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On the Maximum Density of Graphs with Unique-Path Labellings

Abbas Mehrabian* Dieter Mitsche† Paweł Prałat‡

Abstract

A *unique-path labelling* of a simple, finite graph is a labelling of its edges with real numbers such that, for every ordered pair of vertices (u, v) , there is at most one nondecreasing path from u to v . In this paper we prove that any graph on n vertices that admits a unique-path labelling has at most $n \log_2(n)/2$ edges, and that this bound is tight for infinitely many values of n . Thus we significantly improve on the previously best known bounds. The main tool of the proof is a combinatorial lemma which might be of independent interest. For every n we also construct an n -vertex graph that admits a unique-path labelling and has $n \log_2(n)/2 - O(n)$ edges.

1 Introduction

Let G be a finite, simple graph. A *unique-path labelling* (also known as *good edge-labelling*, see, e.g., [1, 3, 6]) of G is a labelling of its edges with real numbers such that, for any ordered pair of vertices (u, v) , there is at most one nondecreasing path from u to v . This notion was introduced in [2] to solve wavelength assignment problems for specific categories of graphs. We say G is *good* if it admits a unique-path labelling.

Let $f(n)$ be the maximum number of edges of a good graph on n vertices. Araújo, Cohen, Giroire, and Havet [1] initiated the study of this function. They observed that hypercube graphs are good and that any graph containing K_3 or $K_{2,3}$ is not good. From

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these observations they concluded that if n is a power of two, then

$$f(n) \geq \frac{n}{2} \log_2(n),$$

and that for all n ,

$$f(n) \leq \frac{n\sqrt{n}}{\sqrt{2}} + O(n^{4/3}).$$

The first author of this paper proved that any good graph whose maximum degree is within a constant factor of its average degree (in particular, any good regular graph) has at most $n^{1+o(1)}$ edges—see [6] for more details.

Before we state the main result of this paper, we need one more definition. Let $b(n)$ be the function that counts the total number of 1's in the binary expansions of all integers from 0 up to $n - 1$. This function was studied in [5]. Our main result is the following theorem.

Theorem 1. *For all positive integers n ,*

$$\frac{n}{2} \log_2 \left(\frac{3n}{4} \right) \leq b(n) \leq f(n) \leq \frac{n}{2} \log_2(n).$$

It follows that the asymptotic value of $f(n)$ is $n \log_2(n)/2 - O(n)$. Note that Theorem 1 implies that *any* good graph on n vertices has at most $n \log_2(n)/2$ edges, significantly improving the previously known upper bounds. Moreover, this bound is tight if n is a power of two. We also give an explicit construction of a good graph with n vertices and $b(n)$ edges for every n .

2 The Proofs

This section is devoted to proving the main result, Theorem 1.

2.1 The upper bound

For a graph G , an edge-labelling $\phi : E(G) \rightarrow \mathbb{R}$, and an integer $t \geq 0$, a *nice t -walk from v_0 to v_t* is a sequence $v_0 v_1 \dots v_t$ of vertices such that $v_{i-1} v_i$ is an edge for $1 \leq i \leq t$, and $v_{i-1} \neq v_{i+1}$ and $\phi(v_{i-1} v_i) \leq \phi(v_i v_{i+1})$ for $1 \leq i \leq t - 1$. We call v_t the *last vertex* of the walk. When t does not play a role, we simply refer to a *nice walk*. The existence of a self-intersecting nice walk implies that the edge-labelling is not a unique-path labelling: let

$v_0v_1 \dots v_t$ be a shortest such walk with $v_0 = v_t$. Then there are two nondecreasing paths $v_0v_1 \dots v_{t-1}$ and v_0v_{t-1} from v_0 to v_{t-1} . Thus if for some pair of distinct vertices (u, v) there are two nice walks from u to v , then the labelling is not a unique-path labelling. Also, if for some vertex v , there is a nice t -walk from v to v with $t > 0$, then the labelling is not a unique-path labelling. Consequently, if the total number of nice walks is larger than $2\binom{n}{2} + n = n^2$, then the labelling is not a unique-path labelling.

The following lemma will be very useful.

Lemma 1. *Let G and H be graphs with unique-path labellings on disjoint vertex sets. Then if we add a matching between the vertices of G and H (i.e., add a set of edges, such that each added edge has exactly one endpoint in $V(G)$ and exactly one endpoint in $V(H)$, and every vertex in $V(G) \cup V(H)$ is incident to at most one added edge), then the resulting graph is good.*

Proof. Consider unique-path labellings of G and H , and let M be a number greater than all existing labels. Then label the matching edges with $M, M + 1, M + 2$, etc. It is not hard to verify that the resulting edge-labelling is still a unique-path labelling. ■

Corollary 2. *We have $f(1) = 0$ and for all $n > 1$,*

$$f(n) \geq \max \left\{ f(n_1) + f(n_2) + \min\{n_1, n_2\} : 1 \leq n_1, 1 \leq n_2, n_1 + n_2 = n \right\}.$$

The proof of the upper bound in Theorem 1 relies on the analysis of a one-player game, which is defined next. The player, who will be called Alice henceforth, starts with n sheets of paper, on each of which a positive integer is written. In every step, Alice performs the following operation. She chooses any two sheets. Assume that the numbers written on them are a and b . She erases these numbers, and writes $a + b$ on both sheets. Clearly, the sum of the numbers increases by $a + b$ after this move. The aim of the game is to keep the sum of the numbers smaller than a certain threshold.

The *configuration* of the game is a multiset of size n , containing the numbers written on the sheets, in which the multiplicity of number x equals the number of sheets on which x is written. Let S be the *starting configuration* of the game, namely, a multiset of size n containing the numbers initially written on the sheets, and let $k \geq 0$ be an integer. We denote by $opt(S, k)$ the smallest sum Alice can get after performing k operations. An intuitively good-looking strategy is the following: in each step, choose two sheets with the smallest numbers. We call this the *greedy* strategy, and show that it is indeed an optimal strategy. Specifically, we prove the following theorem, which may be of independent interest.

Theorem 2. *For any starting configuration S and any nonnegative integer k , if Alice plays the greedy strategy, then the sum of the numbers after k moves equals $\text{opt}(S, k)$.*

Before proving Theorem 2, we show how this implies our upper bound.

Proof of the upper bound of Theorem 1. Let G be a graph with n vertices and $m > n \log_2(n)/2$ edges. We need to show that G does not have a unique-path labelling. Consider an arbitrary edge-labelling $\phi : E(G) \rightarrow \mathbb{R}$. Enumerate the edges of G as e_1, e_2, \dots, e_m such that

$$\phi(e_1) \leq \phi(e_2) \leq \dots \leq \phi(e_m).$$

We may assume that the inequalities are strict. Indeed, if some label L appears $p > 1$ times, we can assign the labels $L, L + 1, \dots, L + (p - 1)$ to the edges originally labelled L , and increase by p the labels of edges with original label larger than L . It is easy to see that the modified edge-labelling is still a unique-path labelling, and by repeatedly applying this operation all ties are broken.

Let us denote by G_i the subgraph of G induced by $\{e_1, e_2, \dots, e_i\}$. For each vertex v and $0 \leq i \leq m$, let $a_v^{(i)}$ be the number of nice walks with last vertex v in G_i . Clearly, $a_v^{(0)} = 1$ for all vertices v . Suppose the graph is initially empty and we add the edges e_1, e_2, \dots, e_m , one by one, in this order. Fix an i with $1 \leq i \leq m$. Let u and v be the endpoints of e_i . After adding the edge e_i , for any t , any nice t -walk with last vertex u (respectively, v) in G_{i-1} can be extended via e_i to a nice $(t + 1)$ -walk with last vertex v (respectively, u) in G_i . So, we have $a_u^{(i)} = a_v^{(i)} = a_u^{(i-1)} + a_v^{(i-1)}$ and $a_w^{(i)} = a_w^{(i-1)}$ for $w \notin \{u, v\}$ (if the walk ends at some other vertex, the additional edge e_i does not help).

Thus the final list of numbers $\{a_v^{(m)}\}_{v \in V(G)}$ can be seen as the end-result of an instance of the one-player game described before, with starting configuration $S = \{1, 1, \dots, 1\}$, so we have

$$\sum_{v \in V(G)} a_v^{(m)} \geq \text{opt}(S, m).$$

Hence, in order to prove that ϕ is not a unique-path labelling, it is sufficient to show that $\text{opt}(S, m) > n^2$.

Let m_0 be the largest number for which $\text{opt}(S, m_0) \leq n^2$, and let $\alpha = \lfloor \log_2(n) \rfloor$. First, assume that n is even. By Theorem 2, we may assume that Alice plays according to the greedy strategy. The smallest number on the sheets is initially 1, and is doubled after every $n/2$ moves. Hence after $\alpha n/2$ moves, the smallest number becomes 2^α , so the sum of the numbers would be $2^\alpha n$. In every subsequent move, the sum is increased by $2^{\alpha+1}$,

so Alice can play at most $(n^2 - 2^\alpha n)/2^{\alpha+1}$ more moves before the sum of the numbers becomes greater than n^2 . Consequently,

$$m_0 \leq \alpha \frac{n}{2} + \frac{n(n - 2^\alpha)}{2^{\alpha+1}}.$$

Now, define $h(x) := \log_2(x) - x + 1$. Then h is concave in $[1, 2]$ and $h(1) = h(2) = 0$, which implies that $h(x) \geq 0$ for all $x \in [1, 2]$. In particular, for $x_0 = n/2^\alpha$, we have

$$\frac{n - 2^\alpha}{2^\alpha} = x_0 - 1 \leq \log_2(x_0) = \log_2\left(\frac{n}{2^\alpha}\right) = \log_2(n) - \alpha.$$

Therefore,

$$m_0 \leq \frac{n}{2} \alpha + \frac{n}{2} \frac{n - 2^\alpha}{2^\alpha} \leq \frac{n}{2} \log_2(n) < m,$$

which completes the proof.

Finally, assume that n is odd. Since $2n$ is even, we have

$$f(2n) \leq n \log_2(2n) = n \log_2(n) + n.$$

On the other hand, by Corollary 2,

$$f(2n) \geq 2f(n) + n.$$

Combining these inequalities gives

$$f(n) \leq \frac{n}{2} \log_2(n),$$

completing the proof of the lemma. ■

The rest of this section is devoted to proving Theorem 2. Let $S = \{s_1, s_2, \dots, s_n\}$ be the starting configuration of the game. Consider a k -step strategy $T = ((i_1, j_1), (i_2, j_2), \dots, (i_k, j_k))$, where i_r and j_r are the indices of the sheets Alice choose in the r -th step. Note that after the k -th step, the sum of the numbers is of the form $\sum_{i=1}^n c_i s_i$ for some positive integers $\{c_i\}_{i=1}^n$. The vector $(c_i)_{i=1}^n$ depends only on $i_1, j_1, i_2, j_2, \dots, i_k, j_k$, and not on $\{s_i\}_{i=1}^n$. We call $(c_i)_{i=1}^n$ the *characteristic vector* of strategy T . Notice that for any permutation π of $\{1, 2, \dots, n\}$, if (c_1, c_2, \dots, c_n) is the characteristic vector of some k -step strategy, then so is $(c_{\pi(1)}, c_{\pi(2)}, \dots, c_{\pi(n)})$. This is because Alice can first permute the sheets according to the permutation π , and then apply the same strategy as before.

Proof of Theorem 2. We use induction over the number of moves k . If $k = 1$, the statement is obvious, so let us assume that $k \geq 2$. Let $S = \{s_t\}_{t=1}^n$ be the starting configuration.

We may assume that $s_1 \leq s_2 \leq \dots \leq s_n$. Let $T = ((i_1, j_1), (i_2, j_2), \dots, (i_k, j_k))$ be an optimal k -step strategy with characteristic vector $(c_t)_{t=1}^n$. We first make an observation and a claim.

First, let $1 \leq t \leq n$ be arbitrary and let r be the first step in which Alice chooses sheet t , say $i_r = t$. Then, observe that $c_{j_r} \geq c_{i_r}$, with equality if and only if r is the first step in which sheet j_r is chosen: indeed, if sheet j_r is chosen for the first time at step r , then from step r onwards the numbers s_t and s_{j_r} are always summed together, hence $c_{j_r} = c_{i_r}$. If on the other hand, sheet j_r had been chosen before, then the corresponding coefficient c_{j_r} is strictly greater. Note that this fact does not depend on the optimality of T .

Second, we claim that $c_1 \geq c_2 \geq \dots \geq c_n$. Assume that this was not true, and consider a permutation π of $\{1, 2, \dots, n\}$ such that $c_{\pi(1)} \geq c_{\pi(2)} \geq \dots \geq c_{\pi(n)}$. Then, by the Rearrangement Inequality (see e.g. [4], inequality (368), p. 261),

$$\sum_{t=1}^n c_{\pi(t)} s_t < \sum_{t=1}^n c_t s_t.$$

However, $(c_{\pi(1)}, c_{\pi(2)}, \dots, c_{\pi(n)})$ is the characteristic vector of some k -step strategy, and this contradicts the optimality of T .

Let r be the first step in which Alice chooses sheet 1, say $i_r = 1$. Then, by the observation above, $c_{j_r} \geq c_1$. However, c_1 is the maximum among $\{c_t\}_{t=1}^n$ by the claim, hence we have $c_{j_r} = c_{j_r-1} = \dots = c_2 = c_1$, and r is the first step in which sheet j_r is chosen. Now, let σ be the permutation on $\{1, 2, \dots, n\}$ obtained from applying the transposition $(2, j_r)$ on the identity permutation. Then $(c_{\sigma(t)})_{t=1}^n$ is the characteristic vector of some k -step strategy T' , which is optimal since $\sum_{t=1}^n c_{\sigma(t)} s_t = \sum_{t=1}^n c_t s_t = \text{opt}(S, k)$. Note that we could possibly have $T' = T$.

In T' , the sheets 1 and 2 are chosen in the r -th step, and none of them has been chosen prior to this step. Thus, the move $(1, 2)$ can be shifted to the beginning of the move sequence without changing the characteristic vector. Hence, there exists an optimal k -step strategy starting with the summation of two minimal numbers, i.e., the same starting move as the greedy strategy. After this first step, we have a new configuration and $k - 1$ more moves, for which, by induction, the greedy strategy is optimal, and this concludes the proof. ■

2.2 The lower bound

In this section we prove the lower bound in Theorem 1. Recall that $b(n)$ is equal to the total number of 1's in the binary expansions of all integers from 0 up to $n - 1$. It is known [5] that $b(1) = 0$ and $b(n)$ satisfies the recursive formula

$$b(n) = \max\{b(n_1) + b(n_2) + \min\{n_1, n_2\} : 1 \leq n_1, 1 \leq n_2, n_1 + n_2 = n\},$$

and the lower bound in Theorem 1 follows by using induction and applying Corollary 2. Moreover, McIlroy [5] proved that $b(n) \geq n \log_2(\frac{3}{4}n) / 2$.

For every n we also give an explicit construction of a good graph with n vertices and $b(n)$ edges. It is easy to see that $b(n)$ equals the number of edges in the graph G_n with vertex set $\{0, 1, \dots, n - 1\}$, and with vertices i and j being adjacent if the binary expansions of i and j differ in exactly one digit. This graph is an induced subgraph of the $\lceil \log_2(n) \rceil$ -dimensional hypercube graph. It can be shown by induction and Lemma 1 that the hypercube graph is good, which implies that G_n is also good (since the restriction of a unique-path labelling for the supergraph to the edges of the subgraph is a unique-path labelling for the subgraph). Hence G_n is a good graph with n vertices and $b(n)$ edges.

3 Concluding Remarks

We proved that any n -vertex graph with a unique-path labelling has at most $n \log_2(n) / 2$ edges, and for every n we constructed a good n -vertex graph with $n \log_2(n) / 2 - O(n)$ edges. Thus we proved $f(n) = n \log_2(n) / 2 - O(n)$. One can try to investigate the second order term of the function $f(n)$. Perhaps it is the case that our construction is best possible; that is, in fact $f(n) = b(n)$?

It would be interesting to further investigate the connection between having a unique-path labelling and other parameters of the graph; in particular, the length of the shortest cycle (known as the girth) of the graph (see, e.g., [3]). Araújo et al. [1] proved that any planar graph with girth at least 6 has a unique-path labelling, and asked whether 6 can be replaced with 5 in this result. The first author [6] proved that any graph with maximum degree Δ and girth at least 40Δ is good. This does not seem to be tight, and improving the dependence on Δ is an interesting research direction.

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