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To cite this version:
Etienne Blanchard. Subtriviality of continuous fields of nuclear C*-algebras, with an appendix by E. Kirchberg. Journal für die reine und angewandte Mathematik, 1997, 498, pp.133—149. hal-00922873

HAL Id: hal-00922873
https://hal.science/hal-00922873
Submitted on 31 Dec 2013

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Subtriviality of continuous fields of nuclear $C^*$-algebras

By Etienne Blanchard

Abstract

We extend in this paper the characterisation of a separable nuclear $C^*$-algebra given by Kirchberg proving that given a unital separable continuous field of nuclear $C^*$-algebras $A$ over a compact metrizable space $X$, the $C(X)$-algebra $A$ is isomorphic to a unital $C(X)$-subalgebra of the trivial continuous field $O_2 \otimes C(X)$, image of $O_2 \otimes C(X)$ by a norm one projection.

AMS classification: 46L05.

0 Introduction

In order to study deformations in the $C^*$-algebraic framework, Dixmier introduced the notion of continuous field of $C^*$-algebras over a locally compact space ([7]). In the same way as there is a faithful representation in a Hilbert space for any $C^*$-algebra thanks to the Gelfand–Naimark–Segal construction, a separable continuous field of $C^*$-algebras $A$ over a compact metrizable space $X$ always admits a continuous field of faithful representations $\pi$ in a Hilbert $C(X)$-module, i.e. there exists a family of representations $\{\pi_x, x \in X\}$, in a separable Hilbert space $H$ which factorize through a faithful representation of the fibre $A_x$ such that for each $a \in A$, the map $x \mapsto \pi_x(a)$ is strongly continuous ([4, théorème 3.3]).

In a work on tensor products over $C(X)$ of continuous fields of $C^*$-algebras over $X$ ([16]), Kirchberg and Wassermann raised the question of whether the continuous field of $C^*$-algebras $A$ could be subtrivialized, i.e. whether one could find a continuous field of faithful representations $\pi$ such that the map $x \mapsto \pi_x(a) \in L(H)$ is actually norm continuous for all $a$ in $A$. In this case, given any $C^*$-algebra $B$, the minimal tensor product $A \otimes B$ is a $C(X)$-subalgebra of the trivial continuous field $[L(H) \otimes B] \otimes C(X)$ and is therefore a continuous field with fibres $(A \otimes B)_x = A_x \otimes B$. They proved that a non-exact continuous field with exact fibres cannot be subtrivialized and they constructed such examples.

The non-trivial example of the continuous field of rotation algebras over the unit circle $\mathbb{T}$ had already been studied by Haagerup and Rørdam in [10]. More precisely, they constructed continuous functions $u, v$ from $\mathbb{T}$ to the unitary group $U(H)$ of the infinite-dimensional separable Hilbert space $H$ satisfying the commutation relation $u_t v_t = t v_t u_t$.
for all \( t \in \mathbb{T} \) and the uniform continuity condition \( \max \{ \| u_t - u_{t'} \|, \| v_t - v_{t'} \| \} < C' |t - t'|^{1/2} \) where \( C' \) is a computable constant.

Our purpose in the present paper is to show that the subtrivialization is always possible in the nuclear separable case through a generalisation of the following theorem of Kirchberg using \( RKK \)-theory arguments:

**Theorem 0.1** ([15]) A unital separable \( C^* \)-algebra \( A \) is exact if and only if it is isomorphic to a \( C^* \)-subalgebra of \( O_2 \). Moreover the \( C^* \)-algebra \( A \) is nuclear if and only if \( A \) is isomorphic to a \( C^* \)-subalgebra of \( O_2 \) containing the unit \( 1_{O_2} \) of \( O_2 \), image of \( O_2 \) by a unital completely positive projection.

As a matter of fact, we get an equivalent characterisation of nuclear separable continuous fields of \( C^* \)-algebras (theorem 3.2) which is made possible thanks to \( C(\mathcal{X}) \)-linear homotopy invariance of the bifunctor \( RKK(\mathcal{X}; -,-) \) (theorem 1.6) and \( C(\mathcal{X}) \)-linear Weyl-von Neumann absorption results (proposition 2.5). This also enables us to have a better understanding of the characterisation of separable continuous fields of nuclear \( C^* \)-algebras given by Bauval in [2].

In an added appendix, the corresponding characterisation of exact separable continuous fields of \( C^* \)-algebras as \( C(\mathcal{X}) \)-subalgebras of \( O_2 \otimes C(\mathcal{X}) \) given by Eberhard Kirchberg is described (theorem A.1).

I would like to thank E. Kirchberg for his enlightenment on the exact case. I also want to express my gratitude to C. Anantharaman-Delaroche and J. Cuntz for fruitful discussions.

## 1 Preliminaries

### 1.1 \( C(\mathcal{X}) \)-algebras

Let \( X \) be a compact Hausdorff space and \( C(\mathcal{X}) \) be the \( C^* \)-algebra of continuous functions on \( X \) with complex values. We start by recalling the following definition.

**Definition 1.1** ([13]) A \( C(\mathcal{X}) \)-algebra is a \( C^* \)-algebra \( A \) endowed with a unital morphism from \( C(\mathcal{X}) \) in the centre of the multiplier algebra \( M(A) \) of \( A \).

**Remark:** We do not assume that \( C(\mathcal{X}) \) embeds into \( M(A) \). For instance, there is a natural structure of \( C([0,2]) \)-algebra on the \( C^* \)-algebra \( C([0,1]) \).

For \( x \in X \), define the kernel \( C_x(\mathcal{X}) \) of the evaluation map \( ev_x : C(\mathcal{X}) \to \mathbb{C} \) at \( x \); denote by \( A_x \) the quotient of a \( C(\mathcal{X}) \)-algebra \( A \) by the closed ideal \( C_x(\mathcal{X})A \) and by \( a_x \) the image of an element \( a \in A \) in the fibre \( A_x \). Then the function

\[
x \mapsto \|a_x\| = \inf \{\|1 - f + f(x)a\|, f \in C(\mathcal{X})\}
\]

is upper semi-continuous for any \( a \in A \) and the \( C(\mathcal{X}) \)-algebra \( A \) is said to be a continuous field of \( C^* \)-algebras over \( X \) if the function \( x \mapsto \|a_x\| \) is actually continuous for every \( a \in A \) ([7]).
Examples 1. If $A$ is a $C(X)$-algebra and $D$ is a $C^*$-algebra, the spatial tensor product
$B = A \otimes D$ is naturally endowed with a structure of $C(X)$-algebra through the map
$f \in C(X) \mapsto f \otimes 1_M(D) \in M(A \otimes D)$. In particular, if $A = C(X)$, the tensor product
$B$ is a trivial continuous field over $X$ with constant fibre $B_x \simeq D$.

2. Given a $C(X)$-algebra $A$, define the unital $C(X)$-algebra $A$ generated by $A$ and
$u[C(X)]$ in $M[A \otimes C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$. It
defines a continuous field of $C^*$-algebras over $X$ if and only if the $C(X)$-algebra $A$ is
continuous ([4, proposition 3.2]).

Remark: If $A$ is a separable continuous field of non-zero $C^*$-algebras (not necessarily
unital) over the compact Hausdorff space $X$, the positive cone $C(X)_+$ and so the $C^*$-
algebra $C(X)$ are separable. Hence, the topological space $X$ is metrizable.

Definition 1.2 ([4, 5]) Given a continuous field of $C^*$-algebras $A$ over the compact
Hausdorff space $X$, a continuous field of representations of a $C(X)$-algebra $D$ in
the multiplier $C^*$-algebra $M(A)$ of $A$ is a $C(X)$-linear morphism $\pi : D \to M(A)$, i.e. for
each $x \in X$, the induced representation $\pi_x$ of $D$ in $M(A_x)$ factorizes through the fibre
$D_x$.

If the $C(X)$-algebra $D$ admits a continuous field of faithful representations $\pi$ in the
$C(X)$-algebra $M(A)$ where $A$ is a continuous field over $X$, i.e. the induced representa-
tion of the fibre $D_x$ in $M(A_x)$ is faithful for every point $x \in X$, the function

$$x \mapsto \|\pi_x(d)\| = \sup\{\|\pi_x(a)\|, a \in A \text{ such that } \|a\| \leq 1\}$$

is lower semi-continuous for all $d \in D$ and the $C(X)$-algebra $D$ is therefore continuous.

In particular a separable $C(X)$-algebra $D$ is continuous if and only if there exists a
Hilbert $C(X)$-module $\mathcal{E}$ such that $D$ admits a continuous field of faithful representations
in the multiplier algebra $M(\mathcal{K}^{\infty}(\mathcal{E})) = \mathcal{L}(\mathcal{E})$ of the continuous field over $X$ of compact
operators $\mathcal{K}^{\infty}(\mathcal{E})$ acting on $\mathcal{E}$ ([4, théorème 3.3]).

Let us also mention the characterisation of separable continuous fields of nuclear
$C^*$-algebras over a compact metrizable space $X$ given by Bauval in [2] using a natural
$C(X)$-linear version of nuclearity introduced by Kasparov and Skandalis in [14]§6.2 : a
$C(X)$-linear completely positive $\sigma$ from a $C(X)$-algebra $A$ into a $C(X)$-algebra $B$ is
said to be $C(X)$-nuclear if and only if given any compact set $F$ in $A$ and any strictly
positive real number $\varepsilon$, there exist an integer $k$ and $C(X)$-linear completely positive
contractions $T : A \to M_k(\mathbb{C}) \otimes C(X)$ and $S : M_k(\mathbb{C}) \otimes C(X) \to B$ such that for all
$a \in F$, one has the inequality

$$\|\sigma(a) - (S \circ T)(a)\| < \varepsilon.$$ 

One can then state the following results. The first assertion is a simple $C(X)$-linear
reformulation of the Choi-Effros theorem and the second one is due to Bauval.

Proposition 1.3 Let $X$ be a compact metrizable space and $A$ be a separable $C(X)$-
algebra.
1. ([14] §6.2) Given a $C(X)$-algebra $B$ and a closed ideal $J \subset B$, any contractive $C(X)$-nuclear map $A \to B/J$ admits a contractive $C(X)$-linear completely positive lift $A \to B$.

2. ([2, théorème 7.2]) The $C(X)$-algebra $A$ is a continuous fields of nuclear $C^*$-algebras over $X$ if and only if the identity map $\text{id}_A : A \to A$ is $C(X)$-nuclear.

**Remark:** In assertion 1., the ideal $J = (C(X)B)J = C(X)J$ is a $C(X)$-algebra.

### 1.2 $C(X)$-extensions

Given a compact Hausdorff space $X$, we introduce a natural $C(X)$-linear version of the semi-group $\text{Ext}(-,-)$ defined by Kasparov ([12, 13]).

Call a morphism of $C(X)$-algebras a $*$-homomorphism between $C(X)$-algebras which is $C(X)$-linear.

**Definition 1.4** A $C(X)$-extension of a $C(X)$-algebra $A$ by a $C(X)$-algebra $B$ is a short exact sequence

$$0 \to B \to D \xrightarrow{\pi} A \to 0$$

in the category of $C(X)$-algebras. The $C(X)$-extension is said to be trivial if the map $\pi$ admits a cross section $s : A \to D$ which is a morphism of $C(X)$-algebras.

As in the $C^*$-algebraic case a $C(X)$-extension $0 \to B \to D \to A \to 0$ of $A$ by $B$ defines unambiguously an homomorphism from $D$ to the multiplier algebra $M(B)$ of $B$, which gives a morphism of $C(X)$-algebras $\sigma : A \to M(B)/B$ (called the Busby invariant of the extension) and the $C(X)$-extension is trivial if and only if the map $\sigma$ lifts to a morphism of $C(X)$-algebras $A \to M(B)$. Conversely, given a morphism of $C(X)$-algebras $\sigma : A \to M(B)/B$, setting $D = \{(a,m) \in A \times M(B), \sigma(a) = q(m)\}$ where $q$ is the quotient map $M(B) \to M(B)/B$, one has a $C(X)$-extension $0 \to B \to D \to A \to 0$ (see [12] §7).

**Remark:** A $C(X)$-extension $0 \to B \to D \to A \to 0$ induces for every $x \in X$ a $C^*$-extension $0 \to B_x \to D_x \to A_x \to 0$.

In order to define the sum of two $C(X)$-extensions, recall that the Cuntz algebra $O_2$ is the unital $C^*$-algebra generated by two orthogonal isometries $s_1$ and $s_2$ subject to the relation $1 = s_1s_1^* + s_2s_2^*$ ([6]). Then if $\mathcal{K}$ is the $C^*$-algebra of compact operators on the infinite-dimensional separable Hilbert space, one defines the sum of two $C(X)$-extensions $\sigma_1$ and $\sigma_2$ of the $C(X)$-algebra $A$ by the stable $C(X)$-algebra $\mathcal{K} \otimes B$ through the choice of a unital copy of $O_2$ in the multiplier algebra $M(\mathcal{K})$ of $\mathcal{K}$ to be the $C(X)$-extension

$$\sigma_1 \oplus \sigma_2 : a \mapsto q(s_1 \otimes 1)\sigma_1(a)q(s_1^* \otimes 1) + q(s_2 \otimes 1)\sigma_2(a)q(s_2^* \otimes 1) \in M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B),$$

where $q$ is the quotient map $M(\mathcal{K} \otimes B) \to M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B).$
Definition 1.5 Given a compact Hausdorff space $X$ and two $C(X)$-algebras $A$ and $B$, $\text{Ext}(X;A,B)$ is the semi-group of $C(X)$-extensions of $A$ by $\mathcal{K} \otimes B$ divided by the equivalence relation $\sim$ where $\sigma_1 \sim \sigma_2$ if there exist a unitary $U \in M(\mathcal{K} \otimes B)$ of image $q(U)$ in the quotient $M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$ and two trivial $C(X)$-extensions $\pi_1$ and $\pi_2$ such that for all $a \in A$,

$$(\sigma_2 \oplus \pi_2)(a) = q(U)^*(\sigma_1 \oplus \pi_1)(a)q(U) \text{ (in } M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)).$$

Let $\text{Ext}(X;A,B)^{-1}$ be the group of invertible elements of $\text{Ext}(X;A,B)$, i.e. the group of classes of $C(X)$-extension $\sigma$ such that there exists a $C(X)$-extension $\theta$ with $\sigma \oplus \theta$ trivial. One can generalise Kasparov’s theorem of homotopy invariance of the group $\text{Ext}(A,B)^{-1}$ to the framework of $C(X)$-algebras as follows.

Theorem 1.6 ([12]) Assume that $A$ is a separable $C(X)$-algebra and that $B$ is a $\sigma$-unital $C(X)$-algebra. Then the group $\text{Ext}(X;A,B)^{-1}$ is isomorphic to the group $\mathcal{R}\mathcal{K}\mathcal{K}^1(X;A,B)$ and is therefore $C(X)$-linear homotopy invariant in both entries $A$ and $B$.

Proof: Let us first make the following observation. Given a $C(X)$-algebra $B$ and a Hilbert $B$-module $\mathcal{E}$, denote by $\mathcal{L}(\mathcal{E})$ the set of bounded $B$-linear operators on $\mathcal{E}$ which admit an adjoint ([11]). Then any operator $T \in \mathcal{L}(\mathcal{E})$ is $B$-linear and so $C(X)$-linear. This argument provides a natural extension of the Stinespring-Kasparov theorem ([12]) to the framework of $C(X)$-algebras. Consequently, if $A$ is a separable $C(X)$-algebra and $B$ is a $\sigma$-unital $C(X)$-algebra, the class of a $C(X)$-extension $\sigma : A \to M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B)$ is invertible in $\text{Ext}(X;A,B)$ if and only if there is a $C(X)$-linear completely positive contractive lift $A \to M(\mathcal{K} \otimes B)$.

Let $\mathcal{R}\mathcal{E}(X;A,B)$ be the set of Kasparov $C(X)$-$A,B$-bimodules ([13, definition 2.19]), i.e. the set of Kasparov $A,B$ bimodules $(\mathcal{E},F)$ such that the representation $A \to \mathcal{L}(\mathcal{E})$ is a $C(X)$-representation. Call a $C(X)$-linear operator homotopy an element $\{\mathcal{E},F_t,0 \leq t \leq 1\} \in \mathcal{R}\mathcal{E}(X;A,B \otimes C([0,1]))$ such that $t \mapsto F_t$ is norm continuous and define on $\mathcal{R}\mathcal{E}(X;A,B)$ the equivalence relation corresponding to the one defined by Skandalis in [18, definition 2]. The constructions given by Kasparov in [12, section 7] imply that, if the $C(X)$-algebra $B$ is $\sigma$-unital, the group of equivalence classes $\mathcal{R}\mathcal{K}\mathcal{K}(X;A,B \otimes \mathcal{C}_1)$ is isomorphic to $\text{Ext}(X;A,B)^{-1}$, where $\mathcal{C}_1$ is the first (graded) Clifford algebra.

On the other hand, given two graded $C(X)$-algebras $A$ and $B$ with $A$ separable, the different steps of the demonstration of [18, theorem 19] provide us with an isomorphism between the two groups $\mathcal{R}\mathcal{K}\mathcal{K}(X;A,B)$ and $\mathcal{R}\mathcal{K}\mathcal{K}(X;A,B)$ since proposition 2.21 of [13] defines an intersection product in $\mathcal{R}\mathcal{K}\mathcal{K}$-theory and lemma 18 of [18] gives us the equality

$$(ev_0 \otimes id_{C(X)})^*(1_{C(X)}) = (ev_1 \otimes id_{C(X)})^*(1_{C(X)}) \text{ in } \mathcal{R}\mathcal{K}\mathcal{K}(X;C([0,1]) \otimes C(X), C(X)),$$

where $1_{C(X)}$ is the Kasparov $C(X), C(X)$-bimodule $(C(X),0)$ and $ev_t : C([0,1]) \to \mathcal{C}$ is the evaluation map at $t \in [0,1]$. □

Remarks: 1. Kuiper’s theorem implies that the law of addition on the abelian group $\text{Ext}(X;A,B)^{-1}$ is independent of the choice of the unital copy of $\mathcal{O}_2$ in $M(\mathcal{K})$. 5
2. If \( A \) is a separable nuclear continuous field of \( \mathcal{C}^* \)-algebras over \( X \) and \( B \) is a \( C(X) \)-algebra, every \( C(X) \)-linear morphism from \( A \) to the quotient \( M(\mathcal{K} \otimes B)/(\mathcal{K} \otimes B) \) is \( C(X) \)-nuclear and therefore admits a \( C(X) \)-linear completely positive lifting \( A \to M(\mathcal{K} \otimes B) \) thanks to proposition 1.3. Accordingly one has the equality
\[
\text{Ext}(X; A, B)^{-1} = \text{Ext}(X; A, B).
\]

2 An absorption result

In this section we prove a continuous generalisation of a statement contained in [15] which will enable us to get a \( C(X) \)-linear Weyl-von Neumann type result (proposition 2.5). Let us start with the following definition of Cuntz.

**Definition 2.1** ([6]) A simple \( \mathcal{C}^* \)-algebra \( A \) distinct from \( \mathbb{C} \) is said to be purely infinite if and only if for any non-zero \( a, b \in A \), there exist elements \( x, y \in A \) such that \( a = xby \).

Then, we can state a proposition from Kirchberg’s classification work, based on Glimm’s lemma ([7], § 11.2). A sketch of proof can also be found in [1, proposition 5.1].

**Proposition 2.2** ([15]) Let \( A \) be a purely infinite simple \( \mathcal{C}^* \)-algebra and \( D \) be a separable \( \mathcal{C}^* \)-subalgebra of \( M(A) \). Assume that \( V : D \to A \) is a nuclear contraction.

Then there exists a sequence \( (a_n) \) of elements in \( A \) of norm less than 1 such that \( V(d) = \lim_{n \to \infty} a_n^* d a_n \) for all \( d \in D \).

**Remark:** A simple ring has by definition exactly two distinct two sided ideals and is therefore non-zero.

**Corollary 2.3** Let \( A \) be a continuous field of purely infinite simple \( \mathcal{C}^* \)-algebras over a compact Hausdorff space \( X \) and assume that \( D \) is a separable \( C(X) \)-subalgebra of the multiplier algebra \( M(A) \) such that there is a unital \( C(X) \)-embedding of the \( \mathcal{C}^* \)-algebra \( \mathcal{O}_\infty \otimes C(X) \) in the commutant \( D' \) of \( D \) in \( M(A) \) and the identity map \( \text{id}_D : D \to M(A) \) is a continuous field of faithful representations.

If \( V : D \to A \) is a \( C(X) \)-nuclear contraction, there exists a sequence \( (a_n) \) in the unit ball of \( A \) with the property that for all \( d \in D \),
\[
V(d) = \lim_{n \to \infty} a_n^* d a_n.
\]

**Proof:** If \( F \) is a compact generating set for \( D \), it is enough to prove that given a strictly positive real number \( \varepsilon > 0 \), there exists an element \( a \) in the unit ball of \( A \) such that \( \|V(d) - a^* d a\| < \varepsilon \) for all \( d \in F \).

For \( x \in X \), the fibre \( A_x \) is a purely infinite simple \( \mathcal{C}^* \)-algebra and the map \( d \mapsto V(d)_x \in A_x \) factorizes through \( D_x \simeq (\text{id}_D)_x(D) \subset M(A_x) \) since \( \text{id}_D \) is a continuous field of faithful representations. As a consequence, the previous proposition implies that we can find an element \( g \in A \) with \( \|g\| \leq 1 \) satisfying for all \( d \in F \) the inequality
\[
\|[V(d) - g^* d g]_x\| < \varepsilon.
\]
Thus, by upper semi-continuity and compactness, there exist a finite open covering \( \{ U_1, \ldots, U_n \} \) of the space \( X \) and elements \( g_1, \ldots, g_n \) in the unit ball of \( A \) such that for all \( d \in F \) and \( x \in U_i, 1 \leq i \leq n \),

\[
\| [V(d) - g_i^* d g_i]_x \| < \varepsilon.
\]

Choose \( n \) orthogonal isometries \( w_1, \ldots, w_n \) in the \( \text{C}^* \)-algebra \( \mathcal{O}_\infty \otimes 1_{C(X)} \subset D' \) and let \( \{ \phi_i \} \) be a partition of the unit \( 1_{C(X)} \) subordinate to the covering \( \{ U_i \} \) of \( X \). The element \( a = \sum_i \phi_i^{1/2} w_i g_i \in A \) verifies:

1. \( a^* a = \sum_{i,j} \sqrt{\phi_i \phi_j} g_i^* w_i^* w_j g_j = \sum_i \phi_i g_i^* g_i \leq 1_{M(A)} \),

2. for \( d \in F \) and \( x \in X \), \( \| [V(d) - a^* da]_x \| \leq \sum_i \phi_i(x) \| [V(d) - g_i^* d g_i]_x \| < \varepsilon \). \( \square \)

Let us mention the following technical corollary which will be needed in theorem 3.2.

**Corollary 2.4** If \( p \in \mathcal{O}_2 \otimes C(X) \) is a projection such that for all points \( x \in X \), \( p_x \) is non-zero, then there exists an isometry \( u \in \mathcal{O}_2 \otimes C(X) \) such that \( p = uu^* \).

**Proof**: Let \( \mathcal{D}_2 = \lim_{n \to \infty} \mathcal{O}_2 \otimes^n \) be the infinite tensor product of \( \mathcal{O}_2 \).

Given a projection \( q \in \mathcal{D}_2 \otimes C(X) \) such that \( \| q_x \| = 1 \) for all \( x \in X \), we first show that there exists an element \( v \in \mathcal{D}_2 \otimes C(X) \) satisfying \( 1_{\mathcal{D}_2 \otimes C(X)} = v^* q v \). Namely, by density of the algebraic tensor product

\[
\bigcup_n \mathcal{O}_2 \otimes^n C(X) = \bigcup_n \left[ \mathcal{O}_2 \otimes^n C(X) \right]
\]

in the \( \text{C}^* \)-algebra \( \mathcal{D}_2 \otimes C(X) \) and functional calculus one can find an integer \( n > 0 \) and a projection \( r \in \mathcal{O}_2 \otimes^n C(X) \subset \mathcal{D}_2 \otimes C(X) \) such that \( \| q - r \| < 1 \), which implies that \( r = s^* q s \) for some element \( s \in \mathcal{D}_2 \otimes C(X) \). Take then a faithful state \( \varphi \) on \( \mathcal{O}_2 \otimes^n \) and consider the \( C(X) \)-linear completely positive map

\[
V : [\mathcal{O}_2 \otimes^n \otimes 1_{\mathcal{O}_2}] \otimes C(X) \to \mathcal{O}_2 \otimes^{n+1} \otimes C(X)
\]

defined by the formula \( V(d) = (\varphi \otimes id_{C(X)})(d)1_{\mathcal{O}_2 \otimes^{n+1} \otimes C(X)} \) for \( d \in [\mathcal{O}_2 \otimes^n \otimes 1_{\mathcal{O}_2}] \otimes C(X) \approx \mathcal{O}_2 \otimes^n \otimes C(X) \). According to corollary 2.3, there exists an element \( t \in \mathcal{O}_2 \otimes^{n+1} \otimes C(X) \) such that

\[
1_{\mathcal{D}_2 \otimes C(X)} = 1_{\mathcal{O}_2 \otimes^{n+1} \otimes C(X)} = t^* r t = (st)^* q (st)
\]

Consider now the set \( \mathcal{P} \) of projections \( p \in \mathcal{O}_2 \otimes C(X) \) such that \( p_x \neq 0 \) for all points \( x \in X \). If \( p \) belongs to \( \mathcal{P} \), there exists an isometry \( v \in \mathcal{O}_2 \otimes C(X) \) such that \( p \geq vv^* \) since the \( K \)-trivial purely infinite separable unital nuclear \( \text{C}^* \)-algebra \( \mathcal{D}_2 \) satisfying the U.C.T. is isomorphic to \( \mathcal{O}_2 \) ([15]). As a consequence, if \( t \) is the isometry \( t = v(s_1 \otimes 1)v^* \), the projection \( r = tt^* \) (Murray-von Neumann equivalent to \( 1_{\mathcal{O}_2 \otimes C(X)} \)) verifies

\[
p - r \geq r' = v(s_2 s_2^* \otimes 1) v^* \in \mathcal{P}.
\]
The non-empty set $\mathcal{P}$ therefore satisfies the conditions $(\pi_1)-(\pi_4)$ defined by Cuntz in [6]. But the $C^*$-algebra $O_2 \otimes C(X)$ is $K_0$-triviality thanks to [6, theorem 2.3] and the theorem 1.4 of [6] enables us to conclude. □

One now deduces from corollary 2.3 the following absorption results ([21, 12, 15]):

**Proposition 2.5** Let $A$ be a $\sigma$-unital continuous field of purely infinite simple nuclear $C^*$-algebras over a compact Hausdorff space $X$ and let $K$ be the $C^*$-algebra of compact operators on the separable Hilbert space $H$. Denote by $q$ the quotient map $M(K \otimes A) \to M(K \otimes A)/(K \otimes A)$.

1. Assume that $D$ is a unital separable $C(X)$-subalgebra of the multiplier algebra $M(K \otimes A)$ with same unit such that there is a unital $C(X)$-embedding of the $C(X)$-algebra $O_\infty \otimes C(X)$ in the commutant of $D$ in $M(K \otimes A)$ and the identity map $id_D$ is a continuous field of faithful representations of $D$ in $M(K \otimes A)$.

   (a) If $V$ is a unital $C(X)$-linear completely positive map from $D$ in $M(K \otimes A)$ which is zero on the intersection $D \cap (K \otimes A)$, there exists a sequence of isometries $s_n$ in $M(K \otimes A)$ such that for every $d \in D$,

   $$V(d) - s_n^* ds_n \in K \otimes A \text{ and } V(d) = \lim_n s_n^* ds_n.$$

   (b) If $\pi$ is a unital morphism of $C(X)$-algebras from $D$ into $M(K \otimes A)$ which is zero on the intersection $D \cap (K \otimes A)$, there exists a sequence of unitaries $u_n$ in $M(K \otimes A)$ such that for every $d \in D$,

   $$(d \oplus \pi(d)) - u_n^* du_n \in K \otimes A \text{ and } (d \oplus \pi(d)) = \lim_n u_n^* du_n.$$

   (c) Let $B$ be a $C(X)$-algebra and assume that the quotient $D/(D \cap (K \otimes A))$ is isomorphic to the $C(X)$-algebra $B$, where $B$ is the unital $C(X)$-algebra generated by $C(X)$ and $B$ in $M[B \oplus C(X)]$ ([4, définition 2.7]).

   Then, if $\pi : B \to M(K \otimes A)$ is a $C(X)$-linear homomorphism, there exists a unitary $U \in M(K \otimes A)$ such that for all $b \in B \subset M(K \otimes A)/(K \otimes A),

   $$b \oplus (q \circ \pi)(b) = q(U)^* b q(U).$$

2. Assume that the continuous field $A$ is separable and let $D$ be a separable $C(X)$-subalgebra of $M(A)$ containing $A$ such that the identity representation $D \to M(A)$ is a continuous field of faithful representations and there is a unital $C(X)$-embedding of the $C(X)$-algebra $O_\infty \otimes C(X)$ in the commutant of $D$ in $M(A)$. Define the quotient $C(X)$-algebra $B = D/A$.

   If $\pi : K \otimes B \to M(K \otimes A)$ is a morphism of $C(X)$-algebras, there exists a unitary $U \in M(K \otimes A)$ such that for all $b \in K \otimes B \subset M(K \otimes A)/(K \otimes A),

   $$b \oplus (q \circ \pi)(b) = q(U)^* b q(U).$$
**Proof:** 1. It derives from corollary 2.3 by the same method as the one developed by Kasparov in [11, theorem 5 and 6]. Nevertheless, for the convenience of the reader we describe the different steps of the demonstration.

1.a) Let $F$ be a compact generating set for $D$ containing the unit $1_{M(K \otimes A)}$. Then given a real number $\varepsilon > 0$, it is enough to find an element $a \in M(K \otimes A)$ such that $V(d) - a^*da \in K \otimes A$ and $\|V(d) - a^*da\| < 3\varepsilon$ for all $d \in F$.

Let $\{e_n\}$ be an increasing, positive, quasicentral, countable approximate unit in the ideal $K \otimes A$ of the $C^*$-algebra generated by $K \otimes A + V(D)$. If we set $f_0 = (e_0)^{1/2}$ and $f_k = (e_k - e_{k-1})^{1/2}$ for $k \geq 1$, we can then assume, passing to a subsequence of $(e_n)$ if necessary, that $\|V(d)f_k - f_k V(d)\| < 2^{-k}\varepsilon$ for all $k \in \mathbb{N}$ and $d \in F$. This implies that the series $\sum_k [V(d)f_k - f_k V(d)]f_k$ is convergent in $K \otimes A$ and its norm is less than $\varepsilon$. Furthermore, the series $\sum_k [f_k V(d)f_k]_k$ is strictly convergent in $M(K \otimes A)$ for all $d \in F$ since $\sum_k f_k^2$ is strictly convergent to $1$.

Notice now that the maps $V_k(d) = f_k V(d)f_k$ are all $C(X)$-nuclear since the separable continuous field $K \otimes A$ is nuclear. The corollary 2.3 therefore enables us to choose by induction $a_k \in K \otimes A$ satisfying the following conditions:

1. $\forall d \in F \quad \|V_k(d) - a_k^*dak\| < 2^{-k}\varepsilon$,
2. $\forall d \in F, \forall l < k \quad \|a_l^*dak\| < 2^{-l-k}\varepsilon$,
3. $\sum_k a_k$ is strictly convergent toward an element $a \in M(A)$.

One then checks as in [11, theorem 5] that the limit $a$ satisfies the desired properties.

1.b) Take a compact generating $F$ for $D$ containing $1_{M(K \otimes A)}$ and consider the homomorphism $\pi' = 1 \otimes \pi : D \to M(K \otimes (K \otimes A)) \simeq M(K \otimes A)$. Given $\delta > 0$, one can find, thank to the previous assertion, an isometry $s \in M(K \otimes A)$ such that

$$s^*ds - \pi'(d) \in K \otimes A \text{ and } \|s^*ds - \pi'(d)\| < \delta \text{ for all } d \in K^*K.$$ 

As a consequence, if we fix $\varepsilon > 0$, the choice of $\delta$ small enough gives us the inequality $\|pd - dp\| < \varepsilon$, and so $\|d - [pd + p^d + p^]|| < 2\varepsilon$ for all $d \in F$, where $p = ss^*$ and $p^+ = 1 - p$.

Define the unital map $\Theta : D \to M(p^+ (K \otimes A)p^+)$ by the formula $\Theta(d) = p^+ dp^+$. According to the stabilisation theorem of Kasparov ([11, theorem 2]), one can construct a unitary $w \in M(K \otimes A)$ verifying for all $d \in F$ the inequality

$$\|d - w^*[\pi'(d) \oplus \Theta(d)]w\| < 3\varepsilon.$$ 

To finish the demonstration, notice that the two homomorphisms $\pi'$ and $\pi' \oplus \pi$ are unitarily equivalent.

1.c) Consider the unital extension $\tilde{\pi}$ of $\pi$ to $B$. Then, the morphism $\tilde{\pi} \circ q : D \to M(K \otimes A)$ reduces the demonstration to the previous assertion.

2. The identity representation of the unital $C(X)$-algebra $D = (K \otimes D) + C(X) \subset M(K \otimes A)$ is clearly a continuous field of faithful representations since the unital $C(X)$-representation $C(X) \to M(A)$ is a continuous field of faithful representations. Extend
the map $\pi : \mathcal{K} \otimes B = (\mathcal{K} \otimes D)/(\mathcal{K} \otimes A) \to M(\mathcal{K} \otimes A)$ to a unital morphism of $C(X)$-algebras $\tilde{\pi} : D/(\mathcal{K} \otimes A) \to M(\mathcal{K} \otimes A)$. Applying assertion 1.b) to the unital homomorphism $d \mapsto (\tilde{\pi} \circ q)(d)$ from the $C(X)$-subalgebra $D \subset M(\mathcal{K} \otimes A)$ to the multiplier algebra $M(\mathcal{K} \otimes A)$ now leads to the desired conclusion. \hfill $\square$

3 The subtriviality

Given a separable continuous field of nuclear $C^*$-algebras $A$ over $X$, the strategy to prove the subtriviality of the $C(X)$-algebra $A$ will be the same as the one developed by Kirchberg in [15] to prove theorem 0.1 whose main ideas of demonstration are also explained in [1, Théorème 6.1]. We associate to $A$ a $C(X)$-extension by an hereditary $C^*$-subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$ (proposition 3.1) and then prove that after stabilisation, this $C(X)$-extension splits by $RKK$-theory arguments (theorem 3.2).

3.1 Let us construct the fundamental $C(X)$-extension associated to an exact separable continuous field of $C^*$-algebras.

**Proposition 3.1** Given a compact Hausdorff space $X$ and a non-zero separable unital exact $C(X)$-algebra $A$, there exist a unital $C(X)$-subalgebra $F$ of $\mathcal{O}_2 \otimes C(X)$ with same unit and an hereditary subalgebra $I$ of $\mathcal{O}_2 \otimes C(X)$ such that $I$ is an ideal in $F$ and the $C(X)$-algebra $A$ is isomorphic to the quotient $C(X)$-algebra $F/I$.

Furthermore, if the topological space $X$ is perfect (i.e. without any isolated point) and the $C(X)$-algebra $A$ is continuous, the canonical map $F \to M(I)$ is a continuous field of faithful representations.

**Proof:** Thanks to the characterisation of separable exact $C^*$-algebras obtained by Kirchberg (theorem 0.1), one may assume that the $C^*$-algebra $A$ is a $C^*$-subalgebra of $\mathcal{O}_2$ containing the unit of $\mathcal{O}_2$.

Let $G \subset \mathcal{O}_2 \otimes C(X)$ be the trivial continuous field $A \otimes C(X)$ over $X$. Then the kernel of the $C(X)$-linear morphism $\pi : G \to A$ defined by $\pi(a \otimes f) = fa$ is the ideal $J = C_\Delta(X \times X)G$ where $C_\Delta(X \times X)$ is the ideal in $C(X \times X)$ of functions which are zero on the diagonal. Indeed suppose that $T \in G$ verifies $\pi(T) = 0$. Then given $\varepsilon > 0$, take a finite number of elements $a_i \in A$, $f_i \in C(X)$ such that $\|T - \sum_i a_i \otimes f_i\| < \varepsilon$; one has $\|T - \sum_i (1 \otimes f_i - f_i \otimes 1)(a_i \otimes 1)\| < \varepsilon + \|\pi(\sum_i a_i \otimes f_i)\| < 2\varepsilon$.

Define then the hereditary subalgebra $I = J[\mathcal{O}_2 \otimes C(X)]J$ in $\mathcal{O}_2 \otimes C(X)$ generated by $J$. It is a $C(X)$-algebra since it is closed by Cohen theorem (see e.g. [4, proposition 1.8]) and the product $(1\mathcal{O}_2 \otimes f)(bc) = b(1\mathcal{O}_2 \otimes f)c$ belongs to $I$ for all $f \in C(X)$ and $b, c \in I$. If we set $F = I + G$, the intersection $G \cap I$ is reduced by construction to the subalgebra $J$, and so we have a $C(X)$-extension

$$0 \to I \to F \to A \to 0.$$  

Assume now that the space $X$ is perfect and that the $C(X)$-algebra $A$ is continuous. We need to prove that the map $F_x \to M(I_x)$ is injective for each $x \in X$. Let $a \in G$ and $b \in I$ be two elements such that the sum $d = a + b \in F$ verifies for a given point $x \in X$ the equality
\[ d_x I_x + I_x d_x = 0 \text{ (in } [O_2 \otimes C(X)]_x \cong O_2 \otimes \mathcal{C}). \]

To end the proof, we have to show that \( d_x \) is zero. For every \( f \in C_\Delta(X \times X) \), one has \( (bf)_x = -(af)_x \in J_x \), whence \( b_x \in J_x \) and so \( d_x \in G_x \). But the representation of \( G_x \cong A \) in \( M(J_x) \cong M(C_x(X)A) \) is injective since \( X \) is perfect and \( A \) is continuous, from which we deduce that \( d_x = 0 \). \( \square \)

**Remark:** With the previous notations, if the \( C(X) \)-algebra \( A \) is nuclear and \( \psi \) is a unital completely positive projection from \( O_2 \) onto \( A \), the map \( \pi \circ (\psi \otimes id_{C(X)}) \) is a unital \( C(X) \)-linear completely positive map from \( O_2 \otimes C(X) \) onto the \( C(X) \)-subalgebra \( A \) which is zero on the nuclear hereditary \( C(X) \)-subalgebra \( I \).

### 3.2 We can now state the main theorem:

**Theorem 3.2** Let \( X \) be a compact metrizable space and \( A \) be a unital separable \( C(X) \)-algebra with a unital embedding of the \( C(X) \)-algebra \( C(X) \) in \( A \).

The following assertions are equivalent:

1. \( A \) is a continuous field of nuclear \( C^* \)-algebras over \( X \);

2. there exist a unital monomorphism of \( C(X) \)-algebras \( \alpha : A \to O_2 \otimes C(X) \) and a unital \( C(X) \)-linear completely positive map \( E : O_2 \otimes C(X) \to A \) such that \( E \circ \alpha = id_A \).

**Proof:** 2 implies 1 By assumption the identity map \( id_A = E \circ id_{O_2 \otimes C(X)} \circ \alpha : A \to A \) is nuclear since the \( C^* \)-algebra \( O_2 \otimes C(X) \) is nuclear and so the \( C^* \)-algebra \( A \) is nuclear. Besides the \( C(X) \)-algebra \( A \) is isomorphic to the \( C(X) \)-subalgebra \( \alpha(A) \) of the continuous field \( O_2 \otimes C(X) \) and is therefore continuous.

1 implies 2 Let us first deal with the case where the space \( X \) is perfect.

Given a unital nuclear separable continuous fields \( A \) over \( X \) which is unitarily embedded in the \( C^* \)-algebra \( O_2 \), consider the \( C(X) \)-extension

\[ 0 \to I \to F \xrightarrow{\pi} A \to 0 \]

constructed in proposition 3.1 and take the associated \( C(X) \)-extension

\[ 0 \to K \otimes I \otimes O_2 \to D = (K \otimes F \otimes 1_{O_2}) + (K \otimes I \otimes O_2) \to K \otimes A \to 0. \]

The \( C(X) \)-nuclear quotient map \( \sigma = \sigma \circ id_{K \otimes A} \) from the separable nuclear continuous field \( K \otimes A \) to the quotient \( D/(K \otimes I \otimes O_2) \subset M(K \otimes I \otimes O_2)/(K \otimes I \otimes O_2) \) admits a \( C(X) \)-linear completely positive lifting \( K \otimes A \to D \) (in \( K \otimes [O_2 \otimes C(X)] \otimes O_2 \)) thanks to proposition 1.3. This means that the class of \( \sigma \) is invertible in \( Ext(X; K \otimes A, I \otimes O_2) \) (see the second remark following theorem 1.6).

But the group \( Ext(X; K \otimes A, I \otimes O_2)^{-1} \) is \( C(X) \)-linear homotopy invariant (theorem 1.6), hence zero since the endomorphism \( \phi_2(a) = s_1 a s_1^* + s_2 a s_2^* \) of \( O_2 \) is homotopic to the identity map \( id_{O_2} \) ([6, proposition 2.2]) and so \( [\theta] = 2[\theta] \) in \( Ext(X; K \otimes A, I \otimes O_2)^{-1} \) for any invertible extension \( \theta \) of \( K \otimes A \) by \( I \otimes O_2 \). As a consequence, the \( C(X) \)-extension defined by \( \sigma \) is stably trivial. Furthermore, the identity representation of...
$D \subset M(K \otimes I \otimes O_2)$ is a continuous field of faithful representations (proposition 3.1) and the assertion 2. of proposition 2.5 implies that the quotient morphism $(id_K \otimes \pi \otimes id_{O_2})$ from $D$ to $K \otimes A$ admits a cross section $\alpha$ which is a morphism of $C(X)$-algebras.

This monomorphism $\alpha$ is going to enable us to conclude by standard arguments, using theorem 0.1 and the result of Elliott ([9]) that the $C^*$-algebra $O_2$ is isomorphic to $O_2 \otimes O_2$.

Choose a non-zero minimal projection $e_{11}$ in the $C^*$-algebra $K$ of compact operators that we embed in $O_2$ and let $\varphi$ be a state on $O_2$ such that $\varphi(e_{11}) = 1$. If we take a unital completely positive projection $\psi$ of $O_2$ onto the nuclear $C^*$-subalgebra $A \subset O_2$ (theorem 0.1), the composed map

$$E = (\varphi \otimes id_A) \circ (id_{O_2} \otimes [\pi \circ (\psi \otimes id_{C(X)})] \otimes \varphi)$$

is a unital $C(X)$-linear completely positive map from $O_2 \otimes [O_2 \otimes C(X)] \otimes O_2$ onto $A$. Take also an isometry $u \in O_2 \otimes C(X)$ such that $\alpha(e_{11} \otimes 1_A) = uu^*$ (corollary 2.4) and define the unital $C(X)$-algebra morphism

$$\beta : A \rightarrow O_2 \otimes [O_2 \otimes C(X)] \otimes O_2 \simeq O_2 \otimes C(X)$$

by the formula $\beta(a) = u^* \alpha(e_{11} \otimes a)u$. If $\tilde{E} : O_2 \otimes C(X) \rightarrow A$ is the completely positive unital map $d \mapsto E(udu^*)$, one gets for all $a \in A$ the equality

$$(\tilde{E} \circ \beta)(a) = (E \circ \alpha)(e_{11} \otimes a) = a$$

- Let us now come back to the general case of a compact space $X$.

Define the continuous field $B = A \otimes C([0, 1])$ over the perfect compact space $Y = X \times [0, 1]$. According to the previous discussion, there exist a unital completely positive map $\tilde{E} : O_2 \otimes C(Y) \rightarrow B$ and a $C(X) \otimes C([0, 1])$-linear monomorphism $\tilde{\alpha} : B \rightarrow O_2 \otimes C(Y)$ such that $\tilde{E} \circ \tilde{\alpha} = id_B$. If $ev_1 : C([0, 1]) \rightarrow \mathbb{C}$ is the evaluation map at $x = 1 \in [0, 1]$, define the two maps $E : O_2 \otimes C(X) \rightarrow A$ and $\alpha : A \rightarrow O_2 \otimes C(X)$ by

$$E(d) = (id_A \otimes ev_1) \circ \tilde{E}(d \otimes 1_{C([0,1])})$$

$$\alpha(a) = (id_{O_2 \otimes C(X)} \otimes ev_1) \circ \tilde{\alpha}(a \otimes 1_{C([0,1])}).$$

Then $E$ is a unital $C(X)$-linear completely positive map, $\alpha$ is a unital $C(X)$-linear monomorphism and one has the identity $E \circ \alpha = id_A$. □

**Remark:** Assume that $X$ is a locally compact metrizable space and that the $C_0(X)$-algebra $A$ is a nuclear continuous field of $C^*$-algebras over $X$, where $C_0(X)$ denotes the algebra of continuous functions on $X$ vanishing at infinity. If $\overline{X}$ is the Alexandroff compactification of $X$, the unital $C(\overline{X})$-algebra $A$ generated by $A$ and $C(\overline{X})$ in the multiplier algebra $M[A \oplus C(\overline{X})]$ is a separable unital continuous field of $C^*$-algebras over $\overline{X}$ ([4, proposition 3.2]). By theorem 3.2, there exists therefore a $C(\overline{X})$-linear monomorphism $\alpha : A \hookrightarrow O_2 \otimes C(\overline{X})$ and the $C_0(X)$-algebra $A$ is isomorphic to the $C_0(X)$-subalgebra $\alpha(A)$ of $O_2 \otimes C_0(X)$. □
4 Concluding remarks

4.1 A $C(X)$-subalgebra of $\mathcal{O}_2 \otimes C(X)$ is by construction exact and continuous. Conversely, if $A$ is a non-zero exact separable unital continuous field of $C^*$-algebras over a perfect metrizable compact space $X$, one has by proposition 3.1 a $C(X)$-extension

\[ 0 \to I \to F \to A \to 0 \]

where $F$ is a $C(X)$-subalgebra of $\mathcal{O}_2 \otimes C(X)$. If the identity map $A \to A = F/I$ admits a $C(X)$-linear completely positive lifting $A \to F$, the same method as the one used in theorem 3.2 will imply that the exact continuous field $A$ is isomorphic to a $C(X)$-subalgebra of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$.

It is therefore interesting to know whether this map admits a $C(X)$-linear completely positive lifting in the not discrete case.

4.2 Let us have a look at one of the technical problems involved, the Hahn-Banach type extension property in the continuous field framework for finite type $C(X)$-submodules.

Let $A$ be a separable unital continuous field of $C^*$-algebras over a compact metrizable space $X$ and let $D$ be a finitely generated $C(X)$-submodule which is an operator system. Assume that $\phi : D \to C(X)$ is a $C(X)$-linear unital completely positive map. Then for $x \in X$, there exists, thanks to [4, proposition 3.13], a continuous field of states $\Phi_x$ on $A$, i.e. a $C(X)$-linear unital positive map from $A$ to $C(X)$, such that for all $d \in D$,

\[ \Phi_x(d)(x) = \phi(d)(x). \]

As a consequence, given $\varepsilon > 0$ and a finite subset $\mathcal{F}$ of $D$, one can build by continuity and compactness a continuous field of states $\Phi$ on $A$ such that

\[ \max\{\|\Phi(d) - \phi(d)\|, d \in \mathcal{F}\} < \varepsilon. \]

But one cannot find in general any continuous field of states on $A$ whose restriction to $D$ is $\phi$. Indeed, consider the $C(\tilde{\mathbb{N}})$-algebra $A = \mathbb{C}^2 \otimes C(\tilde{\mathbb{N}})$ where $\tilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of the space $\mathbb{N}$ of positive integers. Define the positive element $a \in C(\tilde{\mathbb{N}})A \subset A$ by the formulas

\[ a_n = \begin{cases} \left(\frac{1}{n+1}, 0\right) & \text{if } n \text{ even} \\ \left(0, \frac{1}{n+1}\right) & \text{if } n \text{ odd} \end{cases} \]

and let $\phi$ be the $C(\tilde{\mathbb{N}})$-linear unital completely positive map with values in $C(\tilde{\mathbb{N}})$ defined on the $C(\tilde{\mathbb{N}})$-submodule generated by the two $C(\tilde{\mathbb{N}})$-linearly independent elements $1_A$ and $a$ through the formula

\[ \phi(a)(n) = \frac{1}{n+1} \text{ if } n < \infty \text{ and } \phi(a)(\infty) = 0. \]

Suppose that the continuous field of states $\Phi$ is a $C(\tilde{\mathbb{N}})$-linear extension of $\phi$ to $A$. Then as A. Bauval already noticed it, one has the contradiction

\[ 1 = \Phi(1_A)(\infty) = \Phi((1, 0) \otimes 1)(\infty) + \Phi((0, 1) \otimes 1)(\infty) = \lim_{n \to \infty} \Phi((1, 0) \otimes 1)(2n+1) + \lim_{n \to \infty} \Phi((0, 1) \otimes 1)(2n) = 0 + 0 = 0. \]
In this appendix, we solve in proposition A.3 the lifting question raised in paragraph 4.1 through a continuous generalisation of joint work of E.G. Effros and U. Haagerup on lifting problems for C*-algebras ([8], see also [22]). This result enables us to state the following characterisation of separable exact continuous fields of C*-algebras:

**Theorem A.1** Let $X$ be a compact metrizable space and $A$ be a (unital) separable continuous field of C*-algebras over $X$.

Then the C*-algebra $A$ is exact if and only if there exists a (unital) monomorphism of $(X)$-algebras $A \hookrightarrow O_2 \otimes C(X)$.

Let us start with a technical C(X)-linear version of Auerbach’s theorem ([17, proposition 1.3]) for a continuous field of C*-algebras $A$ over $X$ which gives us local bases over $C(X)$ with continuous coordinate maps for particular free C(X)-submodules of finite type in $A$.

Define a C(X)-operator system in $A$ to be a C(X)-submodule which is an operator system.

**Lemma A.2** ([8, lemma 2.4]) Let $A$ be a separable unital continuous field of C*-algebras over a compact metrizable space $X$, $E \subset A$ be a C(X)-operator system and assume that there exists an integer $n \in \mathbb{N}^*$ such that for all $x \in X$, the dimension dim $E_x$ of the operator system $E_x \subset A_x$ equals $n$. Then the following holds.

Given any point $x \in X$, there exist an open neighbourhood $U$ of $x$ in $X$, self-adjoint C(X)-linear contractions $\varphi_i : A \to C(X)$ and self-adjoint elements $f_i \in E$ with $\|f_i\| \leq 2$ for $1 \leq i \leq n$ such that

$$\forall a \in C_0(U)E, \quad a = \sum_i \varphi_i(a)f_i.$$  

Furthermore, there exists a continuous field of states $\Psi : A \to C(X)$ such that the restriction of the map $2n\Psi - id_A$ to the operator system $E$ is completely positive.

**Proof** : Let us fix a point $x \in X$. Then there exist, thanks to Auerbach’s theorem, a normal basis $\{r_1, \ldots, r_n\}$ of the fibre $E_x$ where each $r_i$ is self-adjoint and norm one hermitian functionals $\phi_j : A_x \to \mathbb{C}$, $1 \leq j \leq n$, such that $\phi_j(r_i) = \delta_{i,j}$.

Consider the polar decomposition $\phi_j = \phi_j^+ - \phi_j^-$ where $\phi_j^+$ and $\phi_j^-$ are positive functionals such that $1 = \|\phi_j\| = \|\phi_j^+\| + \|\phi_j^-\|$. By [4] lemme 3.12, there exist C(X)-linear positive maps $\varphi_j^+ : A \to C(X)$ which extend the functionals $\phi_j^+$ and $\phi_j^-$ on the fibre $A_x$ to the C(X)-algebra $A$ with the property that $\varphi_j^+(1) = \|\phi_j^+\|$ and $\varphi_j^-(1) = \|\phi_j^-\|$. Take also $n$ norm 1 self-adjoint elements $e_i \in E$ satisfying the equality $(e_i)_x = r_i$ and define the matrix $T = [\varphi_j(e_i)]_{i,j} \in M_n(\mathbb{R}) \otimes C(X)$.

One has by construction $T_x = 1_{M_n(\mathbb{R})}$: the set $U_1 \subset X$ of points $y \in X$ for which the spectrum of $T_y \in M_n(\mathbb{R})$ is included in the open set $\{z \in \mathbb{C}, |z| > 1/2\}$ is therefore an open neighbourhood of $x$ in $X$ ([4, proposition 2.4 b)]). If $\eta$ is a continuous function on $X$ with values in $[0,1]$ which is 0 outside $U_1$ and 1 on an open neighbourhood $U$ of the point $x \in X$, the self-adjoint elements $f_1, \ldots, f_n$ of norm less than 2 are then well defined in $C_0(U_1)E$ by the formula
\[ T \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \eta \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \]

and satisfy the relation \( \varphi_j(f_i)(y) = \delta_{i,j} \) for each \( y \in \mathcal{U} \) since the matrix \( T_y \) is invertible, whence the desired equality for every \( a \in C_0(\mathcal{U})E \).

Keeping the same fixed point \( x \), define now the continuous field of states \( \Phi = \frac{1}{n} \sum_i (\varphi_i^+ + \varphi_i^-) \). Then one gets for all \( a \in C_0(\mathcal{U})E \) the equality:

\[ (2n\Phi - \text{id}_A)(a) = \sum_{1 \leq i \leq n} \left[ \varphi_i^+(a)(2 - f_i) + \varphi_i^-(a)(2 + f_i) \right]. \]

The restriction of the map \((2n\Phi - \text{id}_A)\) to \( C_0(\mathcal{U})E \) is therefore completely positive and an appropriate partition of the unit \( 1_{C(X)} \) enables us to conclude. \( \square \)

Noticing that a \( C(X) \)-linear map \( \sigma : A \rightarrow B \) between \( C(X) \)-algebras is completely positive if and only if each induced map \( \sigma_x : A_x \rightarrow B_x \) is completely positive (see for instance [4, proposition 2.9]), the lemma A.2 allows us to state a continuous version of theorem 2.5 of [8]. Replacing then the continuous field \( A \) by \( A \oplus M_{2^n}(\mathbb{C}) \otimes C(X) \) (where \( M_{2^n}(\mathbb{C}) = \lim_{n \rightarrow \infty} M_{2^n}(\mathbb{C}) \)) and working with finitely generated \( C(X) \)-operator systems \( E_k \subset A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X) \) for which the function \( x \mapsto \dim(E_k)_x \) is continuous, one derives the following desired \( C(X) \)-linear completely positive lifting result.

**Proposition A.3** ([8, theorem 3.4]) Suppose that \( A \) and \( B \) are two unital separable exact continuous fields of \( C^* \)-algebras over a compact space \( X \) with \( A = B/J \) for some nuclear ideal \( J \) in \( B \).

Then there exists a \( C(X) \)-linear unital completely positive lifting \( A \rightarrow B \) of \( \text{id}_A \).

**Proof:** Let us define the two continuous fields \( \mathcal{A} = A \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X) \) and \( \mathcal{B} = B \oplus M_{2^\infty}(\mathbb{C}) \otimes C(X) \). It is clearly enough to find a \( C(X) \)-linear unital completely positive cross section \( \theta \) of the quotient morphism \( \mathcal{B} \rightarrow \mathcal{A} \) (by [4, theorem 3.3]).

Consider a dense sequence \( \{a_k\} \) in the self-adjoint part of \( \mathcal{A} \) where each \( a_k \) belongs to the dense subalgebra \( A \oplus \bigcup_n M_{2^n}(\mathbb{C}) \otimes C(X) \) of \( \mathcal{A} \) and \( a_1 = 1 \). Let us show that we may assume inductively that \( C(X) \)-operator system \( E_n \) generated by the \( a_k \), \( 1 \leq k \leq n \), satisfies the equality \( \dim(E_n)_x = n \) for every \( n \in \mathbb{N}^* \) and every \( x \in X \). The inductive step is the following. Given \( n \geq 2 \), there exists by construction an integer \( l \) such that \( E_n \subset A \oplus M_{2^l}(\mathbb{C}) \otimes C(X) \). Set \( a'_n = a_n + 2^{-n-1}d_l \otimes 1_{C(X)} \) where

\[ d_l = 1_{M_{2^l}(\mathbb{C})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_{2^l+1}(\mathbb{C}) \subset M_{2^\infty}(\mathbb{C}). \]

Then the \( C(X) \)-module \( E'_n = E_{n-1} + C(X)a'_n \) verifies for each \( x \in X \) the equality \( \dim(E'_n)_x = \dim(E_{n-1})_x + 1 \).

Using proposition 1.3, one can now finish the proof by the same method as the one developed by E.G. Effros and U. Haagerup in [8].3 (see also [22, theorem 6.10]). \( \square \)
References


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