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Tensor products of $C(X)$-algebras over $C(X)$

Etienne Blanchard

Abstract
Given a Hausdorff compact space $X$, we study the $C^*$-(semi)-norms on the algebraic tensor product $A \otimes_{C(X)} B$ of two $C(X)$-algebras $A$ and $B$ over $C(X)$. In particular, if one of the two $C(X)$-algebras defines a continuous field of $C^*$-algebras over $X$, there exist minimal and maximal $C^*$-norms on $A \otimes_{C(X)} B$ but there does not exist any $C^*$-norm on $A \otimes_{C(X)} B$ in general.

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0 Introduction

Tensor products of $C^*$-algebras have been extensively studied over the last decades (see references in [12]). One of the main results was obtained by M. Takesaki in [15] where he proved that the spatial tensor product $A \otimes_{\text{min}} B$ of two $C^*$-algebras $A$ and $B$ always defines the minimal $C^*$-norm on the algebraic tensor product $A \otimes B$ of $A$ and $B$ over the complex field $\mathbb{C}$.

More recently, G.G. Kasparov constructed in [10] a tensor product over $C(X)$ for $C(X)$-algebras. The author was also led to introduce in [3] several notions of tensor products over $C(X)$ for $C(X)$-algebras and to study the links between those objects.

Notice that E. Kirchberg and S. Wassermann have proved in [11] that the subcategory of continuous fields over a Hausdorff compact space is not closed under such tensor products over $C(X)$ and therefore, in order to study tensor products over $C(X)$ of continuous fields, it is natural to work in the $C(X)$-algebras framework.

Let us introduce the following definition:

**Definition 0.1.** Given two $C(X)$-algebras $A$ and $B$, we denote by $\mathcal{I}(A,B)$ the involutive ideal of the algebraic tensor product $A \otimes B$ generated by the elements $(fa) \otimes b - a \otimes (fb)$, where $f \in C(X)$, $a \in A$ and $b \in B$.

Our aim in the present article is to study the $C^*$-norms on the algebraic tensor product $(A \otimes B)/\mathcal{I}(A,B)$ of two $C(X)$-algebras $A$ and $B$ over $C(X)$ and to see how one can enlarge the results of Takesaki to this framework.

We first define an ideal $\mathcal{J}(A,B) \subset A \otimes B$ which contains $\mathcal{I}(A,B)$ such that every $C^*$-semi-norm on $A \otimes B$ which is zero on $\mathcal{I}(A,B)$ is also zero on $\mathcal{J}(A,B)$ and we prove that there always exist a minimal $C^*$-norm $\| \cdot \|_m$ and a maximal $C^*$-norm $\| \cdot \|_M$ on the quotient $(A \otimes B)/\mathcal{J}(A,B)$. 

We then study the following question of G.A. Elliott ([5]): when do the two ideals \( \mathcal{I}(A, B) \) and \( \mathcal{J}(A, B) \) coincide?

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1 Preliminaries

We briefly recall here the basic properties of \( C(X) \)-algebras.

Let \( X \) be a Hausdorff compact space and \( C(X) \) be the \( C^* \)-algebra of continuous functions on \( X \). For \( x \in X \), define the morphism \( e_x : C(X) \to \mathbb{C} \) of evaluation at \( x \) and denote by \( C_x(X) \) the kernel of this map.

**DEFINITION 1.1** ([10]) A \( C(X) \)-algebra is a \( C^* \)-algebra \( A \) endowed with a unital morphism from \( C(X) \) in the center of the multiplier algebra \( M(A) \) of \( A \).

We associate to such an algebra the unital \( C(X) \)-algebra \( A \) generated by \( A \) and \( u[C(X)] \) in \( M[A \oplus C(X)] \) where \( u(g)(a \oplus f) = ga \oplus gf \) for \( a \in A \) and \( f, g \in C(X) \).

For \( x \in X \), denote by \( A_x \) the quotient of \( A \) by the closed ideal \( C_x(X)A \) and by \( a_x \) the image of \( a \in A \) in the fibre \( A_x \). Then, as

\[
\|a_x\| = \inf\{\|1 - f + f(x)\|a\|, f \in C(X)\},
\]

the map \( x \mapsto \|a_x\| \) is upper semi-continuous for all \( a \in A \) ([14]).

Note that the map \( A \to \oplus A_x \) is a monomorphism since if \( a \in A \), there is a pure state \( \phi \) on \( A \) such that \( \phi(a^*a) = \|a\|^2 \). As the restriction of \( \phi \) to \( C(X) \subset M(A) \) is a character, there exists \( x \in X \) such that \( \phi \) factors through \( A_x \) and so \( \phi(a^*a) = \|a_x\|^2 \).

Let \( S(A) \) be the set of states on \( A \) endowed with the weak topology and let \( S_X(A) \) be the subset of states \( \phi \) whose restriction to \( C(X) \subset M(A) \) is a character, i.e. such that there exists an \( x \in X \) (denoted \( x = p(\phi) \)) verifying \( \phi(f) = f(x) \) for all \( f \in C(X) \).

Then the previous paragraph implies that the set of pure states \( P(A) \) on \( A \) is included in \( S_X(A) \).

Let us introduce the following notation: if \( E \) is a Hilbert \( A \)-module where \( A \) is a \( C^* \)-algebra, we will denote by \( \mathcal{L}_A(E) \) or simply \( \mathcal{L}(E) \) the set of bounded \( A \)-linear operators on \( E \) which admit an adjoint ([9]).

**DEFINITION 1.2** ([3]) Let \( A \) be a \( C(X) \)-algebra.

A \( C(X) \)-representation of \( A \) in the Hilbert \( C(X) \)-module \( E \) is a morphism \( \pi : A \to \mathcal{L}(E) \) which is \( C(X) \)-linear, i.e. such that for every \( x \in X \), the representation \( \pi_x = \pi \otimes e_x \) in the Hilbert space \( E_x = E \otimes e_x \subset \mathbb{C} \) factors through a representation of \( A_x \). Furthermore, if \( \pi_x \) is a faithful representation of \( A_x \) for every \( x \in X \), \( \pi \) is said to be a field of faithful representations of \( A \).

A continuous field of states on \( A \) is a \( C(X) \)-linear map \( \varphi : A \to C(X) \) such that for any \( x \in X \), the map \( \varphi_x = e_x \circ \varphi \) defines a state on \( A_x \).
If \( \pi \) is a \( C(X) \)-representation of the \( C(X) \)-algebra \( A \), the map \( x \mapsto \| \pi_x(a) \| \) is lower semi-continuous since \( \langle \xi, \pi(a) \eta \rangle \in C(X) \) for every \( \xi, \eta \in \mathcal{E} \). Therefore, if \( A \) admits a field of faithful representations \( \pi \), the map \( x \mapsto \| a_x \| = \| \pi_x(a) \| \) is continuous for every \( a \in A \), which means that \( A \) is a continuous field of \( C^* \)-algebras over \( X \) ([4]).

The converse is also true ([3] théorème 3.3): given a separable \( C(X) \)-algebra \( A \), the following assertions are equivalent:

1. \( A \) is a continuous field of \( C^* \)-algebras over \( X \),
2. the map \( p : S_X(A) \to X \) is open,
3. \( A \) admits a field of faithful representations.

\section{\( C^* \)-norms on \( (A \odot B)/\mathcal{J}(A, B) \)}

\textbf{Definition 2.1} Given two \( C(X) \)-algebras \( A \) and \( B \), we define the involutive ideal \( \mathcal{J}(A, B) \) of the algebraic tensor product \( A \odot B \) of elements \( \alpha \in A \odot B \) such that \( \alpha_x = 0 \) in \( A_x \odot B_x \) for every \( x \in X \).

By construction, the ideal \( I(A, B) \) is included in \( \mathcal{J}(A, B) \).

\textbf{Proposition 2.2} Assume that \( \| \|_3 \) is a \( C^* \)-semi-norm on the algebraic tensor product \( A \odot B \) of two \( C(X) \)-algebras \( A \) and \( B \).

If \( \| \|_3 \) is zero on the ideal \( I(A, B) \), then

\[ \| \alpha \|_3 = 0 \text{ for all } \alpha \in \mathcal{J}(A, B). \]

\textbf{Proof:} Let \( D_3 \) be the Hausdorff completion of \( A \odot B \) for \( \| \|_3 \). By construction, \( D_3 \) is a quotient of \( A \otimes_{\max} B \). Furthermore, if \( C_{\Delta} \) is the ideal of \( C(X \times X) \) of functions which are zero on the diagonal, the image of \( C_{\Delta} \) in \( M(D_3) \) is zero.

As a consequence, the map from \( A \otimes_{\max} B \) onto \( D_3 \) factors through the quotient \( A_{C(X)} \otimes B \) by \( C_{\Delta} \times (A \otimes_{\max} B). \)

But an easy diagram-chasing argument shows that \( (A_{C(X)} \otimes B)_x = A_x \otimes_{\max} B_x \) for every \( x \in X \) ([3] corollaire 3.17) and therefore the image of \( \mathcal{J}(A, B) \subset A \otimes_{\max} B \) in \( A_{C(X)} \otimes B \) is zero. \( \square \)

\subsection{The maximal \( C^* \)-norm}

\textbf{Definition 2.3} Given two \( C(X) \)-algebras \( A_1 \) and \( A_2 \), we denote by \( \| \|_M \) the \( C^* \)-semi-norm on \( A_1 \odot A_2 \) defined for \( \alpha \in A_1 \odot A_2 \) by

\[ \| \alpha \|_M = \sup \{ \| (\sigma_1^x \otimes_{\max} \sigma_2^x)(\alpha) \|, x \in X \} \]

where \( \sigma_i^x \) is the map \( A_i \to (A_i)_x \).

As \( \| \|_M \) is zero on the ideal \( \mathcal{J}(A_1, A_2) \), if we identify \( \| \|_M \) with the \( C^* \)-semi-norm induced on \( (A_1 \odot A_2)/\mathcal{J}(A_1, A_2) \), we get:
PROPOSITION 2.4 \textit{The semi-norm }$\| \|_M$\textit{ is the maximal C$^*$-norm on the quotient }$(A_1 \odot A_2)/J(A_1, A_2)$.\textit{\textit{Proof:}} By construction, $\| \|_M$ defines a C$^*$-norm on $(A_1 \odot A_2)/J(A_1, A_2)$. Moreover, as the quotient $A_1 \otimes_{C(X)} A_2$ of $A_1 \otimes_{\max} A_2$ by $C_\Delta \times (A_1 \otimes_{\max} A_2)$ maps injectively in

$$\oplus_{x \in X} (A_1 \otimes_{C(X)} A_2)_x = \oplus_{x \in X} ((A_1)_x \otimes_{\max} (A_2)_x),$$

the norm of $A_1 \otimes_{C(X)} A_2$ coincides on the dense subalgebra $(A_1 \odot A_2)/J(A_1, A_2)$ with $\| \|_M$. But we saw in proposition 2.2 that if $\| \|_\beta$ is a C$^*$-norm on the algebra $(A \odot B)/J(A, B)$, the completion of $(A \odot B)/J(A, B)$ for $\| \|_\beta$ is a quotient of $A \otimes_{C(X)} B$. \hfill $\Box$

2.2 The minimal C$^*$-norm

DEFINITION 2.5 \textit{Given two C(X)-algebras }$A_1$\textit{ and }$A_2$, \textit{we define the semi-norm }$\| \|_m$\textit{ on }$A_1 \odot A_2$\textit{ by the formula}

$$\| \alpha \|_m = \sup \{ \| (\sigma^x_1 \otimes_{\min} \sigma^x_2) (\alpha) \|, x \in X \}$$

where $\sigma^x_1$ is the map $A_1 \rightarrow (A_1)_x$ and we denote by $A_1 \otimes_{C(X)}^m A_2$ the Hausdorff completion of $A_1 \odot A_2$ for that semi-norm.

Remark: In general, the canonical map $(A_1 \otimes_{C(X)}^m A_2)_x \rightarrow (A_1)_x \otimes_{\min} (A_2)_x$ is not a monomorphism ([11]).

By construction, $\| \|_m$ induces a C$^*$-norm on $(A_1 \odot A_2)/J(A_1, A_2)$. We are going to prove that this C$^*$-norm defines the minimal C$^*$-norm on the involutive algebra $(A_1 \odot A_2)/J(A_1, A_2)$.

Let us introduce some notation.

Given two unital C(X)-algebras $A_1$ and $A_2$, let $P(A_1) \subset S_X(A_1)$ denote the set of pure states on $A_1$ and let $P(A_1) \times_X P(A_2)$ denote the closed subset of $P(A_1) \times P(A_2)$ of couples $(\omega_1, \omega_2)$ such that $p(\omega_1) = p(\omega_2)$, where $p : P(A_1) \rightarrow X$ is the restriction to $P(A_1)$ of the map $p : S_X(A_1) \rightarrow X$ defined in section 1.

LEMMA 2.6 Assume that $\| \|_\beta$ is a C$^*$-semi-norm on the algebraic tensor product $A_1 \odot A_2$ of two unital C(X)-algebras $A_1$ and $A_2$ which is zero on the ideal $J(A_1, A_2)$ and define the closed subset $S_\beta \subset P(A_1) \times_X P(A_2)$ of couples $(\omega_1, \omega_2)$ such that

$$\| (\omega_1 \otimes \omega_2) (\alpha) \| \leq \| \alpha \|_\beta \text{ for all } \alpha \in A_1 \odot A_2.$$

If $S_\beta \neq P(A_1) \times_X P(A_2)$, there exist self-adjoint elements $a_i \in A_i$ such that $a_1 \otimes a_2 \notin J(A_1, A_2)$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

\textit{Proof:} Define for $i = 1, 2$ the adjoint action $ad$ of the unitary group $U(A_i)$ of $A_i$ on the pure states space $P(A_i)$ by the formula
Then $S_\beta$ is invariant under the product action $ad \times ad$ of $U(A_1) \times U(A_2)$ and we can therefore find non empty open subsets $U_1 \subset P(A_i)$ which are invariant under the action of $U(A_i)$ such that $(U_1 \times U_2) \cap S_\beta = \emptyset$.

Now, if $K_i$ is the complement of $U_i$ in $P(A_i)$, the set
\[ K_i^\perp = \{ a \in A_i \mid \omega(a) = 0 \text{ for all } \omega \in K_i \} \]
is a non empty ideal of $A_i$ and furthermore, if $\omega \in P(A_i)$ is zero on $K_i^\perp$, then $\omega$ belongs to $K_i$ ([8] lemma 8,[15]).

As a consequence, if $(\varphi_1, \varphi_2)$ is a point of $U_1 \times_X U_2$, there exist non zero self-adjoint elements $a_i \in K_i^\perp$ such that $\varphi_i(a_i) = 1$. If $x = p(\varphi_i)$, this implies in particular that $(a_1)_x \otimes (a_2)_x \neq 0$, and hence $a_1 \otimes a_2 \notin J(A_1, A_2)$. \qed

**LEMMA 2.7** ([15] theorem 1) Let $A_1$ and $A_2$ be two unital $C(X)$-algebras.

If the algebra $A_1$ is an abelian algebra, there exists only one $C^*$-norm on the quotient $(A_1 \circledast A_2)/J(A_1, A_2)$.

**Proof:** Let $\| \cdot \|_\beta$ be a $C^*$-semi-norm on $A_1 \circledast A_2$ such that for all $\alpha \in A_1 \circledast A_2$, $\| \alpha \|_\beta = 0$ if and only if $\alpha \in J(A_1, A_2)$.

If $\rho \in P(A_3)$ is a pure state on the Hausdorff completion $A_3$ of $A_1 \circledast A_2$ for the semi-norm $\| \cdot \|_\beta$, then for every $a_1 \otimes a_2 \in A_1 \circledast A_2$,
\[ \rho(a_1 \otimes a_2) = \rho(a_1 \otimes 1)\rho(1 \otimes a_2) \]
since $A_1 \otimes 1$ is included in the center of $M(A_3)$. Moreover, if we define the states $\omega_1$ and $\omega_2$ by the formulas $\omega_1(a_1) = \rho(a_1 \otimes 1)$ and $\omega_2(a_2) = \rho(1 \otimes a_2)$, then $\omega_2$ is pure since $\rho$ is pure, and $(\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)$. It follows that $P(A_3)$ is isomorphic to $S_\beta$.

In particular, if $a_1 \otimes a_2 \in A_1 \circledast A_2$ verifies
\[ (\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0 \text{ for all couples } (\omega_1, \omega_2) \in S_\beta, \]
the element $a_1 \otimes a_2$ is zero in $A_3$ and therefore belongs to the ideal $J(A_1, A_2)$. Accordingly, the previous lemma implies that $P(A_1) \times_X P(A_2) = S_\beta = P(A_3)$.

As a consequence, we get for every $\alpha \in A_1 \circledast A_2$
\[ \| \alpha \|_\beta^2 = \sup\{ \rho(\alpha^* \alpha), \rho \in P(A_3) \} \]
\[ = \sup\{ (\omega_1 \otimes \omega_2)(\alpha^* \alpha), (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2) \} \]
But that last expression does not depend on $\| \cdot \|_\beta$, and hence the unicity. \qed

**PROPOSITION 2.8** ([15] theorem 2) Let $A_1$ and $A_2$ be two unital $C(X)$-algebras.

If $\| \cdot \|_\beta$ is a $C^*$-semi-norm on $A_1 \circledast A_2$ whose kernel is $J(A_1, A_2)$, then
\[ \forall \alpha \in A_1 \circledast A_2, \quad \| \alpha \|_\beta \geq \| \alpha \|_m. \]

**Proof:** If we show that $S_\beta = P(A_1) \times_X P(A_2)$, then for every $\rho \in S_\beta$ and every $\alpha$ in $A_1 \circledast A_2$, we have $\rho(s^* \alpha^* \alpha s) \leq \rho(s^* s)\| \alpha \|_\beta^2$ for all $s \in A_1 \circledast A_2$. Therefore
Remark: As the $(A_1 \otimes A_2)\#$-algebra generated by $C(X)$ and $a_1$ in $A_1$. The preceding lemma implies that $B\otimes_{C(X)} A_2$ maps injectively into the Hausdorff completion $A_\beta$ of $A_1 \otimes A_2$ for $\| \|_\beta$.

Consider pure states $\rho \in P(B_x)$ and $\omega \in P(A_{2,x})$ such that $\rho(\omega_1) \neq 0$ and $\omega_2(\omega_2) \neq 0$ and extend the pure state $\rho \otimes \omega$ on $B\otimes_{C(X)} A_2$ to a pure state $\omega$ on $A_\beta$. If we set $\omega_2(a) = \omega(a \otimes 1)$ for $a \in A_1$, then $\omega$ is pure and $\omega(\alpha) = (\omega_1 \otimes \omega_2)(\alpha)$ for all $\alpha \in A_1 \otimes A_2$ since $\omega$ and $\omega_2$ are pure ([15] lemma 4). As a consequence, $(\omega_1, \omega_2) \in S_\beta$, which is absurd since $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = \rho(\alpha_1)\omega_2(a_2) \neq 0$.

Proposition 2.9 Given two $C(X)$-algebras $A_1$ and $A_2$, the semi-norm $\| \|_m$ defines the minimal $C^*$-norm on the involutive algebra $(A_1 \otimes A_2)/J(A_1, A_2)$.

Proof: Let $\| \|_\beta$ be a $C^*$-norm on $(A_1 \otimes A_2)/J(A_1, A_2)$. Thanks to the previous proposition, all we need to prove is that one can extend $\| \|_\beta$ to a $C^*$-norm $(A_1 \otimes A_2)/J(A_1, A_2)$, where $A_1$ and $A_2$ are the unital $C(X)$-algebras associated to the $C(X)$-algebras $A_1$ and $A_2$ (definition 1.1).

Consider the Hausdorff completion $D_\beta$ of $(A_1 \otimes A_2)/J(A_1, A_2)$ and denote by $\pi_i$ the canonical representation of $A_i$ in $M(D_\beta)$ for $i = 1, 2$. Let us define the representation $\tilde{\pi}_i$ of $A_i$ in $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ by the following formulas:

$$\tilde{\pi}_1(b_1 + u(f))(a \oplus a_1 \oplus a_2 \oplus g) = (\pi_1(b_1) + g)a_1 \oplus f a_2 \oplus f g$$

$$\tilde{\pi}_2(b_2 + u(f))(a \oplus a_1 \oplus a_2 \oplus g) = (\pi_2(b_2) + g)a_1 \oplus f a_1 \oplus (b_2 + f)a_2 \oplus f g$$

For $i = 1, 2$, let $\varepsilon_i : A_i \to C(X)$ be the map defined by

$$\varepsilon_i[a + u(f)] = f \text{ for } a \in A_i \text{ and } f \in C(X).$$

Then using the maps $(\varepsilon_1 \otimes \varepsilon_2)$, $(\varepsilon_1 \otimes id)$ and $(id \otimes \varepsilon_2)$, one proves easily that if $\alpha \in A_1 \otimes A_2$, $(\tilde{\pi}_1 \otimes \tilde{\pi}_2)(\alpha) = 0$ if and only if $\alpha$ belongs to $J(A_1, A_2)$.

Therefore, the norm of $M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X))$ restricted to the subalgebra $(A_1 \otimes A_2)/J(A_1, A_2)$ extends $\| \|_\beta$. □

Remark: As the $C(X)$-algebra $A$ is nuclear if and only if every fibre $A_x$ is nuclear ([12]), $A \otimes_{C(X)} B \simeq A \otimes_{C(X)} B$ for every $C(X)$-algebra $B$ if and only if $A$ is nuclear.
3 When does the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$ hold?

Given two $C(X)$-algebras $A$ and $B$, Giordano and Mingo have studied in [6] the case where the algebra $C(X)$ is a von Neumann algebra: their theorem 3.1 and lemma 1.5 of [10] imply that in that case, we always have the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

Our purpose in this section is to find sufficient conditions on the $C(X)$-algebras $A$ and $B$ in order to ensure this equality and to present a counter-example in the general case.

**PROPOSITION 3.1** Let $X$ be a second countable Hausdorff compact space and let $A$ and $B$ be two $C(X)$-algebras.

If $A$ is a continuous field of $C^*$-algebras over $X$, then $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

**Proof:** Let us prove by induction on the non negative integer $n$ that if

$$s = \sum_{1 \leq i \leq n} a_i \otimes b_i \in \mathcal{J}(A, B),$$

then $s$ belongs to the ideal $\mathcal{I}(A, B)$.

If $n = 0$, there is nothing to prove. Consider therefore an integer $n > 0$ and suppose the result has been proved for any $p < n$.

Fix an element $s = \sum_{1 \leq i \leq n} a_i \otimes b_i \in \mathcal{J}(A, B)$ and define the continuous positive function $h \in C(X)$ by the formula $h(x)^{10} = \sum \|(a_k)_x\|^2$.

The element $a'_k = h^{-1}a_k$ is then well defined in $A$ for every $k$ since $a'_k a_k \leq h^{10}$. Consequently, the function $f_k(x) = \|(a'_k)_x\|$ is continuous.

For $1 \leq k \leq n$, let $D_k$ denote the separable $C(X)$-algebra generated by 1 and the $a'_i a'_j', 1 \leq j \leq n$, in the unital $C(X)$-algebra $A$ associated to $A$ (definition 1.1). Then $D_k$ is a unital continuous field of $C^*$-algebras over $X$ (see for instance [3] proposition 3.2).

Consider the open subset $S^k = \{ \psi \in S_X(D_k) \mid \psi[a'_k a'_k] > \psi(f^2_k/2) \}$. If we apply lemma 3.6 b) of [3] to the restriction of $p : S_X(D_k) \to X$ to $S^k$, we may construct a continuous field of states $\omega_k$ on $D_k$ such that $\omega_k[a'_k a'_k] \geq f^2_k/2$.

Now, if we set $s' = \sum_i a'_i \otimes b_i$, as $(a'_i \otimes 1)s' \in \mathcal{J}(D_k, B)$,

$$(\omega_k \otimes id)[(a'_k \otimes 1)s'] = \omega_k[a'_k a'_k]b_k + \sum_{j \neq k} \omega_k[a'_k a'_j]b_j = 0.$$

Noticing that $f^3_k$ is in the ideal of $C(X)$ generated by $\omega_k[a'_k a'_k]$, we get that $f^3_kb_k$ belongs to the $C(X)$-module generated by the $b_j, j \neq k$, and thanks to the induction hypothesis, it follows that $(f^3_k \otimes 1)s' \in \mathcal{I}(A, B)$ for each $k$.

But $h^2 = \sum_k f^2_k$ and so $h^4 \leq n \sum_k f^4_k$ is in the ideal of $C(X)$ generated by the $f^3_k$, which implies $s = (h^4 \otimes 1)s' \in \mathcal{I}(A, B)$.

**Remarks:** 1. As a matter of fact, it is not necessary to assume that the space $X$ satisfies the second axiom of countability thanks to the following lemma of [11]: if $P(a) \in C(X)$
denotes the map $x \mapsto \|a_x\|$ for $a \in A$, there exists a separable $C^*$-subalgebra $C(Y)$ of $C(X)$ with same unit such that if $D_k$ is the separable unital $C^*$-algebra generated by $C(Y), 1 \in A$ and the $a'_j a''_j$, $1 \leq j \leq n$, then $P(D_k) = C(Y)$. Furthermore, if $\Phi : X \to Y$ is the transpose of the inclusion map $C(Y) \hookrightarrow C(X)$ restricted to pure states, the map $D/(C_\Phi(x)(Y)D) \to A_+$ is a monomorphism for every $x \in X$ since $A$ is continuous.

Consider now a continuous field of states $\omega_k : D_k \to C(Y)$ on the continuous field $D_k$ over $Y$; if $\sum c^i \otimes d^i \in D_k \otimes B$ is zero in $A_x \otimes B_x$ for $x \in X$, then $\sum \omega_k(c^i(x)) d^i_x = 0$ in $B_x$, which enables us to conclude as in the separable case.

2. J. Mingo has drawn the author’s attention to the following result of Glimm ([7] lemma 10): if $C(X)$ is a von Neumann algebra and $A$ is a $C(X)$-algebras, then $P(a)^2 = \min \{z \in C(X)_+, \; z \geq a^* a\}$ is continuous for every $a \in A$. Therefore we always have in that case the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$ thanks to the previous remark.

**COROLLARY 3.2** Let $A$ and $B$ be two $C(X)$-algebras and assume that there exists a finite subset $F = \{x_1, \cdots, x_p\} \subset X$ such that for all $a \in A$, the function $x \mapsto \|a_x\|$ is continuous on $X \setminus F$.

Then the ideals $\mathcal{I}(A, B)$ and $\mathcal{J}(A, B)$ coincide.

**Proof:** Fix an element $\alpha = \sum_{1 \leq i \leq n} a_i \otimes b_i \in A \otimes B$ which belongs to $\mathcal{J}(A, B)$. In particular, we have $\sum_i (a_i)_x \otimes (b_i)_x = 0$ in $A_x \otimes B_x$ for each $x \in F$.

As a consequence, thanks to theorem III of [13], we may find complex matrices $(\lambda_{i,j})_{i,j} \in M_n(\mathbb{C})$ for all $1 \leq m \leq p$ such that, if we define the elements $c^m_k \in A$ and $d^m_k \in B$ by the formulas

$$c^m_k = \sum_i \lambda^m_{i,k} a_i \quad\text{and}\quad d^m_k = b_k - \sum_j \lambda^m_{k,j} b_j,$$

we have $(c^m_k)_m = 0$ and $(d^m_k)_m = 0$ for all $k$ and all $m$.

Consider now a partition $\{f_i\}_{1 \leq i \leq p}$ of $1 \in C(X)$ such that for all $1 \leq l, m \leq p$, $f_l(x_m) = \delta_{i,m}$ where $\delta$ is the Kronecker symbol and define for all $1 \leq l \leq n$ the elements $c_k = \sum_m f_m c^m_k$ and $d_k = \sum_m f_m d^m_k$. Thus,

$$\alpha = (\sum_i a_i \otimes d_i) + \left(\sum_{i,m} \lambda^m_{i,j} a_i \otimes f_m b_j\right)$$

$$= (\sum_i a_i \otimes d_i) + \left(\sum_j c_j \otimes b_j\right) + \left(\sum_{i,j,m} \lambda^m_{i,j} (a_i \otimes f_m b_j - f_m a_i \otimes b_j)\right),$$

and there exists therefore an element $\beta \in \mathcal{I}(A, B)$ such that $\alpha - \beta$ admits a finite decomposition $\sum_i a'_i \otimes b'_i$ with $a'_i \in C_0(X \setminus F) A$ and $b'_i \in C_0(X \setminus F) B$.

But $C_0(X \setminus F) A$ is a continuous field. Accordingly, proposition 3.1 implies that $\alpha - \beta \in \mathcal{I}(C_0(X \setminus F) A, C_0(X \setminus F) B) \subset \mathcal{I}(A, B).$ \(\Box\)

**Remark:** If $\tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of $\mathbb{N}$ and if $A$ and $B$ are two $C(\tilde{\mathbb{N}})$-algebras, the corollary 3.2 implies the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

Let us now introduce a counter-example in the general case.

Consider a dense countable subset $X = \{a_n\}_{n \in \mathbb{N}}$ of the interval $[0, 1]$. The $C^*$-algebra $C_0(\mathbb{N})$ of sequences with values in $\mathbb{C}$ vanishing at infinity is then endowed with the $C([0, 1])$-algebra structure defined by:
If we call $A$ this $C([0,1])$-algebra, then $A_x = 0$ for all $x \not\in X$.

Indeed, assume that $x \not\in X$ and take $\alpha \in A$. If $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\alpha_n| < \varepsilon$ for all $n \geq N$. Consider a continuous function $f \in C([0,1])$ such that $0 \leq f \leq 1$, $f(x) = 0$ and $f(a_i) = 1$ for every $1 \leq i \leq N$; we then have:

$$
\|\alpha_x\| \leq \|(1 - f)\alpha\| < \varepsilon.
$$

Let $Y = \{b_n\}_{n \in \mathbb{N}}$ be another dense countable subset of $[0,1]$ and denote by $B$ the associated $C([0,1])$-algebra whose underlying algebra is $C_0(\mathbb{N})$.

Then, if $X \cap Y = \emptyset$, the previous remark implies that for every $x \in [0,1]$, $A_x \otimes B_x = 0$ and hence, $\mathcal{J}(A, B) = A \otimes B$. What we therefore need to prove is that the ideal $\mathcal{I}(A, B)$ is strictly included in $A \otimes B$.

Let us fix two sequences $\alpha \in A$ and $\beta \in B$ whose terms are all non zero and suppose that $\alpha \otimes \beta$ admits a decomposition $\sum_{1 \leq i \leq k} (f_i \alpha_i^i \otimes \beta_i^k - \alpha_i \otimes (f_i \beta_i^k))$ in $A \otimes B$. Then for every $n, m \in \mathbb{N}$,

$$
\alpha_n \beta_m = \sum_{1 \leq i \leq k} \alpha_i \beta_i \gamma_i [f_i(a_n) - f_i(b_m)].
$$

Now, if we set $\phi_i(a_n) = \alpha_i \alpha_n$ and $\psi_i(b_n) = \beta_i \beta_n$ for $1 \leq i \leq k$ and $n \in \mathbb{N}$, this equality means that for all $(x, y) \in X \times Y$,

$$
1 = \sum_{1 \leq i \leq k} \psi_i(x) \phi_i(y) [f_i(x) - f_i(y)].
$$

But this is impossible because of the following proposition:

**PROPOSITION 3.3** Let $X$ and $Y$ be two dense subsets of the interval $[0,1]$ and let $n$ be a non negative integer.

Given continuous functions $f_i$ on $[0,1]$ and numerical functions $\psi_i : X \to \mathbb{C}$ and $\varphi_i : Y \to \mathbb{C}$ for $1 \leq i \leq n$, if there exists a constant $c \in \mathbb{C}$ such that

$$
\forall (x, y) \in X \times Y, \quad \sum_{1 \leq i \leq n} \psi_i(x) \varphi_i(y) [f_i(x) - f_i(y)] = c
$$

then $c = 0$.

**Proof:** We shall prove the proposition by induction on $n$.

If $n = 0$, the result is trivial. Take therefore $n > 0$ and assume that the proposition is true for any $k < n$.

Suppose then that the subsets $X$ and $Y$ of $[0,1]$, the functions $f_i$, $\psi_i$ and $\varphi_i$, $1 \leq i \leq n$, satisfy the hypothesis of the proposition for the constant $c$.

For $x \in X$, let $p(x) \leq n$ be the dimension of the vector space generated in $\mathbb{C}^n$ by the $\{\varphi_i(y) [f_i(x) - f_i(y)]\}_{1 \leq i \leq n}$, $y \in Y$.

If $p(x) < n$, there exists a subset $F(x) \subset \{1, \ldots, n\}$ of cardinal $p(x)$ such that for every $j \not\in F(x)$,
\[ \varphi_j(y)[f_j(x) - f_j(y)] = \sum_{i \in F(x)} \lambda_i^j(x)\varphi_i(y)[f_i(x) - f_i(y)] \text{ for all } y \in Y, \]

where the \( \lambda_i^j(x) \in C \) are given by the Cramer formulas. As a consequence,

\[ \sum_{i \in F(x)} \left( \psi_i(x) + \sum_{j \notin F(x)} [\lambda_i^j(x)] \psi_j(x) \right) \varphi_i(y)[f_i(x) - f_i(y)] = c. \]

Now, if \( p(x) < n \) for every \( x \in X \), there exists a subset \( F \subset \{1, \ldots, n\} \) of cardinal \( p < n \) such that the interior of the closure of the set of those \( x \) for which \( F(x) = F \) is not empty and contains therefore a closed interval homeomorphic to \([0, 1]\). The induction hypothesis for \( k = p \) implies that \( c = 0 \).

Assume on the other hand that \( x_0 \in X \) verifies \( p(x_0) = n \). We may then find \( y_1, \ldots, y_n \) in \( Y \) such that if we set

\[ a_{i,j}(x) = \varphi_i(y_j)[f_i(x) - f_i(y_j)], \]

the matrix \( (a_{i,j}(x_0)) \) is invertible. There exists therefore a closed connected neighborhood \( I \) of \( x_0 \) on which the matrix \( (a_{i,j}(x)) \) remains invertible.

But for each \( 1 \leq j \leq n \), \( \sum a_{i,j}(x)\psi_i(x) = c \) and therefore the \( \psi_i(x) \) extend by the Cramer formulas to continuous functions on the closed interval \( I \).

For \( y \in Y \cap I \), let \( q(y) \) denote the dimension of the vector space generated in \( C^n \) by the \( \left( \psi_i(x)[f_i(x) - f_i(y)] \right)_{1 \leq i \leq n}, x \in X \cap I. \)

If \( q(y) < n \) for every \( y \), then the induction hypothesis implies \( c = 0 \). But if there exists \( y_0 \) such that \( q(y_0) = n \), we may find an interval \( J \subset I \) homeomorphic to \([0, 1]\) on which the \( \varphi_i \) extend to continuous functions; evaluating the starting formula at a point \((x, x) \in J \times J \), we get \( c = 0 \). \( \square \)

4 The associativity

Given three \( C(X) \)-algebras \( A_1, A_2 \) and \( A_3 \), we deduce from [3] corollaire 3.17:

\[ [(A_1^M \otimes_{C(X)} A_2^M) \otimes_{C(X)} A_3^M]_x = (A_1^M \otimes_{C(X)} A_2^M) \otimes_{\max} A_3^M = (A_1^M \otimes_{\max} A_2^M) \otimes_{\max} A_3^M, \]

which implies the associativity of the tensor product \( M \otimes_{C(X)} \) over \( C(X) \).

On the contrary, the minimal tensor product \( m \otimes_{C(X)} \) over \( C(X) \) is not in general associative. Indeed, Kirchberg and Wassermann have shown in [11] that if \( \tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \) is the Alexandroff compactification of \( \mathbb{N} \), there exist separable continuous fields \( A \) and \( B \) such that

\[ (A^m \otimes_{C(\tilde{\mathbb{N})}} B)_{\infty} \neq A_{\infty} \otimes_{\min} B_{\infty}. \]

If we now endow the \( C^* \)-algebra \( D = C \) with the \( C(\tilde{\mathbb{N}}) \)-algebra structure defined by \( f.a = f(\infty)a \), then for all \( C(\tilde{\mathbb{N}}) \)-algebra \( D' \), we have
Therefore, \([A^m \otimes C(\tilde{\eta}) B^m \otimes C(\tilde{\eta}) D]_\infty \simeq (A^m \otimes C(\tilde{\eta}) B^m)\) whereas \([A^m \otimes C(\tilde{\eta}) (B^m \otimes C(\tilde{\eta}) D)]_\infty \) is isomorphic to \(A_\infty \otimes_{\min} B_\infty\).

However, in the case of (separable) continuous fields, we can deduce the associativity of \(\otimes\), from the following proposition:

**PROPOSITION 4.1** Let \(A\) and \(B\) be two \(C(X)\)-algebras.

Assume \(\pi\) is a field of faithful representations of \(A\) in the Hilbert \(C(X)\)-module \(E\), then the morphism \(a \otimes b \rightarrow \pi(a) \otimes b\) induces a faithful \(C(X)\)-linear representation of \(A^m \otimes_{\min} C(X)\) in the Hilbert \(B\)-module \(E \otimes C(X) B\).

**Proof:** Notice that for all \(x \in X\), we have \((E \otimes C(X) B) \otimes_B B_x = E_x \otimes B_x\).

Now, as \(B\) maps injectively in \(B_d = \oplus_{x \in X} B_x\), \(L_B(E \otimes C(X) B)\) maps injectively in \(\oplus_{x \in X} L_{B_d}(E_x \otimes B_x) \subset L_{B_d}(E \otimes C(X) B \otimes_B B_d)\) and therefore if \(\alpha \in A \otimes B\), we have \(\|(\pi \otimes \text{id})(\alpha)\| = \sup_{x \in X} \|(\pi_x \otimes \text{id})(\alpha_x)\| = \|\alpha\|_m\).

Accordingly, if for \(1 \leq i \leq 3\), \(A_i\) is a separable continuous field of \(C^*\)-algebras over \(X\) which admits a field of faithful representations in the \(C(X)\)-module \(E_i\), the \(C(X)\)-representations of \((A_1 \otimes_{\min} A_2) \otimes_{\min} A_3\) and \((A_1 \otimes_{\min} A_2 \otimes_{\min} A_3)\) in the Hilbert \(C(X)\)-module \((E_1 \otimes C(X)) E_2 \otimes C(X) E_3 = E_1 \otimes C(X) (E_2 \otimes C(X)) E_3\) are faithful, and hence the maps \(A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \rightarrow (A_1 \otimes_{\min} A_2) \otimes_{\min} A_3\) and \(A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \rightarrow A_1 \otimes_{\min} (A_2 \otimes_{\min} A_3)\) have the same kernel.

**References**


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