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# Comparison of Simulations of Convective Flows

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**Abstract.** We show that a single particle distribution for the D2Q13 lattice Boltzmann scheme can simulate coupled effects involving advection and diffusion of velocity and temperature. We consider various test cases: non-linear waves with periodic boundary conditions, a test case with buoyancy, propagation of transverse waves, Couette and Poiseuille flows. We test various boundary conditions and propose to mix bounce-back and anti-bounce-back numerical boundary conditions to take into account velocity and temperature Dirichlet conditions. We present also first results for the de Vahl Davis heated cavity. Our results are compared with the coupled D2Q9-D2Q5 lattice Boltzmann approach for the Boussinesq system and with an elementary finite differences solver for the compressible Navier-Stokes equations.

**Keywords:** bounce-back, natural convection, adiabatic wall, de Vahl Davis.

**AMS classification:** 6505, 76N15, 80A20, 82C20.

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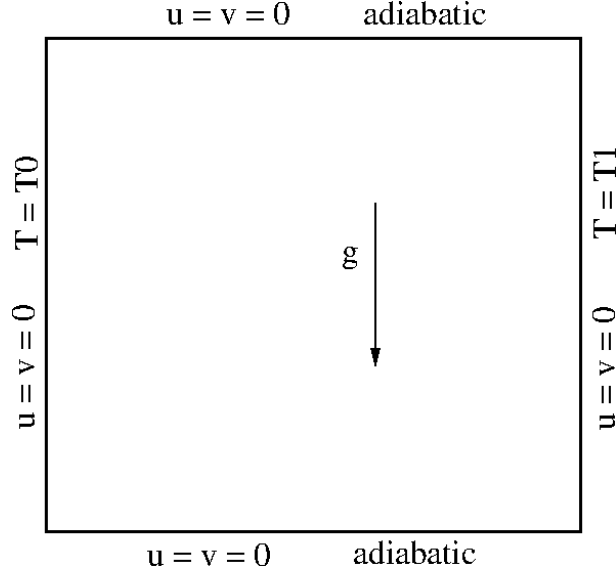
## Introduction

- Lattice Boltzmann schemes have proven their efficiency for the computation of quasi-incompressible flows. We refer *e.g.* to [2, 7, 9] among others. In these cases, the physical conservations of mass and momentum are implemented in the framework of lattice Boltzmann schemes. When compressible effects are taken into account, it is necessary to add the conservation of energy. A classical approach is to begin with weakly compressible effects that can be modelled with the so-called Boussinesq approximation. In that case, the incompressibility condition remains a good approximation and coupled effects between conservations of momentum and energy are taken into account with a precise thermodynamical analysis. We refer to Landau [12] or Batchelor [1] for the derivation of the Boussinesq approximation. The implementation of the Boussinesq approximation is possible with the lattice Boltzmann approach with the introduction of two particle distributions. This idea has been also proposed in the context of finite volumes by the team of Perthame [8], and with lattice Boltzmann schemes by Shan and Chen [18], Dellar [3], Mezrhab *et al* [14] and Wang *et al* [20] among others.

- In this contribution, we study a direct approximation of the compressible Navier Stokes equations with a lattice Boltzmann scheme using a single particle distribution. A first tentative [10] has shown that for a critical value of the Prandtl number, the thermal wave and the viscous one merge together, the physics is badly represented and an instability occurs in general. In consequence, no satisfying compressible flows have been obtained with this direct numerical modelling. In a second tentative [11], we have analyzed with great details several lattice Boltzmann schemes with four conservation laws in two space dimensions. With a adequate fitting of the parameters of the scheme, it is possible to enlarge the zone in the spectral space where the thermal and viscous waves remain decoupled. Moreover, these parameters guarantee also the isotropy of the acoustic waves. Our objective is to enlarge the domain of validity of our previous study: incorporate the treatment of boundary conditions with rigid walls with a given temperature or adiabatic boundaries, study several couplings between velocity and temperature for elementary Couette and Poiseuille flows, study the possibility of Dirichlet and Neumann boundary conditions. Finally, our objective is the simulation of the de Vahl Davis test case [19] described in Figure 1.

- The outlook of the article is the following. In Section 1, we recall fundamental aspects of the coupled D2Q9-D2Q5 lattice Boltzmann approach. We present our actual choices for the implementation of the lattice Boltzmann approach with the D2Q13 stencil and to treat all the physical fields with single particle distribution and the D2Q13 scheme. In Section 3, we develop a very elementary finite-difference approach for the compressible Navier-Stokes equations. With this tool, we can compare our new D2Q13 approach with a classical reference. In Section 4, we consider a simple test case for non-linear waves. We study the buoyancy in Section 5, the propagation of transverse waves in Section 6, the simulation of Couette flows in Section 7 and Poiseuille flows in Section 8. In Section 9,

we consider a test case to take into account various temperature and flux-type boundary conditions. First results for the de Vahl Davis heated cavity are presented in Section 10.



**Figure 1.** De Vahl Davis test case for natural convection

### 1) Coupled D2Q9-D2Q5 lattice Boltzmann scheme

- The Boussinesq approximation of the compressible Navier-Stokes equations can be written as a system of coupled partial differential equations. The unknowns are the velocity  $u$ , the temperature  $T$  and the pressure  $p$ . The parameters are the shear viscosity  $\nu$ , the temperature dissipation rate  $\kappa$ , the thermal expansion coefficient  $\beta$  and  $g$  the Earth's gravity. The buoyancy term  $(1 - \beta(T - T_0))g$  is a source term for the momentum equation and the velocity field  $u$  directly imposes strong constraints for the transport of temperature. The equations of the Boussinesq system are

$$(1) \quad \begin{cases} \operatorname{div} u = 0, \\ \frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p - \nu \Delta u = (1 - \beta(T - T_0))g, \\ \frac{\partial T}{\partial t} + u \cdot \nabla T - \kappa \Delta T = 0. \end{cases}$$

The Rayleigh number is defined according to

$$(2) \quad R_a \equiv \frac{|g| \beta \Delta T L^3}{\nu \kappa}.$$

- A difficult stationary test case is the computation of the velocity and temperature fields for  $R_a = 10^6$ . The references are the original contribution of de Vahl Davis [19], the Le Quéré [13] and the associated workshop in the 2000's with various Navier-Stokes solvers, the introduction of the D2Q9-D2Q5 coupled approximation by Mezrhab *et al.* [14] and the very precise results of Wang *et al* [20] with the same approach.

- Recall that the discrete velocities of a D2Q5 lattice Boltzmann scheme follow the axis of coordinates:

$$(3) \quad v_j \in \{(0, 0), (1, 0), (0, 1), (-1, 0), (0, -1)\}, \quad 0 \leq j \leq 4.$$

For a D2Q9 scheme we add to the previous D2Q5 velocities (3) the four ones along the diagonals:

$$(4) \quad v_j \in \{(1, 1), (-1, 1), (-1, -1), (1, -1)\}, \quad 5 \leq j \leq 8.$$

The flow is simulated with a D2Q9 lattice Boltzmann scheme with 3 conserved moments, the density and the two components of the momentum:

$$(5) \quad \rho \equiv \sum_{j=0}^8 f_j, \quad (j_x, j_y) \equiv \sum_{j=0}^8 v_j f_j.$$

The six other moments of the fluid are presented in the reference [9]. The equilibrium values for the moments of order two have to take into account the compressible effects:

$$(6) \quad E^{\text{eq}} = \alpha \rho + 3 \frac{j_x^2 + j_y^2}{\rho}, \quad XX^{\text{eq}} = \frac{j_x^2 - j_y^2}{\rho}, \quad XY^{\text{eq}} = \frac{j_x j_y}{\rho}.$$

The equilibrium values of the three last moments have no influence on the physical properties. The sound velocity  $c_s$ , the shear viscosity  $\mu$  and the bulk viscosity  $\zeta$  are given from the previous equilibria according to

$$(7) \quad c_s = \sqrt{\frac{4 + \alpha}{6}}, \quad \mu = \frac{1}{3} \left( \frac{1}{s_{XX}} - \frac{1}{2} \right), \quad \zeta = -\alpha \left( \frac{1}{s_E} - \frac{1}{2} \right).$$

- The temperature is simulated with a simple D2Q5 scheme with only one conserved moment

$$(8) \quad T \equiv \sum_{j=0}^4 g_j.$$

The other nontrivial equilibrium values follow the relations

$$(9) \quad E^{\text{eq}} = \beta \rho, \quad j_x^{\text{eq}} = \rho V_x, \quad j_y^{\text{eq}} = \rho V_y.$$

The diffusion coefficient  $\kappa$  is easy to identify:

$$(10) \quad \kappa = \frac{\beta + 4}{10} \left( \frac{1}{s_E} - \frac{1}{2} \right).$$

This D2Q5 model as defined does not satisfy the Galilean invariance with respect to advection at uniform speed  $\{V_x, V_y\}$  (see Qian and Zhou [16]); the equivalent equation for the D2Q5 scalar scheme is equal to

$$\frac{\partial T}{\partial t} + V_x \frac{\partial T}{\partial x} + V_y \frac{\partial T}{\partial y} - \kappa \Delta T + \left( \frac{1}{s_E} - \frac{1}{2} \right) \left( V_x^2 \frac{\partial^2 T}{\partial x^2} + 2 V_x V_y \frac{\partial^2 T}{\partial x \partial y} + V_y^2 \frac{\partial^2 T}{\partial y^2} \right) = 0$$

and can be easily identified with the methods developed *e.g.* in [5].

## 2) Compressible D2Q13 lattice Boltzmann scheme

• The stencil of the D2Q13 lattice Boltzmann scheme is built (see *e.g.* [10, 11]) on the D2Q9 scheme with the following complementary velocity set:

$$(11) \quad v_j \in \{(2, 0), (0, 2), (-2, 0), (0, -2)\}, \quad 9 \leq j \leq 12.$$

A family of 13 orthogonal moments are generated by the condition

$$(12) \quad m_k = p_k(v_j), \quad 0 \leq j, k \leq 8,$$

with the following set  $\{p_k\}$  of polynomials

$$(13) \quad \left\{ \begin{array}{ll} \text{scalars} & \begin{array}{l} \rho \\ E \\ \epsilon \\ \varpi \end{array} \\ \text{vectors} & \begin{array}{l} j_x \\ j_y \\ q_x \\ q_y \\ r_x \\ r_y \end{array} \\ \text{tensors} & \begin{array}{l} XX \\ XY \\ XX_e \end{array} \end{array} \right. \begin{array}{l} 1 \\ -28 + 13(x^2 + y^2) \\ 140 + (x^2 + y^2)(-361/2 + 77(x^2 + y^2)/2) \\ -12 + (x^2 + y^2)(\frac{581}{12} + (x^2 + y^2)(-\frac{273}{8} + \frac{137}{24}(x^2 + y^2))) \\ x \\ y \\ x(3 + x^2 + y^2) \\ y(3 + x^2 + y^2) \\ x(\frac{101}{6} + (x^2 + y^2)(-\frac{63}{4} + \frac{35}{12}(x^2 + y^2))) \\ y(\frac{101}{6} + (x^2 + y^2)(-\frac{63}{4} + \frac{35}{12}(x^2 + y^2))) \\ x^2 - y^2 \\ x y \\ (x^2 - y^2)(-\frac{65}{12} + \frac{17}{12}(x^2 + y^2)). \end{array}$$

• The collisions conserve two scalars  $\rho$  and  $E$  and two vector components  $j_x$  and  $j_y$ . The equilibrium values of the other moments and the relaxation rates are constrained by the result of a linearized analysis of the four hydrodynamic modes. The four modes show isotropic behaviour for their attenuation and propagation velocity meaning that Galilean invariance is achieved. In this contribution, we search the equilibrium values of the relevant moments with the following framework:

$$(14) \quad \left\{ \begin{array}{ll} q_x^{\text{eq}} = j_x(c_1 + h_1\rho + k_1E), & r_x^{\text{eq}} = j_x(c_2 + h_2\rho + k_2E), \\ \epsilon^{\text{eq}} = c_{\epsilon\rho}\rho + c_{\epsilon E}E, & \varpi^{\text{eq}} = c_{\varpi\rho}\rho + c_{\varpi E}E, \\ XX^{\text{eq}} = \frac{j_x^2 - j_y^2}{\rho}, & XY^{\text{eq}} = \frac{j_x j_y}{\rho}, \quad XX_e^{\text{eq}} = 0. \end{array} \right.$$

• With a classical notation for the Hénon coefficient:

$$(15) \quad \sigma_i \equiv \frac{1}{s_i} - \frac{1}{2},$$

we can recover the physical waves : two acoustics, one transverse and one longitudinal diffusion:

$$(16) \quad \left\{ \begin{array}{l} h_1 = \frac{17}{26} - \frac{c_1}{2} - \frac{E_0}{13}, \quad k_1 = \frac{2}{13}, \\ h_2 = -\left(\frac{39}{2} + \frac{13}{2}c_1 + E_0\right) k_2, \\ -\frac{7}{624}(13c_1 + 95 + 2E_0) \frac{874481 + 459459c_1 - 103428c_{\epsilon E} + 70686E_0}{114404 + 51051c_1 - 11492c_{\epsilon E} + 7854E_0}. \end{array} \right.$$

In order to enforce isotropy at second order around a null velocity, we have to set

$$(17) \quad \begin{cases} \sigma_{qx} = -\frac{1309}{2} \frac{\sigma_{XX}(13 c_1 + 95 + 2 E_0)}{114404 + 51051 c_1 - 11492 c_{\epsilon E} + 7854 E_0}, \\ c_{\epsilon\rho} = 140 + 28 c_{\epsilon E}, \\ \quad + \frac{(13 c_1 + 95 + 2 E_0)(114404 + 51051 c_1 - 11492 c_{\epsilon E} + 7854 E_0)}{22984 Pr}, \\ c_2 = -\frac{65}{24} - \frac{21}{8}(c_1 + h_1 + k_1 E_0) - k_2 E_0 - h_2. \end{cases}$$

These constraints leave as independent parameters :  $Pr$ ,  $E_0$ ,  $c_1$ ,  $k_2$ ,  $c_{\epsilon E}$ ,  $c_{\varpi\rho}$ ,  $c_{\varpi E}$  and the relaxation rates  $s_{XX}$ ,  $s_{rx}$ ,  $s_\epsilon$ ,  $s_\varpi$  and  $s_{XX\epsilon}$ .

### 3) Navier Stokes solver for a compressible gas

- This approach starts from the conservation equations of mass and momentum:

$$(18) \quad \begin{cases} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho v_x + \frac{\partial}{\partial y}\rho v_y = 0 \\ \frac{\partial}{\partial t}\rho v_x + \frac{\partial}{\partial x}\rho v_x v_x + \frac{\partial}{\partial y}\rho v_x v_y + \frac{\partial}{\partial x}P - \nu \Delta v_x - \xi \frac{\partial}{\partial x}(\text{div } v_x) = 0 \\ \frac{\partial}{\partial t}\rho v_y + \frac{\partial}{\partial x}\rho v_x v_y + \frac{\partial}{\partial y}\rho v_y v_y + \frac{\partial}{\partial y}P - \nu \Delta v_y - \xi \frac{\partial}{\partial y}(\text{div } v_y) = 0. \end{cases}$$

We add also the conservation of total energy. We assume the fluid is a perfect gas, then the pressure  $P$  is given according to

$$(19) \quad P = \rho R T$$

and the internal energy per unit mass  $e$  is related to  $T$  by

$$(20) \quad e = \frac{RT}{\gamma - 1}.$$

Then the evolution equation for the internal energy takes the form

$$(21) \quad \begin{cases} \frac{\partial}{\partial t}\rho e + \frac{\partial}{\partial x}\rho e v_x + \frac{\partial}{\partial y}\rho e v_y + P \left[ \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y \right] - \kappa \left[ \frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T \right] \\ - \nu \left[ \left( \frac{\partial}{\partial x}u_x - \frac{\partial}{\partial y}u_y \right)^2 + \left( \frac{\partial}{\partial x}u_y + \frac{\partial}{\partial y}u_x \right)^2 \right] - \xi \left( \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y \right)^2 = 0. \end{cases}$$

- A linearized analysis gives the propagation and damping of the four hydrodynamic modes. We deduce an algebraic expression for the sound velocity  $c_s$ , the relaxation  $\nu_\tau$  of the transverse mode, the relaxation  $\nu_{\text{diff}}$  of the diffusive mode, and the damping  $\nu_{\text{acous}}$  of the sound modes:

$$(22) \quad c_s = \sqrt{\gamma RT}, \quad \nu_\tau = \nu, \quad \nu_{\text{diff}} = \kappa \frac{\gamma - 1}{R\gamma}, \quad \nu_{\text{acous}} = \frac{1}{2}(\nu + \xi) + \gamma \frac{(\gamma - 1)^2}{2R\gamma}.$$

The non-linear terms allow to show that a uniform advection speed  $\{V_x, V_y\}$  leads to phase shifts compatible with Galilean invariance.

- The model can be approximately simulated with simple finite difference expressions for the space derivatives. We have developed a compressible Navier-Stokes solver for the

numerical resolution of the mathematical model (18) - (21). We use a cell vertex approach (with the nomenclature of Roache [17]). All the differential operators are discretized with centered finite differences. The discrete evolution in times is obtained with an elementary forward Euler first order explicit scheme. The Dirichlet boundary conditions for velocity and temperature are implemented in a clear way by forcing the given value on the boundary node vertex. For the adiabatic wall where  $\frac{\partial T}{\partial n}$  is null, a Neumann homogeneous boundary condition is enforced with mirror techniques described in the classical reference [17].

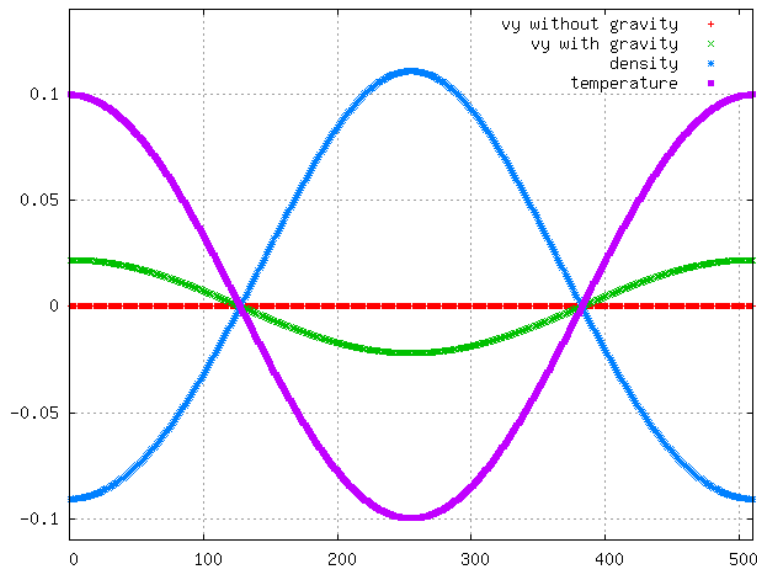
#### 4) A simple test case

- This test case has been studied in our contribution [11]. The domain is a  $N_x \times N_y$  rectangle with periodic boundary conditions. This test case is error-free as far as boundary conditions are concerned. The initial condition is a fluid at rest:  $V_x = V_y = 0$ . The initial temperature  $T(x, y) = T_0 + \delta T_0 \cos k \cdot x$  is associated with a wave number  $k = 2\pi K/N_x$ . Then density, pressure or energy are such that no acoustic wave is excited. This is possible with the following conditions:

$$(23) \quad \begin{cases} \text{For D2Q9-D2Q5 :} & \rho = 1, \\ \text{For D2Q13 :} & \rho = 1 - 28 (T(x, y) - T_0), \\ \text{For Navier Stokes :} & P = RT_0, \rho E = \frac{P}{\gamma-1} + \frac{\rho(V_x^2 + V_y^2)}{2}. \end{cases}$$

One verifies that  $T(x, y, t)$  relaxes exponentially in time.

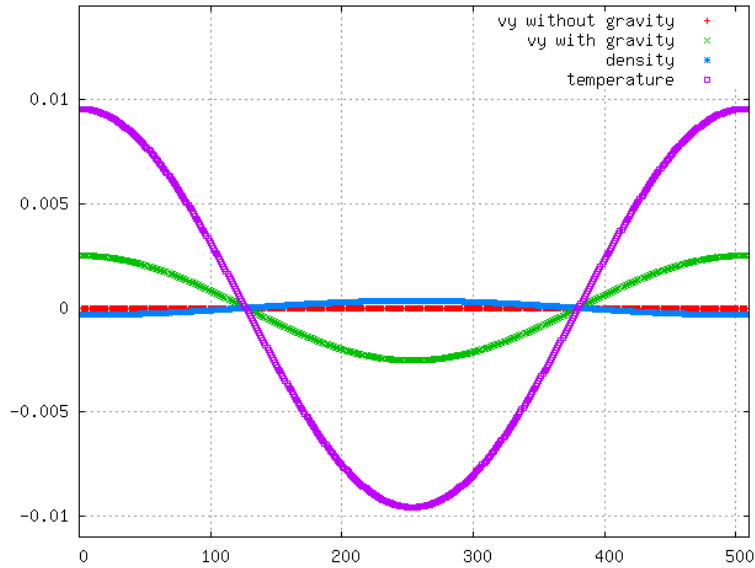
#### 5) Buoyancy



**Figure 2.** Buoyancy flow with the compressible Navier-Stokes solver. The nonlinear exchanges between temperature and density are not affected by the gravity.



- For the first scheme D2Q9-D2Q5 we simulate buoyancy by adding a vertical force ( $V_y$ ) proportional to  $T - T_0$ . For the D2Q13 scheme and the direct approach of Navier-Stokes equations with finite differences, we add vertical force in the  $V_y$  momentum equation proportional to  $\rho - \rho_0$ . Then the vertical speed increases approximately linearly with time and there is essentially no horizontal velocity. With the Navier-Stokes solver, we use a domain composed by 510 mesh points in width, and periodic in height. The temperature is periodic relative to the  $x$  direction. With the D2Q13 lattice Boltzmann solver, we use the same domain as previously: a domain of 510 meshes in width and periodic in height, with an (initial) temperature periodic in  $x$  (see Fig. 3).



**Figure 3.** Buoyancy flow with the D2Q13 direct lattice Boltzmann solver. The non-linear exchanges between temperature and density are not affected by the gravity.

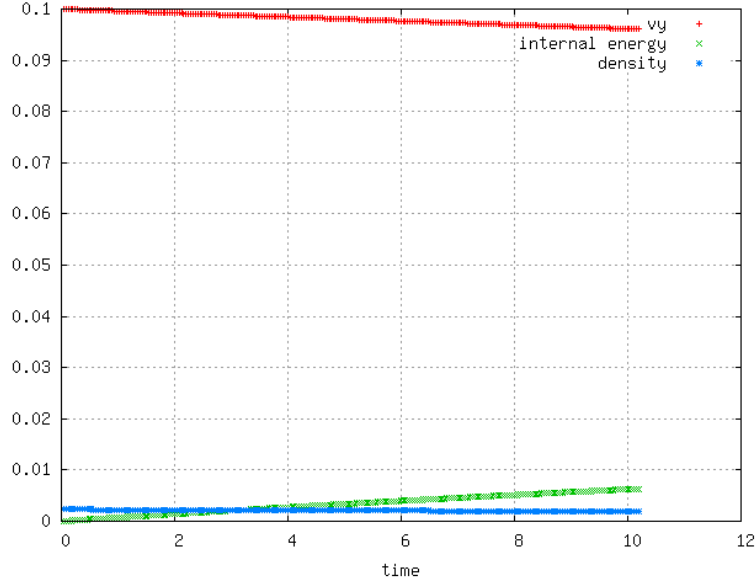
## 6) Transverse waves

- With the D2Q13 stencil, the non linear behaviour for transverse waves is operating as follows. The initial conditions  $v_y(x, y, 0) \equiv V_{y0} \cos kx$  leads to density waves of wave vector  $2k$ . We modify the initial conditions by  $\rho(x, y, 0) = \rho_0 + a v_y(x, y, 0)^2$ . We measure the following global aggregates relative to time:

$$(24) \quad \tilde{V}_y(t) = \sum_x v_y \cos(kx), \quad \tilde{\rho}(t) = \sum_x \rho \cos(2kx), \quad \tilde{E}(t) = \sum_x E \cos(2kx).$$

The typical result for D2Q13 is summarized in Fig. 4: the growth of  $\tilde{E}$  in time is proportional to  $\nu k^2 V_y^2$ .

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**Figure 4.** Transverse waves with the D2Q13 direct lattice Boltzmann solver. The growth of  $\tilde{E}$  in time is proportional to  $\nu k^2 V_y^2$ .

- The interpretation of this evolution can be stated as follows. The linearized equivalent equations at order 1 with space derivatives can be written in matrix form

$$(25) \quad \begin{vmatrix} \partial_t & \partial_r & 0 \\ (\frac{14}{13} + \frac{1}{2}V^2) \partial_r & \partial_t & \frac{1}{26}\partial_r \\ 0 & \frac{1}{2}(39 + 13c_1 + 2E_0) \partial_r & \partial_t \end{vmatrix} = 0.$$

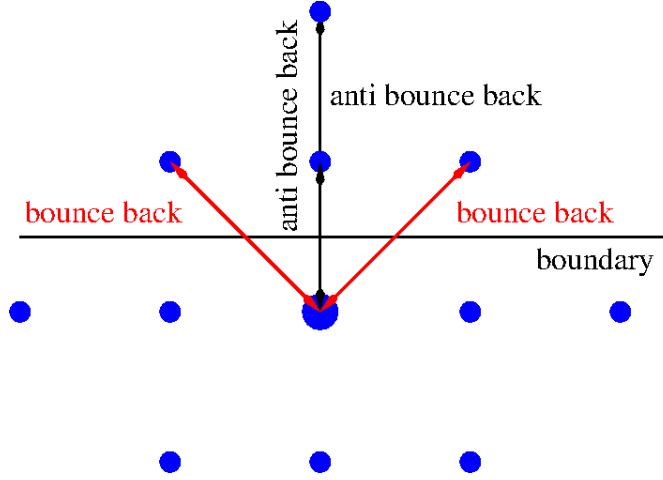
Without advective velocity, the diffusive mode is  $(1, 0, -28)^t$  *id est*  $E = -28\rho$ . With a transverse velocity  $V$ , the diffusive mode is equal to  $(1, 0, -28 - 13V^2)^t$ , *id est*  $E = -28\tilde{\rho}$  with a density  $\rho$  replaced by  $\tilde{\rho} \equiv 1 + \frac{13}{28}V^2$ .

### 7) Couette flows

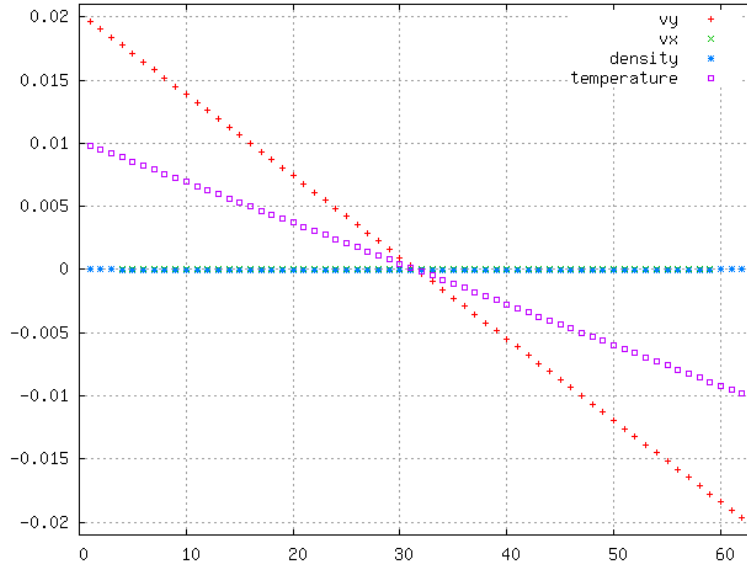
- Typical boundary conditions for Couette flows with the D2Q9-D2Q5 scheme are stated as follows. For  $x = 1$  and  $x = N_x$  the velocity is known:  $V_x = 0$  and  $V_y$  is given. This type of boundary condition is classically achieved by a “bounce-back” condition. Assuming a zero value for the mean temperature, for  $x = 1$  the temperature is imposed:  $T = +\Delta T$  and when  $x = N_x$  it has the opposite sign:  $T = -\Delta T$  for  $x = N_x$ . This boundary condition is achieved by an “anti-bounce-back”. In consequence, the way we implement the boundary conditions is not straightforward. For the unit velocities with non-zero component parallel to the boundary, we use a bounce-back boundary condition. For the other velocities, an “anti-bounce-back” is implemented. We consider for example:

$$(26) \quad f_1 + f_3 = 2 (p_\rho \rho + p_{XX} XX + p_E E)$$

with  $E$  and  $XX$  imposed and  $\rho$  estimated by extrapolation from values measured in the fluid.



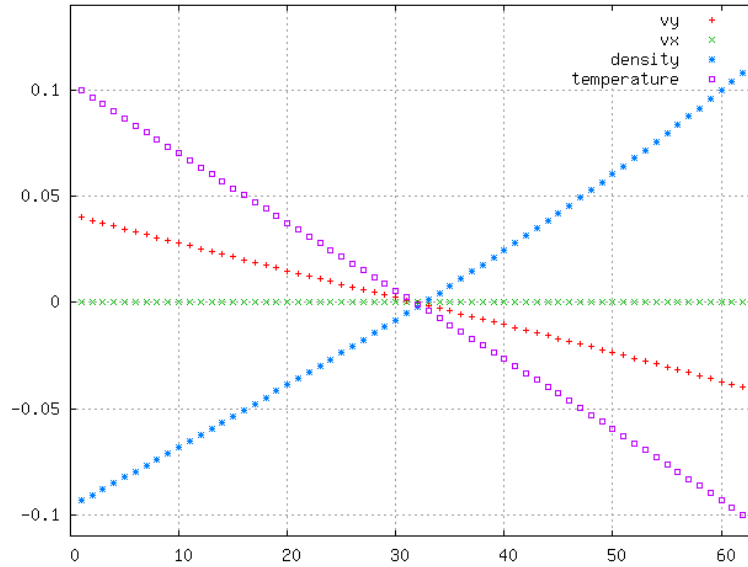
**Figure 5.** Mixed “bounce-back” and “anti-bounce-back” boundary conditions for a flow simulated with the D2Q13 lattice Boltzmann scheme. In this case, both velocity and temperature are imposed.



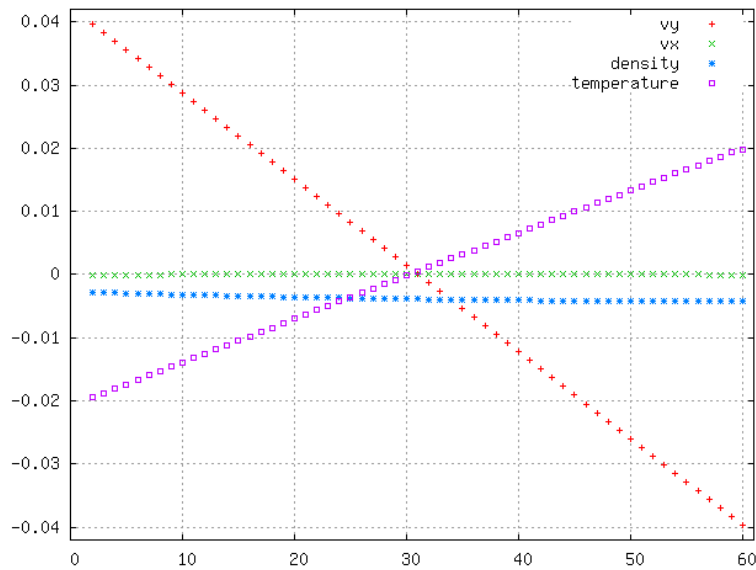
**Figure 6.** Couette flow with the D2Q9-D2Q5 coupled scheme. Result for the temperature field without motion of the lateral plates.

- With the coupled D2Q9-D2Q5 scheme, the density is unchanged (see Fig. 6). For the discretization of the compressible Navier Stokes equations, the pressure remains constant. Then, due to the equation of state (19), the variation of density and temperature are coupled. This effect is clearly visible on the Fig. 7 (direct Navier Stokes solver) With the D2Q13 lattice Boltzmann scheme, the internal energy  $e$  (proportional to the temperature) can be recovered thanks to the relation  $e = E - \frac{13}{2}(V_x^2 + V_y^2)$  as displayed on Fig. 8.

## COMPARISON OF SIMULATIONS OF CONVECTIVE FLOWS



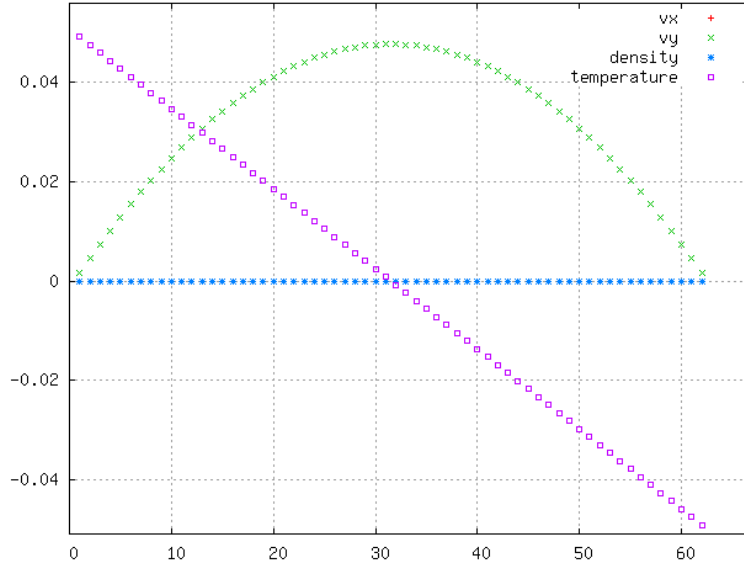
**Figure 7.** Couette flow with a finite difference direct Navier-Stokes solver. Same result for the temperature field without motion of the plates.



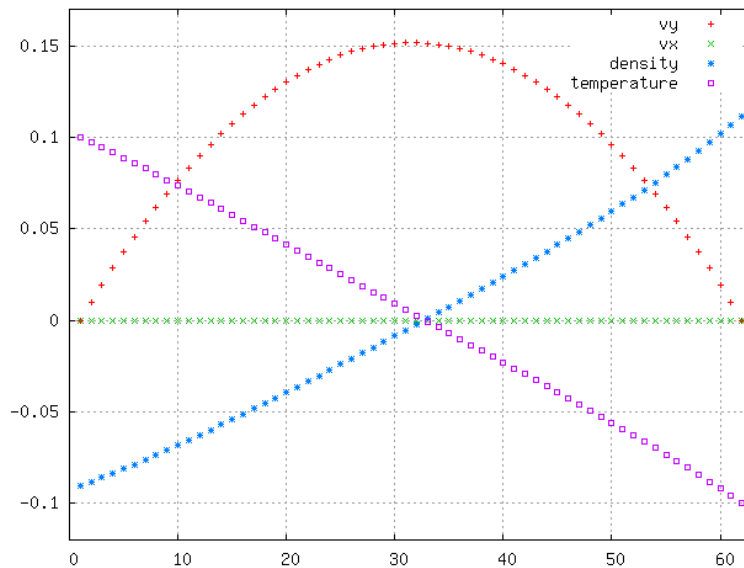
**Figure 8.** Couette flow with the D2Q13 lattice Boltzmann solver. The dashed curves show the “temperature”  $T \equiv E - \frac{13}{2}(V_x^2 + V_y^2)$ . There is no variation of the temperature when no gradient is imposed between the plates.

### 8) Poiseuille flow

- A Poiseuille flow is realized by adding an external term to take into account the gradient of pressure. Then a parabolic velocity profile is obtained as usual. Moreover, we add a Couette-type temperature profile between the lateral plates. In all our simulations, we do not observe any variation of the temperature when no gradient is imposed between the plates. Moreover, when a gradient of temperature is imposed, we observe a regular evolution of the temperature without destruction of the parabolic profile.

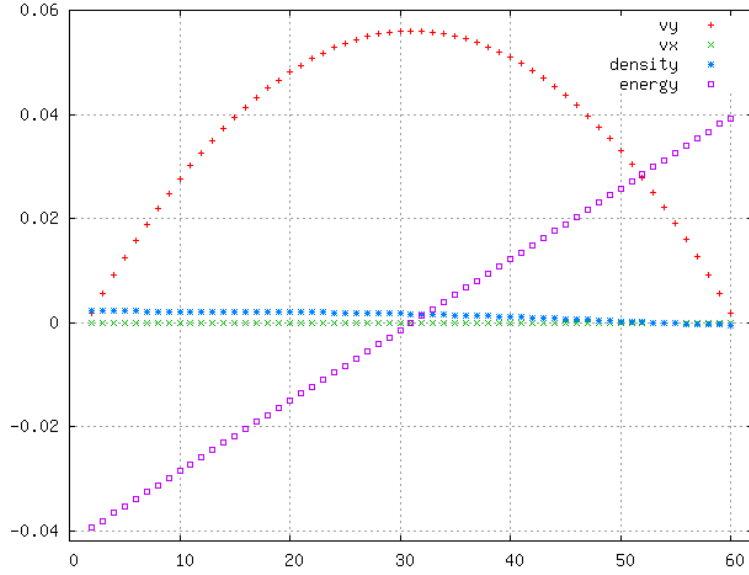


**Figure 9.** Poiseuille flow with the D2Q9-D2Q5 coupled scheme. No variation of the temperature when no gradient is imposed between the plates.



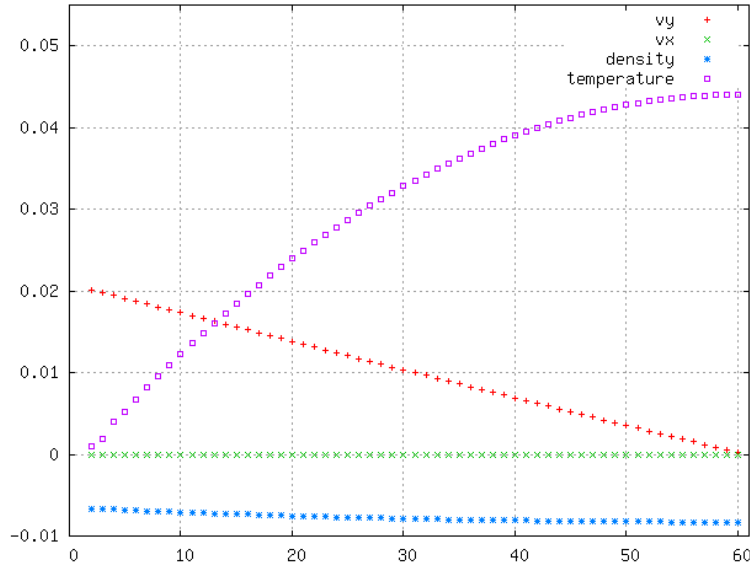
**Figure 10.** Poiseuille flow with a finite difference direct Navier-Stokes solver. No variation of the temperature when no gradient is imposed between the plates.

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**Figure 11.** Poiseuille flow with the D2Q13 lattice Boltzmann scheme. No variation of the temperature when no gradient is imposed between the plates.

### 9) Test of an adiabatic boundary for the D2Q13 scheme



**Figure 12.** Test of an adiabatic boundary with the D2Q13 lattice Boltzmann solver. A uniform source of energy is applied. The  $y$ -velocity is null when the left boundary is fixed. The other fields are unchanged.

- In order to implement correctly a null flux Neumann boundary condition relative to the temperature, we have tested our schemes for a uniform volumic source of energy. A homogeneous temperature given at the left boundary and a homogeneous Neumann condition for temperature at the right boundary. At the left boundary the velocity is

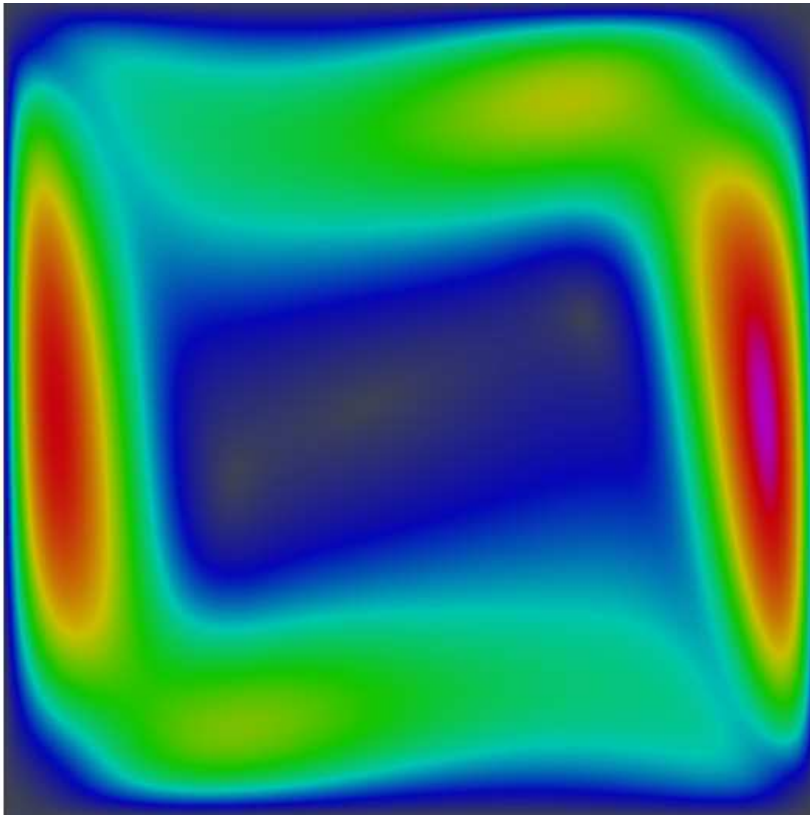
given as homogeneous or inhomogeneous:  $v_y = 0$  or  $v_y = 0.02$ . We impose a null velocity at the right boundary. The solution is a “semi-parabol” and is correctly simulated as described in Fig. 12.

## 10) Thermal test case of de Vahl Davis

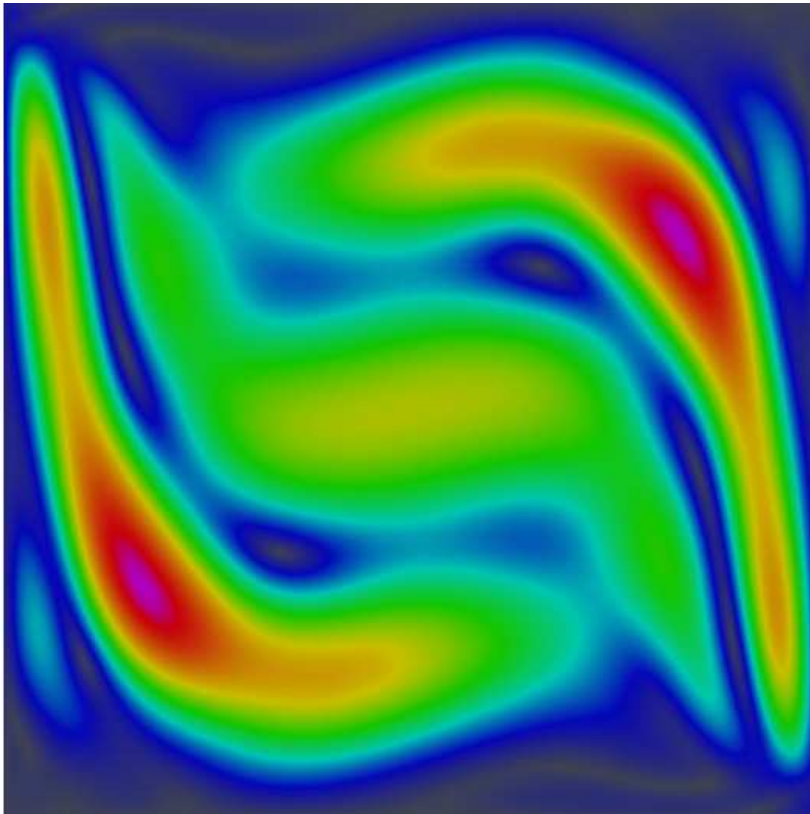
- The de Vahl Davis [19] test has been described in the introduction. We have used a  $187 \times 187$  domain with a Prandtl number equal to 0.71 with the lattice Boltzmann simulations and a grid with  $256 \times 256$  mesh points with the simple Navier-Stokes solver. Our results are summarized in the following table.

Rayleigh	de Vahl Davis	Le Quéré	Mezrhab	D2Q9-D2Q5	Navier Stokes	D2Q13
$10^5$	4.519		4.521	4.521	4.57	4.50
$10^6$	8.800	8.8252	8.824	8.828	8.88	8.73

**Table 1.** Comparison of Nusselt number integrated in all the domain for two Rayleigh numbers.



**Figure 13.** De Vahl Davis thermal test case for natural convection with the D2Q13 direct lattice Boltzmann solver. Rayleigh number =  $10^5$ . Iso-velocity curves for the modulus of the fluid speed. The maximum velocity is  $5.5 \cdot 10^{-3}$ .



**Figure 14.** De Vahl Davis thermal test case for natural convection with the D2Q13 direct lattice Boltzmann solver. Rayleigh number =  $10^5$ . Modulus of the asymmetry of the Fluid Speed. Curves for the departure from center symmetry. The maximum difference of velocity  $|V(x, y) + V(x_0 - x, y_0 - y)|$  is  $0.28 \cdot 10^{-3}$  (5 %).

- We have compared our results with those of de Vahl Davis [19], Le Quéré [13], Mezhrab *et al.* [14], Wang *et al.* [20] with the coupled approach D2Q9-D2Q5 and our simple finite differences Navier-Stokes solver. The results “D2Q13” obtained with a single particle distribution are correct but must be considered as preliminary compared to the other results. Inspection of the thermal and velocity fields obtained with the D2Q9+D2Q5 shows that they are symmetric with respect to the center of the cavity. Similar inspection for the fields obtained either with D2Q13 or the simple compressible Navier-Stokes code used here show disymmetries that increase with the Rayleigh number. A detailed analysis of these asymmetries will be performed later and checked with data obtained with more sophisticated Navier-Stokes codes.

## Conclusion

In this contribution, we have shown that coupled fluid and thermal flows that characterize natural convection can be simulated in two space dimensions with a D2Q13 lattice Boltzmann scheme. We have tested our approach by a progressive complexification of the test cases. The de Vahl Davis test case gives encouraging results when we compare our result



to previous ones obtained with a D2Q9-D2Q5 coupled approach or with a direct simulation of the compressible Navier Stokes equations with finite differences. Nevertheless, complementary studies are necessary to improve this method of simulation.

## References

- [1] G.K. Batchelor. *An Introduction to Fluid Dynamics*, Cambridge University Press, 1967.
- [2] R. Benzi, S. Succi, Vergassola “The lattice Boltzmann equation: theory and applications”, *Physics Reports*, vol. **222**, p. 145-197, 1992.
- [3] P. Dellar. “Lattice Kinetic Schemes for Magnetohydrodynamics”, *Journal of Computational Physics*, vol. **179**, p. 95-126, 2002.
- [4] F. Dubois. “Stable lattice Boltzmann schemes with a dual entropy approach for monodimensional nonlinear waves”, *Computers and Mathematics with Applications*, vol. **65**, p. 142-159, 2013.
- [5] F. Dubois, P. Lallemand. “Towards higher order lattice Boltzmann schemes”, *Journal of Statistical Mechanics: Theory and Experiment*, P06006, 2009.
- [6] D. d’Humières. “Generalized Lattice-Boltzmann Equations”, in *Rarefied Gas Dynamics: Theory and Simulations*, vol. **159** of *AIAA Progress in Astronautics and Astronautics*, p. 450-458, 1992.
- [7] M. Junk, A. Klar, L.S. Luo. “Asymptotic analysis of the lattice Boltzmann equation”, *Journal of Computational Physics*, vol. **210**, p. 676-704, 2005.
- [8] B. Khobalatte, B. Perthame. “Maximum principle on the entropy and second-order kinetic schemes”, *Mathematics of Computation*, vol. **62**, p. 119-131, 1994.
- [9] P. Lallemand, L.-S. Luo. “Theory of the lattice Boltzmann method: Dispersion, dissipation, isotropy, Galilean invariance, and stability”, *Physical Review E*, vol. **61**, p. 6546-6562, June 2000.
- [10] P. Lallemand, L.-S. Luo. “Theory of the lattice Boltzmann method: Acoustic and thermal properties in two and three dimensions”, *Physical Review E*, vol. **68**, p. 036706, 2003. vol. 68, no. 3, 2003
- [11] P. Lallemand, F. Dubois. “Some results on energy-conserving lattice Boltzmann models”, *Computers and Mathematics with Applications*, vol. **65**, p. 831-844, 2013.
- [12] L.D. Landau, E.M. Lifshitz. *Fluid Mechanics*, Pergamon Press, 1959.
- [13] P. Le Quéré. “Accurate solutions to the square thermally driven cavity at high Rayleigh number”, *Computers and Fluids*, vol. **20**, p. 29-41, 1991.

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- [14] A. Mezrhab, A. Moussaoui, M. Jami, H. Naji. “Double MRT thermal lattice Boltzmann method for simulating convective flows”, *Physics Letters A* vol. **374**, p. 3499-3507, 2010.
- [15] X. Nie, X. Shan, H. Chen. “Thermal lattice Boltzmann model for gases with internal degrees of freedom”, *Physical Review E*, vol. **77**, p. 035701(R), 2008.
- [16] Y. Qian, Y. Zhou. “Complete Galilean-invariant lattice BGK models for the Navier-Stokes equation”, *Europhysics Letters*, vol. **42**, p. 359-364, 1998.
- [17] P.J. Roache. *Computational Fluid Dynamics*, 446 pages, Hermosa Publishers, Albuquerque, 1976.
- [18] X. Shan and H. Chen. “Lattice Boltzmann model for simulating flows with multiple phases and components”, *Physical Review E*, vol. **47**, p. 1815–1819, 1993.
- [19] G. De Vahl Davis. “Natural convection of air in a square cavity: A benchmark numerical solution”, *Int. J. of Num. Meth. in fluids*, vol. **3**, p. 249-264, 1983.
- [20] J. Wang, D. Wang, P. Lallemand, L-S. Luo. “Lattice Boltzmann simulations of thermal convective flows in two dimensions”, *Computers and Mathematics with Applications*, vol. **65**, p. 262-286, 2013.