Tensor products of $C(X)$-algebras over $C(X)$
Etienne Blanchard

To cite this version:
Etienne Blanchard. Tensor products of $C(X)$-algebras over $C(X)$. Asterisque, 1995, 232, pp.81–92. hal-00922902v2

HAL Id: hal-00922902
https://hal.science/hal-00922902v2
Submitted on 3 Jan 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Tensor products of $C(X)$-algebras over $C(X)$

Etienne Blanchard

Abstract

Given a Hausdorff compact space $X$, we study the $C^*$-(semi)-norms on the algebraic tensor product $A \otimes_{C(X)} B$ of two $C(X)$-algebras $A$ and $B$ over $C(X)$. In particular, if one of the two $C(X)$-algebras defines a continuous field of $C^*$-algebras over $X$, there exist minimal and maximal $C^*$-norms on $A \otimes_{C(X)} B$ but there does not exist any $C^*$-norm on $A \otimes_{C(X)} B$ in general.

AMS classification: 46L05, 46M05.

0 Introduction

Tensor products of $C^*$-algebras have been extensively studied over the last decades (see references in [12]). One of the main results was obtained by M. Takesaki in [15] where he proved that the spatial tensor product $A \otimes_{\min} B$ of two $C^*$-algebras $A$ and $B$ always defines the minimal $C^*$-norm on the algebraic tensor product $A \otimes B$ of $A$ and $B$ over the complex field $\mathbb{C}$.

More recently, G.G. Kasparov constructed in [10] a tensor product over $C(X)$ for $C(X)$-algebras. The author was also led to introduce in [3] several notions of tensor products over $C(X)$ for $C(X)$-algebras and to study the links between those objects.

Notice that E. Kirchberg and S. Wassermann have proved in [11] that the subcategory of continuous fields over a Hausdorff compact space is not closed under such tensor products over $C(X)$ and therefore, in order to study tensor products over $C(X)$ of continuous fields, it is natural to work in the $C(X)$-algebras framework.

Let us introduce the following definition:

**DEFINITION 0.1.** Given two $C(X)$-algebras $A$ and $B$, we denote by $\mathcal{I}(A, B)$ the involutive ideal of the algebraic tensor product $A \otimes B$ generated by the elements $(fa) \otimes b - a \otimes (fb)$, where $f \in C(X)$, $a \in A$ and $b \in B$.

Our aim in the present article is to study the $C^*$-norms on the algebraic tensor product $(A \otimes B)/\mathcal{I}(A, B)$ of two $C(X)$-algebras $A$ and $B$ over $C(X)$ and to see how one can enlarge the results of Takesaki to this framework.

We first define an ideal $\mathcal{J}(A, B) \subset A \otimes B$ which contains $\mathcal{I}(A, B)$ such that every $C^*$-semi-norm on $A \otimes B$ which is zero on $\mathcal{I}(A, B)$ is also zero on $\mathcal{J}(A, B)$ and we prove that there always exist a minimal $C^*$-norm $\| \cdot \|_m$ and a maximal $C^*$-norm $\| \cdot \|_M$ on the quotient $(A \otimes B)/\mathcal{J}(A, B)$.
We then study the following question of G.A. Elliott ([5]): when do the two ideals $I(A, B)$ and $J(A, B)$ coincide?

The author would like to express his gratitude to C. Anantharaman-Delaroche and G. Skandalis for helpful comments. He is also very indebted to S. Wassermann for sending him a preliminary version of [11] and to J. Cuntz who invited him to the Mathematical Institute of Heidelberg.

1 Preliminaries

We briefly recall here the basic properties of $C(X)$-algebras.

Let $X$ be a Hausdorff compact space and $C(X)$ be the $C^*$-algebra of continuous functions on $X$. For $x \in X$, define the morphism $e_x : C(X) \to \mathbb{C}$ of evaluation at $x$ and denote by $C_x(X)$ the kernel of this map.

**DEFINITION 1.1** ([10]) A $C(X)$-algebra is a $C^*$-algebra $A$ endowed with a unital morphism from $C(X)$ in the center of the multiplier algebra $M(A)$ of $A$.

We associate to such an algebra the unital $C(X)$-algebra $A$ generated by $A$ and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a \otimes f) = ga \otimes gf$ for $a \in A$ and $f, g \in C(X)$.

For $x \in X$, denote by $A_x$ the quotient of $A$ by the closed ideal $C_x(X)A$ and by $a_x$ the image of $a \in A$ in the fibre $A_x$. Then, as

$$
\|a_x\| = \inf\{\|1 - f + f(x)\|a, f \in C(X)\},
$$

the map $x \mapsto \|a_x\|$ is upper semi-continuous for all $a \in A$ ([14]).

Note that the map $A \to \oplus A_x$ is a monomorphism since if $a \in A$, there is a pure state $\phi$ on $A$ such that $\phi(a^*a) = \|a\|^2$. As the restriction of $\phi$ to $C(X) \subset M(A)$ is a character, there exists $x \in X$ such that $\phi$ factors through $A_x$ and so $\phi(a^*a) = \|a_x\|^2$.

Let $S(A)$ be the set of states on $A$ endowed with the weak topology and let $S_X(A)$ be the subset of states $\varphi$ whose restriction to $C(X) \subset M(A)$ is a character, i.e. such that there exists $x \in X$ (denoted $x = p(\varphi)$) verifying $\varphi(f) = f(x)$ for all $f \in C(X)$. Then the previous paragraph implies that the set of pure states $P(A)$ on $A$ is included in $S_X(A)$.

Let us introduce the following notation: if $E$ is a Hilbert $A$-module where $A$ is a $C^*$-algebra, we will denote by $L_A(E)$ or simply $L(E)$ the set of bounded $A$-linear operators on $E$ which admit an adjoint ([9]).

**DEFINITION 1.2** ([3]) Let $A$ be a $C(X)$-algebra.

A $C(X)$-representation of $A$ in the Hilbert $C(X)$-module $E$ is a morphism $\pi : A \to L(E)$ which is $C(X)$-linear, i.e. such that for every $x \in X$, the representation $\pi_x = \pi \otimes e_x$ in the Hilbert space $E_x = E \otimes e_x \subset E$ factors through a representation of $A_x$. Furthermore, if $\pi_x$ is a faithful representation of $A_x$ for every $x \in X$, $\pi$ is said to be a field of faithful representations of $A$.

A continuous field of states on $A$ is a $C(X)$-linear map $\varphi : A \to C(X)$ such that for any $x \in X$, the map $\varphi_x = e_x \circ \varphi$ defines a state on $A_x$. 

2
If $\pi$ is a $C(X)$-representation of the $C(X)$-algebra $A$, the map $x \mapsto \|\pi_x(a)\|$ is lower semi-continuous since $(\xi, \pi(a)\eta) \in C(X)$ for every $\xi, \eta \in E$. Therefore, if $A$ admits a field of faithful representations $\pi$, the map $x \mapsto \|a_x\| = \|\pi_x(a)\|$ is continuous for every $a \in A$, which means that $A$ is a continuous field of $C^*$-algebras over $X$ ([4]).

The converse is also true ([3] théorème 3.3): given a separable $C(X)$-algebra $A$, the following assertions are equivalent:

1. $A$ is a continuous field of $C^*$-algebras over $X$,
2. the map $p : S_X(A) \to X$ is open,
3. $A$ admits a field of faithful representations.

2 $C^*$-norms on $(A \otimes B)/J(A, B)$

**DEFINITION 2.1** Given two $C(X)$-algebras $A$ and $B$, we define the involutive ideal $J(A, B)$ of the algebraic tensor product $A \otimes B$ of elements $\alpha \in A \otimes B$ such that $\alpha_x = 0$ in $A_x \otimes B_x$ for every $x \in X$.

By construction, the ideal $I(A, B)$ is included in $J(A, B)$.

**PROPOSITION 2.2** Assume that $\| \|_\beta$ is a $C^*$-semi-norm on the algebraic tensor product $A \otimes B$ of two $C(X)$-algebras $A$ and $B$.

If $\| \|_\beta$ is zero on the ideal $I(A, B)$, then

$$\|\alpha\|_\beta = 0 \text{ for all } \alpha \in J(A, B).$$

**Proof:** Let $D_\beta$ be the Hausdorff completion of $A \otimes B$ for $\| \|_\beta$. By construction, $D_\beta$ is a quotient of $A \otimes_{\max} B$. Furthermore, if $C_\Delta$ is the ideal of $C(X \times X)$ of functions which are zero on the diagonal, the image of $C_\Delta$ in $M(D_\beta)$ is zero.

As a consequence, the map from $A \otimes_{\max} B$ onto $D_\beta$ factors through the quotient $A^{M}_{\otimes_{C(X)}} B$ of $A \otimes_{\max} B$ by $C_\Delta \times (A \otimes_{\max} B).

But an easy diagram-chasing argument shows that $(A^{M}_{\otimes_{C(X)}} B)_x = A_x \otimes_{\max} B_x$ for every $x \in X$ ([3] corollaire 3.17) and therefore the image of $J(A, B) \subset A \otimes_{\max} B$ in $A^{M}_{\otimes_{C(X)}} B$ is zero. $\square$

2.1 The maximal $C^*$-norm

**DEFINITION 2.3** Given two $C(X)$-algebras $A_1$ and $A_2$, we denote by $\| \|_M$ the $C^*$-semi-norm on $A_1 \otimes A_2$ defined for $\alpha \in A_1 \otimes A_2$ by

$$\|\alpha\|_M = \sup\{\|\sigma^*_1 \otimes_{\max} \sigma^*_2\|, x \in X\},$$

where $\sigma^*_i$ is the map $A_i \to (A_i)_x$.

As $\| \|_M$ is zero on the ideal $J(A_1, A_2)$, if we identify $\| \|_M$ with the $C^*$-semi-norm induced on $(A_1 \otimes A_2)/J(A_1, A_2)$, we get:
PROPOSITION 2.4 The semi-norm $\| \cdot \|_M$ is the maximal C*-norm on the quotient $(A_1 \odot A_2)/\mathcal{J}(A_1, A_2)$.

Proof: By construction, $\| \cdot \|_M$ defines a C*-norm on $(A_1 \odot A_2)/\mathcal{J}(A_1, A_2)$. Moreover, as the quotient $A_1 \otimes_{C(X)} A_2$ of $A_1 \otimes_{\text{max}} A_2$ by $C_\Delta \times (A_1 \otimes_{\text{max}} A_2)$ maps injectively in

$$\bigoplus_{x \in X} (A_1 \otimes_{C(X)} A_2)_x = \bigoplus_{x \in X} (A_1)_x \otimes_{\text{max}} (A_2)_x,$$

the norm of $A_1 \otimes_{C(X)} A_2$ coincides on the dense subalgebra $(A_1 \otimes A_2)/\mathcal{J}(A_1, A_2)$ with $\| \cdot \|_M$. But we saw in proposition 2.2 that if $\| \cdot \|_\beta$ is a C*-norm on the algebra $(A \odot B)/\mathcal{J}(A, B)$, the completion of $(A \odot B)/\mathcal{J}(A, B)$ for $\| \cdot \|_\beta$ is a quotient of $A \otimes_{C(X)} B$. □

2.2 The minimal C*-norm

DEFINITION 2.5 Given two $C(X)$-algebras $A_1$ and $A_2$, we define the semi-norm $\| \cdot \|_m$ on $A_1 \odot A_2$ by the formula

$$\| \alpha \|_m = \sup \{ \|(\sigma^*_x \otimes_{\text{min}} \sigma^*_y)(\alpha)\|, x \in X \}$$

where $\sigma^*_x$ is the map $A_1 \rightarrow (A_1)_x$ and we denote by $A_1 \otimes_{C(X)} A_2$ the Hausdorff completion of $A_1 \odot A_2$ for that semi-norm.

Remark: In general, the canonical map $(A_1 \otimes_{C(X)} A_2)_x \rightarrow (A_1)_x \otimes_{\text{min}} (A_2)_x$ is not a monomorphism ([11]).

By construction, $\| \cdot \|_m$ induces a C*-norm on $(A_1 \odot A_2)/\mathcal{J}(A_1, A_2)$. We are going to prove that this C*-norm defines the minimal C*-norm on the involutive algebra $(A_1 \odot A_2)/\mathcal{J}(A_1, A_2)$.

Let us introduce some notation.

Given two unital $C(X)$-algebras $A_1$ and $A_2$, let $P(A_i) \subseteq S_X(A_i)$ denote the set of pure states on $A_i$ and let $P(A_1) \times_X P(A_2)$ denote the closed subset of $P(A_1) \times P(A_2)$ of couples $(\omega_1, \omega_2)$ such that $p(\omega_1) = p(\omega_2)$, where $p : P(A_i) \rightarrow X$ is the restriction to $P(A_i)$ of the map $p : S_X(A_i) \rightarrow X$ defined in section 1.

LEMMA 2.6 Assume that $\| \cdot \|_\beta$ is a C*-semi-norm on the algebraic tensor product $A_1 \odot A_2$ of two unital $C(X)$-algebras $A_1$ and $A_2$ which is zero on the ideal $\mathcal{J}(A_1, A_2)$ and define the closed subset $S_\beta \subseteq P(A_1) \times_X P(A_2)$ of couples $(\omega_1, \omega_2)$ such that

$$\left| (\omega_1 \otimes \omega_2)(\alpha) \right| \leq \| \alpha \|_\beta \quad \text{for all } \alpha \in A_1 \odot A_2.$$

If $S_\beta \neq P(A_1) \times_X P(A_2)$, there exist self-adjoint elements $a_i \in A_i$ such that $a_1 \otimes a_2 \notin \mathcal{J}(A_1, A_2)$ but $(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0$ for all couples $(\omega_1, \omega_2) \in S_\beta$.

Proof: Define for $i = 1, 2$ the adjoint action $ad$ of the unitary group $U(A_i)$ of $A_i$ on the pure states space $P(A_i)$ by the formula

4
[(ad_u)\omega](a) = \omega(u^*au).

Then \(S_\beta\) is invariant under the product action \(ad \times ad\) of \(U(A_1) \times U(A_2)\) and we can therefore find non empty open subsets \(U_i \subset P(A_i)\) which are invariant under the action of \(U(A_i)\) such that \((U_1 \times U_2) \cap S_\beta = \emptyset\).

Now, if \(K_i\) is the complement of \(U_i\) in \(P(A_i)\), the set
\[
K_i^\perp = \{a \in A_i \mid \omega(a) = 0 \text{ for all } \omega \in K_i\}
\]
is a non empty ideal of \(A_i\) and furthermore, if \(\omega \in P(A_i)\) is zero on \(K_i^\perp\), then \(\omega\) belongs to \(K_i\) ([8] lemma 8,[15]).

As a consequence, if \((\varphi_1, \varphi_2)\) is a point of \(U_1 \times_X U_2\), there exist non zero self-adjoint elements \(a_i \in K_i^\perp\) such that \(\varphi_i(a_i) = 1\). If \(x = p(\varphi_i)\), this implies in particular that \((a_1)_x \otimes (a_2)_x \neq 0\), and hence \(a_1 \otimes a_2 \notin J(A_1, A_2)\). □

**LEMMA 2.7** ([15] theorem 1) Let \(A_1\) and \(A_2\) be two unital \(C(X)\)-algebras.

If the algebra \(A_1\) is an abelian algebra, there exists only one \(C^*\)-norm on the quotient \((A_1 \odot A_2) / J(A_1, A_2)\).

**Proof:** Let \(\| \|_\beta\) be a \(C^*\)-semi-norm on \(A_1 \odot A_2\) such that for all \(\alpha \in A_1 \odot A_2, \| \alpha \|_\beta = 0\) if and only if \(\alpha \in J(A_1, A_2)\).

If \(\rho \in P(A_\beta)\) is a pure state on the Hausdorff completion \(A_\beta\) of \(A_1 \odot A_2\) for the semi-norm \(\| \|_\beta\), then for every \(a_1 \otimes a_2 \in A_1 \odot A_2\),
\[
\rho(a_1 \otimes a_2) = \rho(a_1 \odot 1)\rho(1 \otimes a_2)
\]
since \(A_1 \odot 1\) is included in the center of \(M(A_\beta)\). Moreover, if we define the states \(\omega_1\) and \(\omega_2\) by the formulas \(\omega_1(a_1) = \rho(a_1 \odot 1)\) and \(\omega_2(a_2) = \rho(1 \otimes a_2)\), then \(\omega_2\) is pure since \(\rho\) is pure, and \((\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)\). It follows that \(P(A_\beta)\) is isomorphic to \(S_\beta\).

In particular, if \(a_1 \otimes a_2 \in A_1 \odot A_2\) verifies
\[
(\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0 \text{ for all couples } (\omega_1, \omega_2) \in S_\beta,
\]
the element \(a_1 \otimes a_2\) is zero in \(A_\beta\) and therefore belongs to the ideal \(J(A_1, A_2)\). Accordingly, the previous lemma implies that \(P(A_1) \times_X P(A_2) = S_\beta = P(A_\beta)\).

As a consequence, we get for every \(\alpha \in A_1 \odot A_2\)
\[
\| \alpha \|_\beta^2 = \sup\{\rho(\alpha^*\alpha), \rho \in P(A_\beta)\} = \sup\{(\omega_1 \otimes \omega_2)(\alpha^*\alpha), (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2)\}
\]
But that last expression does not depend on \(\| \|_\beta\), and hence the unicity. □

**PROPOSITION 2.8** ([15] theorem 2) Let \(A_1\) and \(A_2\) be two unital \(C(X)\)-algebras.

If \(\| \|_\beta\) is a \(C^*\)-semi-norm on \(A_1 \odot A_2\) whose kernel is \(J(A_1, A_2)\), then
\[
\forall \alpha \in A_1 \odot A_2, \quad \| \alpha \|_\beta \geq \| \alpha \|_m.
\]

**Proof:** If we show that \(S_\beta = P(A_1) \times_X P(A_2)\), then for every \(\rho \in S_\beta\) and every \(\alpha\) in \(A_1 \odot A_2\), we have \(\rho(s^*\alpha^*\alpha s) \leq \rho(s^*s)\| \alpha \|_\beta^2\) for all \(s \in A_1 \odot A_2\). Therefore
\[ \|\alpha\|_m^2 = \sup\left\{ \| (\sigma_1^* \otimes \sigma_2^*) (\alpha) \|_2^2, x \in X \right\} \]
\[ = \sup\left\{ \frac{(\omega_1 \otimes \omega_2)(s^* \alpha s)}{(\omega_1 \otimes \omega_2)(s^* s)}, (\omega_1, \omega_2) \in P(A_1) \times_X P(A_2) \text{ and } s \in A_1 \oplus A_2 \text{ such that } (\omega_1 \otimes \omega_2)(s^* s) \neq 0 \right\} \]
\[ \leq \|\alpha\|_\beta^2. \]

Suppose that \( S_\beta \neq P(A_1) \times_X P(A_2) \). Then there exist thanks to lemma 2.6 self-adjoint elements \( a_i \in A_i \) and a point \( x \in X \) such that \( (a_1)_x \otimes (a_2)_x \neq 0 \) but \( (\omega_1 \otimes \omega_2)(a_1 \otimes a_2) = 0 \) for all couples \((\omega_1, \omega_2) \in S_\beta\).

Let \( B \) be the unital abelian \( C(X) \)-algebra generated by \( C(X) \) and \( a_1 \) in \( A_1 \). The preceding lemma implies that \( B \otimes C(X) A_2 \) maps injectively into the Hausdorff completion \( A_\beta \) of \( A_1 \otimes A_2 \) for \( \|\|_\beta \).

Consider pure states \( \rho \in P(B_\beta) \) and \( \omega_2 \in P((A_2)_x) \) such that \( \rho(a_1) \neq 0 \) and \( \omega_2(a_2) \neq 0 \) and extend the pure state \( \rho \otimes \omega_2 \) on \( B \otimes C(X) A_2 \) to a pure state \( \omega \) on \( A_\beta \). If we set \( \omega_1(a) = \omega(a \otimes 1) \) for \( a \in A_1 \), then \( \omega_1 \) is pure and \( \omega(\alpha) = (\omega_1 \otimes \omega_2)(\alpha) \) for all \( \alpha \in A_1 \otimes A_2 \) since \( \omega \) and \( \omega_2 \) are pure ([15] lemma 4). As a consequence, \( (\omega_1, \omega_2) \in S_\beta \), which is absurd since \( (\omega_1 \otimes \omega_2)(a_1 \otimes a_2) \neq \rho(a_1)\omega_2(a_2) \neq 0 \). \( \square \)

**PROPOSITION 2.9** Given two \( C(X) \)-algebras \( A_1 \) and \( A_2 \), the semi-norm \( \|\|_m \) defines the minimal \( C^* \)-norm on the involutive algebra \( (A_1 \otimes A_2)/J(A_1, A_2) \).

**Proof:** Let \( \|\|_\beta \) be a \( C^* \)-norm on \( (A_1 \otimes A_2)/J(A_1, A_2) \). Thanks to the previous proposition, all we need to prove is that one can extend \( \|\|_\beta \) to a \( C^* \)-norm on \( (A_1 \otimes A_2)/J(A_1, A_2) \), where \( A_1 \) and \( A_2 \) are the unital \( C(X) \)-algebras associated to the \( C(X) \)-algebras \( A_1 \) and \( A_2 \) (definition 1.1).

Consider the Hausdorff completion \( D_\beta \) of \( (A_1 \otimes A_2)/J(A_1, A_2) \) and denote by \( \pi_i \) the canonical representation of \( A_i \) in \( M(D_\beta) \) for \( i = 1, 2 \). Let us define the representation \( \tilde{\pi}_i \) of \( A_i \) in \( M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X)) \) by the following formulas:

\[ \tilde{\pi}_1(b_1 + u(f)) (\alpha + a_1 \oplus a_2 \oplus g) = (\pi_1(b_1) + g) (\alpha + (b_1 + f) a_1 + f a_2 + f g) \]
\[ \tilde{\pi}_2(b_2 + u(f)) (\alpha + a_1 \oplus a_2 \oplus g) = (\pi_2(b_2) + g) (\alpha + f a_1 \oplus (b_2 + f) a_2 + f g) \]

For \( i = 1, 2 \), let \( \varepsilon_i : A_i \to C(X) \) be the map defined by

\[ \varepsilon_i[a + u(f)] = f \text{ for } a \in A_i \text{ and } f \in C(X). \]

Then using the maps \( (\varepsilon_1 \otimes \varepsilon_2), (\varepsilon_1 \otimes i d) \) and \( (i d \otimes \varepsilon_2) \), one proves easily that if \( \alpha \in A_1 \otimes A_2 \), \( (\tilde{\pi}_1 \oplus \tilde{\pi}_2)(\alpha) = 0 \) if and only if \( \alpha \) belongs to \( J(A_1, A_2) \).

Therefore, the norm of \( M(D_\beta \oplus A_1 \oplus A_2 \oplus C(X)) \) restricted to the subalgebra \( (A_1 \otimes A_2)/J(A_1, A_2) \) extends \( \|\|_\beta \). \( \square \)

**Remark:** As the \( C(X) \)-algebra \( A \) is nuclear if and only if every fibre \( A_x \) is nuclear ([12]), \( A \otimes_{C(X)} B \simeq A_{C(X)} \otimes B \) for every \( C(X) \)-algebra \( B \) if and only if \( A \) is nuclear.
3 When does the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$ hold?

Given two $C(X)$-algebras $A$ and $B$, Giordano and Mingo have studied in [6] the case where the algebra $C(X)$ is a von Neumann algebra; their theorem 3.1 and lemma 1.5 of [10] imply that in that case, we always have the equality $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

Our purpose in this section is to find sufficient conditions on the $C(X)$-algebras $A$ and $B$ in order to ensure this equality and to present a counter-example in the general case.

**PROPOSITION 3.1** Let $X$ be a second countable Hausdorff compact space and let $A$ and $B$ be two $C(X)$-algebras.

If $A$ is a continuous field of $C^*$-algebras over $X$, then $\mathcal{I}(A, B) = \mathcal{J}(A, B)$.

**Proof:** Let us prove by induction on the non negative integer $n$ that if

$$s = \sum_{1 \leq i \leq n} a_i \otimes b_i \in \mathcal{J}(A, B),$$

then $s$ belongs to the ideal $\mathcal{I}(A, B)$.

If $n = 0$, there is nothing to prove. Consider therefore an integer $n > 0$ and suppose the result has been proved for any $p < n$.

Fix an element $s = \sum_{1 \leq i \leq n} a_i \otimes b_i \in \mathcal{J}(A, B)$ and define the continuous positive function $h \in C(X)$ by the formula $h(x)^{10} = \sum \|(a_k)_x\|^2$.

The element $a_k' = h^{-1}a_k$ is then well defined in $A$ for every $k$ since $a_k^*a_k \leq h^{10}$. Consequently, the function $f_k(x) = \|(a_k')_x\|$ is continuous.

For $1 \leq k \leq n$, let $D_k$ denote the separable $C(X)$-algebra generated by 1 and the $a_k^*a_j'$, $1 \leq j \leq n$, in the unital $C(X)$-algebra $A$ associated to $A$ (definition 1.1). Then $D_k$ is a unital continuous field of $C^*$-algebras over $X$ (see for instance [3] proposition 3.2).

Consider the open subset $\mathcal{S}^k = \{ \psi \in \mathcal{S}_X(D_k) \mid \psi[a_k^*a_k'] > \psi(f_k^3/2) \}$. If we apply lemma 3.6 b) of [3] to the restriction of $p : \mathcal{S}_X(D_k) \rightarrow X$ to $\mathcal{S}^k$, we may construct a continuous field of states $\omega_k$ on $D_k$ such that $\omega_k[a_k^*a_k'] \geq f_k^3/2$.

Now, if we set $s' = \sum_i a_i' \otimes b_i$, as $(a_i^* \otimes 1)s'$ belongs to $\mathcal{J}(D_k, B)$,

$$\omega_k \otimes id)[(a_k^* \otimes 1)s'] = \omega_k[a_k^*a_k']b_k + \sum_{j \neq k} \omega_k[a_k^*a_j']b_j = 0.$$

Noticing that $f_k^3$ is in the ideal of $C(X)$ generated by $\omega_k[a_k^*a_k']$, we get that $f_k^3b_k$ belongs to the $C(X)$-module generated by the $b_i$, $j \neq k$, and thanks to the induction hypothesis, it follows that $(f_k^3 \otimes 1)s' \in \mathcal{I}(A, B)$ for each $k$.

But $h^2 = \sum_k f_k^2$ and so $h^4 \leq n \sum_k f_k^3$ is in the ideal of $C(X)$ generated by the $f_k^3$, which implies $s = (h^4 \otimes 1)s' \in \mathcal{I}(A, B)$.

**Remarks:** 1. As a matter of fact, it is not necessary to assume that the space $X$ satisfies the second axiom of countability thanks to the following lemma of [11]: if $P(a) \in C(X)$
denotes the map \( x \mapsto \|a_x\| \) for \( a \in \mathcal{A} \), there exists a separable \( C^* \)-subalgebra \( C(Y) \) of \( C(X) \) with same unit such that if \( D_k \) is the separable unital \( C^* \)-algebra generated by \( C(Y) \). If \( D_k \subset \mathcal{A} \) and the \( a'_k a'_j \), \( 1 \leq j \leq n \), then \( P(D_k) = C(Y) \). Furthermore, if \( \Phi : X \to Y \) is the transpose of the inclusion map \( C(Y) \hookrightarrow C(X) \) restricted to pure states, the map \( D/(C_\Phi(x)(Y)D) \to \mathcal{A}_x \) is a monomorphism for every \( x \in X \) since \( \mathcal{A} \) is continuous.

Consider now a continuous field of states \( \omega_k : D_k \to C(Y) \) on the continuous field \( D_k \) over \( Y \); if \( \sum c^i \otimes d^i \in D_k \otimes B \) is zero in \( A_x \otimes B_x \) for \( x \in X \), then \( \sum \omega_k(c^i)(x)d^i_x = 0 \) in \( B_x \), which enables us to conclude as in the separable case.

2. J. Mingo has drawn the author’s attention to the following result of Glimm ([7] lemma 10): if \( C(X) \) is a von Neumann algebra and \( A \) is a \( C(X) \)-algebras, then \( P(a)^2 = \min\{\|z \in C(X)\|, \|z \geq a^*a\}\) is continuous for every \( a \in \mathcal{A} \). Therefore we always have in that case the equality \( \mathcal{I}(A, B) = \mathcal{J}(A, B) \) thanks to the previous remark.

**COROLLARY 3.2** Let \( A \) and \( B \) be two \( C(X) \)-algebras and assume that there exists a finite subset \( F = \{x_1, \cdots, x_p\} \subset X \) such that for all \( a \in \mathcal{A} \), the function \( x \mapsto \|a_x\| \) is continuous on \( X \setminus F \).

Then the ideals \( \mathcal{I}(A, B) \) and \( \mathcal{J}(A, B) \) coincide.

**Proof:** Fix an element \( \alpha = \sum_{1 \leq i \leq n} a_i \otimes b_i \in A \otimes B \) which belongs to \( \mathcal{J}(A, B) \). In particular, we have \( \sum_i (a_i)_x \otimes (b_i)_x = 0 \) in \( A_x \otimes B_x \) for each \( x \in F \).

As a consequence, thanks to theorem III of [13], we may find complex matrices \( (\lambda_{i,j})_{i,j} \in M_n(\mathbb{C}) \) for all \( 1 \leq m \leq p \) such that, if we define the elements \( c^m_k \in A \) and \( d^m_k \in B \) by the formulas

\[
\begin{align*}
c^m_k &= \sum_i \lambda^m_{i,k} a_i, \\
d^m_k &= b_k - \sum_j \lambda^m_{j,k} b_j,
\end{align*}
\]

we have \( (c^m_k)_x = 0 \) and \( (d^m_k)_x = 0 \) for all \( k \) and all \( m \).

Consider now a partition \( \{f_l\}_{1 \leq l \leq p} \) of \( 1 \in C(X) \) such that for all \( 1 \leq l, m \leq p \), \( f_l(x_m) = \delta_{l,m} \) where \( \delta \) is the Kronecker symbol and define for all \( 1 \leq k \leq n \) the elements \( c_k = \sum m f_m c^m_k \) and \( d_k = \sum m f_m d^m_k \). Thus,

\[
\begin{align*}
\alpha &= (\sum_i a_i \otimes d_i) + \left( \sum_{i,j,m} \lambda_{i,j} a_i \otimes f_m b_j \right) \\
&= (\sum_i a_i \otimes d_i) + \left( \sum_j c_j \otimes b_j \right) + \left( \sum_{i,j,m} \lambda_{i,j} (a_i \otimes f_m b_j - f_m a_i \otimes b_j) \right)
\end{align*}
\]

and there exists therefore an element \( \beta \in \mathcal{I}(A, B) \) such that \( \alpha - \beta \) admits a finite decomposition \( \sum_i a'_i \otimes b'_i \) with \( a'_i \in C_0(X \setminus F) \) and \( b'_i \in C_0(X \setminus F) \).

But \( C_0(X \setminus F) \) is a continuous field. Accordingly, proposition 3.1 implies that \( \alpha - \beta \in \mathcal{I}(C_0(X \setminus F) \, A, C_0(X \setminus F) \, B) \subset \mathcal{I}(A, B) \). \( \square \)

**Remark:** If \( \bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \) is the Alexandroff compactification of \( \mathbb{N} \) and if \( A \) and \( B \) are two \( C(\bar{\mathbb{N}}) \)-algebras, the corollary 3.2 implies the equality \( \mathcal{I}(A, B) = \mathcal{J}(A, B) \).

Let us now introduce a counter-example in the general case.

Consider a dense countable subset \( X = \{a_n\}_{n \in \mathbb{N}} \) of the interval \([0, 1]\). The \( C^* \)-algebra \( C_0(\mathbb{N}) \) of sequences with values in \( \mathbb{C} \) vanishing at infinity is then endowed with the \( C([0, 1]) \)-algebra structure defined by:
\[ \forall f \in C([0,1]), \forall \alpha = (\alpha_n) \in C_0(\mathbb{N}), \quad (f.\alpha)_n = f(a_n)\alpha_n \text{ for } n \in \mathbb{N}. \]

If we call \textit{PROPOSITION 3.3} that \( |n, m| \leq 0 \) for every \( 0 \leq \phi \leq 1 \) and hence, \( \leq \) is strictly included in \( A \).

Proof: We shall prove the proposition by induction on \( c \).

Indeed, assume that \( x \notin X \) and take \( \alpha \in A \). If \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) such that \( |\alpha_n| < \varepsilon \) for all \( n \geq N \). Consider a continuous function \( f \in C([0,1]) \) such that \( 0 \leq f \leq 1 \), \( f(x) = 0 \) and \( f(a_i) = 1 \) for every \( 1 \leq i \leq N \); we then have:

\[ \|\alpha_x\| \leq \| (1 - f) \alpha \| < \varepsilon. \]

Let \( Y = \{ b_n \}_{n \in \mathbb{N}} \) be another dense countable subset of \([0,1]\) and denote by \( B \) the associated \( C([0,1]) \)-algebra whose underlying algebra is \( C_0(\mathbb{N}) \).

Then, if \( X \cap Y = \emptyset \), the previous remark implies that for every \( x \in [0,1] \), \( A_x \cap B_x = 0 \) and hence, \( \mathcal{J}(A, B) = A \cap B \). What we therefore need to prove is that the ideal \( \mathcal{I}(A, B) \) is strictly included in \( A \cap B \).

Let us fix two sequences \( \alpha \in A \) and \( \beta \in B \) whose terms are all non zero and suppose that \( \alpha \otimes \beta \) admits a decomposition \( \sum_{1 \leq i \leq k} \left[ \left( f_i \alpha^i \right) \otimes \beta^i - \alpha^i \otimes (f_i b^i) \right] \) in \( A \cap B \). Then for every \( n, m \in \mathbb{N} \),

\[ \alpha_n \beta_m = \sum_{1 \leq i \leq k} \alpha_n^i \beta_m^i \left[ f_i(a_n) - f_i(b_m) \right]. \]

Now, if we set \( \phi_i(a_n) = \alpha_n^i / \alpha_n \) and \( \psi_i(b_n) = \beta_n^i / \beta_n \) for \( 1 \leq i \leq k \) and \( n \in \mathbb{N} \), this equality means that for all \( (x, y) \in X \times Y \),

\[ 1 = \sum_{1 \leq i \leq k} \psi_i(x) \varphi_i(y) \left[ f_i(x) - f_i(y) \right]. \]

But this is impossible because of the following proposition:

**PROPOSITION 3.3** Let \( X \) and \( Y \) be two dense subsets of the interval \([0,1]\) and let \( n \) be a non negative integer.

Given continuous functions \( f_i \) on \([0,1]\) and numerical functions \( \psi_i : X \to \mathbb{C} \) and \( \varphi_i : Y \to \mathbb{C} \) for \( 1 \leq i \leq n \), if there exists a constant \( c \in \mathbb{C} \) such that

\[ \forall (x, y) \in X \times Y, \quad \sum_{1 \leq i \leq n} \psi_i(x) \varphi_i(y) \left[ f_i(x) - f_i(y) \right] = c \]

then \( c = 0 \).

**Proof:** We shall prove the proposition by induction on \( n \).

If \( n = 0 \), the result is trivial. Take therefore \( n > 0 \) and assume that the proposition is true for any \( k < n \).

Suppose then that the subsets \( X \) and \( Y \) of \([0,1]\), the functions \( f_i, \psi_i \) and \( \varphi_i \), \( 1 \leq i \leq n \), satisfy the hypothesis of the proposition for the constant \( c \).

For \( x \in X \), let \( p(x) \leq n \) be the dimension of the vector space generated in \( \mathbb{C}^n \) by the \( \left\{ \varphi_i(y) [f_i(x) - f_i(y)] \right\}_{1 \leq i \leq n} \), \( y \in Y \).

If \( p(x) < n \), there exists a subset \( F(x) \subset \{1, \ldots, n\} \) of cardinal \( p(x) \) such that for every \( j \notin F(x) \):
\[ \varphi_j(y)[f_j(x) - f_j(y)] = \sum_{i \in F(x)} \lambda_i^j(x) \varphi_i(y)[f_i(x) - f_i(y)] \text{ for all } y \in Y, \]

where the \( \lambda_i^j(x) \in \mathbb{C} \) are given by the Cramer formulas. As a consequence,

\[ \sum_{i \in F(x)} \left( \psi_i(x) + \sum_{j \not\in F(x)} [\lambda_i^j(x) \psi_j(x)] \right) \varphi_i(y)[f_i(x) - f_j(y)] = c. \]

Now, if \( p(x) < n \) for every \( x \in X \), there exists a subset \( F \subset \{1, \ldots, n\} \) of cardinal \( p < n \) such that the interior of the closure of the set of those \( x \) for which \( F(x) = F \) is not empty and contains therefore a closed interval homeomorphic to \([0,1]\). The induction hypothesis for \( k = p \) implies that \( c = 0 \).

Assume on the other hand that \( x_0 \in X \) verifies \( p(x_0) = n \). We may then find \( y_1, \ldots, y_n \) in \( Y \) such that if we set

\[ a_{i,j}(x) = \varphi_i(y_j)[f_i(x) - f_i(y_j)], \]

the matrix \( (a_{i,j}(x_0)) \) is invertible. There exists therefore a closed connected neighborhood \( I \) of \( x_0 \) on which the matrix \( (a_{i,j}(x)) \) remains invertible.

But for each \( 1 \leq j \leq n \), \( \sum a_{i,j}(x) \psi_i(x) = c \) and therefore the \( \psi_i(x) \) extend by the Cramer formulas to continuous functions on the closed interval \( I \).

For \( y \in Y \cap I \), let \( q(y) \) denote the dimension of the vector space generated in \( \mathbb{C}^n \) by the \( \left( \psi_i(x)[f_i(x) - f_i(y)] \right)_{1 \leq i \leq n}, \ x \in X \cap I \).

If \( q(y) < n \) for every \( y \), then the induction hypothesis implies \( c = 0 \). But if there exists \( y_0 \) such that \( q(y_0) = n \), we may find an interval \( J \subset I \) homeomorphic to \([0,1]\) on which the \( \varphi_i \) extend to continuous functions; evaluating the starting formula at a point \( (x,x) \in J \times J \), we get \( c = 0 \). \( \square \)

## 4 The associativity

Given three \( C(X) \)-algebras \( A_1, A_2 \) and \( A_3 \), we deduce from [3] corollaire 3.17:

\[ [(A_1 \otimes_{C(X)} A_2) \otimes_{C(X)} A_3]_x = (A_1 \otimes_{C(X)} A_2)_x \otimes_{\text{max}} (A_3)_x = (A_1)_x \otimes_{\text{max}} (A_2)_x \otimes_{\text{max}} (A_3)_x, \]

which implies the associativity of the tensor product \( \otimes_{C(X)} \) over \( C(X) \).

On the contrary, the minimal tensor product \( \otimes_{C(X)} \) over \( C(X) \) is not in general associative. Indeed, Kirchberg and Wassermann have shown in [11] that if \( \tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \) is the Alexandroff compactification of \( \mathbb{N} \), there exist separable continuous fields \( A \) and \( B \) such that

\[ (A \otimes_{C(\tilde{\mathbb{N}})} B)_\infty \neq A_\infty \otimes_{\text{min}} B_\infty. \]

If we now endow the \( \mathcal{C}^* \)-algebra \( D = \mathbb{C} \) with the \( C(\tilde{\mathbb{N}}) \)-algebra structure defined by \( f.a = f(\infty)a \), then for all \( C(\tilde{\mathbb{N}}) \)-algebra \( D' \), we have
\[ D^m_{\otimes C(N)} D'_n = \begin{cases} 0 & \text{if } n \text{ is finite,} \\ (D')_\infty & \text{if } n = \infty. \end{cases} \]

Therefore, \((A^m_{\otimes C(N)} B^m_{\otimes C(N)} D)_{\infty} \simeq (A^m_{\otimes C(N)} B)_{\infty}\) whereas \([A^m_{\otimes C(N)} (B^m_{\otimes C(N)} D)]_{\infty}\) is isomorphic to \(A_{\infty \otimes \min} B_{\infty}\).

However, in the case of (separable) continuous fields, we can deduce the associativity of \(\otimes_{C(X)}\) from the following proposition:

**PROPOSITION 4.1** Let \(A\) and \(B\) be two \((X)\)-algebras.

Assume \(\pi\) is a field of faithful representations of \(A\) in the Hilbert \((X)\)-module \(E\), then the morphism \(a \otimes b \to \pi(a) \otimes b\) induces a faithful \((X)\)-linear representation of \(A_{\otimes C(X)} B\) in the Hilbert \(B\)-module \(E \otimes C(X) B\).

**Proof:** Notice that for all \(x \in X\), we have \((E \otimes C(X) B) \otimes_B B_x = E_x \otimes B_x\).

Now, as \(B\) maps injectively in \(B_d = \oplus_{x \in X} B_x\), \(L_B(E \otimes C(X) B)\) maps injectively in \(\oplus_{x \in X} L_{B_x}(E_x \otimes B_x) \subset L_{B_d}(E \otimes C(X) B \otimes_B B_d)\) and therefore if \(\alpha \in A \otimes B\), we have \(\|((\pi \otimes id)(\alpha))\| = \sup_{x \in X} \|((\pi_x \otimes id)(\alpha_x))\| = \|\alpha\|_m\). \(\square\)

Accordingly, if for \(1 \leq i \leq 3\), \(A_i\) is a separable continuous field of \((X)\)-algebras over \(X\) which admits a field of faithful representations in the \((X)\)-module \(E_i\), the \((X)\)-representations of \((A_1^m \otimes_{C(X)} A_2^m \otimes_{C(X)} A_3^m)\) and \((A_1^m \otimes_{C(X)} (A_2^m \otimes_{C(X)} A_3^m))\) in the Hilbert \((X)\)-module \((E_1 \otimes_{C(X)} E_2) \otimes_{C(X)} E_3\) are faithful, and hence the maps \(A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \to (A_1^m \otimes_{C(X)} A_2^m \otimes_{C(X)} A_3^m)\) and \(A_1 \otimes_{\min} A_2 \otimes_{\min} A_3 \to A_1^m \otimes_{C(X)} (A_2^m \otimes_{C(X)} A_3^m)\) have the same kernel.

**References**


Etienne Blanchard, Mathematisches Institut, Universität Heidelberg
Im Neuenheimer Feld 288, D – 69120 Heidelberg
e-mail: blanchar@mathi.uni-heidelberg.de