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► **To cite this version:**

Etienne Blanchard. Tensor products of  $C(X)$ -algebras over  $C(X)$ . *Asterisque*, 1995, 232, pp.81–92.  
hal-00922902v1

**HAL Id: hal-00922902**

**<https://hal.science/hal-00922902v1>**

Submitted on 31 Dec 2013 (v1), last revised 3 Jan 2014 (v2)

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# Subtriviality of continuous fields of nuclear $C^*$ -algebras

By Étienne Blanchard

Last update 15/08/04

## Abstract

We extend in this paper the characterisation of a separable nuclear  $C^*$ -algebra given by Kirchberg proving that given a unital separable continuous field of nuclear  $C^*$ -algebras  $A$  over a compact metrizable space  $X$ , the  $C(X)$ -algebra  $A$  is isomorphic to a unital  $C(X)$ -subalgebra of the trivial continuous field  $\mathcal{O}_2 \otimes C(X)$ , image of  $\mathcal{O}_2 \otimes C(X)$  by a norm one projection.

**AMS classification:** 46L05.

## 0 Introduction

In order to study deformations in the  $C^*$ -algebraic framework, Dixmier introduced the notion of continuous field of  $C^*$ -algebras over a locally compact space ([7]). In the same way as there is a faithful representation in a Hilbert space for any  $C^*$ -algebra thanks to the Gelfand–Naimark–Segal construction, a separable continuous field of  $C^*$ -algebras  $A$  over a compact metrizable space  $X$  always admits a continuous field of faithful representations  $\pi$  in a Hilbert  $C(X)$ -module, i.e. there exists a family of representations  $\{\pi_x, x \in X\}$ , in a separable Hilbert space  $H$  which factorize through a faithful representation of the fibre  $A_x$  such that for each  $a \in A$ , the map  $x \mapsto \pi_x(a)$  is strongly continuous ([4, théorème 3.3]).

In a work on tensor products over  $C(X)$  of continuous fields of  $C^*$ -algebras over  $X$  ([16]), Kirchberg and Wassermann raised the question of whether the continuous field of  $C^*$ -algebras  $A$  could be subtrivialized, i.e. whether one could find a continuous field of faithful representations  $\pi$  such that the map  $x \mapsto \pi_x(a) \in L(H)$  is actually norm continuous for all  $a$  in  $A$ . In this case, given any  $C^*$ -algebra  $B$ , the minimal tensor product  $A \otimes B$  is a  $C(X)$ -subalgebra of the trivial continuous field  $[L(H) \otimes B] \otimes C(X)$  and is therefore a continuous field with fibres  $(A \otimes B)_x = A_x \otimes B$ . They proved that a non-exact continuous field with exact fibres cannot be subtrivialized and they constructed such examples.

The non-trivial example of the continuous field of rotation algebras over the unit circle  $\mathbb{T}$  had already been studied by Haagerup and Rørdam in [10]. More precisely, they constructed continuous functions  $u, v$  from  $\mathbb{T}$  to the unitary group  $U(H)$  of the infinite-dimensional separable Hilbert space  $H$  satisfying the commutation relation  $u_t v_t = v_t u_t$

for all  $t \in \mathbb{T}$  and the uniform continuity condition  $\max\{\|u_t - u_{t'}\|, \|v_t - v_{t'}\|\} < C'|t - t'|^{1/2}$  where  $C'$  is a computable constant.

Our purpose in the present paper is to show that the subtrivialization is always possible in the nuclear separable case through a generalisation of the following theorem of Kirchberg using  $\mathcal{R}KK$ -theory arguments:

**Theorem 0.1** ([15]) *A unital separable  $C^*$ -algebra  $A$  is exact if and only if it is isomorphic to a  $C^*$ -subalgebra of  $\mathcal{O}_2$ . Moreover the  $C^*$ -algebra  $A$  is nuclear if and only if  $A$  is isomorphic to a  $C^*$ -subalgebra of  $\mathcal{O}_2$  containing the unit  $1_{\mathcal{O}_2}$  of  $\mathcal{O}_2$ , image of  $\mathcal{O}_2$  by a unital completely positive projection.*

As a matter of fact, we get an equivalent characterisation of nuclear separable continuous fields of  $C^*$ -algebras (theorem 3.2) which is made possible thanks to  $C(X)$ -linear homotopy invariance of the bifunctor  $\mathcal{R}KK(X; -, -)$  (theorem 1.6) and  $C(X)$ -linear Weyl-von Neumann absorption results (proposition 2.5). This also enables us to have a better understanding of the characterisation of separable continuous fields of nuclear  $C^*$ -algebras given by Bauval in [2].

In an added appendix, the corresponding characterisation of exact separable continuous fields of  $C^*$ -algebras as  $C(X)$ -subalgebras of  $\mathcal{O}_2 \otimes C(X)$  given by Eberhard Kirchberg is described (theorem A.1).

*I would like to thank E. Kirchberg for his enlightenment on the exact case. I also want to express my gratitude to C. Anantharaman-Delaroche and J. Cuntz for fruitful discussions.*

## 1 Preliminaries

### 1.1 $C(X)$ -algebras

Let  $X$  be a compact Hausdorff space and  $C(X)$  be the  $C^*$ -algebra of continuous functions on  $X$  with complex values. We start by recalling the following definition.

**Definition 1.1** ([13]) *A  $C(X)$ -algebra is a  $C^*$ -algebra  $A$  endowed with a unital morphism from  $C(X)$  in the centre of the multiplier algebra  $M(A)$  of  $A$ .*

**Remark:** We do not assume that  $C(X)$  embeds into  $M(A)$ . For instance, there is a natural structure of  $C([0, 2])$ -algebra on the  $C^*$ -algebra  $C([0, 1])$ .

For  $x \in X$ , define the kernel  $C_x(X)$  of the evaluation map  $ev_x : C(X) \rightarrow \mathbb{C}$  at  $x$ ; denote by  $A_x$  the quotient of a  $C(X)$ -algebra  $A$  by the closed ideal  $C_x(X)A$  and by  $a_x$  the image of an element  $a \in A$  in the fibre  $A_x$ . Then the function

$$x \mapsto \|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\}$$

is upper semi-continuous for any  $a \in A$  and the  $C(X)$ -algebra  $A$  is said to be a continuous field of  $C^*$ -algebras over  $X$  if the function  $x \mapsto \|a_x\|$  is actually continuous for every  $a \in A$  ([7]).

**Examples 1.** If  $A$  is a  $C(X)$ -algebra and  $D$  is a  $C^*$ -algebra, the spatial tensor product  $B = A \otimes D$  is naturally endowed with a structure of  $C(X)$ -algebra through the map  $f \in C(X) \mapsto f \otimes 1_{M(D)} \in M(A \otimes D)$ . In particular, if  $A = C(X)$ , the tensor product  $B$  is a trivial continuous field over  $X$  with constant fibre  $B_x \simeq D$

2. Given a  $C(X)$ -algebra  $A$ , define the unital  $C(X)$ -algebra  $\mathcal{A}$  generated by  $A$  and  $u[C(X)]$  in  $M[A \oplus C(X)]$  where  $u(g)(a \oplus f) = ga \oplus gf$  for  $a \in A$  and  $f, g \in C(X)$ . It defines a continuous field of  $C^*$ -algebras over  $X$  if and only if the  $C(X)$ -algebra  $A$  is continuous ([4, proposition 3.2]).

**Remark:** If  $A$  is a separable continuous field of non-zero  $C^*$ -algebras (not necessarily unital) over the compact Hausdorff space  $X$ , the positive cone  $C(X)_+$  and so the  $C^*$ -algebra  $C(X)$  are separable. Hence, the topological space  $X$  is metrizable.

**Definition 1.2** ([4, 5]) *Given a continuous field of  $C^*$ -algebras  $A$  over the compact Hausdorff space  $X$ , a continuous field of representations of a  $C(X)$ -algebra  $D$  in the multiplier  $C^*$ -algebra  $M(A)$  of  $A$  is a  $C(X)$ -linear morphism  $\pi : D \rightarrow M(A)$ , i.e. for each  $x \in X$ , the induced representation  $\pi_x$  of  $D$  in  $M(A_x)$  factorizes through the fibre  $D_x$ .*

If the  $C(X)$ -algebra  $D$  admits a continuous field of faithful representations  $\pi$  in the  $C(X)$ -algebra  $M(A)$  where  $A$  is a continuous field over  $X$ , i.e. the induced representation of the fibre  $D_x$  in  $M(A_x)$  is faithful for every point  $x \in X$ , the function

$$x \mapsto \|\pi_x(d)\| = \sup\{\|(\pi(d)a)_x\|, a \in A \text{ such that } \|a\| \leq 1\}$$

is lower semi-continuous for all  $d \in D$  and the  $C(X)$ -algebra  $D$  is therefore continuous.

In particular a separable  $C(X)$ -algebra  $D$  is continuous if and only if there exists a Hilbert  $C(X)$ -module  $\mathcal{E}$  such that  $D$  admits a continuous field of faithful representations in the multiplier algebra  $M(\mathcal{K}(\mathcal{E})) = \mathcal{L}(\mathcal{E})$  of the continuous field over  $X$  of compact operators  $\mathcal{K}(\mathcal{E})$  acting on  $\mathcal{E}$  ([4, théorème 3.3]).

Let us also mention the characterisation of separable continuous fields of nuclear  $C^*$ -algebras over a compact metrizable space  $X$  given by Bauval in [2] using a natural  $C(X)$ -linear version of nuclearity introduced by Kasparov and Skandalis in [14]§6.2 : a  $C(X)$ -linear completely positive  $\sigma$  from a  $C(X)$ -algebra  $A$  into a  $C(X)$ -algebra  $B$  is said to be  $C(X)$ -nuclear if and only if given any compact set  $F$  in  $A$  and any strictly positive real number  $\varepsilon$ , there exist an integer  $k$  and  $C(X)$ -linear completely positive contractions  $T : A \rightarrow M_k(\mathbb{C}) \otimes C(X)$  and  $S : M_k(\mathbb{C}) \otimes C(X) \rightarrow B$  such that for all  $a \in F$ , one has the inequality

$$\|\sigma(a) - (S \circ T)(a)\| < \varepsilon.$$

One can then state the following results. The first assertion is a simple  $C(X)$ -linear reformulation of the Choi-Effros theorem and the second one is due to Bauval.

**Proposition 1.3** *Let  $X$  be a compact metrizable space and  $A$  be a separable  $C(X)$ -algebra.*

1. ([14]§6.2) Given a  $C(X)$ -algebra  $B$  and a closed ideal  $J \subset B$ , any contractive  $C(X)$ -nuclear map  $A \rightarrow B/J$  admits a contractive  $C(X)$ -linear completely positive lift  $A \rightarrow B$ .
2. ([2, théorème 7.2]) The  $C(X)$ -algebra  $A$  is a continuous fields of nuclear  $C^*$ -algebras over  $X$  if and only if the identity map  $id_A : A \rightarrow A$  is  $C(X)$ -nuclear.

**Remark:** In assertion 1., the ideal  $J = (C(X)B)J = C(X)J$  is a  $C(X)$ -algebra.

## 1.2 $C(X)$ -extensions

Given a compact Hausdorff space  $X$ , we introduce a natural  $C(X)$ -linear version of the semi-group  $Ext(-, -)$  defined by Kasparov ([12, 13]).

Call a morphism of  $C(X)$ -algebras a  $*$ -homomorphism between  $C(X)$ -algebras which is  $C(X)$ -linear.

**Definition 1.4** A  $C(X)$ -extension of a  $C(X)$ -algebra  $A$  by a  $C(X)$ -algebra  $B$  is a short exact sequence

$$0 \rightarrow B \rightarrow D \xrightarrow{\pi} A \rightarrow 0$$

in the category of  $C(X)$ -algebras. The  $C(X)$ -extension is said to be trivial if the map  $\pi$  admits a cross section  $s : A \rightarrow D$  which is a morphism of  $C(X)$ -algebras.

As in the  $C^*$ -algebraic case a  $C(X)$ -extension  $0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0$  of  $A$  by  $B$  defines unambiguously an homomorphism from  $D$  to the multiplier algebra  $M(B)$  of  $B$ , which gives a morphism of  $C(X)$ -algebras  $\sigma : A \rightarrow M(B)/B$  (called the Busby invariant of the extension) and the  $C(X)$ -extension is trivial if and only if the map  $\sigma$  lifts to a morphism of  $C(X)$ -algebras  $A \rightarrow M(B)$ . Conversely, given a morphism of  $C(X)$ -algebras  $\sigma : A \rightarrow M(B)/B$ , setting  $D = \{(a, m) \in A \times M(B), \sigma(a) = q(m)\}$  where  $q$  is the quotient map  $M(B) \rightarrow M(B)/B$ , one has a  $C(X)$ -extension  $0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0$  (see [12]§7).

**Remark:** A  $C(X)$ -extension  $0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0$  induces for every  $x \in X$  a  $C^*$ -extension  $0 \rightarrow B_x \rightarrow D_x \rightarrow A_x \rightarrow 0$ .

In order to define the sum of two  $C(X)$ -extensions, recall that the Cuntz algebra  $\mathcal{O}_2$  is the unital  $C^*$ -algebra generated by two orthogonal isometries  $s_1$  and  $s_2$  subject to the relation  $1 = s_1 s_1^* + s_2 s_2^*$  ([6]). Then if  $\mathcal{K}$  is the  $C^*$ -algebra of compact operators on the infinite-dimensional separable Hilbert space, one defines the sum of two  $C(X)$ -extensions  $\sigma_1$  and  $\sigma_2$  of the  $C(X)$ -algebra  $A$  by the stable  $C(X)$ -algebra  $\mathcal{K} \otimes B$  through the choice of a unital copy of  $\mathcal{O}_2$  in the multiplier algebra  $M(\mathcal{K})$  of  $\mathcal{K}$  to be the  $C(X)$ -extension

$$\sigma_1 \oplus \sigma_2 : a \mapsto q(s_1 \otimes 1)\sigma_1(a)q(s_1^* \otimes 1) + q(s_2 \otimes 1)\sigma_2(a)q(s_2^* \otimes 1) \in M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B),$$

where  $q$  is the quotient map  $M(\mathcal{K} \otimes B) \rightarrow M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$ .

**Definition 1.5** Given a compact Hausdorff space  $X$  and two  $C(X)$ -algebras  $A$  and  $B$ ,  $Ext(X; A, B)$  is the semi-group of  $C(X)$ -extensions of  $A$  by  $\mathcal{K} \otimes B$  divided by the equivalence relation  $\sim$  where  $\sigma_1 \sim \sigma_2$  if there exist a unitary  $U \in M(\mathcal{K} \otimes B)$  of image  $q(U)$  in the quotient  $M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$  and two trivial  $C(X)$ -extensions  $\pi_1$  and  $\pi_2$  such that for all  $a \in A$ ,

$$(\sigma_2 \oplus \pi_2)(a) = q(U)^*(\sigma_1 \oplus \pi_1)(a)q(U) \text{ (in } M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)\text{)}.$$

Let  $Ext(X; A, B)^{-1}$  be the group of invertible elements of  $Ext(X; A, B)$ , i.e. the group of classes of  $C(X)$ -extension  $\sigma$  such that there exists a  $C(X)$ -extension  $\theta$  with  $\sigma \oplus \theta$  trivial. One can generalise Kasparov's theorem of homotopy invariance of the group  $Ext(A, B)^{-1}$  to the framework of  $C(X)$ -algebras as follows.

**Theorem 1.6** ([12]) Assume that  $A$  is a separable  $C(X)$ -algebra and that  $B$  is a  $\sigma$ -unital  $C(X)$ -algebra. Then the group  $Ext(X; A, B)^{-1}$  is isomorphic to the group  $\mathcal{R}KK^1(X; A, B)$  and is therefore  $C(X)$ -linear homotopy invariant in both entries  $A$  and  $B$ .

**Proof :** Let us first make the following observation. Given a  $C(X)$ -algebra  $B$  and a Hilbert  $B$ -module  $\mathcal{E}$ , denote by  $\mathcal{L}(\mathcal{E})$  the set of bounded  $B$ -linear operators on  $\mathcal{E}$  which admit an adjoint ([11]). Then any operator  $T \in \mathcal{L}(\mathcal{E})$  is  $B$ -linear and so  $C(X)$ -linear. This argument provides a natural extension of the Stinespring-Kasparov theorem ([12]) to the framework of  $C(X)$ -algebras. Consequently, if  $A$  is a separable  $C(X)$ -algebra and  $B$  is a  $\sigma$ -unital  $C(X)$ -algebra, the class of a  $C(X)$ -extension  $\sigma : A \rightarrow M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$  is invertible in  $Ext(X; A, B)$  if and only if there is a  $C(X)$ -linear completely positive contractive lift  $A \rightarrow M(\mathcal{K} \otimes B)$ .

Let  $\mathcal{R}\mathcal{E}(X; A, B)$  be the set of Kasparov  $C(X)$ - $A, B$ -bimodules ([13, definition 2.19]), i.e. the set of Kasparov  $A, B$  bimodules  $(\mathcal{E}, F)$  such that the representation  $A \rightarrow \mathcal{L}(\mathcal{E})$  is a  $C(X)$ -representation. Call a  $C(X)$ -linear operator homotopy an element  $\{(\mathcal{E}, F_t), 0 \leq t \leq 1\} \in \mathcal{R}\mathcal{E}(X; A, B \otimes C([0, 1]))$  such that  $t \mapsto F_t$  is norm continuous and define on  $\mathcal{R}\mathcal{E}(X; A, B)$  the equivalence relation corresponding to the one defined by Skandalis in [18, definition 2]. The constructions given by Kasparov in [12, section 7] imply that, if the  $C(X)$ -algebra  $B$  is  $\sigma$ -unital, the group of equivalence classes  $\mathcal{R}\widetilde{K}K(X; A, B \otimes \mathcal{C}_1)$  is isomorphic to  $Ext(X; A, B)^{-1}$ , where  $\mathcal{C}_1$  is the first (graded) Clifford algebra.

On the other hand, given two graded  $C(X)$ -algebras  $A$  and  $B$  with  $A$  separable, the different steps of the demonstration of [18, theorem 19] provide us with an isomorphism between the two groups  $\mathcal{R}\widetilde{K}K(X; A, B)$  and  $\mathcal{R}KK(X; A, B)$  since proposition 2.21 of [13] defines an intersection product in  $\mathcal{R}\widetilde{K}K$ -theory and lemma 18 of [18] gives us the equality

$$(ev_0 \otimes id_{C(X)})^*(1_{C(X)}) = (ev_1 \otimes id_{C(X)})^*(1_{C(X)}) \text{ in } \mathcal{R}\widetilde{K}K(X; C([0, 1]) \otimes C(X), C(X)),$$

where  $1_{C(X)}$  is the Kasparov  $C(X), C(X)$ -bimodule  $(C(X), 0)$  and  $ev_t : C([0, 1]) \rightarrow \mathbb{C}$  is the evaluation map at  $t \in [0, 1]$ .  $\square$

**Remarks:** 1. Kuiper's theorem implies that the law of addition on the abelian group  $Ext(X; A, B)^{-1}$  is independent of the choice of the unital copy of  $\mathcal{O}_2$  in  $M(\mathcal{K})$ .

2. If  $A$  is a separable nuclear continuous field of  $C^*$ -algebras over  $X$  and  $B$  is a  $C(X)$ -algebra, every  $C(X)$ -linear morphism from  $A$  to the quotient  $M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$  is  $C(X)$ -nuclear and therefore admits a  $C(X)$ -linear completely positive lifting  $A \rightarrow M(\mathcal{K} \otimes B)$  thanks to proposition 1.3. Accordingly one has the equality

$$\text{Ext}(X; A, B)^{-1} = \text{Ext}(X; A, B).$$

## 2 An absorption result

In this section we prove a continuous generalisation of a statement contained in [15] which will enable us to get a  $C(X)$ -linear Weyl-von Neumann type result (proposition 2.5). Let us start with the following definition of Cuntz.

**Definition 2.1** ([6]) *A simple  $C^*$ -algebra  $A$  distinct from  $\mathbb{C}$  is said to be purely infinite if and only if for any non-zero  $a, b \in A$ , there exist elements  $x, y \in A$  such that  $a = xby$ .*

Then, we can state a proposition from Kirchberg's classification work, based on Glimm's lemma ([7], § 11.2). A sketch of proof can also be found in [1, proposition 5.1].

**Proposition 2.2** ([15]) *Let  $A$  be a purely infinite simple  $C^*$ -algebra and  $D$  be a separable  $C^*$ -subalgebra of  $M(A)$ . Assume that  $V : D \rightarrow A$  is a nuclear contraction.*

*Then there exists a sequence  $(a_n)$  of elements in  $A$  of norm less than 1 such that  $V(d) = \lim_{n \rightarrow \infty} a_n^* d a_n$  for all  $d \in D$ .*

**Remark:** A simple ring has by definition exactly two distinct two sided ideals and is therefore non-zero.

**Corollary 2.3** *Let  $A$  be a continuous field of purely infinite simple  $C^*$ -algebras over a compact Hausdorff space  $X$  and assume that  $D$  is a separable  $C(X)$ -subalgebra of the multiplier algebra  $M(A)$  such that there is a unital  $C(X)$ -embedding of the  $C(X)$ -algebra  $\mathcal{O}_\infty \otimes C(X)$  in the commutant  $D'$  of  $D$  in  $M(A)$  and the identity map  $id_D : D \rightarrow M(A)$  is a continuous field of faithful representations.*

*If  $V : D \rightarrow A$  is a  $C(X)$ -nuclear contraction, there exists a sequence  $(a_n)$  in the unit ball of  $A$  with the property that for all  $d \in D$ ,*

$$V(d) = \lim_{n \rightarrow \infty} a_n^* d a_n.$$

**Proof :** If  $F$  is a compact generating set for  $D$ , it is enough to prove that given a strictly positive real number  $\varepsilon > 0$ , there exists an element  $a$  in the unit ball of  $A$  such that  $\|V(d) - a^* d a\| < \varepsilon$  for all  $d \in F$ .

For  $x \in X$ , the fibre  $A_x$  is a purely infinite simple  $C^*$ -algebra and the map  $d \mapsto V(d)_x \in A_x$  factorizes through  $D_x \simeq (id_D)_x(D) \subset M(A_x)$  since  $id_D$  is a continuous field of faithful representations. As a consequence, the previous proposition implies that we can find an element  $g \in A$  with  $\|g\| \leq 1$  satisfying for all  $d \in F$  the inequality

$$\| [V(d) - g^* d g]_x \| < \varepsilon.$$

Thus, by upper semi-continuity and compactness, there exist a finite open covering  $\{U_1, \dots, U_n\}$  of the space  $X$  and elements  $g_1, \dots, g_n$  in the unit ball of  $A$  such that for all  $d \in F$  and  $x \in U_i$ ,  $1 \leq i \leq n$ ,

$$\| [V(d) - g_i^* d g_i]_x \| < \varepsilon.$$

Choose  $n$  orthogonal isometries  $w_1, \dots, w_n$  in the  $C^*$ -algebra  $\mathcal{O}_\infty \otimes 1_{C(X)} \subset D'$  and let  $\{\phi_i\}$  be a partition of the unit  $1_{C(X)}$  subordinate to the covering  $\{U_i\}$  of  $X$ . The element  $a = \sum_i \phi_i^{1/2} w_i g_i \in A$  verifies:

1.  $a^* a = \sum_{i,j} \sqrt{\phi_i \phi_j} g_i^* w_i^* w_j g_j = \sum_i \phi_i g_i^* g_i \leq 1_{M(A)}$ ,
2. for  $d \in F$  and  $x \in X$ ,  $\| [V(d) - a^* d a]_x \| \leq \sum_i \phi_i(x) \| [V(d) - g_i^* d g_i]_x \| < \varepsilon$ .  $\square$

Let us mention the following technical corollary which will be needed in theorem 3.2.

**Corollary 2.4** *If  $p \in \mathcal{O}_2 \otimes C(X)$  is a projection such that for all points  $x \in X$ ,  $p_x$  is non-zero, then there exists an isometry  $u \in \mathcal{O}_2 \otimes C(X)$  such that  $p = uu^*$ .*

**Proof :** Let  $\mathcal{D}_2 = \lim_{n \rightarrow \infty} \mathcal{O}_2^{\otimes n}$  be the infinite tensor product of  $\mathcal{O}_2$ .

Given a projection  $q \in \mathcal{D}_2 \otimes C(X)$  such that  $\|q_x\| = 1$  for all  $x \in X$ , we first show that there exists an element  $v \in \mathcal{D}_2 \otimes C(X)$  satisfying  $1_{\mathcal{D}_2 \otimes C(X)} = v^* q v$ . Namely, by density of the algebraic tensor product

$$\left[ \bigcup_n \mathcal{O}_2^{\otimes n} \right] \odot C(X) = \bigcup_n \left[ \mathcal{O}_2^{\otimes n} \odot C(X) \right]$$

in the  $C^*$ -algebra  $\mathcal{D}_2 \otimes C(X)$  and functional calculus one can find an integer  $n > 0$  and a projection  $r \in \mathcal{O}_2^{\otimes n} \otimes C(X) \subset \mathcal{D}_2 \otimes C(X)$  such that  $\|q - r\| < 1$ , which implies that  $r = s^* q s$  for some element  $s \in \mathcal{D}_2 \otimes C(X)$ . Take then a faithful state  $\varphi$  on  $\mathcal{O}_2^{\otimes n}$  and consider the  $C(X)$ -linear completely positive map

$$V : [\mathcal{O}_2^{\otimes n} \otimes 1_{\mathcal{O}_2}] \otimes C(X) \rightarrow \mathcal{O}_2^{\otimes n+1} \otimes C(X)$$

defined by the formula  $V(d) = (\varphi \otimes id_{C(X)})(d) 1_{\mathcal{O}_2^{\otimes n+1} \otimes C(X)}$  for  $d \in [\mathcal{O}_2^{\otimes n} \otimes 1_{\mathcal{O}_2}] \otimes C(X) \simeq \mathcal{O}_2^{\otimes n} \otimes C(X)$ . According to corollary 2.3, there exists an element  $t \in \mathcal{O}_2^{\otimes n+1} \otimes C(X)$  such that

$$1_{\mathcal{D}_2 \otimes C(X)} = 1_{\mathcal{O}_2^{\otimes n+1} \otimes C(X)} = t^* r t = (st)^* q (st).$$

Consider now the set  $\mathcal{P}$  of projections  $p$  in  $\mathcal{O}_2 \otimes C(X)$  such that  $p_x \neq 0$  for all points  $x \in X$ . If  $p$  belongs to  $\mathcal{P}$ , there exists an isometry  $v \in \mathcal{O}_2 \otimes C(X)$  such that  $p \geq vv^*$  since the  $K$ -trivial purely infinite separable unital nuclear  $C^*$ -algebra  $\mathcal{D}_2$  satisfying the U.C.T. is isomorphic to  $\mathcal{O}_2$  ([15]). As a consequence, if  $t$  is the isometry  $t = v(s_1 \otimes 1)v^*$ , the projection  $r = tt^*$  (Murray-von Neumann equivalent to  $1_{\mathcal{O}_2 \otimes C(X)}$ ) verifies

$$p - r \geq r' = v(s_2 s_2^* \otimes 1)v^* \in \mathcal{P}.$$



The non-empty set  $\mathcal{P}$  therefore satisfies the conditions  $(\pi_1)$ – $(\pi_4)$  defined by Cuntz in [6]. But the  $C^*$ -algebra  $\mathcal{O}_2 \otimes C(X)$  is  $K_0$ -triviality thanks to [6, theorem 2.3] and the theorem 1.4 of [6] enables us to conclude.  $\square$

One now deduces from corollary 2.3 the following absorption results ([21, 12, 15]):

**Proposition 2.5** *Let  $A$  be a  $\sigma$ -unital continuous field of purely infinite simple nuclear  $C^*$ -algebras over a compact Hausdorff space  $X$  and let  $\mathcal{K}$  be the  $C^*$ -algebra of compact operators on the separable Hilbert space  $H$ . Denote by  $q$  the quotient map  $M(\mathcal{K} \otimes A) \rightarrow M(\mathcal{K} \otimes A)/(\mathcal{K} \otimes A)$ .*

1. *Assume that  $D$  is a unital separable  $C(X)$ -subalgebra of the multiplier algebra  $M(\mathcal{K} \otimes A)$  with same unit such that there is a unital  $C(X)$ -embedding of the  $C(X)$ -algebra  $\mathcal{O}_\infty \otimes C(X)$  in the commutant of  $D$  in  $M(\mathcal{K} \otimes A)$  and the identity map  $id_D$  is a continuous field of faithful representations of  $D$  in  $M(\mathcal{K} \otimes A)$ .*

(a) *If  $V$  is a unital  $C(X)$ -linear completely positive map from  $D$  in  $M(\mathcal{K} \otimes A)$  which is zero on the intersection  $D \cap (\mathcal{K} \otimes A)$ , there exists a sequence of isometries  $s_n$  in  $M(\mathcal{K} \otimes A)$  such that for every  $d \in D$ ,*

$$V(d) - s_n^* d s_n \in \mathcal{K} \otimes A \text{ and } V(d) = \lim_n s_n^* d s_n.$$

(b) *If  $\pi$  is a unital morphism of  $C(X)$ -algebras from  $D$  into  $M(\mathcal{K} \otimes A)$  which is zero on the intersection  $D \cap (\mathcal{K} \otimes A)$ , there exists a sequence of unitaries  $u_n$  in  $M(\mathcal{K} \otimes A)$  such that for every  $d \in D$ ,*

$$(d \oplus \pi(d)) - u_n^* d u_n \in \mathcal{K} \otimes A \text{ and } (d \oplus \pi(d)) = \lim_n u_n^* d u_n.$$

(c) *Let  $B$  be a  $C(X)$ -algebra and assume that the quotient  $D/(D \cap (\mathcal{K} \otimes A))$  is isomorphic to the  $C(X)$ -algebra  $\mathcal{B}$ , where  $\mathcal{B}$  is the unital  $C(X)$ -algebra generated by  $C(X)$  and  $B$  in  $M[B \oplus C(X)]$  ([4, définition 2.7]).*

*Then, if  $\pi : B \rightarrow M(\mathcal{K} \otimes A)$  is a  $C(X)$ -linear homomorphism, there exists a unitary  $U \in M(\mathcal{K} \otimes A)$  such that for all  $b \in B \subset M(\mathcal{K} \otimes A)/(\mathcal{K} \otimes A)$ ,*

$$b \oplus (q \circ \pi)(b) = q(U)^* b q(U).$$

2. *Assume that the continuous field  $A$  is separable and let  $D$  be a separable  $C(X)$ -subalgebra of  $M(A)$  containing  $A$  such that the identity representation  $D \rightarrow M(A)$  is a continuous field of faithful representations and there is a unital  $C(X)$ -embedding of the  $C(X)$ -algebra  $\mathcal{O}_\infty \otimes C(X)$  in the commutant of  $D$  in  $M(A)$ . Define the quotient  $C(X)$ -algebra  $B = D/A$ .*

*If  $\pi : \mathcal{K} \otimes B \rightarrow M(\mathcal{K} \otimes A)$  is a morphism of  $C(X)$ -algebras, there exists a unitary  $U \in M(\mathcal{K} \otimes A)$  such that for all  $b \in \mathcal{K} \otimes B \subset M(\mathcal{K} \otimes A)/(\mathcal{K} \otimes A)$ ,*

$$b \oplus (q \circ \pi)(b) = q(U)^* b q(U).$$

**Proof :** 1. It derives from corollary 2.3 by the same method as the one developed by Kasparov in [11, theorem 5 and 6]. Nevertheless, for the convenience of the reader we describe the different steps of the demonstration.

1.a) Let  $F$  be a compact generating set for  $D$  containing the unit  $1_{M(\mathcal{K} \otimes A)}$ . Then given a real number  $\varepsilon > 0$ , it is enough to find an element  $a \in M(\mathcal{K} \otimes A)$  such that  $V(d) - a^*da \in \mathcal{K} \otimes A$  and  $\|V(d) - a^*da\| < 3\varepsilon$  for all  $d \in F$ .

Let  $\{e_n\}$  be an increasing, positive, quasicontral, countable approximate unit in the ideal  $\mathcal{K} \otimes A$  of the  $C^*$ -algebra generated by  $\mathcal{K} \otimes A + V(D)$ . If we set  $f_0 = (e_0)^{1/2}$  and  $f_k = (e_k - e_{k-1})^{1/2}$  for  $k \geq 1$ , we can then assume, passing to a subsequence of  $(e_n)$  if necessary, that  $\|V(d)f_k - f_kV(d)\| < 2^{-k}\varepsilon$  for all  $k \in \mathbb{N}$  and  $d \in F$ . This implies that the series  $\sum_k [V(d)f_k - f_kV(d)]f_k$  is convergent in  $\mathcal{K} \otimes A$  and its norm is less than  $\varepsilon$ . Furthermore, the series  $\sum_k [f_kV(d)f_k]$  is strictly convergent in  $M(\mathcal{K} \otimes A)$  for all  $d \in F$  since  $\sum_k f_k^2$  is strictly convergent to 1.

Notice now that the maps  $V_k(d) = f_kV(d)f_k$  are all  $C(X)$ -nuclear since the separable continuous field  $\mathcal{K} \otimes A$  is nuclear. The corollary 2.3 therefore enables us to choose by induction  $a_k \in \mathcal{K} \otimes A$  satisfying the following conditions:

1.  $\forall d \in F \quad \|V_k(d) - a_k^*da_k\| < 2^{-k}\varepsilon,$
2.  $\forall d \in F, \forall l < k \quad \|a_l^*da_k\| < 2^{-l-k}\varepsilon,$
3.  $\sum_k a_k$  is strictly convergent toward an element  $a \in M(A)$ .

One then checks as in [11, theorem 5] that the limit  $a$  satisfies the desired properties.

1.b) Take a compact generating  $F$  for  $D$  containing  $1_{M(\mathcal{K} \otimes A)}$  and consider the homomorphism  $\pi' = 1 \otimes \pi : D \rightarrow M(\mathcal{K} \otimes (\mathcal{K} \otimes A)) \simeq M(\mathcal{K} \otimes A)$ . Given  $\delta > 0$ , one can find, thank to the previous assertion, an isometry  $s \in M(\mathcal{K} \otimes A)$  such that

$$s^*ds - \pi'(d) \in \mathcal{K} \otimes A \text{ and } \|s^*ds - \pi'(d)\| < \delta \text{ for all } d \in K^*K.$$

As a consequence, if we fix  $\varepsilon > 0$ , the choice of  $\delta$  small enough gives us the inequality  $\|pd - dp\| < \varepsilon$ , and so  $\|d - [pdp + p^\perp dp^\perp]\| < 2\varepsilon$  for all  $d \in F$ , where  $p = ss^*$  and  $p^\perp = 1 - p$ .

Define the unital map  $\Theta : D \rightarrow M(p^\perp(\mathcal{K} \otimes A)p^\perp)$  by the formula  $\Theta(d) = p^\perp dp^\perp$ . According to the stabilisation theorem of Kasparov ([11, theorem 2]), one can construct a unitary  $w \in M(\mathcal{K} \otimes A)$  verifying for all  $d \in F$  the inequality

$$\|d - w^*[\pi'(d) \oplus \Theta(d)]w\| < 3\varepsilon.$$

To finish the demonstration, notice that the two homomorphisms  $\pi'$  and  $\pi' \oplus \pi$  are unitarily equivalent.

1.c) Consider the unital extension  $\tilde{\pi}$  of  $\pi$  to  $\mathcal{B}$ . Then, the morphism  $\tilde{\pi} \circ q : D \rightarrow M(\mathcal{K} \otimes A)$  reduces the demonstration to the previous assertion.

2. The identity representation of the unital  $C(X)$ -algebra  $\mathcal{D} = (\mathcal{K} \otimes D) + C(X) \subset M(\mathcal{K} \otimes A)$  is clearly a continuous field of faithful representations since the unital  $C(X)$ -representation  $C(X) \rightarrow M(A)$  is a continuous field of faithful representations. Extend

the map  $\pi : \mathcal{K} \otimes B = (\mathcal{K} \otimes D)/(\mathcal{K} \otimes A) \rightarrow M(\mathcal{K} \otimes A)$  to a unital morphism of  $C(X)$ -algebras  $\tilde{\pi} : \mathcal{D}/(\mathcal{K} \otimes A) \rightarrow M(\mathcal{K} \otimes A)$ . Applying assertion 1.b) to the unital homomorphism  $d \mapsto (\tilde{\pi} \circ q)(d)$  from the  $C(X)$ -subalgebra  $\mathcal{D} \subset M(\mathcal{K} \otimes A)$  to the multiplier algebra  $M(\mathcal{K} \otimes A)$  now leads to the desired conclusion.  $\square$

### 3 The subtriviality

Given a separable continuous field of nuclear  $C^*$ -algebras  $A$  over  $X$ , the strategy to prove the subtriviality of the  $C(X)$ -algebra  $A$  will be the same as the one developed by Kirchberg in [15] to prove theorem 0.1 whose main ideas of demonstration are also explained in [1, Théorème 6.1]. We associate to  $A$  a  $C(X)$ -extension by an hereditary  $C^*$ -subalgebra of the trivial continuous field  $\mathcal{O}_2 \otimes C(X)$  (proposition 3.1) and then prove that after stabilisation, this  $C(X)$ -extension splits by  $\mathcal{R}KK$ -theory arguments (theorem 3.2).

**3.1** Let us construct the fundamental  $C(X)$ -extension associated to an exact separable continuous field of  $C^*$ -algebras.

**Proposition 3.1** *Given a compact Hausdorff space  $X$  and a non-zero separable unital exact  $C(X)$ -algebra  $A$ , there exist a unital  $C(X)$ -subalgebra  $F$  of  $\mathcal{O}_2 \otimes C(X)$  with same unit and an hereditary subalgebra  $I$  of  $\mathcal{O}_2 \otimes C(X)$  such that  $I$  is an ideal in  $F$  and the  $C(X)$ -algebra  $A$  is isomorphic to the quotient  $C(X)$ -algebra  $F/I$ .*

*Furthermore, if the topological space  $X$  is perfect (i.e. without any isolated point) and the  $C(X)$ -algebra  $A$  is continuous, the canonical map  $F \rightarrow M(I)$  is a continuous field of faithful representations.*

**Proof :** Thanks to the characterisation of separable exact  $C^*$ -algebras obtained by Kirchberg (theorem 0.1), one may assume that the  $C^*$ -algebra  $A$  is a  $C^*$ -subalgebra of  $\mathcal{O}_2$  containing the unit of  $\mathcal{O}_2$ .

Let  $G \subset \mathcal{O}_2 \otimes C(X)$  be the trivial continuous field  $A \otimes C(X)$  over  $X$ . Then the kernel of the  $C(X)$ -linear morphism  $\pi : G \rightarrow A$  defined by  $\pi(a \otimes f) = fa$  is the ideal  $J = C_\Delta(X \times X)G$  where  $C_\Delta(X \times X)$  is the ideal in  $C(X \times X)$  of functions which are zero on the diagonal. Indeed suppose that  $T \in G$  verifies  $\pi(T) = 0$ . Then given  $\varepsilon > 0$ , take a finite number of elements  $a_i \in A$ ,  $f_i \in C(X)$  such that  $\|T - \sum_i a_i \otimes f_i\| < \varepsilon$ ; one has  $\|T - \sum_i (1 \otimes f_i - f_i \otimes 1)(a_i \otimes 1)\| < \varepsilon + \|\pi(\sum_i a_i \otimes f_i)\| < 2\varepsilon$ .

Define then the hereditary subalgebra  $I = J[\mathcal{O}_2 \otimes C(X)]J$  in  $\mathcal{O}_2 \otimes C(X)$  generated by  $J$ . It is a  $C(X)$ -algebra since it is closed by Cohen theorem (see e.g. [4, proposition 1.8]) and the product  $(1_{\mathcal{O}_2} \otimes f)(bc) = b(1_{\mathcal{O}_2} \otimes f)c$  belongs to  $I$  for all  $f \in C(X)$  and  $b, c \in I$ . If we set  $F = I + G$ , the intersection  $G \cap I$  is reduced by construction to the subalgebra  $J$ , and so we have a  $C(X)$ -extension

$$0 \rightarrow I \rightarrow F \rightarrow A \rightarrow 0.$$

Assume now that the space  $X$  is perfect and that the  $C(X)$ -algebra  $A$  is continuous. We need to prove that the map  $F_x \rightarrow M(I_x)$  is injective for each  $x \in X$ . Let  $a \in G$  and  $b \in I$  be two elements such that the sum  $d = a + b \in F$  verifies for a given point  $x \in X$  the equality

$$d_x I_x + I_x d_x = 0 \text{ (in } [\mathcal{O}_2 \otimes C(X)]_x \simeq \mathcal{O}_2 \otimes \mathbb{C} \text{)}.$$

To end the proof, we have to show that  $d_x$  is zero. For every  $f \in C_\Delta(X \times X)$ , one has  $(bf)_x = -(af)_x \in J_x$ , whence  $b_x \in J_x$  and so  $d_x \in G_x$ . But the representation of  $G_x \simeq A$  in  $M(J_x) \simeq M(C_x(X)A)$  is injective since  $X$  is perfect and  $A$  is continuous, from which we deduce that  $d_x = 0$ .  $\square$

**Remark:** With the previous notations, if the  $C(X)$ -algebra  $A$  is nuclear and  $\psi$  is a unital completely positive projection from  $\mathcal{O}_2$  onto  $A$ , the map  $\pi \circ (\psi \otimes id_{C(X)})$  is a unital  $C(X)$ -linear completely positive map from  $\mathcal{O}_2 \otimes C(X)$  onto the  $C(X)$ -subalgebra  $A$  which is zero on the nuclear hereditary  $C(X)$ -subalgebra  $I$ .

**3.2** We can now state the main theorem:

**Theorem 3.2** *Let  $X$  be a compact metrizable space and  $A$  be a unital separable  $C(X)$ -algebra with a unital embedding of the  $C(X)$ -algebra  $C(X)$  in  $A$ .*

*The following assertions are equivalent:*

1.  *$A$  is a continuous field of nuclear  $C^*$ -algebras over  $X$ ;*
2. *there exist a unital monomorphism of  $C(X)$ -algebras  $\alpha : A \hookrightarrow \mathcal{O}_2 \otimes C(X)$  and a unital  $C(X)$ -linear completely positive map  $E : \mathcal{O}_2 \otimes C(X) \rightarrow A$  such that  $E \circ \alpha = id_A$ .*

**Proof :**  $2 \Rightarrow 1$  By assumption the identity map  $id_A = E \circ id_{\mathcal{O}_2 \otimes C(X)} \circ \alpha : A \rightarrow A$  is nuclear since the  $C^*$ -algebra  $\mathcal{O}_2 \otimes C(X)$  is nuclear and so the  $C^*$ -algebra  $A$  is nuclear. Besides the  $C(X)$ -algebra  $A$  is isomorphic to the  $C(X)$ -subalgebra  $\alpha(A)$  of the continuous field  $\mathcal{O}_2 \otimes C(X)$  and is therefore continuous.

$1 \Rightarrow 2$  • Let us first deal with the case where the space  $X$  is perfect.

Given a unital nuclear separable continuous fields  $A$  over  $X$  which is unitaly embedded in the  $C^*$ -algebra  $\mathcal{O}_2$ , consider the  $C(X)$ -extension

$$0 \rightarrow I \rightarrow F \xrightarrow{\pi} A \rightarrow 0$$

constructed in proposition 3.1 and take the associated  $C(X)$ -extension

$$0 \rightarrow \mathcal{K} \otimes I \otimes \mathcal{O}_2 \rightarrow D = (\mathcal{K} \otimes F \otimes 1_{\mathcal{O}_2}) + (\mathcal{K} \otimes I \otimes \mathcal{O}_2) \rightarrow \mathcal{K} \otimes A \rightarrow 0.$$

The  $C(X)$ -nuclear quotient map  $\sigma = \sigma \circ id_{\mathcal{K} \otimes A}$  from the separable nuclear continuous field  $\mathcal{K} \otimes A$  to the quotient  $D/(\mathcal{K} \otimes I \otimes \mathcal{O}_2) \subset M(\mathcal{K} \otimes I \otimes \mathcal{O}_2)/(\mathcal{K} \otimes I \otimes \mathcal{O}_2)$  admits a  $C(X)$ -linear completely positive lifting  $\mathcal{K} \otimes A \rightarrow D$  ( $\subset \mathcal{K} \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2$ ) thanks to proposition 1.3. This means that the class of  $\sigma$  is invertible in  $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)$  (see the second remark following theorem 1.6).

But the group  $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)^{-1}$  is  $C(X)$ -linear homotopy invariant (theorem 1.6), hence zero since the endomorphism  $\phi_2(a) = s_1 a s_1^* + s_2 a s_2^*$  of  $\mathcal{O}_2$  is homotopic to the identity map  $id_{\mathcal{O}_2}$  ([6, proposition 2.2]) and so  $[\theta] = 2[\theta]$  in  $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)^{-1}$  for any invertible extension  $\theta$  of  $\mathcal{K} \otimes A$  by  $I \otimes \mathcal{O}_2$ . As a consequence, the  $C(X)$ -extension defined by  $\sigma$  is stably trivial. Furthermore, the identity representation of

$D \subset M(\mathcal{K} \otimes I \otimes \mathcal{O}_2)$  is a continuous field of faithful representations (proposition 3.1) and the assertion 2. of proposition 2.5 implies that the quotient morphism  $(id_{\mathcal{K}} \otimes \pi \otimes id_{\mathcal{O}_2})$  from  $D$  to  $\mathcal{K} \otimes A$  admits a cross section  $\alpha$  which is a morphism of  $C(X)$ -algebras.

This monomorphism  $\alpha$  is going to enable us to conclude by standard arguments, using theorem 0.1 and the result of Elliott ([9]) that the  $C^*$ -algebra  $\mathcal{O}_2$  is isomorphic to  $\mathcal{O}_2 \otimes \mathcal{O}_2$ .

Choose a non-zero minimal projection  $e_{11}$  in the  $C^*$ -algebra  $\mathcal{K}$  of compact operators that we embed in  $\mathcal{O}_2$  and let  $\varphi$  be a state on  $\mathcal{O}_2$  such that  $\varphi(e_{11}) = 1$ . If we take a unital completely positive projection  $\psi$  of  $\mathcal{O}_2$  onto the nuclear  $C^*$ -subalgebra  $A \subset \mathcal{O}_2$  (theorem 0.1), the composed map

$$E = (\varphi \otimes id_A) \circ (id_{\mathcal{O}_2} \otimes [\pi \circ (\psi \otimes id_{C(X)})] \otimes \varphi)$$

is a unital  $C(X)$ -linear completely positive map from  $\mathcal{O}_2 \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2$  onto  $A$ . Take also an isometry  $u \in \mathcal{O}_2 \otimes C(X)$  such that  $\alpha(e_{11} \otimes 1_A) = uu^*$  (corollary 2.4) and define the unital  $C(X)$ -algebra morphism

$$\beta : A \longrightarrow \mathcal{O}_2 \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2 \simeq \mathcal{O}_2 \otimes C(X)$$

by the formula  $\beta(a) = u^* \alpha(e_{11} \otimes a) u$ . If  $\tilde{E} : \mathcal{O}_2 \otimes C(X) \rightarrow A$  is the completely positive unital map  $d \mapsto E(udu^*)$ , one gets for all  $a \in A$  the equality

$$(\tilde{E} \circ \beta)(a) = (E \circ \alpha)(e_{11} \otimes a) = a$$

• Let us now come back to the general case of a compact space  $X$ .

Define the continuous field  $B = A \otimes C([0, 1])$  over the perfect compact space  $Y = X \times [0, 1]$ . According to the previous discussion, there exist a unital completely positive map  $\tilde{E} : \mathcal{O}_2 \otimes C(Y) \rightarrow B$  and a  $C(X) \otimes C([0, 1])$ -linear monomorphism  $\tilde{\alpha} : B \rightarrow \mathcal{O}_2 \otimes C(Y)$  such that  $\tilde{E} \circ \tilde{\alpha} = id_B$ . If  $ev_1 : C([0, 1]) \rightarrow \mathbb{C}$  is the evaluation map at  $x = 1 \in [0, 1]$ , define the two maps  $E : \mathcal{O}_2 \otimes C(X) \rightarrow A$  and  $\alpha : A \rightarrow \mathcal{O}_2 \otimes C(X)$  by

$$E(d) = (id_A \otimes ev_1) \circ \tilde{E}(d \otimes 1_{C([0,1])}) \text{ and } \alpha(a) = (id_{\mathcal{O}_2 \otimes C(X)} \otimes ev_1) \circ \tilde{\alpha}(a \otimes 1_{C([0,1])}).$$

Then  $E$  is a unital  $C(X)$ -linear completely positive map,  $\alpha$  is a unital  $C(X)$ -linear monomorphism and one has the identity  $E \circ \alpha = id_A$ .  $\square$

**Remark:** Assume that  $X$  is a locally compact metrizable space and that the  $C_0(X)$ -algebra  $A$  is a nuclear continuous field of  $C^*$ -algebras over  $X$ , where  $C_0(X)$  denotes the algebra of continuous functions on  $X$  vanishing at infinity. If  $\tilde{X}$  is the Alexandroff compactification of  $X$ , the unital  $C(\tilde{X})$ -algebra  $\mathcal{A}$  generated by  $A$  and  $C(\tilde{X})$  in the multiplier algebra  $M[A \oplus C(\tilde{X})]$  is a separable unital continuous field of  $C^*$ -algebras over  $\tilde{X}$  ([4, proposition 3.2]). By theorem 3.2, there exists therefore a  $C(\tilde{X})$ -linear monomorphism  $\alpha : \mathcal{A} \hookrightarrow \mathcal{O}_2 \otimes C(\tilde{X})$  and the  $C_0(X)$ -algebra  $A$  is isomorphic to the  $C_0(X)$ -subalgebra  $\alpha(A)$  of  $\mathcal{O}_2 \otimes C_0(X)$ .

## 4 Concluding remarks

**4.1** A  $C(X)$ -subalgebra of  $\mathcal{O}_2 \otimes C(X)$  is by construction exact and continuous. Conversely, if  $A$  is a non-zero exact separable unital continuous field of  $C^*$ -algebras over a perfect metrizable compact space  $X$ , one has by proposition 3.1 a  $C(X)$ -extension

$$0 \rightarrow I \rightarrow F \rightarrow A \rightarrow 0$$

where  $F$  is a  $C(X)$ -subalgebra of  $\mathcal{O}_2 \otimes C(X)$ . If the identity map  $A \rightarrow A = F/I$  admits a  $C(X)$ -linear completely positive lifting  $A \rightarrow F$ , the same method as the one used in theorem 3.2 will imply that the exact continuous field  $A$  is isomorphic to a  $C(X)$ -subalgebra of the trivial continuous field  $\mathcal{O}_2 \otimes C(X)$ .

It is therefore interesting to know whether this map admits a  $C(X)$ -linear completely positive lifting in the not discrete case.

**4.2** Let us have a look at one of the technical problems involved, the Hahn-Banach type extension property in the continuous field framework for finite type  $C(X)$ -submodules.

Let  $A$  be a separable unital continuous field of  $C^*$ -algebras over a compact metrizable space  $X$  and let  $D$  be a finitely generated  $C(X)$ -submodule which is an operator system. Assume that  $\phi : D \rightarrow C(X)$  is a  $C(X)$ -linear unital completely positive map. Then for  $x \in X$ , there exists, thanks to [4, proposition 3.13], a continuous field of states  $\Phi^x$  on  $A$ , i.e. a  $C(X)$ -linear unital positive map from  $A$  to  $C(X)$ , such that for all  $d \in D$ ,

$$\Phi^x(d)(x) = \phi(d)(x).$$

As a consequence, given  $\varepsilon > 0$  and a finite subset  $\mathcal{F}$  of  $D$ , one can build by continuity and compactness a continuous field of states  $\Phi$  on  $A$  such that

$$\max\{\|\Phi(d) - \phi(d)\|, d \in \mathcal{F}\} < \varepsilon.$$

But one cannot find in general any continuous field of states on  $A$  whose restriction to  $D$  is  $\phi$ . Indeed, consider the  $C(\tilde{\mathbb{N}})$ -algebra  $A = \mathbb{C}^2 \otimes C(\tilde{\mathbb{N}})$  where  $\tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$  is the Alexandroff compactification of the space  $\mathbb{N}$  of positive integers. Define the positive element  $a \in C_\infty(\tilde{\mathbb{N}})A \subset A$  by the formulas

$$a_n = \begin{cases} (\frac{1}{n+1}, 0) & \text{if } n \text{ even} \\ (0, \frac{1}{n+1}) & \text{if } n \text{ odd} \end{cases}$$

and let  $\phi$  be the  $C(\tilde{\mathbb{N}})$ -linear unital completely positive map with values in  $C(\tilde{\mathbb{N}})$  defined on the  $C(\tilde{\mathbb{N}})$ -submodule generated by the two  $C(\tilde{\mathbb{N}})$ -linearly independent elements  $1_A$  and  $a$  through the formula

$$\phi(a)(n) = \frac{1}{n+1} \text{ if } n < \infty \text{ and } \phi(a)(\infty) = 0.$$

Suppose that the continuous field of states  $\Phi$  is a  $C(\tilde{\mathbb{N}})$ -linear extension of  $\phi$  to  $A$ . Then as A. Bauval already noticed it, one has the contradiction

$$\begin{aligned} 1 &= \Phi(1_A)(\infty) = \Phi((1, 0) \otimes 1)(\infty) + \Phi((0, 1) \otimes 1)(\infty) \\ &= \lim_{n \rightarrow \infty} \Phi((1, 0) \otimes 1)(2n+1) + \lim_{n \rightarrow \infty} \Phi((0, 1) \otimes 1)(2n) \\ &= 0 + 0 = 0. \end{aligned}$$

# Appendix *by Eberhard Kirchberg*

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In this appendix, we solve in proposition A.3 the lifting question raised in paragraph 4.1 through a continuous generalisation of joint work of E.G. Effros and U. Haagerup on lifting problems for  $C^*$ -algebras ([8], see also [22]). This result enables us to state the following characterisation of separable exact continuous fields of  $C^*$ -algebras:

**Theorem A.1** *Let  $X$  be a compact metrizable space and  $A$  be a (unital) separable continuous field of  $C^*$ -algebras over  $X$ .*

*Then the  $C^*$ -algebra  $A$  is exact if and only if there exists a (unital) monomorphism of  $C(X)$ -algebras  $A \hookrightarrow \mathcal{O}_2 \otimes C(X)$ .*

Let us start with a technical  $C(X)$ -linear version of Auerbach's theorem ([17, proposition 1.c.3]) for a continuous field of  $C^*$ -algebras  $A$  over  $X$  which gives us local bases over  $C(X)$  with continuous coordinate maps for particular free  $C(X)$ -submodules of finite type in  $A$ .

Define a  $C(X)$ -operator system in  $A$  to be a  $C(X)$ -submodule which is an operator system.

**Lemma A.2** *([8, lemma 2.4]) Let  $A$  be a separable unital continuous field of  $C^*$ -algebras over a compact metrizable space  $X$ ,  $E \subset A$  be a  $C(X)$ -operator system and assume that there exists an integer  $n \in \mathbb{N}^*$  such that for all  $x \in X$ , the dimension  $\dim E_x$  of the operator system  $E_x \subset A_x$  equals  $n$ . Then the following holds.*

*Given any point  $x \in X$ , there exist an open neighbourhood  $\mathcal{U}$  of  $x$  in  $X$ , self-adjoint  $C(X)$ -linear contractions  $\varphi_i : A \rightarrow C(X)$  and self-adjoint elements  $f_i \in E$  with  $\|f_i\| \leq 2$  for  $1 \leq i \leq n$  such that*

$$\forall a \in C_0(\mathcal{U})E, a = \sum_i \varphi_i(a) f_i.$$

*Furthermore, there exists a continuous field of states  $\Psi : A \rightarrow C(X)$  such that the restriction of the map  $2n\Psi - id_A$  to the operator system  $E$  is completely positive.*

**Proof :** Let us fix a point  $x \in X$ . Then there exist, thanks to Auerbach's theorem, a normal basis  $\{r_1, \dots, r_n\}$  of the fibre  $E_x$  where each  $r_i$  is self adjoint and norm one hermitian functionals  $\phi_j : A_x \rightarrow \mathbb{C}$ ,  $1 \leq j \leq n$ , such that  $\phi_j(r_i) = \delta_{i,j}$ .

Consider the polar decomposition  $\phi_j = \phi_j^+ - \phi_j^-$  where  $\phi_j^+$  and  $\phi_j^-$  are positive functionals such that  $1 = \|\phi_j\| = \|\phi_j^+\| + \|\phi_j^-\|$ . By [4] lemme 3.12, there exist  $C(X)$ -linear positive maps  $\varphi_j^+$  and  $\varphi_j^- : A \rightarrow C(X)$  which extend the functionals  $\phi_j^+$  and  $\phi_j^-$  on the fibre  $A_x$  to the  $C(X)$ -algebra  $A$  with the property that  $\varphi_j^+(1) = \|\phi_j^+\|$  and  $\varphi_j^-(1) = \|\phi_j^-\|$ . Take also  $n$  norm 1 self-adjoint elements  $e_i \in E$  satisfying the equality  $(e_i)_x = r_i$  and define the matrix  $T = [\varphi_j(e_i)]_{i,j} \in M_n(\mathbb{R}) \otimes C(X)$ .

One has by construction  $T_x = 1_{M_n(\mathbb{R})}$ ; the set  $\mathcal{U}_1 \subset X$  of points  $y \in X$  for which the spectrum of  $T_y \in M_n(\mathbb{R})$  is included in the open set  $\{z \in \mathbb{C}, |z| > 1/2\}$  is therefore an open neighbourhood of  $x$  in  $X$  ([4, proposition 2.4 b)). If  $\eta$  is a continuous function on  $X$  with values in  $[0, 1]$  which is 0 outside  $\mathcal{U}_1$  and 1 on an open neighbourhood  $\mathcal{U}$  of the point  $x \in X$ , the self-adjoint elements  $f_1, \dots, f_n$  of norm less than 2 are then well defined in  $C_0(\mathcal{U}_1)E$  by the formula

$$T \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \eta \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

and satisfy the relation  $\varphi_j(f_i)(y) = \delta_{i,j}$  for each  $y \in \mathcal{U}$  since the matrix  $T_y$  is invertible, whence the desired equality for every  $a \in C_0(\mathcal{U})E$ .

Keeping the same fixed point  $x$ , define now the continuous field of states  $\Phi = \frac{1}{n} \sum_i (\varphi_i^+ + \varphi_i^-)$ . Then one gets for all  $a \in C_0(\mathcal{U})E$  the equality:

$$(2n\Phi - id_A)(a) = \sum_{1 \leq i \leq n} [\varphi_i^+(a)(2 - f_i) + \varphi_i^-(a)(2 + f_i)].$$

The restriction of the map  $(2n\Phi - id_A)$  to  $C_0(\mathcal{U})E$  is therefore completely positive and an appropriate partition of the unit  $1_{C(X)}$  enables us to conclude.  $\square$

Noticing that a  $C(X)$ -linear map  $\sigma : A \rightarrow B$  between  $C(X)$ -algebras is completely positive if and only if each induced map  $\sigma_x : A_x \rightarrow B_x$  is completely positive (see for instance [4, proposition 2.9]), the lemma A.2 allows us to state a continuous version of theorem 2.5 of [8]. Replacing then the continuous field  $A$  by  $A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$  (where  $M_{2^\infty}(\mathbb{C}) = \lim_{n \rightarrow \infty} M_{2^n}(\mathbb{C})$ ) and working with finitely generated  $C(X)$ -operator systems  $E_k \subset A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$  for which the function  $x \mapsto \dim(E_k)_x$  is continuous, one derives the following desired  $C(X)$ -linear completely positive lifting result.

**Proposition A.3** ([8, theorem 3.4]) *Suppose that  $A$  and  $B$  are two unital separable exact continuous fields of  $C^*$ -algebras over a compact space  $X$  with  $A = B/J$  for some nuclear ideal  $J$  in  $B$ .*

*Then there exists a  $C(X)$ -linear unital completely positive lifting  $A \rightarrow B$  of  $id_A$ .*

**Proof :** Let us define the two continuous fields  $\mathcal{A} = A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$  and  $\mathcal{B} = B \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$ . It is clearly enough to find a  $C(X)$ -linear unital completely positive cross section  $\theta$  of the quotient morphism  $\mathcal{B} \rightarrow \mathcal{A}$  (by [4, theorem 3.3]).

Consider a dense sequence  $\{a_k\}$  in the self-adjoint part of  $\mathcal{A}$  where each  $a_k$  belongs to the dense subalgebra  $A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$  of  $\mathcal{A}$  and  $a_1 = 1$ . Let us show that we may assume inductively that  $C(X)$ -operator system  $E_n$  generated by the  $a_k$ ,  $1 \leq k \leq n$ , satisfies the equality  $\dim(E_n)_x = n$  for every  $n \in \mathbb{N}^*$  and every  $x \in X$ . The inductive step is the following. Given  $n \geq 2$ , there exists by construction an integer  $l$  such that  $E_n \subset A \oplus M_{2^l}(\mathbb{C}) \otimes C(X)$ . Set  $a'_n = a_n + 2^{-n-1}d_l \otimes 1_{C(X)}$  where

$$d_l = 1_{M_{2^l}(\mathbb{C})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2^{l+1}}(\mathbb{C}) \subset M_{2^\infty}(\mathbb{C}).$$

Then the  $C(X)$ -module  $E'_n = E_{n-1} + C(X)a'_n$  verifies for each  $x \in X$  the equality  $\dim(E'_n)_x = \dim(E_{n-1})_x + 1$ .

Using proposition 1.3, one can now finish the proof by the same method as the one developed by E.G. Effros and U. Haagerup in [8].3 (see also [22, theorem 6.10]).  $\square$



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