}_+$  let  $\Delta_b$  be the sublaplacian

$$(1) \quad \Delta_b u = -\operatorname{div}(\nabla^H u)$$

of  $(M, \theta)$  acting on smooth real valued functions  $u \in C^\infty(M, \mathbb{R})$ . As  $\Delta_b$  is a subelliptic operator (of order  $1/2$ ) it has a discrete spectrum

$$(2) \quad 0 = \lambda_0(\theta) < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \dots \uparrow +\infty$$

(the eigenvalues of  $\Delta_b$  are counted with their multiplicities). Each eigenvalue  $\lambda_\nu(\theta)$ ,  $\nu = 0, 1, 2, \dots$ , is thought of as a function of  $\theta \in \mathcal{P}_+$ . We shall deal mainly with the following problem: *Is there a natural topology on  $\mathcal{P}_+$  such that each eigenvalue function  $\lambda_\nu : \mathcal{P}_+ \rightarrow \mathbb{R}$  is continuous?* The analogous problem for the spectrum of the Laplace-Beltrami operator on a compact Riemannian manifold was solved by S. Bando & H. Urakawa, [2], and our main result is imitative of their Theorem 2.2 (cf. *op. cit.*, p. 155). We shall establish

**Corollary 1.** *For every compact strictly pseudoconvex CR manifold  $M$  the space of positively oriented contact forms  $\mathcal{P}_+$  admits a natural complete*

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<sup>1</sup>Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, Campus Macchia Romana, 85100 Potenza, Italy, sorin.dragomir@unibas.it

<sup>2</sup>Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais, Tours, France, amine.aribi@lmpt.univ-tours.fr, Ahmad.Elsoufi@lmpt.univ-tours.fr

distance function  $d : \mathcal{P}_+ \times \mathcal{P}_+ \rightarrow [0, +\infty)$  such that each eigenvalue function  $\lambda_k : \mathcal{P}_+ \rightarrow \mathbb{R}$  is continuous relative to the  $d$ -topology.

By a result of J.M. Lee, [8], for every  $\theta \in \mathcal{P}_+$  there is a Lorentzian metric  $F_\theta \in \text{Lor}(C(M))$  (the Fefferman metric) on the total space  $C(M)$  of the canonical circle bundle  $S^1 \rightarrow C(M) \xrightarrow{\pi} M$ . Also if  $\square$  is the Laplace-Beltrami operator of  $F_\theta$  (the wave operator) then  $\sigma(\Delta_b) \subset \sigma(\square)$ . Therefore the eigenvalues  $\lambda_k$  may be thought of as functions  $\lambda_k^\uparrow : C \rightarrow \mathbb{R}$  on the set  $C = \{F_\theta \in \text{Lor}(C(M)) : \theta \in \mathcal{P}_+\}$  of all Fefferman metrics on  $C(M)$ . On the other hand  $\text{Lor}(C(M))$  may be endowed with the distance function  $d_g^\infty$  considered by P. Mounoud, [10] (associated to a fixed Riemannian metric  $g$  on  $C(M)$ ) and hence  $(C, d_g^\infty)$  is itself a metric space. It is then a natural question whether  $\lambda_k^\uparrow$  are continuous functions relative to the  $d_g^\infty$ -topology.

The paper is organized as follows. In § 2 we recall the needed material on CR and pseudohermitian geometry. The distance function  $d$  (in Corollary 1) is built in § 3. In § 4 we establish a Max-Mini principle (cf. Proposition 2) for the eigenvalues of a sublaplacian. Then Corollary 1 follows from Theorem 1 in § 5. In § 6 we prove the continuity of the eigenvalues with respect to the Fefferman metric (cf. Corollary 2) though only as functions on  $C_+ = \{e^{u\circ\pi}F_{\theta_0} : u \in C^\infty(M, \mathbb{R}), u > 0\}$ .

## 2. REVIEW OF CR AND PSEUDOHERMITIAN GEOMETRY

Let  $(M, T_{1,0}(M))$  be a CR manifold, of CR dimension  $n$ , where  $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$  is its CR structure. Cf. e.g. [5], p. 3-4. The *Levi distribution* is  $H(M) = \text{Re}\{T_{1,0}(M) \oplus \overline{T_{1,0}(M)}\}$ . The Levi distribution carries the complex structure  $J : H(M) \rightarrow H(M)$  given by  $J(Z - \bar{Z}) = i(Z - \bar{Z})$  for any  $Z \in T_{1,0}(M)$  (here  $i = \sqrt{-1}$ ). A *pseudohermitian structure* is a globally defined nowhere zero section  $\theta \in C^\infty(H(M)^\perp)$  in the conormal bundle  $H(M)^\perp \subset T^*(M)$ . Pseudohermitian structures do exist by the mere assumption that  $M$  be orientable. Let  $\mathcal{P}$  be the set of all pseudohermitian structures on  $M$ . As  $H(M)^\perp \rightarrow M$  is a real line bundle for any  $\theta, \theta_0 \in \mathcal{P}$  there is a  $C^\infty$  function  $\lambda : M \rightarrow \mathbb{R} \setminus \{0\}$  such that  $\theta = \lambda\theta_0$ . Given  $\theta \in \mathcal{P}$  the *Levi form* is  $G_\theta(X, Y) = (d\theta)(X, JY)$  for every  $X, Y \in \mathfrak{X}(M)$ . Then  $G_{\lambda\theta_0} = \lambda G_{\theta_0}$ . The CR manifold  $M$  is *strictly pseudoconvex* if  $G_\theta$  is positive definite (write  $G_\theta > 0$ ) for some  $\theta \in \mathcal{P}$ . If  $M$  is strictly pseudoconvex then each  $\theta \in \mathcal{P}$  is a contact form i.e.  $\Psi_\theta = \theta \wedge (d\theta)^n$  is a volume form on  $M$ . Clearly, if  $G_\theta$  is positive definite then  $G_{-\theta}$  is negative definite. Hence  $\mathcal{P}$  admits a natural orientation  $\mathcal{P}_+$  ( $G_\theta > 0$  for each  $\theta \in \mathcal{P}_+$ ). Let  $M$  be a strictly pseudoconvex CR manifold and  $\theta \in \mathcal{P}_+$ . The *Reeb vector field* is the globally defined, nowhere zero, tangent vector field  $T \in \mathfrak{X}(M)$ , transverse to  $H(M)$ , determined by  $\theta(T) = 1$  and  $(d\theta)(T, X) = 0$  for any  $X \in \mathfrak{X}(M)$  (cf. Proposition 1.2 in [5], p. 8). The

*Webster metric* is the Riemannian metric  $g_\theta$  on  $M$  given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for every  $X, Y \in H(M)$ . Let  $S^1 \rightarrow C(M) \xrightarrow{\pi} M$  be the canonical circle bundle (cf. Definition 2.9 in [5], p. 119). For every  $\theta \in \mathcal{P}_+$  there is a Lorentzian metric  $F_\theta$  on  $C(M)$  (the *Fefferman metric*, cf. Definition 2.15 in [5], p. 128) such that the set  $\mathcal{C} = \{F_\theta : \theta \in \mathcal{P}_+\}$  of all Fefferman metrics is given by  $\mathcal{C} = \{e^{u \circ \pi} F_\theta : u \in C^\infty(M, \mathbb{R})\}$  for each fixed contact form  $\theta \in \mathcal{P}_+$  (by a result of J.M. Lee, [8], or Theorem 2.3 in [5], p. 128).  $\mathcal{C}$  is also referred to as the *restricted conformal class* of  $F_\theta$  and it is a CR invariant.

If  $u \in C^\infty(M, \mathbb{R})$  then the *horizontal gradient*  $\nabla^H u \in C^\infty(H(M))$  is given by  $\nabla^H u = \Pi_H \nabla u$ . Here  $\Pi_H : T(M) \rightarrow H(M)$  is the projection relative to the decomposition  $T(M) = H(M) \oplus \mathbb{R}T$  and  $\nabla u$  is the gradient of  $u$  with respect to the Webster metric i.e.  $g_\theta(\nabla u, X) = X(u)$  for any  $X \in \mathfrak{X}(M)$ . The divergence operator  $\text{div} : \mathfrak{X}(M) \rightarrow C^\infty(M, \mathbb{R})$  is meant with respect to the volume form  $\Psi_\theta$  i.e.  $\mathcal{L}_X \Psi_\theta = \text{div}(X) \Psi_\theta$  for any  $X \in \mathfrak{X}(M)$ . The *sublaplacian*  $\Delta_b$  of  $(M, \theta)$  is then the formally self-adjoint, second order, degenerate elliptic (in the sense of J.M. Bony, [4]) operator given by  $\Delta_b u = -\text{div}(\nabla^H u)$  for any  $u \in C^\infty(M, \mathbb{R})$ . A systematic application of functional analysis methods to the study of sublaplacians (on domains in strictly pseudoconvex CR manifolds) was started in [3]. By a result following essentially from work in [9] (cf. also [12]) if  $M$  is compact then  $\Delta_b$  has a discrete spectrum  $\sigma(\Delta_b) = \{\lambda_\nu : \nu \geq 0\}$  such that  $\lambda_0 = 0$  and  $\lambda_\nu \uparrow +\infty$  as  $\nu \rightarrow \infty$ .

### 3. A TOPOLOGY ON THE SPACE OF ORIENTED CONTACT FORMS

Let  $\{U_\lambda\}_{\lambda \in \Lambda}$  be a finite open covering of  $M$  such that the closure of each  $U_\lambda$  is contained in a larger open set  $V_\lambda$  which is both the domain of a local frame  $\{X_a : 1 \leq a \leq 2n\} \subset C^\infty(V_\lambda, H(M))$  with  $X_{\alpha+n} = JX_\alpha$  for any  $1 \leq \alpha \leq n$ , and a coordinate neighborhood with the local coordinates  $(x^1, \dots, x^{2n+1})$ . For each point  $x \in M$  let  $P_x$  (respectively  $S_x$ ) be the set of all symmetric positive definite (respectively merely symmetric) bilinear forms on  $T_x(M)$ . Let us consider the anti-reflexive partial order relation on  $S_x$  defined by

$$\varphi < \psi \iff \psi - \varphi \in P_x, \quad \varphi, \psi \in S_x.$$

Next let  $\rho'_x : P_x \times P_x \rightarrow [0, +\infty)$  be the distance function given by

$$\rho'_x(\varphi, \psi) = \inf \{\delta > 0 : \exp(-\delta) \varphi < \psi < \exp(\delta) \varphi\}$$

for any  $\varphi, \psi \in P_x$ . Then  $(P_x, \rho'_x)$  is a complete metric space (by (iii) of Lemma 1.1 in [2], p. 158).

Let  $\mathcal{M}$  be the set of all Riemannian metrics on  $M$ , so that  $g_\theta \in \mathcal{M}$  for every  $\theta \in \mathcal{P}_+$ . Following [2] one may endow  $\mathcal{M}$  with a complete distance

function  $\rho$ . Indeed as  $M$  is compact one may set

$$\rho''(g_1, g_2) = \sup_{x \in M} \rho'_x(g_{1,x}, g_{2,x}), \quad g_1, g_2 \in \mathcal{M}.$$

Also let  $S(M)$  be the space of all  $C^\infty$  symmetric  $(0, 2)$ -tensor fields on  $M$ , organized as a Fréchet space by the family of seminorms  $\{|\cdot|_k : k \in \mathbb{N} \cup \{0\}\}$  where

$$|g|_k = \sum_{\lambda \in \Lambda} |g|_{\lambda,k}, \quad |g|_{\lambda,k} = \sup_{x \in \bar{U}_\lambda} \sum_{|\alpha| \leq k} |D^\alpha g_{ij}(x)|,$$

where

$$D^\alpha = \partial^{|\alpha|} / \partial (x^1)^{\alpha_1} \dots \partial (x^{2n+1})^{\alpha_{2n+1}}, \quad g_{ij} = g(\partial / \partial x^i, \partial / \partial x^j) \in C^\infty(V_\lambda, \mathbb{R}),$$

for any  $g \in S(M)$ . The topology of  $S(M)$  as a locally convex space is compatible to the distance function

$$\rho'(g_1, g_2) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|g_1 - g_2|_k}{1 + |g_1 - g_2|_k}, \quad g_1, g_2 \in S(M).$$

In particular  $(S(M), \rho')$  is a complete metric space. If

$$\rho(g_1, g_2) = \rho'(g_1, g_2) + \rho''(g_1, g_2)$$

then  $(\mathcal{M}, \rho)$  is a complete metric space (cf. Proposition 2 in [2], p. 158). Each metric  $g \in \mathcal{M}$  determines a Laplace-Beltrami operator  $\Delta_g$  hence the eigenvalues of  $\Delta_g$  may be thought of as functions of  $g$  and as such the eigenvalues are (by Theorem 2.2 in [2], p. 161) continuous functions on  $(\mathcal{M}, \rho)$ . To deal with the similar problem for the spectrum of a sublaplacian, we start by observing that the natural counterpart of  $\mathcal{M}$  in the category of strictly pseudoconvex CR manifolds is the set  $\mathcal{M}_H$  of all sub-Riemannian metrics on  $(M, H(M))$ . Nevertheless only a particular sort of sub-Riemannian metric gives rise to a sublaplacian i.e.  $\Delta_b$  is associated to  $G_\theta \in \mathcal{M}_H$  for some positively oriented contact form  $\theta \in \mathcal{P}_+$ . Of course  $\mathcal{P}_+ \subset \Omega^1(M)$  and one may endow  $\Omega^1(M)$  with the  $C^\infty$  topology. One may then attempt to repeat the arguments in [2] (by replacing  $S(M)$  with  $\Omega^1(M)$ ). The situation at hand is however much simpler since, once a contact form  $\theta_0 \in \mathcal{P}_+$  is fixed, all others are parametrized by  $C^\infty(M, \mathbb{R})$  i.e. for any  $\theta \in \mathcal{P}_+$  there is a unique  $u \in C^\infty(M, \mathbb{R})$  such that  $\theta = e^u \theta_0$ . We may then use the canonical Fréchet space structure (and corresponding complete distance function) of  $C^\infty(M, \mathbb{R})$ . Precisely, for every  $u \in C^\infty(M, \mathbb{R})$ ,  $\lambda \in \Lambda$  and  $k \in \mathbb{N} \cup \{0\}$  we set

$$p_{\lambda,k}(u) = \sup_{x \in \bar{U}_\lambda} \sum_{|\alpha| \leq k} |D^\alpha u(x)|,$$

$$p_k(u) = \sum_{\lambda \in \Lambda} p_{\lambda,k}(u), \quad |u|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{p_k(u)}{1 + p_k(u)}.$$

If  $\theta_0 \in \mathcal{P}_+$  is a fixed contact form then we set

$$d'(\theta_1, \theta_2) = |u_1 - u_2|_{C^\infty}, \quad \theta_1, \theta_2 \in \mathcal{P}_+,$$

where  $u_i \in C^\infty(M, \mathbb{R})$  are given by  $\theta_i = e^{u_i}\theta_0$  for any  $i \in \{1, 2\}$ . The definition of  $d'$  doesn't depend upon the choice of  $\theta_0 \in \mathcal{P}_+$ .

**Lemma 1.**  $(\mathcal{P}_+, d')$  is a complete metric space.

*Proof.* Let  $\{\theta_\nu\}_{\nu \geq 1}$  be a Cauchy sequence in  $(\mathcal{P}_+, d')$ . If  $u_\nu \in C^\infty(M, \mathbb{R})$  is the function determined by  $\theta_\nu = e^{u_\nu}\theta_0$  then (by the very definition of  $d'$ )  $\{u_\nu\}_{\nu \geq 1}$  is a Cauchy sequence in  $C^\infty(M, \mathbb{R})$ . Here  $C^\infty(M, \mathbb{R})$  is organized as a Fréchet space by the (countable, separating) family of seminorms  $\{p_k : k \in \mathbb{N} \cup \{0\}\}$ . Hence there is  $u \in C^\infty(M, \mathbb{R})$  such that  $|u_\nu - u|_{C^\infty} \rightarrow 0$  as  $\nu \rightarrow \infty$ . Finally if  $\theta = e^u\theta_0 \in \mathcal{P}_+$  then  $d'(\theta_\nu, \theta) \rightarrow 0$  as  $\nu \rightarrow \infty$ . Q.e.d.

Let  $S(H) \subset H(M)^* \otimes H(M)^*$  be the subbundle of all bilinear symmetric forms on  $H(M)$ . For every  $G \in C^\infty(S(H))$ ,  $k \in \mathbb{Z}$ ,  $k \geq 0$ , and  $\lambda \in \Lambda$  we set

$$|G|_{\lambda, k} = \sup_{x \in \overline{U}_\lambda} \sum_{|\alpha| \leq k} \sum_{a, b=1}^{2n} |D^\alpha G_{ab}(x)|,$$

$$|G|_k = \sum_{\lambda \in \Lambda} |G|_{\lambda, k}, \quad |G|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|G|_k}{1 + |G|_k},$$

where  $G_{ab} = G(X_a, X_b) \in C^\infty(V_\lambda, \mathbb{R})$ . Moreover we set

$$\rho'_H(G_1, G_2) = |G_1 - G_2|_{C^\infty}, \quad G_1, G_2 \in C^\infty(S(H)).$$

**Lemma 2.**  $\{|\cdot|_k : k \in \mathbb{N} \cup \{0\}\}$  is a countable separating family of seminorms organizing  $\mathfrak{X} = C^\infty(S(H))$  as a Fréchet space. In particular  $(\mathfrak{X}, \rho'_H)$  is a complete metric space.

*Proof.* For each  $k \in \mathbb{N} \cup \{0\}$  and  $N \in \mathbb{N}$  we set

$$(3) \quad V(k, N) = \{G \in \mathfrak{X} : |G|_k < 1/N\}.$$

Let  $\mathcal{B}$  be the collection of all finite intersections of sets (3). Then  $\mathcal{B}$  is (cf. e.g. Theorem 1.37 in [11], p. 27) a convex balanced local base for a topology  $\tau$  on  $\mathfrak{X}$  which makes  $\mathfrak{X}$  into a locally convex space such that every seminorm  $|\cdot|_k$  is continuous and a set  $E \subset \mathfrak{X}$  is bounded if and only if every  $|\cdot|_k$  is bounded on  $E$ .  $\tau$  is compatible with the distance function  $\rho'_H$ . Let  $\{G_m\}_{m \geq 1} \subset \mathfrak{X}$  be a Cauchy sequence relative to  $\rho'_H$ . Thus for every fixed  $k \in \mathbb{N} \cup \{0\}$  and  $N \in \mathbb{N}$  one has  $G_m - G_p \in V(k, N)$  for  $m, p$  sufficiently large. Consequently

$$\begin{aligned} & |D^\alpha(G_m)_{ab}(x) - D^\alpha(G_p)_{ab}(x)| < 1/N, \\ & x \in \overline{U}_\lambda, \quad \lambda \in \Lambda, \quad |\alpha| \leq k, \quad 1 \leq a, b \leq 2n. \end{aligned}$$

It follows that each sequence  $\{D^\alpha(G_m)_{ab}\}_{m \geq 1}$  converges uniformly on  $\overline{U}_\lambda$  to a function  $G_{ab}^\alpha$ . In particular for  $\alpha = \mathbf{0}$  one has  $(G_m)_{ab}(x) \rightarrow G_{ab}^{\mathbf{0}}(x)$  as  $m \rightarrow \infty$ , uniformly in  $x \in \overline{U}_\lambda$ . If  $\lambda, \lambda' \in \Lambda$  are such that  $U_\lambda \cap U_{\lambda'} \neq \emptyset$  and

$$X'_b = A_b^a X_a, \quad A \equiv [A_b^a] : U_\lambda \cap U_{\lambda'} \rightarrow \text{GL}(2n, \mathbb{R}),$$

is a local transformation of the frame in  $H(M)$  then

$$(G_m)'_{ab} = A_a^c A_b^d (G_m)_{cd} \quad \text{on } U_\lambda \cap U_{\lambda'}$$

so that (for  $m \rightarrow \infty$ )  $G'_{ab}{}^{\mathbf{0}} = A_a^c A_b^d G_{cd}^{\mathbf{0}}$  on  $U_\lambda \cap U_{\lambda'}$ . Thus  $G_{ab}^{\mathbf{0}} \in C^\infty(U_\lambda)$  glue up to a (globally defined) bilinear symmetric form  $G^{\mathbf{0}}$  on  $H(M)$  and  $G_m \rightarrow G^{\mathbf{0}}$  in  $\mathfrak{X}$  as  $m \rightarrow \infty$ . Q.e.d.

For each point  $x \in M$  let  $P(H)_x$  be the set of all symmetric positive definite bilinear forms on  $H(M)_x$ . We endow  $S(H)_x$  with the anti-reflexive partial order relation

$$\varphi < \psi \iff \psi - \varphi \in P(H)_x, \quad \varphi, \psi \in S(H)_x.$$

Next let  $\rho''_x : P(H)_x \times P(H)_x \rightarrow [0, +\infty)$  be given by

$$\rho''_x(\varphi, \psi) = \inf \{ \delta > 0 : \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi \}$$

for any  $\varphi, \psi \in P(H)_x$ .

**Lemma 3.**  $\rho''_x$  is a distance function on  $P(H)_x$ .

*Proof.* As  $e^{-\delta}\varphi < \psi < e^\delta\varphi$  is equivalent to  $e^{-\delta}\psi < \varphi < e^\delta\psi$ , it follows that  $\rho''_x$  is symmetric. To prove the triangle inequality we assume that  $\rho''_x(\varphi, \psi) > \rho''_x(\varphi, \chi) + \rho''_x(\chi, \psi)$  for some  $\varphi, \psi, \chi \in P(H)_x$ . Then

$$\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \inf \{ \delta > 0 : \exp(-\delta)\chi < \psi < \exp(\delta)\chi \}$$

hence there is  $\delta_2 > 0$  such that  $e^{-\delta_2}\chi < \psi < e^{\delta_2}\chi$  and  $\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \delta_2$ . Similarly

$$\rho''_x(\varphi, \psi) - \delta_2 > \inf \{ \delta > 0 : \exp(-\delta)\varphi < \chi < \exp(\delta)\varphi \}$$

yields the existence of a number  $\delta_1 > 0$  such that  $e^{-\delta_1}\varphi < \chi < e^{\delta_1}\varphi$  and  $\rho''_x(\varphi, \psi) - \delta_2 > \delta_1$ . Let us set  $\delta \equiv \delta_1 + \delta_2$ . The inequalities written so far show that  $e^{-\delta}\varphi < \psi < e^\delta\varphi$  and  $\rho''_x(\varphi, \psi) > \delta$ , a contradiction. Finally, let us assume that  $\rho''_x(\varphi, \psi) = 0$  so that for any  $k \in \mathbb{N}$

$$\inf \{ \delta > 0 : \exp(-\delta)\varphi < \psi < \exp(\delta)\varphi \} < 1/k$$

i.e. there is  $\delta_k > 0$  such that  $e^{-\delta_k}\varphi < \psi < e^{\delta_k}\varphi$  and  $\delta_k < 1/k$ . Thus  $\lim_{k \rightarrow \infty} \delta_k = 0$  and  $\psi - e^{-\delta_k}\varphi \in P(H)_x$  shows (by passing to the limit with  $k \rightarrow \infty$  in  $\psi(v, v) - e^{-\delta_k}\varphi(v, v) > 0$ ,  $v \in H(M)_x \setminus \{0\}$ ) that  $\varphi < \psi$ . Similarly  $e^{\delta_k}\varphi - \psi \in P(H)_x$  yields in the limit  $\psi < \varphi$ , and we may conclude that  $\varphi = \psi$ . Viceversa, if  $\varphi \in P(H)_x$  then

$$\{ \delta > 0 : (1 - e^{-\delta})\varphi, (e^\delta - 1)\varphi \in P(H)_x \} = (0, +\infty)$$



hence  $\rho'_x(\varphi, \varphi) = 0$ . Q.e.d.

**Lemma 4.** i)  $(P(H)_x, \rho'_x)$  is a complete metric space.

ii) Let  $\{\varphi_j\}_{j \in \mathbb{N}} \subset P(H)_x$  such that  $\lim_{j \rightarrow \infty} \varphi_j = \varphi \in P(H)_x$  in the  $\rho'_x$ -topology. Then  $\lim_{j \rightarrow \infty} \varphi_j(v, w) = \varphi(v, w)$  for any  $v, w \in H(M)_x$ .

*Proof.* i) Let  $\{\varphi_j\}_{j \in \mathbb{N}} \subset P(H)_x$  be a Cauchy sequence in the  $\rho'_x$ -topology i.e. for any  $\epsilon > 0$  there is  $j_\epsilon \in \mathbb{N}$  such that  $\rho'_x(\varphi_{j+p}, \varphi_j) > \epsilon$  for any  $j \geq j_\epsilon$  and any  $p = 1, 2, \dots$ . Hence there is  $\delta_\epsilon > 0$  such that  $e^{-\delta_\epsilon} \varphi_j < \varphi_{j+p} < e^{\delta_\epsilon} \varphi_j$  and  $\delta_\epsilon < \epsilon$ . Consequently

$$|\log \varphi_{j+p}(v, v) - \log \varphi_j(v, v)| < \delta_\epsilon < \epsilon$$

for any  $v \in H(M)_x \setminus \{0\}$ . Therefore if

$$\xi_j \equiv (\log \varphi_j(v, v), \dots, \log \varphi_j(v, v)) \in \mathbb{R}^{2n}$$

then  $\{\xi_j\}_{j \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{R}^{2n}$ . Let then  $\xi = \lim_{j \rightarrow \infty} \xi_j$  and let  $\varphi : H(M)_x \times H(M)_x \rightarrow \mathbb{R}$  be the bilinear form given by  $\varphi(v, v) = \exp(\xi^a)$  for any  $v \in H(M)_x \setminus \{0\}$  followed by polarization. Here  $\xi = (\xi^1, \dots, \xi^{2n})$ . Then  $\varphi \in P(H)_x$  and  $\lim_{j \rightarrow \infty} \varphi_j = \varphi$  in the  $\rho'_x$ -topology.

ii) If  $\varphi_j \rightarrow \varphi$  as  $j \rightarrow \infty$  then  $\log \varphi_j(v, v) \rightarrow \log \varphi(v, v)$  as  $j \rightarrow \infty$ , for any  $v \in H(M)_x \setminus \{0\}$ . Then  $\lim_{j \rightarrow \infty} \varphi_j(v, v) = \varphi(v, v)$  uniformly in  $v$  and statement (ii) follows by polarization. Q.e.d.

As  $M$  is compact we may set

$$\rho''_H(G_1, G_2) = \sup_{x \in M} \rho''_x(G_{1,x}, G_{2,x}),$$

$$\rho_H(G_1, G_2) = \rho'_H(G_1, G_2) + \rho''_H(G_1, G_2), \quad G_1, G_2 \in \mathcal{M}_H.$$

Also let  $d$  be the distance function on  $\mathcal{P}_+$  given by

$$d(\theta_1, \theta_2) = d'(\theta_1, \theta_2) + \rho''_H(G_{\theta_1}, G_{\theta_2}), \quad \theta_1, \theta_2 \in \mathcal{P}_+.$$

**Proposition 1.** i)  $(\mathcal{M}_H, \rho_H)$  is a complete metric space.

ii) The map  $\theta \in \mathcal{P}_+ \mapsto G_\theta \in \mathcal{M}_H$  of  $(\mathcal{P}_+, d)$  into  $(\mathcal{M}_H, \rho_H)$  is continuous.

iii)  $(\mathcal{P}_+, d)$  is a complete metric space.

*Proof.* i) Let  $\{G_j\}_{j \geq 1}$  be a Cauchy sequence in  $(\mathcal{M}_H, \rho_H)$ . Then  $\{G_j\}_{j \geq 1}$  is a Cauchy sequence in both  $(\mathfrak{X}, \rho'_H)$  and  $(\mathcal{M}_H, \rho''_H)$ . Yet  $(\mathfrak{X}, \rho'_H)$  is complete (by Lemma 2). Thus  $\rho'_H(G_j, G) \rightarrow 0$  as  $j \rightarrow \infty$  for some  $G \in \mathfrak{X}$ . In particular

$$(4) \quad \lim_{j \rightarrow \infty} G_{j,x}(v, w) = G_x(v, w)$$

for every  $x \in M$  and  $v, w \in H(M)_x$ . On the other hand, as  $\{G_j\}_{j \geq 1}$  is Cauchy in  $(\mathcal{M}_H, \rho''_H)$ , for every  $\epsilon > 0$  there is  $N_\epsilon \geq 1$  such that

$$(5) \quad \rho''_x(G_{i,x}, G_{j,x}) \leq \rho''_H(G_i, G_j) < \epsilon$$



for every  $i, j \geq N_\epsilon$  and  $x \in M$ . Thus  $\{G_{j,x}\}_{j \geq 1}$  is Cauchy in the complete (by Lemma 4) metric space  $(P(H)_x, \rho_x'')$  so that  $\rho_x''(G_{j,x}, \varphi) \rightarrow 0$  as  $j \rightarrow \infty$  for some  $\varphi \in P(H)_x$ . Then (by (iii) in Lemma 4)  $\lim_{j \rightarrow \infty} G_{j,x}(v, w) = \varphi(v, w)$  for every  $v, w \in H(M)_x$  hence  $G_x = \varphi$  yielding  $G \in \mathcal{M}_H$ .

ii) Let  $\{\theta_\nu\}_{\nu \geq 1} \subset \mathcal{P}_+$  such that  $d(\theta_\nu, \theta) \rightarrow 0$  for  $\nu \rightarrow \infty$  for some  $\theta \in \mathcal{P}_+$ . If  $\theta_\nu = e^{u_\nu} \theta_0$  and  $\theta = e^u \theta_0$  then  $|u_\nu - u|_{C^\infty} \rightarrow 0$  as  $\nu \rightarrow \infty$ . Then  $G_{\theta_\nu} = e^{u_\nu} G_{\theta_0}$  and  $G_\theta = e^u G_{\theta_0}$ . Since  $D^\alpha u_\nu \rightarrow D^\alpha u$  as  $\nu \rightarrow \infty$ , uniformly on  $\overline{U}_\lambda$ , for any  $\lambda \in \Lambda$ ,  $|\alpha| \leq k$  and  $k \in \mathbb{N} \cup \{0\}$ , it follows that  $D^\alpha(G_{\theta_\nu})_{ab} \rightarrow D^\alpha(G_\theta)_{ab}$  as  $\nu \rightarrow \infty$  uniformly on  $\overline{U}_\lambda$  for any  $1 \leq a, b \leq 2n$ . Hence  $G_{\theta_\nu} \rightarrow G_\theta$  in  $\mathfrak{X}$  so that (by the very definition of  $d$  and  $\rho_H$ )  $\rho_H(G_{\theta_\nu}, G_\theta) \rightarrow 0$ . Q.e.d.

iii) If  $\{\theta_\nu\}_{\nu \geq 1}$  is a Cauchy sequence in  $(\mathcal{P}_+, d)$  then  $\{u_\nu\}_{\nu \geq 1}$  is Cauchy in  $(\mathcal{P}_+, d')$  as well. Yet (by Lemma 1)  $(\mathcal{P}_+, d')$  is complete hence  $d'(\theta_\nu, \theta) \rightarrow 0$  for some  $\theta \in \mathcal{P}_+$ . Then, as a byproduct of the proof of statement (ii), one has  $G_{\theta_\nu} \rightarrow G_\theta$  in  $\mathfrak{X}$ . Finally, the *verbatim* repetition of the arguments in the proof of statement (i) yields  $\rho_H''(G_{\theta_\nu}, G_\theta) \rightarrow 0$  so that  $d(\theta_\nu, \theta) \rightarrow 0$ . Q.e.d.

#### 4. A MAX-MINI PRINCIPLE

For each  $k \in \mathbb{N} \cup \{0\}$  we consider a  $(k + 1)$ -dimensional real subspace  $L_{k+1} \subset C^\infty(M, \mathbb{R})$  and set

$$\Lambda_\theta(L_{k+1}) = \sup \left\{ \frac{\|\nabla^H f\|_{L^2}^2}{\|f\|_{L^2}^2} : f \in L_{k+1} \setminus \{0\} \right\}.$$

Here

$$\|f\|_{L^2} = \left( \int_M f^2 \Psi_\theta \right)^{\frac{1}{2}}, \quad \|X\|_{L^2} = \left( \int_M g_\theta(X, X) \Psi_\theta \right)^{\frac{1}{2}},$$

for any  $f \in C^\infty(M, \mathbb{R})$  and any  $X \in \mathfrak{X}(M)$ . Let  $\{u_\nu\}_{\nu \geq 0} \subset C^\infty(M, \mathbb{R})$  be a complete orthonormal system relative to the  $L^2$  inner product  $(f, g)_{L^2} = \int_M f g \Psi_\theta$  such that  $u_\nu \in \text{Eigen}(\Delta_b; \lambda_\nu(\theta))$  for every  $\nu \geq 0$ . If  $f \in C^\infty(M, \mathbb{R})$  then  $f = \sum_{\nu=0}^\infty a_\nu(f) u_\nu$  ( $L^2$  convergence) for some  $a_\nu(f) \in \mathbb{R}$ . Let  $L_{k+1}^0$  be the subspace of  $C^\infty(M, \mathbb{R})$  spanned by  $\{u_\nu : 0 \leq \nu \leq k\}$ . Let  $(\nabla^H)^*$  be the formal adjoint of  $\nabla^H$  i.e.

$$(\nabla^H f, X)_{L^2} = (f, (\nabla^H)^* X)_{L^2}$$

for any  $f \in C^\infty(M, \mathbb{R})$  and  $X \in C^\infty(H(M))$ . Mere integration by parts shows that

$$(\nabla^H)^* X = -\text{div}(X), \quad X \in C^\infty(H(M)),$$

implying (by (1)) the useful identity

$$(6) \quad \|\nabla^H f\|_{L^2}^2 = (f, \Delta_b f)_{L^2}, \quad f \in C^\infty(M, \mathbb{R}).$$

Let  $f \in L_{k+1}^0 \setminus \{0\}$  so that  $f = \sum_{v=0}^k a_v u_v$  for some  $a_v \in \mathbb{R}$ . Then (by (6))

$$\|\nabla^H f\|_{L^2}^2 = \sum_{v=0}^k a_v^2 \lambda_v(\theta) \leq \lambda_k(\theta) \sum_{v=0}^k a_v^2 = \lambda_k(\theta) \|f\|_{L^2}^2$$

hence

$$(7) \quad \Lambda_\theta(L_{k+1}^0) \leq \lambda_k(\theta).$$

Our purpose in this section is to establish

**Proposition 2.** *Let  $M$  be a compact strictly pseudoconvex CR manifold and  $\theta \in \mathcal{P}_+$  a positively oriented contact form. Then*

$$(8) \quad \lambda_k(\theta) = \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$$

where the g.l.b. is taken over all subspaces  $L_{k+1} \subset C^\infty(M, \mathbb{R})$  with  $\dim_{\mathbb{R}} L_{k+1} = k + 1$ .

So far (by (7))  $\lambda_k(\theta) \geq \Lambda_\theta(L_{k+1}^0) \geq \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$ . The proof of Proposition 2 is by contradiction. We assume that  $\lambda_k(\theta) > \inf_{L_{k+1}} \Lambda_\theta(L_{k+1})$  i.e. there is a  $(k + 1)$ -dimensional subspace  $L_{k+1} \subset C^\infty(M, \mathbb{R})$  such that  $\Lambda_\theta(L_{k+1}) < \lambda_k(\theta)$ . Then  $\Lambda_\theta(L_{k+1})$  is finite and

$$\|f\|_{L^2}^2 \Lambda_\theta(L_{k+1}) \geq \|\nabla^H f\|_{L^2}^2, \quad f \in L_{k+1}.$$

Then (by (6))

$$\sum_{v=0}^{\infty} a_v(f)^2 \Lambda_\theta(L_{k+1}) \geq \sum_{v=0}^{\infty} \lambda_v(\theta) a_v(f)^2$$

so that

$$(9) \quad \begin{aligned} & \sum_{\Lambda_\theta(L_{k+1}) \geq \Lambda_\nu(\theta)} a_\nu(f)^2 [\Lambda_\theta(L_{k+1}) - \lambda_\nu(\theta)] \geq \\ & \geq \sum_{\Lambda_\theta(L_{k+1}) < \lambda_\nu(\theta)} a_\nu(f)^2 [\lambda_\nu(\theta) - \Lambda_\theta(L_{k+1})]. \end{aligned}$$

Let  $\Phi : L_{k+1} \rightarrow C^\infty(M, \mathbb{R})$  be the linear map given by

$$\Phi(f) = \sum_{v=0}^m a_v(f) u_v, \quad f \in L_{k+1},$$

where  $m = \max\{v \geq 0 : \lambda_v(\theta) \leq \Lambda_\theta(L_{k+1})\}$ . Note that  $0 \leq m \leq k - 1$  (by the contradiction assumption). We claim that

$$(10) \quad \text{Ker}(\Phi) \neq (0).$$

Of course (10) is only true within the contradiction loop. The statement follows from  $\dim_{\mathbb{R}} \Phi(L_{k+1}) \leq m + 1 \leq k < k + 1$  (hence  $\Phi$  cannot be injective). Let (by (10))  $f_0 \in L_{k+1}$  such that  $\Phi(f_0) = 0$  and  $f_0 \neq 0$ . Then  $a_\nu(f_0) = 0$  for any  $0 \leq \nu \leq m$  i.e. whenever  $\Lambda_\theta(L_{k+1}) \geq \lambda_\nu(\theta)$ . Applying

(9) to  $f = f_0$  yields  $a_\nu(f_0) = 0$  whenever  $\Lambda_\theta(L_{k+1}) < \lambda_\nu(\theta)$ . Thus  $f_0 = 0$ , a contradiction.

## 5. CONTINUITY OF EIGENVALUES

The scope of § 5 is to establish

**Theorem 1.** *Let  $M$  be a compact strictly pseudoconvex CR manifold. If  $\delta > 0$  and  $\theta, \hat{\theta} \in \mathcal{P}_+$  are two contact forms on  $M$  such that  $d(\theta, \hat{\theta}) < \delta$  then  $e^{-\delta} \lambda_k(\theta) \leq \lambda_k(\hat{\theta}) \leq e^\delta \lambda_k(\theta)$  for any  $k \geq 0$ .*

*Proof.* For any  $x \in M$

$$\delta > \inf \left\{ \epsilon > 0 : e^{-\epsilon} G_{\theta,x} < G_{\hat{\theta},x} < e^\epsilon G_{\theta,x} \right\}$$

i.e. there is  $0 < \epsilon < \delta$  such that  $G_{\hat{\theta},x} - e^{-\epsilon} G_{\theta,x} \in P(H)_x$  and  $e^\epsilon G_{\theta,x} - G_{\hat{\theta},x} \in P(H)_x$ . There is a unique  $u \in C^\infty(M, \mathbb{R})$  such that  $\hat{\theta} = e^u \theta$ . Consequently

$$(11) \quad \hat{\theta} \wedge (d\hat{\theta})^n = e^{(n+1)u} \theta \wedge (d\theta)^n.$$

On the other hand  $e^{-\delta} G_{\theta,x}(v, v) < G_{\hat{\theta},x}(v, v) < e^\delta G_{\theta,x}(v, v)$  for any  $v \in H(M)_x \setminus \{0\}$  implies  $|u| < \delta$ . Then for every  $f \in C^\infty(M)$  (by (11))

$$(12) \quad e^{-(n+1)\delta} \int_M f^2 \Psi_\theta \leq \int_M f^2 \Psi_{\hat{\theta}} \leq e^{(n+1)\delta} \int_M f^2 \Psi_\theta.$$

Moreover

$$(13) \quad \hat{\nabla}^H f = e^{-u} \nabla^H f$$

where  $\hat{\nabla}^H f$  is the horizontal gradient of  $f$  with respect to  $\hat{\theta}$ . Thus (by (13))  $\|\hat{\nabla}^H f\|_\theta^2 = e^{-u} \|\nabla^H f\|_\theta^2 < e^\delta \|\nabla^H f\|_\theta^2$  so that (by (11))

$$(14) \quad \begin{aligned} e^{-(n+2)\delta} \int_M \|\nabla^H f\|_\theta^2 \Psi_\theta &\leq \int_M \|\hat{\nabla}^H f\|_\theta^2 \Psi_{\hat{\theta}} \leq \\ &\leq e^{(n+2)\delta} \int_M \|\nabla^H f\|_\theta^2 \Psi_\theta. \end{aligned}$$

Finally (by (12)-(13))

$$e^{-\delta} \frac{\|\nabla^H f\|_{L^2}^2}{\|f\|_{L^2}^2} \leq \frac{\int_M \|\hat{\nabla}^H f\|_\theta^2 \Psi_{\hat{\theta}}}{\int_M f^2 \Psi_\theta} \leq e^\delta \frac{\|\nabla^H f\|_{L^2}^2}{\|f\|_{L^2}^2}$$

so that (by the Max-Mini principle)

$$(15) \quad e^{-\delta} \lambda_k(\theta) \leq \lambda_k(\hat{\theta}) \leq e^\delta \lambda_k(\theta).$$

Theorem 1 is proved. Corollary 1 follows from (15).

6. SPECTRA OF  $\Delta_b$  AND  $\square$ 

Let  $F_\theta$  be the Fefferman metric of  $(M, \theta)$  and  $\square$  the corresponding wave operator (the Laplace-Beltrami operator of  $(C(M), F_\theta)$ ). We set  $\mathfrak{M} = C(M)$  for simplicity. Let  $g$  be a fixed Riemannian metric on  $\mathfrak{M}$ . The space  $S(\mathfrak{M})$  of all symmetric tensor fields may be identified with the space of all fields of endomorphisms of  $T(\mathfrak{M})$  which are symmetric with respect to  $g$  i.e. for each  $h \in S(\mathfrak{M})$  let  $\tilde{h} \in C^\infty(\text{End}(T(\mathfrak{M})))$  be given by

$$g(\tilde{h}X, Y) = h(X, Y), \quad X, Y \in \mathfrak{X}(\mathfrak{M}).$$

From now on we assume that  $M$  is compact. Then  $\mathfrak{M}$  is compact as well (as  $\mathfrak{M}$  is the total space of a principal bundle with compact base and compact fibres) and we endow  $S(\mathfrak{M})$  with the distance function

$$d_g^\infty(h_1, h_2) = \sup_{z \in \mathfrak{M}} \left[ \text{trace} \left( \varphi_z^2 \right) \right]^{1/2}, \quad h_1, h_2 \in S(\mathfrak{M}),$$

where  $\varphi = \tilde{h}_1 - \tilde{h}_2$  and  $\varphi_z^2 = \varphi_z \circ \varphi_z$ . The set  $\text{Lor}(\mathfrak{M})$  of all Lorentz metrics on  $\mathfrak{M}$  is an open set of  $(S(\mathfrak{M}), d_g^\infty)$  and for any pair  $g_1, g_2$  of Riemannian metrics on  $\mathfrak{M}$  the distance functions  $d_{g_1}$  and  $d_{g_2}$  are uniformly equivalent (cf. e.g. [10], p. 49). We shall use the topology induced by  $d_g^\infty$  on  $\text{Lor}(\mathfrak{M})$  (and therefore on  $C \subset \text{Lor}(\mathfrak{M})$ ). By a result of J.M. Lee, [8], the sublaplacian  $\Delta_b$  of  $(M, \theta)$  is the pushforward of the wave operator i.e.  $\pi_* \square = \Delta_b$ . In particular  $\sigma(\Delta_b) \subset \sigma(\square)$ . Thus each  $\lambda_k : \mathcal{P}_+ \rightarrow \mathbb{R}$  may be thought of as a function  $\lambda_k^\uparrow : C \rightarrow \mathbb{R}$  such that  $\lambda_k^\uparrow \circ F = \lambda_k$  for every  $k \geq 0$ , where  $F : \mathcal{P}_+ \rightarrow C$  is the map given by  $F(\theta) = F_\theta$  for every  $\theta \in \mathcal{P}_+$ . As another consequence of Theorem 1 we establish

**Corollary 2.** *Let  $M$  be a compact strictly pseudoconvex CR manifold and let  $g$  be an arbitrary Riemannian metric on  $\mathfrak{M} = C(M)$ . Let  $\theta_0 \in \mathcal{P}_+$  be a fixed contact form and  $\mathcal{P}_{++} = \{e^u \theta_0 : u \in C^\infty(M, \mathbb{R}), u > 0\}$ . If  $C_+ = \{F_\theta : \theta \in \mathcal{P}_{++}\}$  then for every  $k \in \mathbb{N} \cup \{0\}$  the function  $\lambda_k^\uparrow : C_+ \rightarrow \mathbb{R}$  is continuous relative to the  $d_g^\infty$ -topology.*

*Proof.* Let  $\theta_i \in \mathcal{P}_+$ ,  $i \in \{1, 2\}$ , and let us set  $\varphi = \tilde{F}_{\theta_1} - \tilde{F}_{\theta_2}$ . Let  $\{E_p : 1 \leq p \leq 2n + 2\}$  be a local  $g$ -orthonormal frame on  $T(\mathfrak{M})$ , defined on the open set  $\mathcal{U} \subset \mathfrak{M}$ . Then

$$\text{trace}(\varphi^2) = \sum_{p=1}^{2n+2} g(\varphi^2 E_p, E_p) = \sum_p \left\{ F_{\theta_1}(\varphi E_p, E_p) - F_{\theta_2}(\varphi E_p, E_p) \right\}$$

on  $\mathcal{U}$ . On the other hand if  $\varphi E_p = \varphi_p^q E_q$  then  $\varphi_p^q = F(\theta_1)(E_p, E_q) - F(\theta_2)(E_p, E_q)$  hence

$$(16) \quad \text{trace}(\varphi^2) = (e^{u_1 \circ \pi} - e^{u_2 \circ \pi})^2 \|F_{\theta_0}\|_g^2$$

where  $u_i \in C^\infty(M, \mathbb{R})$  is given by  $\theta_i = e^{u_i} \theta_0$  and  $\|F_{\theta_0}\|_g$  is the norm of  $F_{\theta_0}$  as a  $(0, 2)$ -tensor field on  $\mathfrak{M}$  with respect to  $g$ . Then (by (16))

$$(17) \quad d_g^\infty(F_{\theta_1}, F_{\theta_2}) = \sup_{\mathfrak{M}} |e^{u_1 \circ \pi} - e^{u_2 \circ \pi}| \|F_{\theta_0}\|_g.$$

As  $\mathfrak{M}$  is compact  $a = \inf_{z \in \mathfrak{M}} \|F_{\theta_0}\|_{g,z} > 0$ . Indeed (by compactness)  $a = \|F_{\theta_0}\|_{g,z_0}$  for some  $z_0 \in \mathfrak{M}$ . If  $a = 0$  then  $F_{\theta_0, z_0} = 0$ , a contradiction (as  $F_{\theta_0}$  is Lorentzian, and hence nondegenerate). Let  $\epsilon > 0$  such that  $d_g^\infty(F_{\theta_1}, F_{\theta_2}) < \epsilon$ . Then  $|e^{u_1} - e^{u_2}| < \epsilon/a$  everywhere on  $M$ . As both  $u_1 > 0$  and  $u_2 > 0$  it follows that  $|u_1 - u_2| < \log(1 + \epsilon/a)$ . Indeed  $e^{u_1} - e^{u_2} < \epsilon/a$  is equivalent to  $e^{u_1 - u_2} < 1 + (\epsilon/a)e^{-u_2}$  hence (as  $u_2 > 0$ )

$$u_1 - u_2 < \log[1 + (\epsilon/a)e^{-u_2}] < \log(1 + \epsilon/a).$$

Therefore

$$(1 + \epsilon/a)^{-1} G_{\theta_1, x}(v, v) < G_{\theta_2, x}(v, v) < (1 + \epsilon/a) G_{\theta_1, x}(v, v)$$

for any  $v \in H(M)_x \setminus \{0\}$  and any  $x \in M$ . Consequently  $\rho_H''(G_{\theta_1}, G_{\theta_2}) < \log(1 + \epsilon/a)$ . The arguments in § 5 then yield

$$(1 + \epsilon/a)^{-1} \lambda_k^\uparrow(F_{\theta_1}) \leq \lambda_k^\uparrow(F_{\theta_2}) \leq (1 + \epsilon/a) \lambda_k^\uparrow(F_{\theta_1})$$

and Corollary 2 follows. The problem of the behavior of  $\lambda_k^\uparrow : \mathcal{C} \rightarrow \mathbb{R}$  is open. So does the more general problem of the behavior of the spectrum of the wave operator on  $\mathfrak{M}$  with respect to a change of  $F \in \text{Lor}(\mathfrak{M})$ . Further work (cf. [1]) on the behavior of  $\sigma(\Delta_b)$  under analytic 1-parameter deformations  $\{\theta(t)\}_{t \in \mathbb{R}}$  of a given contact form  $\theta_0 \in \mathcal{P}_+$  builds on the Riemannian counterpart in [6] and the functional analysis results in [7].

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