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Subtriviality of continuous fields of nuclear C^* -algebras

By Etienne Blanchard

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Abstract

We extend in this paper the characterisation of a separable nuclear C^* -algebra given by Kirchberg proving that given a unital separable continuous field of nuclear C^* -algebras A over a compact metrizable space X , the $C(X)$ -algebra A is isomorphic to a unital $C(X)$ -subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$, image of $\mathcal{O}_2 \otimes C(X)$ by a norm one projection.

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0 Introduction

In order to study deformations in the C^* -algebraic framework, Dixmier introduced the notion of continuous field of C^* -algebras over a locally compact space ([7]). In the same way as there is a faithful representation in a Hilbert space for any C^* -algebra thanks to the Gelfand–Naimark–Segal construction, a separable continuous field of C^* -algebras A over a compact metrizable space X always admits a continuous field of faithful representations π in a Hilbert $C(X)$ -module, i.e. there exists a family of representations $\{\pi_x, x \in X\}$, in a separable Hilbert space H which factorize through a faithful representation of the fibre A_x such that for each $a \in A$, the map $x \mapsto \pi_x(a)$ is strongly continuous ([4, théorème 3.3]).

In a work on tensor products over $C(X)$ of continuous fields of C^* -algebras over X ([16]), Kirchberg and Wassermann raised the question of whether the continuous field of C^* -algebras A could be subtrivialized, i.e. whether one could find a continuous field of faithful representations π such that the map $x \mapsto \pi_x(a) \in L(H)$ is actually norm continuous for all a in A . In this case, given any C^* -algebra B , the minimal tensor product $A \otimes B$ is a $C(X)$ -subalgebra of the trivial continuous field $[L(H) \otimes B] \otimes C(X)$ and is therefore a continuous field with fibres $(A \otimes B)_x = A_x \otimes B$. They proved that a non-exact continuous field with exact fibres cannot be subtrivialized and they constructed such examples.

The non-trivial example of the continuous field of rotation algebras over the unit circle \mathbb{T} had already been studied by Haagerup and Rørdam in [10]. More precisely, they constructed continuous functions u, v from \mathbb{T} to the unitary group $U(H)$ of the infinite-dimensional separable Hilbert space H satisfying the commutation relation $u_t v_t = v_t u_t$

for all $t \in \mathbb{T}$ and the uniform continuity condition $\max\{\|u_t - u_{t'}\|, \|v_t - v_{t'}\|\} < C'|t - t'|^{1/2}$ where C' is a computable constant.

Our purpose in the present paper is to show that the subtrivialization is always possible in the nuclear separable case through a generalisation of the following theorem of Kirchberg using $\mathcal{R}KK$ -theory arguments:

Theorem 0.1 ([15]) *A unital separable C^* -algebra A is exact if and only if it is isomorphic to a C^* -subalgebra of \mathcal{O}_2 . Moreover the C^* -algebra A is nuclear if and only if A is isomorphic to a C^* -subalgebra of \mathcal{O}_2 containing the unit $1_{\mathcal{O}_2}$ of \mathcal{O}_2 , image of \mathcal{O}_2 by a unital completely positive projection.*

As a matter of fact, we get an equivalent characterisation of nuclear separable continuous fields of C^* -algebras (theorem 3.2) which is made possible thanks to $C(X)$ -linear homotopy invariance of the bifunctor $\mathcal{R}KK(X; -, -)$ (theorem 1.6) and $C(X)$ -linear Weyl-von Neumann absorption results (proposition 2.5). This also enables us to have a better understanding of the characterisation of separable continuous fields of nuclear C^* -algebras given by Bauval in [2].

In an added appendix, the corresponding characterisation of exact separable continuous fields of C^* -algebras as $C(X)$ -subalgebras of $\mathcal{O}_2 \otimes C(X)$ given by Eberhard Kirchberg is described (theorem A.1).

I would like to thank E. Kirchberg for his enlightenment on the exact case. I also want to express my gratitude to C. Anantharaman-Delaroche and J. Cuntz for fruitful discussions.

1 Preliminaries

1.1 $C(X)$ -algebras

Let X be a compact Hausdorff space and $C(X)$ be the C^* -algebra of continuous functions on X with complex values. We start by recalling the following definition.

Definition 1.1 ([13]) *A $C(X)$ -algebra is a C^* -algebra A endowed with a unital morphism from $C(X)$ in the centre of the multiplier algebra $M(A)$ of A .*

Remark: We do not assume that $C(X)$ embeds into $M(A)$. For instance, there is a natural structure of $C([0, 2])$ -algebra on the C^* -algebra $C([0, 1])$.

For $x \in X$, define the kernel $C_x(X)$ of the evaluation map $ev_x : C(X) \rightarrow \mathbb{C}$ at x ; denote by A_x the quotient of a $C(X)$ -algebra A by the closed ideal $C_x(X)A$ and by a_x the image of an element $a \in A$ in the fibre A_x . Then the function

$$x \mapsto \|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\}$$

is upper semi-continuous for any $a \in A$ and the $C(X)$ -algebra A is said to be a continuous field of C^* -algebras over X if the function $x \mapsto \|a_x\|$ is actually continuous for every $a \in A$ ([7]).

Examples 1. If A is a $C(X)$ -algebra and D is a C^* -algebra, the spatial tensor product $B = A \otimes D$ is naturally endowed with a structure of $C(X)$ -algebra through the map $f \in C(X) \mapsto f \otimes 1_{M(D)} \in M(A \otimes D)$. In particular, if $A = C(X)$, the tensor product B is a trivial continuous field over X with constant fibre $B_x \simeq D$

2. Given a $C(X)$ -algebra A , define the unital $C(X)$ -algebra \mathcal{A} generated by A and $u[C(X)]$ in $M[A \oplus C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$. It defines a continuous field of C^* -algebras over X if and only if the $C(X)$ -algebra A is continuous ([4, proposition 3.2]).

Remark: If A is a separable continuous field of non-zero C^* -algebras (not necessarily unital) over the compact Hausdorff space X , the positive cone $C(X)_+$ and so the C^* -algebra $C(X)$ are separable. Hence, the topological space X is metrizable.

Definition 1.2 ([4, 5]) *Given a continuous field of C^* -algebras A over the compact Hausdorff space X , a continuous field of representations of a $C(X)$ -algebra D in the multiplier C^* -algebra $M(A)$ of A is a $C(X)$ -linear morphism $\pi : D \rightarrow M(A)$, i.e. for each $x \in X$, the induced representation π_x of D in $M(A_x)$ factorizes through the fibre D_x .*

If the $C(X)$ -algebra D admits a continuous field of faithful representations π in the $C(X)$ -algebra $M(A)$ where A is a continuous field over X , i.e. the induced representation of the fibre D_x in $M(A_x)$ is faithful for every point $x \in X$, the function

$$x \mapsto \|\pi_x(d)\| = \sup\{\|(\pi(d)a)_x\|, a \in A \text{ such that } \|a\| \leq 1\}$$

is lower semi-continuous for all $d \in D$ and the $C(X)$ -algebra D is therefore continuous.

In particular a separable $C(X)$ -algebra D is continuous if and only if there exists a Hilbert $C(X)$ -module \mathcal{E} such that D admits a continuous field of faithful representations in the multiplier algebra $M(\mathcal{K}(\mathcal{E})) = \mathcal{L}(\mathcal{E})$ of the continuous field over X of compact operators $\mathcal{K}(\mathcal{E})$ acting on \mathcal{E} ([4, théorème 3.3]).

Let us also mention the characterisation of separable continuous fields of nuclear C^* -algebras over a compact metrizable space X given by Bauval in [2] using a natural $C(X)$ -linear version of nuclearity introduced by Kasparov and Skandalis in [14]§6.2 : a $C(X)$ -linear completely positive σ from a $C(X)$ -algebra A into a $C(X)$ -algebra B is said to be $C(X)$ -nuclear if and only if given any compact set F in A and any strictly positive real number ε , there exist an integer k and $C(X)$ -linear completely positive contractions $T : A \rightarrow M_k(\mathbb{C}) \otimes C(X)$ and $S : M_k(\mathbb{C}) \otimes C(X) \rightarrow B$ such that for all $a \in F$, one has the inequality

$$\|\sigma(a) - (S \circ T)(a)\| < \varepsilon.$$

One can then state the following results. The first assertion is a simple $C(X)$ -linear reformulation of the Choi-Effros theorem and the second one is due to Bauval.

Proposition 1.3 *Let X be a compact metrizable space and A be a separable $C(X)$ -algebra.*

1. ([14]§6.2) Given a $C(X)$ -algebra B and a closed ideal $J \subset B$, any contractive $C(X)$ -nuclear map $A \rightarrow B/J$ admits a contractive $C(X)$ -linear completely positive lift $A \rightarrow B$.
2. ([2, théorème 7.2]) The $C(X)$ -algebra A is a continuous fields of nuclear C^* -algebras over X if and only if the identity map $id_A : A \rightarrow A$ is $C(X)$ -nuclear.

Remark: In assertion 1., the ideal $J = (C(X)B)J = C(X)J$ is a $C(X)$ -algebra.

1.2 $C(X)$ -extensions

Given a compact Hausdorff space X , we introduce a natural $C(X)$ -linear version of the semi-group $Ext(-, -)$ defined by Kasparov ([12, 13]).

Call a morphism of $C(X)$ -algebras a $*$ -homomorphism between $C(X)$ -algebras which is $C(X)$ -linear.

Definition 1.4 A $C(X)$ -extension of a $C(X)$ -algebra A by a $C(X)$ -algebra B is a short exact sequence

$$0 \rightarrow B \rightarrow D \xrightarrow{\pi} A \rightarrow 0$$

in the category of $C(X)$ -algebras. The $C(X)$ -extension is said to be trivial if the map π admits a cross section $s : A \rightarrow D$ which is a morphism of $C(X)$ -algebras.

As in the C^* -algebraic case a $C(X)$ -extension $0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0$ of A by B defines unambiguously an homomorphism from D to the multiplier algebra $M(B)$ of B , which gives a morphism of $C(X)$ -algebras $\sigma : A \rightarrow M(B)/B$ (called the Busby invariant of the extension) and the $C(X)$ -extension is trivial if and only if the map σ lifts to a morphism of $C(X)$ -algebras $A \rightarrow M(B)$. Conversely, given a morphism of $C(X)$ -algebras $\sigma : A \rightarrow M(B)/B$, setting $D = \{(a, m) \in A \times M(B), \sigma(a) = q(m)\}$ where q is the quotient map $M(B) \rightarrow M(B)/B$, one has a $C(X)$ -extension $0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0$ (see [12]§7).

Remark: A $C(X)$ -extension $0 \rightarrow B \rightarrow D \rightarrow A \rightarrow 0$ induces for every $x \in X$ a C^* -extension $0 \rightarrow B_x \rightarrow D_x \rightarrow A_x \rightarrow 0$.

In order to define the sum of two $C(X)$ -extensions, recall that the Cuntz algebra \mathcal{O}_2 is the unital C^* -algebra generated by two orthogonal isometries s_1 and s_2 subject to the relation $1 = s_1 s_1^* + s_2 s_2^*$ ([6]). Then if \mathcal{K} is the C^* -algebra of compact operators on the infinite-dimensional separable Hilbert space, one defines the sum of two $C(X)$ -extensions σ_1 and σ_2 of the $C(X)$ -algebra A by the stable $C(X)$ -algebra $\mathcal{K} \otimes B$ through the choice of a unital copy of \mathcal{O}_2 in the multiplier algebra $M(\mathcal{K})$ of \mathcal{K} to be the $C(X)$ -extension

$$\sigma_1 \oplus \sigma_2 : a \mapsto q(s_1 \otimes 1)\sigma_1(a)q(s_1^* \otimes 1) + q(s_2 \otimes 1)\sigma_2(a)q(s_2^* \otimes 1) \in M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B),$$

where q is the quotient map $M(\mathcal{K} \otimes B) \rightarrow M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$.

Definition 1.5 Given a compact Hausdorff space X and two $C(X)$ -algebras A and B , $Ext(X; A, B)$ is the semi-group of $C(X)$ -extensions of A by $\mathcal{K} \otimes B$ divided by the equivalence relation \sim where $\sigma_1 \sim \sigma_2$ if there exist a unitary $U \in M(\mathcal{K} \otimes B)$ of image $q(U)$ in the quotient $M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$ and two trivial $C(X)$ -extensions π_1 and π_2 such that for all $a \in A$,

$$(\sigma_2 \oplus \pi_2)(a) = q(U)^*(\sigma_1 \oplus \pi_1)(a)q(U) \text{ (in } M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)\text{)}.$$

Let $Ext(X; A, B)^{-1}$ be the group of invertible elements of $Ext(X; A, B)$, i.e. the group of classes of $C(X)$ -extension σ such that there exists a $C(X)$ -extension θ with $\sigma \oplus \theta$ trivial. One can generalise Kasparov's theorem of homotopy invariance of the group $Ext(A, B)^{-1}$ to the framework of $C(X)$ -algebras as follows.

Theorem 1.6 ([12]) Assume that A is a separable $C(X)$ -algebra and that B is a σ -unital $C(X)$ -algebra. Then the group $Ext(X; A, B)^{-1}$ is isomorphic to the group $\mathcal{R}KK^1(X; A, B)$ and is therefore $C(X)$ -linear homotopy invariant in both entries A and B .

Proof : Let us first make the following observation. Given a $C(X)$ -algebra B and a Hilbert B -module \mathcal{E} , denote by $\mathcal{L}(\mathcal{E})$ the set of bounded B -linear operators on \mathcal{E} which admit an adjoint ([11]). Then any operator $T \in \mathcal{L}(\mathcal{E})$ is B -linear and so $C(X)$ -linear. This argument provides a natural extension of the Stinespring-Kasparov theorem ([12]) to the framework of $C(X)$ -algebras. Consequently, if A is a separable $C(X)$ -algebra and B is a σ -unital $C(X)$ -algebra, the class of a $C(X)$ -extension $\sigma : A \rightarrow M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$ is invertible in $Ext(X; A, B)$ if and only if there is a $C(X)$ -linear completely positive contractive lift $A \rightarrow M(\mathcal{K} \otimes B)$.

Let $\mathcal{R}\mathcal{E}(X; A, B)$ be the set of Kasparov $C(X)$ - A, B -bimodules ([13, definition 2.19]), i.e. the set of Kasparov A, B bimodules (\mathcal{E}, F) such that the representation $A \rightarrow \mathcal{L}(\mathcal{E})$ is a $C(X)$ -representation. Call a $C(X)$ -linear operator homotopy an element $\{(\mathcal{E}, F_t), 0 \leq t \leq 1\} \in \mathcal{R}\mathcal{E}(X; A, B \otimes C([0, 1]))$ such that $t \mapsto F_t$ is norm continuous and define on $\mathcal{R}\mathcal{E}(X; A, B)$ the equivalence relation corresponding to the one defined by Skandalis in [18, definition 2]. The constructions given by Kasparov in [12, section 7] imply that, if the $C(X)$ -algebra B is σ -unital, the group of equivalence classes $\mathcal{R}\widetilde{K}K(X; A, B \otimes \mathcal{C}_1)$ is isomorphic to $Ext(X; A, B)^{-1}$, where \mathcal{C}_1 is the first (graded) Clifford algebra.

On the other hand, given two graded $C(X)$ -algebras A and B with A separable, the different steps of the demonstration of [18, theorem 19] provide us with an isomorphism between the two groups $\mathcal{R}\widetilde{K}K(X; A, B)$ and $\mathcal{R}KK(X; A, B)$ since proposition 2.21 of [13] defines an intersection product in $\mathcal{R}\widetilde{K}K$ -theory and lemma 18 of [18] gives us the equality

$$(ev_0 \otimes id_{C(X)})^*(1_{C(X)}) = (ev_1 \otimes id_{C(X)})^*(1_{C(X)}) \text{ in } \mathcal{R}\widetilde{K}K(X; C([0, 1]) \otimes C(X), C(X)),$$

where $1_{C(X)}$ is the Kasparov $C(X), C(X)$ -bimodule $(C(X), 0)$ and $ev_t : C([0, 1]) \rightarrow \mathbb{C}$ is the evaluation map at $t \in [0, 1]$. \square

Remarks: 1. Kuiper's theorem implies that the law of addition on the abelian group $Ext(X; A, B)^{-1}$ is independent of the choice of the unital copy of \mathcal{O}_2 in $M(\mathcal{K})$.

2. If A is a separable nuclear continuous field of C^* -algebras over X and B is a $C(X)$ -algebra, every $C(X)$ -linear morphism from A to the quotient $M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$ is $C(X)$ -nuclear and therefore admits a $C(X)$ -linear completely positive lifting $A \rightarrow M(\mathcal{K} \otimes B)$ thanks to proposition 1.3. Accordingly one has the equality

$$\text{Ext}(X; A, B)^{-1} = \text{Ext}(X; A, B).$$

2 An absorption result

In this section we prove a continuous generalisation of a statement contained in [15] which will enable us to get a $C(X)$ -linear Weyl-von Neumann type result (proposition 2.5). Let us start with the following definition of Cuntz.

Definition 2.1 ([6]) *A simple C^* -algebra A distinct from \mathbb{C} is said to be purely infinite if and only if for any non-zero $a, b \in A$, there exist elements $x, y \in A$ such that $a = xby$.*

Then, we can state a proposition from Kirchberg's classification work, based on Glimm's lemma ([7], § 11.2). A sketch of proof can also be found in [1, proposition 5.1].

Proposition 2.2 ([15]) *Let A be a purely infinite simple C^* -algebra and D be a separable C^* -subalgebra of $M(A)$. Assume that $V : D \rightarrow A$ is a nuclear contraction.*

Then there exists a sequence (a_n) of elements in A of norm less than 1 such that $V(d) = \lim_{n \rightarrow \infty} a_n^ d a_n$ for all $d \in D$.*

Remark: A simple ring has by definition exactly two distinct two sided ideals and is therefore non-zero.

Corollary 2.3 *Let A be a continuous field of purely infinite simple C^* -algebras over a compact Hausdorff space X and assume that D is a separable $C(X)$ -subalgebra of the multiplier algebra $M(A)$ such that there is a unital $C(X)$ -embedding of the $C(X)$ -algebra $\mathcal{O}_\infty \otimes C(X)$ in the commutant D' of D in $M(A)$ and the identity map $id_D : D \rightarrow M(A)$ is a continuous field of faithful representations.*

If $V : D \rightarrow A$ is a $C(X)$ -nuclear contraction, there exists a sequence (a_n) in the unit ball of A with the property that for all $d \in D$,

$$V(d) = \lim_{n \rightarrow \infty} a_n^* d a_n.$$

Proof : If F is a compact generating set for D , it is enough to prove that given a strictly positive real number $\varepsilon > 0$, there exists an element a in the unit ball of A such that $\|V(d) - a^* d a\| < \varepsilon$ for all $d \in F$.

For $x \in X$, the fibre A_x is a purely infinite simple C^* -algebra and the map $d \mapsto V(d)_x \in A_x$ factorizes through $D_x \simeq (id_D)_x(D) \subset M(A_x)$ since id_D is a continuous field of faithful representations. As a consequence, the previous proposition implies that we can find an element $g \in A$ with $\|g\| \leq 1$ satisfying for all $d \in F$ the inequality

$$\| [V(d) - g^* d g]_x \| < \varepsilon.$$

Thus, by upper semi-continuity and compactness, there exist a finite open covering $\{U_1, \dots, U_n\}$ of the space X and elements g_1, \dots, g_n in the unit ball of A such that for all $d \in F$ and $x \in U_i$, $1 \leq i \leq n$,

$$\| [V(d) - g_i^* d g_i]_x \| < \varepsilon.$$

Choose n orthogonal isometries w_1, \dots, w_n in the C^* -algebra $\mathcal{O}_\infty \otimes 1_{C(X)} \subset D'$ and let $\{\phi_i\}$ be a partition of the unit $1_{C(X)}$ subordinate to the covering $\{U_i\}$ of X . The element $a = \sum_i \phi_i^{1/2} w_i g_i \in A$ verifies:

1. $a^* a = \sum_{i,j} \sqrt{\phi_i \phi_j} g_i^* w_i^* w_j g_j = \sum_i \phi_i g_i^* g_i \leq 1_{M(A)}$,
2. for $d \in F$ and $x \in X$, $\| [V(d) - a^* d a]_x \| \leq \sum_i \phi_i(x) \| [V(d) - g_i^* d g_i]_x \| < \varepsilon$. \square

Let us mention the following technical corollary which will be needed in theorem 3.2.

Corollary 2.4 *If $p \in \mathcal{O}_2 \otimes C(X)$ is a projection such that for all points $x \in X$, p_x is non-zero, then there exists an isometry $u \in \mathcal{O}_2 \otimes C(X)$ such that $p = uu^*$.*

Proof : Let $\mathcal{D}_2 = \lim_{n \rightarrow \infty} \mathcal{O}_2^{\otimes n}$ be the infinite tensor product of \mathcal{O}_2 .

Given a projection $q \in \mathcal{D}_2 \otimes C(X)$ such that $\|q_x\| = 1$ for all $x \in X$, we first show that there exists an element $v \in \mathcal{D}_2 \otimes C(X)$ satisfying $1_{\mathcal{D}_2 \otimes C(X)} = v^* q v$. Namely, by density of the algebraic tensor product

$$\left[\bigcup_n \mathcal{O}_2^{\otimes n} \right] \odot C(X) = \bigcup_n \left[\mathcal{O}_2^{\otimes n} \odot C(X) \right]$$

in the C^* -algebra $\mathcal{D}_2 \otimes C(X)$ and functional calculus one can find an integer $n > 0$ and a projection $r \in \mathcal{O}_2^{\otimes n} \otimes C(X) \subset \mathcal{D}_2 \otimes C(X)$ such that $\|q - r\| < 1$, which implies that $r = s^* q s$ for some element $s \in \mathcal{D}_2 \otimes C(X)$. Take then a faithful state φ on $\mathcal{O}_2^{\otimes n}$ and consider the $C(X)$ -linear completely positive map

$$V : [\mathcal{O}_2^{\otimes n} \otimes 1_{\mathcal{O}_2}] \otimes C(X) \rightarrow \mathcal{O}_2^{\otimes n+1} \otimes C(X)$$

defined by the formula $V(d) = (\varphi \otimes id_{C(X)})(d) 1_{\mathcal{O}_2^{\otimes n+1} \otimes C(X)}$ for $d \in [\mathcal{O}_2^{\otimes n} \otimes 1_{\mathcal{O}_2}] \otimes C(X) \simeq \mathcal{O}_2^{\otimes n} \otimes C(X)$. According to corollary 2.3, there exists an element $t \in \mathcal{O}_2^{\otimes n+1} \otimes C(X)$ such that

$$1_{\mathcal{D}_2 \otimes C(X)} = 1_{\mathcal{O}_2^{\otimes n+1} \otimes C(X)} = t^* r t = (st)^* q (st).$$

Consider now the set \mathcal{P} of projections p in $\mathcal{O}_2 \otimes C(X)$ such that $p_x \neq 0$ for all points $x \in X$. If p belongs to \mathcal{P} , there exists an isometry $v \in \mathcal{O}_2 \otimes C(X)$ such that $p \geq vv^*$ since the K -trivial purely infinite separable unital nuclear C^* -algebra \mathcal{D}_2 satisfying the U.C.T. is isomorphic to \mathcal{O}_2 ([15]). As a consequence, if t is the isometry $t = v(s_1 \otimes 1)v^*$, the projection $r = tt^*$ (Murray-von Neumann equivalent to $1_{\mathcal{O}_2 \otimes C(X)}$) verifies

$$p - r \geq r' = v(s_2 s_2^* \otimes 1)v^* \in \mathcal{P}.$$

The non-empty set \mathcal{P} therefore satisfies the conditions (π_1) – (π_4) defined by Cuntz in [6]. But the C^* -algebra $\mathcal{O}_2 \otimes C(X)$ is K_0 -triviality thanks to [6, theorem 2.3] and the theorem 1.4 of [6] enables us to conclude. \square

One now deduces from corollary 2.3 the following absorption results ([21, 12, 15]):

Proposition 2.5 *Let A be a σ -unital continuous field of purely infinite simple nuclear C^* -algebras over a compact Hausdorff space X and let \mathcal{K} be the C^* -algebra of compact operators on the separable Hilbert space H . Denote by q the quotient map $M(\mathcal{K} \otimes A) \rightarrow M(\mathcal{K} \otimes A)/(\mathcal{K} \otimes A)$.*

1. *Assume that D is a unital separable $C(X)$ -subalgebra of the multiplier algebra $M(\mathcal{K} \otimes A)$ with same unit such that there is a unital $C(X)$ -embedding of the $C(X)$ -algebra $\mathcal{O}_\infty \otimes C(X)$ in the commutant of D in $M(\mathcal{K} \otimes A)$ and the identity map id_D is a continuous field of faithful representations of D in $M(\mathcal{K} \otimes A)$.*

(a) *If V is a unital $C(X)$ -linear completely positive map from D in $M(\mathcal{K} \otimes A)$ which is zero on the intersection $D \cap (\mathcal{K} \otimes A)$, there exists a sequence of isometries s_n in $M(\mathcal{K} \otimes A)$ such that for every $d \in D$,*

$$V(d) - s_n^* d s_n \in \mathcal{K} \otimes A \text{ and } V(d) = \lim_n s_n^* d s_n.$$

(b) *If π is a unital morphism of $C(X)$ -algebras from D into $M(\mathcal{K} \otimes A)$ which is zero on the intersection $D \cap (\mathcal{K} \otimes A)$, there exists a sequence of unitaries u_n in $M(\mathcal{K} \otimes A)$ such that for every $d \in D$,*

$$(d \oplus \pi(d)) - u_n^* d u_n \in \mathcal{K} \otimes A \text{ and } (d \oplus \pi(d)) = \lim_n u_n^* d u_n.$$

(c) *Let B be a $C(X)$ -algebra and assume that the quotient $D/(D \cap (\mathcal{K} \otimes A))$ is isomorphic to the $C(X)$ -algebra \mathcal{B} , where \mathcal{B} is the unital $C(X)$ -algebra generated by $C(X)$ and B in $M[B \oplus C(X)]$ ([4, définition 2.7]).*

Then, if $\pi : B \rightarrow M(\mathcal{K} \otimes A)$ is a $C(X)$ -linear homomorphism, there exists a unitary $U \in M(\mathcal{K} \otimes A)$ such that for all $b \in B \subset M(\mathcal{K} \otimes A)/(\mathcal{K} \otimes A)$,

$$b \oplus (q \circ \pi)(b) = q(U)^* b q(U).$$

2. *Assume that the continuous field A is separable and let D be a separable $C(X)$ -subalgebra of $M(A)$ containing A such that the identity representation $D \rightarrow M(A)$ is a continuous field of faithful representations and there is a unital $C(X)$ -embedding of the $C(X)$ -algebra $\mathcal{O}_\infty \otimes C(X)$ in the commutant of D in $M(A)$. Define the quotient $C(X)$ -algebra $B = D/A$.*

If $\pi : \mathcal{K} \otimes B \rightarrow M(\mathcal{K} \otimes A)$ is a morphism of $C(X)$ -algebras, there exists a unitary $U \in M(\mathcal{K} \otimes A)$ such that for all $b \in \mathcal{K} \otimes B \subset M(\mathcal{K} \otimes A)/(\mathcal{K} \otimes A)$,

$$b \oplus (q \circ \pi)(b) = q(U)^* b q(U).$$

Proof : 1. It derives from corollary 2.3 by the same method as the one developed by Kasparov in [11, theorem 5 and 6]. Nevertheless, for the convenience of the reader we describe the different steps of the demonstration.

1.a) Let F be a compact generating set for D containing the unit $1_{M(\mathcal{K} \otimes A)}$. Then given a real number $\varepsilon > 0$, it is enough to find an element $a \in M(\mathcal{K} \otimes A)$ such that $V(d) - a^*da \in \mathcal{K} \otimes A$ and $\|V(d) - a^*da\| < 3\varepsilon$ for all $d \in F$.

Let $\{e_n\}$ be an increasing, positive, quasicontral, countable approximate unit in the ideal $\mathcal{K} \otimes A$ of the C^* -algebra generated by $\mathcal{K} \otimes A + V(D)$. If we set $f_0 = (e_0)^{1/2}$ and $f_k = (e_k - e_{k-1})^{1/2}$ for $k \geq 1$, we can then assume, passing to a subsequence of (e_n) if necessary, that $\|V(d)f_k - f_kV(d)\| < 2^{-k}\varepsilon$ for all $k \in \mathbb{N}$ and $d \in F$. This implies that the series $\sum_k [V(d)f_k - f_kV(d)]f_k$ is convergent in $\mathcal{K} \otimes A$ and its norm is less than ε . Furthermore, the series $\sum_k [f_kV(d)f_k]$ is strictly convergent in $M(\mathcal{K} \otimes A)$ for all $d \in F$ since $\sum_k f_k^2$ is strictly convergent to 1.

Notice now that the maps $V_k(d) = f_kV(d)f_k$ are all $C(X)$ -nuclear since the separable continuous field $\mathcal{K} \otimes A$ is nuclear. The corollary 2.3 therefore enables us to choose by induction $a_k \in \mathcal{K} \otimes A$ satisfying the following conditions:

1. $\forall d \in F \quad \|V_k(d) - a_k^*da_k\| < 2^{-k}\varepsilon$,
2. $\forall d \in F, \forall l < k \quad \|a_l^*da_k\| < 2^{-l-k}\varepsilon$,
3. $\sum_k a_k$ is strictly convergent toward an element $a \in M(A)$.

One then checks as in [11, theorem 5] that the limit a satisfies the desired properties.

1.b) Take a compact generating F for D containing $1_{M(\mathcal{K} \otimes A)}$ and consider the homomorphism $\pi' = 1 \otimes \pi : D \rightarrow M(\mathcal{K} \otimes (\mathcal{K} \otimes A)) \simeq M(\mathcal{K} \otimes A)$. Given $\delta > 0$, one can find, thank to the previous assertion, an isometry $s \in M(\mathcal{K} \otimes A)$ such that

$$s^*ds - \pi'(d) \in \mathcal{K} \otimes A \text{ and } \|s^*ds - \pi'(d)\| < \delta \text{ for all } d \in K^*K.$$

As a consequence, if we fix $\varepsilon > 0$, the choice of δ small enough gives us the inequality $\|pd - dp\| < \varepsilon$, and so $\|d - [pdp + p^\perp dp^\perp]\| < 2\varepsilon$ for all $d \in F$, where $p = ss^*$ and $p^\perp = 1 - p$.

Define the unital map $\Theta : D \rightarrow M(p^\perp(\mathcal{K} \otimes A)p^\perp)$ by the formula $\Theta(d) = p^\perp dp^\perp$. According to the stabilisation theorem of Kasparov ([11, theorem 2]), one can construct a unitary $w \in M(\mathcal{K} \otimes A)$ verifying for all $d \in F$ the inequality

$$\|d - w^*[\pi'(d) \oplus \Theta(d)]w\| < 3\varepsilon.$$

To finish the demonstration, notice that the two homomorphisms π' and $\pi' \oplus \pi$ are unitarily equivalent.

1.c) Consider the unital extension $\tilde{\pi}$ of π to \mathcal{B} . Then, the morphism $\tilde{\pi} \circ q : D \rightarrow M(\mathcal{K} \otimes A)$ reduces the demonstration to the previous assertion.

2. The identity representation of the unital $C(X)$ -algebra $\mathcal{D} = (\mathcal{K} \otimes D) + C(X) \subset M(\mathcal{K} \otimes A)$ is clearly a continuous field of faithful representations since the unital $C(X)$ -representation $C(X) \rightarrow M(A)$ is a continuous field of faithful representations. Extend

the map $\pi : \mathcal{K} \otimes B = (\mathcal{K} \otimes D)/(\mathcal{K} \otimes A) \rightarrow M(\mathcal{K} \otimes A)$ to a unital morphism of $C(X)$ -algebras $\tilde{\pi} : \mathcal{D}/(\mathcal{K} \otimes A) \rightarrow M(\mathcal{K} \otimes A)$. Applying assertion 1.b) to the unital homomorphism $d \mapsto (\tilde{\pi} \circ q)(d)$ from the $C(X)$ -subalgebra $\mathcal{D} \subset M(\mathcal{K} \otimes A)$ to the multiplier algebra $M(\mathcal{K} \otimes A)$ now leads to the desired conclusion. \square

3 The subtriviality

Given a separable continuous field of nuclear C^* -algebras A over X , the strategy to prove the subtriviality of the $C(X)$ -algebra A will be the same as the one developed by Kirchberg in [15] to prove theorem 0.1 whose main ideas of demonstration are also explained in [1, Théorème 6.1]. We associate to A a $C(X)$ -extension by an hereditary C^* -subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$ (proposition 3.1) and then prove that after stabilisation, this $C(X)$ -extension splits by $\mathcal{R}KK$ -theory arguments (theorem 3.2).

3.1 Let us construct the fundamental $C(X)$ -extension associated to an exact separable continuous field of C^* -algebras.

Proposition 3.1 *Given a compact Hausdorff space X and a non-zero separable unital exact $C(X)$ -algebra A , there exist a unital $C(X)$ -subalgebra F of $\mathcal{O}_2 \otimes C(X)$ with same unit and an hereditary subalgebra I of $\mathcal{O}_2 \otimes C(X)$ such that I is an ideal in F and the $C(X)$ -algebra A is isomorphic to the quotient $C(X)$ -algebra F/I .*

Furthermore, if the topological space X is perfect (i.e. without any isolated point) and the $C(X)$ -algebra A is continuous, the canonical map $F \rightarrow M(I)$ is a continuous field of faithful representations.

Proof : Thanks to the characterisation of separable exact C^* -algebras obtained by Kirchberg (theorem 0.1), one may assume that the C^* -algebra A is a C^* -subalgebra of \mathcal{O}_2 containing the unit of \mathcal{O}_2 .

Let $G \subset \mathcal{O}_2 \otimes C(X)$ be the trivial continuous field $A \otimes C(X)$ over X . Then the kernel of the $C(X)$ -linear morphism $\pi : G \rightarrow A$ defined by $\pi(a \otimes f) = fa$ is the ideal $J = C_\Delta(X \times X)G$ where $C_\Delta(X \times X)$ is the ideal in $C(X \times X)$ of functions which are zero on the diagonal. Indeed suppose that $T \in G$ verifies $\pi(T) = 0$. Then given $\varepsilon > 0$, take a finite number of elements $a_i \in A$, $f_i \in C(X)$ such that $\|T - \sum_i a_i \otimes f_i\| < \varepsilon$; one has $\|T - \sum_i (1 \otimes f_i - f_i \otimes 1)(a_i \otimes 1)\| < \varepsilon + \|\pi(\sum_i a_i \otimes f_i)\| < 2\varepsilon$.

Define then the hereditary subalgebra $I = J[\mathcal{O}_2 \otimes C(X)]J$ in $\mathcal{O}_2 \otimes C(X)$ generated by J . It is a $C(X)$ -algebra since it is closed by Cohen theorem (see e.g. [4, proposition 1.8]) and the product $(1_{\mathcal{O}_2} \otimes f)(bc) = b(1_{\mathcal{O}_2} \otimes f)c$ belongs to I for all $f \in C(X)$ and $b, c \in I$. If we set $F = I + G$, the intersection $G \cap I$ is reduced by construction to the subalgebra J , and so we have a $C(X)$ -extension

$$0 \rightarrow I \rightarrow F \rightarrow A \rightarrow 0.$$

Assume now that the space X is perfect and that the $C(X)$ -algebra A is continuous. We need to prove that the map $F_x \rightarrow M(I_x)$ is injective for each $x \in X$. Let $a \in G$ and $b \in I$ be two elements such that the sum $d = a + b \in F$ verifies for a given point $x \in X$ the equality

$$d_x I_x + I_x d_x = 0 \text{ (in } [\mathcal{O}_2 \otimes C(X)]_x \simeq \mathcal{O}_2 \otimes \mathbb{C} \text{)}.$$

To end the proof, we have to show that d_x is zero. For every $f \in C_\Delta(X \times X)$, one has $(bf)_x = -(af)_x \in J_x$, whence $b_x \in J_x$ and so $d_x \in G_x$. But the representation of $G_x \simeq A$ in $M(J_x) \simeq M(C_x(X)A)$ is injective since X is perfect and A is continuous, from which we deduce that $d_x = 0$. \square

Remark: With the previous notations, if the $C(X)$ -algebra A is nuclear and ψ is a unital completely positive projection from \mathcal{O}_2 onto A , the map $\pi \circ (\psi \otimes id_{C(X)})$ is a unital $C(X)$ -linear completely positive map from $\mathcal{O}_2 \otimes C(X)$ onto the $C(X)$ -subalgebra A which is zero on the nuclear hereditary $C(X)$ -subalgebra I .

3.2 We can now state the main theorem:

Theorem 3.2 *Let X be a compact metrizable space and A be a unital separable $C(X)$ -algebra with a unital embedding of the $C(X)$ -algebra $C(X)$ in A .*

The following assertions are equivalent:

1. *A is a continuous field of nuclear C^* -algebras over X ;*
2. *there exist a unital monomorphism of $C(X)$ -algebras $\alpha : A \hookrightarrow \mathcal{O}_2 \otimes C(X)$ and a unital $C(X)$ -linear completely positive map $E : \mathcal{O}_2 \otimes C(X) \rightarrow A$ such that $E \circ \alpha = id_A$.*

Proof : $2 \Rightarrow 1$ By assumption the identity map $id_A = E \circ id_{\mathcal{O}_2 \otimes C(X)} \circ \alpha : A \rightarrow A$ is nuclear since the C^* -algebra $\mathcal{O}_2 \otimes C(X)$ is nuclear and so the C^* -algebra A is nuclear. Besides the $C(X)$ -algebra A is isomorphic to the $C(X)$ -subalgebra $\alpha(A)$ of the continuous field $\mathcal{O}_2 \otimes C(X)$ and is therefore continuous.

$1 \Rightarrow 2$ • Let us first deal with the case where the space X is perfect.

Given a unital nuclear separable continuous fields A over X which is unitaly embedded in the C^* -algebra \mathcal{O}_2 , consider the $C(X)$ -extension

$$0 \rightarrow I \rightarrow F \xrightarrow{\pi} A \rightarrow 0$$

constructed in proposition 3.1 and take the associated $C(X)$ -extension

$$0 \rightarrow \mathcal{K} \otimes I \otimes \mathcal{O}_2 \rightarrow D = (\mathcal{K} \otimes F \otimes 1_{\mathcal{O}_2}) + (\mathcal{K} \otimes I \otimes \mathcal{O}_2) \rightarrow \mathcal{K} \otimes A \rightarrow 0.$$

The $C(X)$ -nuclear quotient map $\sigma = \sigma \circ id_{\mathcal{K} \otimes A}$ from the separable nuclear continuous field $\mathcal{K} \otimes A$ to the quotient $D/(\mathcal{K} \otimes I \otimes \mathcal{O}_2) \subset M(\mathcal{K} \otimes I \otimes \mathcal{O}_2)/(\mathcal{K} \otimes I \otimes \mathcal{O}_2)$ admits a $C(X)$ -linear completely positive lifting $\mathcal{K} \otimes A \rightarrow D$ ($\subset \mathcal{K} \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2$) thanks to proposition 1.3. This means that the class of σ is invertible in $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)$ (see the second remark following theorem 1.6).

But the group $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)^{-1}$ is $C(X)$ -linear homotopy invariant (theorem 1.6), hence zero since the endomorphism $\phi_2(a) = s_1 a s_1^* + s_2 a s_2^*$ of \mathcal{O}_2 is homotopic to the identity map $id_{\mathcal{O}_2}$ ([6, proposition 2.2]) and so $[\theta] = 2[\theta]$ in $Ext(X; \mathcal{K} \otimes A, I \otimes \mathcal{O}_2)^{-1}$ for any invertible extension θ of $\mathcal{K} \otimes A$ by $I \otimes \mathcal{O}_2$. As a consequence, the $C(X)$ -extension defined by σ is stably trivial. Furthermore, the identity representation of

$D \subset M(\mathcal{K} \otimes I \otimes \mathcal{O}_2)$ is a continuous field of faithful representations (proposition 3.1) and the assertion 2. of proposition 2.5 implies that the quotient morphism $(id_{\mathcal{K}} \otimes \pi \otimes id_{\mathcal{O}_2})$ from D to $\mathcal{K} \otimes A$ admits a cross section α which is a morphism of $C(X)$ -algebras.

This monomorphism α is going to enable us to conclude by standard arguments, using theorem 0.1 and the result of Elliott ([9]) that the C^* -algebra \mathcal{O}_2 is isomorphic to $\mathcal{O}_2 \otimes \mathcal{O}_2$.

Choose a non-zero minimal projection e_{11} in the C^* -algebra \mathcal{K} of compact operators that we embed in \mathcal{O}_2 and let φ be a state on \mathcal{O}_2 such that $\varphi(e_{11}) = 1$. If we take a unital completely positive projection ψ of \mathcal{O}_2 onto the nuclear C^* -subalgebra $A \subset \mathcal{O}_2$ (theorem 0.1), the composed map

$$E = (\varphi \otimes id_A) \circ (id_{\mathcal{O}_2} \otimes [\pi \circ (\psi \otimes id_{C(X)})] \otimes \varphi)$$

is a unital $C(X)$ -linear completely positive map from $\mathcal{O}_2 \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2$ onto A . Take also an isometry $u \in \mathcal{O}_2 \otimes C(X)$ such that $\alpha(e_{11} \otimes 1_A) = uu^*$ (corollary 2.4) and define the unital $C(X)$ -algebra morphism

$$\beta : A \longrightarrow \mathcal{O}_2 \otimes [\mathcal{O}_2 \otimes C(X)] \otimes \mathcal{O}_2 \simeq \mathcal{O}_2 \otimes C(X)$$

by the formula $\beta(a) = u^* \alpha(e_{11} \otimes a) u$. If $\tilde{E} : \mathcal{O}_2 \otimes C(X) \rightarrow A$ is the completely positive unital map $d \mapsto E(udu^*)$, one gets for all $a \in A$ the equality

$$(\tilde{E} \circ \beta)(a) = (E \circ \alpha)(e_{11} \otimes a) = a$$

• Let us now come back to the general case of a compact space X .

Define the continuous field $B = A \otimes C([0, 1])$ over the perfect compact space $Y = X \times [0, 1]$. According to the previous discussion, there exist a unital completely positive map $\tilde{E} : \mathcal{O}_2 \otimes C(Y) \rightarrow B$ and a $C(X) \otimes C([0, 1])$ -linear monomorphism $\tilde{\alpha} : B \rightarrow \mathcal{O}_2 \otimes C(Y)$ such that $\tilde{E} \circ \tilde{\alpha} = id_B$. If $ev_1 : C([0, 1]) \rightarrow \mathbb{C}$ is the evaluation map at $x = 1 \in [0, 1]$, define the two maps $E : \mathcal{O}_2 \otimes C(X) \rightarrow A$ and $\alpha : A \rightarrow \mathcal{O}_2 \otimes C(X)$ by

$$E(d) = (id_A \otimes ev_1) \circ \tilde{E}(d \otimes 1_{C([0,1])}) \text{ and } \alpha(a) = (id_{\mathcal{O}_2 \otimes C(X)} \otimes ev_1) \circ \tilde{\alpha}(a \otimes 1_{C([0,1])}).$$

Then E is a unital $C(X)$ -linear completely positive map, α is a unital $C(X)$ -linear monomorphism and one has the identity $E \circ \alpha = id_A$. \square

Remark: Assume that X is a locally compact metrizable space and that the $C_0(X)$ -algebra A is a nuclear continuous field of C^* -algebras over X , where $C_0(X)$ denotes the algebra of continuous functions on X vanishing at infinity. If \tilde{X} is the Alexandroff compactification of X , the unital $C(\tilde{X})$ -algebra \mathcal{A} generated by A and $C(\tilde{X})$ in the multiplier algebra $M[A \oplus C(\tilde{X})]$ is a separable unital continuous field of C^* -algebras over \tilde{X} ([4, proposition 3.2]). By theorem 3.2, there exists therefore a $C(\tilde{X})$ -linear monomorphism $\alpha : \mathcal{A} \hookrightarrow \mathcal{O}_2 \otimes C(\tilde{X})$ and the $C_0(X)$ -algebra A is isomorphic to the $C_0(X)$ -subalgebra $\alpha(A)$ of $\mathcal{O}_2 \otimes C_0(X)$.

4 Concluding remarks

4.1 A $C(X)$ -subalgebra of $\mathcal{O}_2 \otimes C(X)$ is by construction exact and continuous. Conversely, if A is a non-zero exact separable unital continuous field of C^* -algebras over a perfect metrizable compact space X , one has by proposition 3.1 a $C(X)$ -extension

$$0 \rightarrow I \rightarrow F \rightarrow A \rightarrow 0$$

where F is a $C(X)$ -subalgebra of $\mathcal{O}_2 \otimes C(X)$. If the identity map $A \rightarrow A = F/I$ admits a $C(X)$ -linear completely positive lifting $A \rightarrow F$, the same method as the one used in theorem 3.2 will imply that the exact continuous field A is isomorphic to a $C(X)$ -subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$.

It is therefore interesting to know whether this map admits a $C(X)$ -linear completely positive lifting in the not discrete case.

4.2 Let us have a look at one of the technical problems involved, the Hahn-Banach type extension property in the continuous field framework for finite type $C(X)$ -submodules.

Let A be a separable unital continuous field of C^* -algebras over a compact metrizable space X and let D be a finitely generated $C(X)$ -submodule which is an operator system. Assume that $\phi : D \rightarrow C(X)$ is a $C(X)$ -linear unital completely positive map. Then for $x \in X$, there exists, thanks to [4, proposition 3.13], a continuous field of states Φ^x on A , i.e. a $C(X)$ -linear unital positive map from A to $C(X)$, such that for all $d \in D$,

$$\Phi^x(d)(x) = \phi(d)(x).$$

As a consequence, given $\varepsilon > 0$ and a finite subset \mathcal{F} of D , one can build by continuity and compactness a continuous field of states Φ on A such that

$$\max\{\|\Phi(d) - \phi(d)\|, d \in \mathcal{F}\} < \varepsilon.$$

But one cannot find in general any continuous field of states on A whose restriction to D is ϕ . Indeed, consider the $C(\tilde{\mathbb{N}})$ -algebra $A = \mathbb{C}^2 \otimes C(\tilde{\mathbb{N}})$ where $\tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of the space \mathbb{N} of positive integers. Define the positive element $a \in C_\infty(\tilde{\mathbb{N}})A \subset A$ by the formulas

$$a_n = \begin{cases} (\frac{1}{n+1}, 0) & \text{if } n \text{ even} \\ (0, \frac{1}{n+1}) & \text{if } n \text{ odd} \end{cases}$$

and let ϕ be the $C(\tilde{\mathbb{N}})$ -linear unital completely positive map with values in $C(\tilde{\mathbb{N}})$ defined on the $C(\tilde{\mathbb{N}})$ -submodule generated by the two $C(\tilde{\mathbb{N}})$ -linearly independent elements 1_A and a through the formula

$$\phi(a)(n) = \frac{1}{n+1} \text{ if } n < \infty \text{ and } \phi(a)(\infty) = 0.$$

Suppose that the continuous field of states Φ is a $C(\tilde{\mathbb{N}})$ -linear extension of ϕ to A . Then as A. Bauval already noticed it, one has the contradiction

$$\begin{aligned} 1 &= \Phi(1_A)(\infty) = \Phi((1, 0) \otimes 1)(\infty) + \Phi((0, 1) \otimes 1)(\infty) \\ &= \lim_{n \rightarrow \infty} \Phi((1, 0) \otimes 1)(2n+1) + \lim_{n \rightarrow \infty} \Phi((0, 1) \otimes 1)(2n) \\ &= 0 + 0 = 0. \end{aligned}$$

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In this appendix, we solve in proposition A.3 the lifting question raised in paragraph 4.1 through a continuous generalisation of joint work of E.G. Effros and U. Haagerup on lifting problems for C^* -algebras ([8], see also [22]). This result enables us to state the following characterisation of separable exact continuous fields of C^* -algebras:

Theorem A.1 *Let X be a compact metrizable space and A be a (unital) separable continuous field of C^* -algebras over X .*

Then the C^ -algebra A is exact if and only if there exists a (unital) monomorphism of $C(X)$ -algebras $A \hookrightarrow \mathcal{O}_2 \otimes C(X)$.*

Let us start with a technical $C(X)$ -linear version of Auerbach's theorem ([17, proposition 1.c.3]) for a continuous field of C^* -algebras A over X which gives us local bases over $C(X)$ with continuous coordinate maps for particular free $C(X)$ -submodules of finite type in A .

Define a $C(X)$ -operator system in A to be a $C(X)$ -submodule which is an operator system.

Lemma A.2 *([8, lemma 2.4]) Let A be a separable unital continuous field of C^* -algebras over a compact metrizable space X , $E \subset A$ be a $C(X)$ -operator system and assume that there exists an integer $n \in \mathbb{N}^*$ such that for all $x \in X$, the dimension $\dim E_x$ of the operator system $E_x \subset A_x$ equals n . Then the following holds.*

Given any point $x \in X$, there exist an open neighbourhood \mathcal{U} of x in X , self-adjoint $C(X)$ -linear contractions $\varphi_i : A \rightarrow C(X)$ and self-adjoint elements $f_i \in E$ with $\|f_i\| \leq 2$ for $1 \leq i \leq n$ such that

$$\forall a \in C_0(\mathcal{U})E, a = \sum_i \varphi_i(a) f_i.$$

Furthermore, there exists a continuous field of states $\Psi : A \rightarrow C(X)$ such that the restriction of the map $2n\Psi - id_A$ to the operator system E is completely positive.

Proof : Let us fix a point $x \in X$. Then there exist, thanks to Auerbach's theorem, a normal basis $\{r_1, \dots, r_n\}$ of the fibre E_x where each r_i is self adjoint and norm one hermitian functionals $\phi_j : A_x \rightarrow \mathbb{C}$, $1 \leq j \leq n$, such that $\phi_j(r_i) = \delta_{i,j}$.

Consider the polar decomposition $\phi_j = \phi_j^+ - \phi_j^-$ where ϕ_j^+ and ϕ_j^- are positive functionals such that $1 = \|\phi_j\| = \|\phi_j^+\| + \|\phi_j^-\|$. By [4] lemme 3.12, there exist $C(X)$ -linear positive maps φ_j^+ and $\varphi_j^- : A \rightarrow C(X)$ which extend the functionals ϕ_j^+ and ϕ_j^- on the fibre A_x to the $C(X)$ -algebra A with the property that $\varphi_j^+(1) = \|\phi_j^+\|$ and $\varphi_j^-(1) = \|\phi_j^-\|$. Take also n norm 1 self-adjoint elements $e_i \in E$ satisfying the equality $(e_i)_x = r_i$ and define the matrix $T = [\varphi_j(e_i)]_{i,j} \in M_n(\mathbb{R}) \otimes C(X)$.

One has by construction $T_x = 1_{M_n(\mathbb{R})}$; the set $\mathcal{U}_1 \subset X$ of points $y \in X$ for which the spectrum of $T_y \in M_n(\mathbb{R})$ is included in the open set $\{z \in \mathbb{C}, |z| > 1/2\}$ is therefore an open neighbourhood of x in X ([4, proposition 2.4 b)). If η is a continuous function on X with values in $[0, 1]$ which is 0 outside \mathcal{U}_1 and 1 on an open neighbourhood \mathcal{U} of the point $x \in X$, the self-adjoint elements f_1, \dots, f_n of norm less than 2 are then well defined in $C_0(\mathcal{U}_1)E$ by the formula

$$T \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \eta \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

and satisfy the relation $\varphi_j(f_i)(y) = \delta_{i,j}$ for each $y \in \mathcal{U}$ since the matrix T_y is invertible, whence the desired equality for every $a \in C_0(\mathcal{U})E$.

Keeping the same fixed point x , define now the continuous field of states $\Phi = \frac{1}{n} \sum_i (\varphi_i^+ + \varphi_i^-)$. Then one gets for all $a \in C_0(\mathcal{U})E$ the equality:

$$(2n\Phi - id_A)(a) = \sum_{1 \leq i \leq n} [\varphi_i^+(a)(2 - f_i) + \varphi_i^-(a)(2 + f_i)].$$

The restriction of the map $(2n\Phi - id_A)$ to $C_0(\mathcal{U})E$ is therefore completely positive and an appropriate partition of the unit $1_{C(X)}$ enables us to conclude. \square

Noticing that a $C(X)$ -linear map $\sigma : A \rightarrow B$ between $C(X)$ -algebras is completely positive if and only if each induced map $\sigma_x : A_x \rightarrow B_x$ is completely positive (see for instance [4, proposition 2.9]), the lemma A.2 allows us to state a continuous version of theorem 2.5 of [8]. Replacing then the continuous field A by $A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$ (where $M_{2^\infty}(\mathbb{C}) = \lim_{n \rightarrow \infty} M_{2^n}(\mathbb{C})$) and working with finitely generated $C(X)$ -operator systems $E_k \subset A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$ for which the function $x \mapsto \dim(E_k)_x$ is continuous, one derives the following desired $C(X)$ -linear completely positive lifting result.

Proposition A.3 ([8, theorem 3.4]) *Suppose that A and B are two unital separable exact continuous fields of C^* -algebras over a compact space X with $A = B/J$ for some nuclear ideal J in B .*

Then there exists a $C(X)$ -linear unital completely positive lifting $A \rightarrow B$ of id_A .

Proof : Let us define the two continuous fields $\mathcal{A} = A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$ and $\mathcal{B} = B \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X)$. It is clearly enough to find a $C(X)$ -linear unital completely positive cross section θ of the quotient morphism $\mathcal{B} \rightarrow \mathcal{A}$ (by [4, theorem 3.3]).

Consider a dense sequence $\{a_k\}$ in the self-adjoint part of \mathcal{A} where each a_k belongs to the dense subalgebra $A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X)$ of \mathcal{A} and $a_1 = 1$. Let us show that we may assume inductively that $C(X)$ -operator system E_n generated by the a_k , $1 \leq k \leq n$, satisfies the equality $\dim(E_n)_x = n$ for every $n \in \mathbb{N}^*$ and every $x \in X$. The inductive step is the following. Given $n \geq 2$, there exists by construction an integer l such that $E_n \subset A \oplus M_{2^l}(\mathbb{C}) \otimes C(X)$. Set $a'_n = a_n + 2^{-n-1}d_l \otimes 1_{C(X)}$ where

$$d_l = 1_{M_{2^l}(\mathbb{C})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2^{l+1}}(\mathbb{C}) \subset M_{2^\infty}(\mathbb{C}).$$

Then the $C(X)$ -module $E'_n = E_{n-1} + C(X)a'_n$ verifies for each $x \in X$ the equality $\dim(E'_n)_x = \dim(E_{n-1})_x + 1$.

Using proposition 1.3, one can now finish the proof by the same method as the one developed by E.G. Effros and U. Haagerup in [8].3 (see also [22, theorem 6.10]). \square

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