HAL
open science

On finiteness of the N-dimensional Hopf C*-algebras

Etienne Blanchard

To cite this version:

Etienne Blanchard. On finiteness of the N-dimensional Hopf C*-algebras. Operator theoretical methods (Timisoara, 1998), Theta Found., 2000, pp.39–46. hal-00922867

HAL Id: hal-00922867

https://hal.science/hal-00922867v1

Submitted on 31 Dec 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On finiteness of the number of N-dimensional Hopf C*–algebras

Etienne Blanchard

Abstract

Given an algebraically closed field k and an integer N, D. Ştefan has proved that there exists only a finite number of Hopf k-algebras which are both semi-simple and co-semi-simple. In the C*–algebraic framework, we provide in this note explicit upper-bounds for the number of Hopf C*–algebra structures on a given finite dimensional C*–algebra.

Operator theoretical methods (Timisoara, 1998), Theta Found., Bucharest, (2000), pp 39–46. (with an added appendix)

Keywords: Hopf C*-algebra, Finite dimensional Kac algebra, Multiplicative unitary.
AMS Subject Classification: 16W30, 46L05, 47d35.

1 Introduction.

Given an algebraically closed field k and an integer N, D. Ştefan has proved through cohomological arguments that there exists only a finite number of Hopf k-algebras which are both semi-simple and co-semi-simple ([4, Theorem2.2]).

In order to provide an insight into his proof in the C*–algebraic case, let us introduce the following algebraic formulation of the definition of a finite dimension Hopf C*–algebra, a particular case of the definition of a compact Hopf C*–algebra given by Woronowicz in [6].

Definition 1. *Given a non zero integer N, an N-dimensional Hopf C*–algebra is a pair (A, δ), where A is a C*–algebra of dimension N and $\delta : A \to A \otimes A$ is a unital co-associative bisimplifiable C*-morphism, i.e. a C*-morphism such that :*

a) for all $a \in A$, one has $(\delta \otimes id) \circ \delta(a) = (id \otimes \delta) \circ \delta(a)$ (in $A \otimes A \otimes A$),

b) the linear spaces $lin\,\{\delta(A)(A \otimes 1)\}$ and $lin\,\{\delta(A)(1 \otimes A)\}$ are both equal to $A \otimes A$.

The demonstration of D. Ştefan for the finiteness of the number of Hopf C*–algebras with given finite dimension then depends on the following proposition.

Proposition 2 ([4], Theorem 2.1). *Assume that (A, δ) is a finite dimensional Hopf C*–algebra and that the operator $T \in Hom(A, A \otimes A)$ satisfies the two conditions :*

a) $\forall a, b \in A$, $T(ab) = T(a)\delta(b) + \delta(a)T(b)$,

b) $(T \otimes id) \circ \delta + (\delta \otimes id) \circ T = (id \otimes T) \circ \delta + (id \otimes \delta) \circ T$ *in* $Hom(A, A^{\otimes 3})$.

Then $T = 0$.

The proof of finiteness goes as follows. Suppose that there exists an infinite number of distinct Hopf C*–algebra structures (A, δ_n) on the finite dimensional C*–algebra A. Taking a subsequence if necessary, one may assume that the sequence of coproducts $\delta_n \in Hom(A, A \otimes A)$ converges toward a coproduct δ satisfying the relation $\delta \neq \delta_n$ for all $n \in \mathbb{N}$. Then $\delta \in Hom(A, A \otimes A)$ also defines by continuity a Hopf C*–algebra structure on A and the sequence $T_n = \|\delta_n - \delta\|^{-1}(\delta_n - \delta)$ of norm 1 operators admits at least one accumulation point $T \in Hom(A, A \otimes A)$. Then T satisfies the conditions of the previous proposition for the Hopf C*–algebra (A, δ) and is therefore zero, a conclusion which is absurd.

Let us remark that this proof of finiteness of the number of Hopf C*–algebra structures on a finite dimensional C*–algebra does not provides us with any upper bound of this finite number, a question that we shall study in this note using the framework of multiplicative unitaries.

Note also more generally that A. Ocneanu has shown that there are only finitely many classes of inclusions of subfactors with given finite index and finite depth.

I would like to express my gratitude to G. Skandalis for fruitful discussions. I also thank the referee for his motivating comments.

2 Preliminaries.

We describe in this section the one-to-one correspondence between finite dimensional Hopf C*–algebras and multiplicity 1 multiplicative unitaries acting on the tensor square of a finite dimensional Hilbert space (the reader may find a more refined presentation of this equivalence in [1] or [2]). We also provide two technical lemmas which will be used in the demonstration of the main proposition 5.

2.1 Assume that (A, δ) is a finite dimensional Hopf C*–algebra. Then the tracial state $\varphi\colon A = \oplus_i M_{k_i}(\mathbb{C}) \to \mathbb{C}$ defined by $\varphi(p) = k_i/N$ for any minimal projection $p \in M_{k_i}(\mathbb{C}) \subset A$ is a Haar state *i.e.* such that

$$(\varphi \otimes id) \circ \delta(a) = (id \otimes \varphi) \circ \delta(a) = \varphi(a) 1_A$$

for all $a \in A$ ([6], see also [5, theorem 2.3]). Consider the G.N.S. construction $(\mathcal{H}, L, e)$ associated to this state φ and define the operator $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ through the formula

$$V(L(a)e \otimes L(b)e) = (L \otimes L)(\delta(a)(1 \otimes b)\,)(e \otimes e) \text{ for } a, b \in A.$$

As $\varphi = \omega_{e,e} \circ L$ defines the Haar state on (A, δ), the operator V is unitary and one derives from the coassociativity condition the pentagonal relation

$$V_{12} V_{13} V_{23} = V_{23} V_{12} \text{ (in } \mathcal{L}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})\,),$$

i.e. V is a multiplicative unitary ([1]). Furthermore, one has for all $a \in A$ the relation

$$(L \otimes L)\delta(a) = V(L(a) \otimes 1)V^* .$$

2.2 Conversely let $\mathcal{H}$ be a finite dimensional Hilbert space and let $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ be a multiplicative unitary. Then the vector space $L(S)$ generated by the operators $(\omega \otimes id)(V)$, $\omega \in \mathcal{L}(\mathcal{H})^*$, is naturally endowed with a structure of bisimplifiable Hopf C*–algebra of coproduct $\delta(a) = V(a \otimes 1)V^*$ for $a \in L(S)$ ([1, théorème 4.10]). If τ is the normalised trace on $\mathcal{L}(\mathcal{H})$, the operator $\hat{p}_\tau = (id \otimes \tau)(V)$ is a non zero projection on the vector space of *fix* vectors by V, *i.e.* the vectors $\xi \in \mathcal{H}$ such that $V(\xi \otimes \eta) = \xi \otimes \eta$ for all $\eta \in \mathcal{H}$. One calls *multiplicity of* V the rank of this projector.

If $\Sigma \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ is the volt operator (defined by the relation $\Sigma(\xi \otimes \eta) = \eta \otimes \xi$ for $\xi, \eta \in \mathcal{H}$), one checks that the unitary $\Sigma V^* \Sigma$ is also multiplicative. Define also for $\omega \in \mathcal{L}(\mathcal{H})^*$ the linear form ω^* by $\omega^*(a) = \omega(a^*)^*$ for $a \in \mathcal{L}(\mathcal{H})$. Then the vector space $\rho(\widehat{S})$ generated by the elements $(id \otimes \omega)(V) = (\omega^* \otimes id)(\Sigma V^* \Sigma)^*$, $\omega \in \mathcal{L}(H)^*$, is endowed with a structure of bisimplifiable Hopf C*–algebra of coproduct $\widehat{\delta}(b) = V^*(1 \otimes b)V$ for $b \in \widehat{S}$. The vectors $\eta \in \mathcal{H}$ in the image of the projector $p_\tau = (id \otimes \tau)(\Sigma V^* \Sigma) = (\tau \otimes id)(V)$ are said to be *cofix* vectors by V; they satisfy the equality $V(\xi \otimes \eta) = \xi \otimes \eta$ for all $\xi \in \mathcal{H}$.

Note that the rank of the projector p_τ is the same as the one of $\hat{p}_\tau$ since $\tau\big(\hat{p}_\tau\big) = (\tau \otimes \tau)(V) = \tau\big(p_\tau\big)$. As a consequence, if the multiplicity of V is 1, one may find a unit fix vector $e \in \mathcal{H}$ and a unit cofix vector $\hat{e} \in \mathcal{H}$ such that $\langle e, \hat{e} \rangle = (\dim \mathcal{H})^{-1/2}$ since $p_\tau \in S$ is the rank 1 projection $\theta_{\hat{e},\hat{e}}$ and $\tau(\hat{p}_\tau) = \langle e, \hat{p}_\tau e \rangle = |\langle e, \hat{e} \rangle|^2$.

Remarks. *a)* The restriction of the trace τ to the algebra $L(S)$ is equal by uniqueness to the Haar state $\omega_{e,e}$ on the Hopf C*–algebra $L(S)$, where $e \in \mathcal{H}$ is a unit fix vector.

b) The multiplicative unitary V belongs the C*–algebra $\rho(\widehat{S}) \otimes L(S) \subset \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ ([1]).

2.3 Let us investigate more precisely the properties of a given multiplicity 1 multiplicative unitary $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ acting on the tensor square of a finite N–dimensional Hilbert space $\mathcal{H}$.

Assume that the two norm 1 vectors $e, \hat{e} \in \mathcal{H}$ are fix and cofix vectors by V satisfying the equality $\langle e, \hat{e} \rangle = N^{-1/2}$. As the vector e is a cyclic separating vector for the C*–algebra $L(S)$, one can define an operator $U \in \mathcal{L}(\mathcal{H})$ through the formula

$$U\Big((\omega \otimes id)(V).e\Big) = (\omega^* \otimes id)(V)^*.e \qquad \text{for } \omega \in \mathcal{L}(\mathcal{H})^*.$$

Then U is an involutive unitary and the commutant of the C*–algebra $L(S)$ (resp. $\rho(\widehat{S})$) is equal to the C*–algebra $R(S) = UL(S)U$ (resp. $\lambda(\widehat{S}) = U\rho(\widehat{S})U$). Moreover the two unitaries $\widehat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma \in L(S) \otimes \lambda(\widehat{S})$ and $\widetilde{V} = (U \otimes U)\widehat{V}(U \otimes U)$ are multiplicative and one has the equality

$$\Sigma \widehat{V} V \widetilde{V}(1 \otimes U) = 1 \text{ (in } \mathcal{L}(\mathcal{H} \otimes \mathcal{H})). \tag{1}$$

These properties exactly mean that the multiplicity 1 multiplicative unitary V is always irreducible ([1] section 6) and this implies the following lemma.

Lemma 3. *Given two unit vectors $e, \hat{e}$ in a Hilbert space $\mathcal{H}$ of dimension N such that $\langle e, \hat{e}\rangle = N^{-1/2}$, define the map*

$$\pi : T \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \mapsto N^{1/2}(id \otimes \omega_{\hat{e},e})(\Sigma\, T) \in \mathcal{L}(\mathcal{H})\,.$$

Then $\pi(V) = 1$ for every multiplicity 1 multiplicative unitary $V \in \mathcal{L}(H \otimes \mathcal{H})$ which admits the vector e (resp. $\hat{e}$) as a fix (resp. cofix) vector.

Proof. As the two vectors e and $\hat{e}$ are fixed by U, formula (1) entails for all $\xi_1, \xi_2 \in \mathcal{H}$ the sequence of equalities :

$$\langle \xi_1, \pi(V)\xi_2\rangle = \langle \hat{e} \otimes \xi_1, \widehat{V} V \widetilde{V}(1 \otimes U)(\xi_2 \otimes e)\rangle = \langle \xi_1 \otimes \hat{e}, \xi_2 \otimes Ue\rangle = N^{-1/2}\langle \xi_1, \xi_2\rangle\,. \quad \square$$

Lemma 4. *Let $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ be a multiplicative unitary of multiplicity 1 acting on the tensor square of a finite dimensional Hilbert space $\mathcal{H}$ and denote by τ the normalised trace on $\mathcal{L}(\mathcal{H})$.*

a) If $x \in \lambda(\widehat{S})$ and $y \in L(S)$, then $\tau(xy) = \tau(x)\,\tau(y)$.

b) One has for all $a \in \mathcal{L}(\mathcal{H})$ the equality

$$(\tau \otimes id)(V(a \otimes 1)V^*) = \tau(a)\,1 = (id \otimes \tau)(V^*(1 \otimes a)V)\,.$$

Proof. *a)* The element $\lambda(x) \otimes 1$ commutes by construction with $V \in \rho(\widehat{S}) \otimes L(S)$ and for all $y \in L(S)$, one has the equality $\delta(y) = V(y \otimes 1)V^* \in L(S) \otimes L(S)$. As the form τ defines by restriction the Haar state on the Hopf C*–algebra $L(S)$, one has the following sequence of equalities.

$$\begin{aligned}\tau(xy) &= (\tau \otimes \tau)(V(xy \otimes 1)V^*)\\ &= (\tau \otimes \tau)[(x \otimes 1)\,\delta(y)]\\ &= \tau[x\,(id \otimes \tau) \circ \delta(y)]\\ &= \tau(x)\tau(y)\,.\end{aligned}$$

b) It is enough to check the first equality since the second one can then be deduced replacing the multiplicative unitary V by $\Sigma V^* \Sigma$.

If the operator $a \in \mathcal{L}(\mathcal{H})$ admits the decomposition $a = xy$ with $x \in \lambda(\widehat{S})$ and $y \in L(S)$, assertion *a)* implies that

$$\begin{aligned}(\tau \otimes id)(V(a \otimes 1)V^*) &= (\tau \otimes id)[(x \otimes 1)\delta(y)]\\ &= \tau(x)\,(\tau \otimes id)[\delta(y)]\\ &= \tau(x)\,\tau(y)\,1 = \tau(a)\,1\,.\end{aligned}$$

The multiplicity 1 of the multiplicative unitary $\widehat{V}$ then allows us to conclude since the vector space generated by the products xy, $x \in \lambda(\widehat{S})$ and $y \in L(S)$, is all the algebra $\mathcal{L}(\mathcal{H})$. $\square$

3 The approach of multiplicative unitaries.

The purpose of this section is to reformulate the finiteness problem in terms of multiplicative unitaries. In order to do so, recall that two multiplicative unitaries $V, W \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ are said to be equivalent if there exists a unitary $Z \in \mathcal{L}(\mathcal{H})$ (corresponding to a change of orthonormal basis of $\mathcal{H}$) such that $V(Z \otimes Z) = (Z \otimes Z)W$.

3.1 Let e, $\hat{e} \in \mathcal{H}$ be two unit vectors satisfying the relation $\langle e, \hat{e} \rangle = N^{-1/2}$. Given any multiplicity 1 multiplicative unitary $V' \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, one can always assume, up to equivalence by **2.2**, that the vector e (resp. $\hat{e}$) is a fix (resp. cofix) vector for V'. Indeed if f and $\hat{f}$ are fix and cofix unit vectors for V' satisfying $\langle f, \hat{f} \rangle = N^{-1/2}$ and $Z \in \mathcal{L}(\mathcal{H})$ is a unitary such that $Ze = f$, $Z\hat{e} = \hat{f}$, then the operator $(Z \otimes Z)^* V'(Z \otimes Z)$ has the requested properties.

One can therefore state without loss of generality the following proposition.

Proposition 5. *Given a Hilbert space $\mathcal{H}$ of finite dimension N and two unit vectors $e, \hat{e} \in \mathcal{H}$ satisfying the relation $\langle e, \hat{e} \rangle = N^{-1/2}$ there exists a constant $k > 1$ satisfying the following properties.*

If $V, V' \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ are two multiplicity 1 multiplicative unitaries admitting the vector e (resp. $\hat{e}$) as a fix (resp. cofix) vector and satisfying the inequality $\|V - V'\| < 1/k$, then

a) there exists a unitary $Z \in \mathcal{L}(\mathcal{H})$ such that

$$\|V' - (Z \otimes Z)^* V (Z \otimes Z)\| < k \|V' - V\|^2 ,$$

b) there exists a unitary $\mathbf{Z} \in \mathcal{L}(\mathcal{H})$ such that $V' = (\mathbf{Z} \otimes \mathbf{Z})^* V (\mathbf{Z} \otimes \mathbf{Z})$.

Proof. *a)* **Step 1.** Let us first expand the pentagonal relation satisfied by V' as a formula depending on the difference $R = V' - V$.

If one sets $f(R) = R_{12}R_{13}V_{23} + V_{12}R_{13}R_{23} + R_{12}V_{13}R_{23} - R_{23}R_{12} + R_{12}R_{13}R_{23}$, then

$$\begin{aligned} 0 &= (V+R)_{12}(V+R)_{13}(V+R)_{23} - (V+R)_{23}(V+R)_{12} \\ &= 0 + R_{12}V_{13}V_{23} + V_{12}R_{13}V_{23} + V_{12}V_{13}R_{23} - R_{23}V_{12} - V_{23}R_{12} + f(R) . \end{aligned} \tag{2}$$

Using the normalised trace τ on $\mathcal{L}(\mathcal{H})$, define the two operators $K = -(\tau \otimes id)(RV^*)$, $K' = (id \otimes \tau)(V^* R)$ in $\mathcal{L}(\mathcal{H})$ and the map $\alpha = (id \otimes \tau \otimes id) \colon \mathcal{L}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \to \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$. One can then write the following sequence of equalities.

$$\begin{aligned} &\alpha\Big(V_{12}{}^* \Big[R_{12}V_{13}V_{23} + V_{12}R_{13}V_{23} + V_{12}V_{13}R_{23} - R_{23}V_{12} - V_{23}R_{12} \Big] V_{23}^* \Big) \\ &= \alpha\Big((V^*R)_{12}V_{13} + R_{13} + V_{13}(RV^*)_{23} - V_{12}^*(RV^*)_{23}V_{12}V_{13} - V_{13}V_{23}(V^*R)_{12}V_{23}^* \Big) \\ &= (K' \otimes 1)V + R - V(1 \otimes K) - \alpha\Big(V_{12}^*(RV^*)_{23}V_{12} \Big) . V - V . \alpha\Big(V_{23}(V^*R)_{12}V_{23}^* \Big) \\ &= R - V(K' \otimes 1 + 1 \otimes K) + (K' \otimes 1 + 1 \otimes K)V \quad \text{by lemma 4.b)} . \end{aligned}$$

This gives the inequality $\|R - V(K' \otimes 1 + 1 \otimes K) + (K' \otimes 1 + 1 \otimes K)V\| \leq \|f(R)\| \leq 4\|R\|^2 + \|R\|^3$ by above equation (2).

The unitarity of the operator V' means that $1 = (V')^*V' = 1 + R^*V + V^*R + R^*R$, so that $\|K + K^*\| = \|(\tau \otimes id)(R^*V + V^*R)\| \leq \|R\|^2$ and $\|K' + (K')^*\| \leq \|R\|^2$. Thus the two skew–ajoint operators $\mathbf{K}_\iota = \frac{1}{2}(K - K^*)$ and $\mathbf{K}'_\iota = \frac{1}{2}(K' - (K')^*)$ provide us with the upper estimate

$$\begin{aligned} &\|R - V(\mathbf{K}'_\iota \otimes 1 + 1 \otimes \mathbf{K}_\iota) + (\mathbf{K}'_\iota \otimes 1 + 1 \otimes \mathbf{K}_\iota)V\| \\ &\qquad \leq 4\|R\|^2 + \|R\|^3 + 2\|\mathbf{K}_\iota - K\| + 2\|\mathbf{K}'_\iota - K'\| \leq 6\|R\|^2 + \|R\|^3 \,. \end{aligned} \tag{3}$$

Step 2. In order to derive a unitary approximation from (3), define for $t \in [0,1]$ the unitaries $v_t = exp(t\,\mathbf{K}_\iota)$, $w_t = expt(t\,\mathbf{K}'_\iota)$ in $\mathcal{L}(\mathcal{H})$ and $g(t) = (w_t \otimes v_t)^*V(w_t \otimes v_t) \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$. The function g is C^∞ and for all $t \in [0,1]$, one has the formula $g'(t) = (w_t \otimes v_t)^* \Big[(\mathbf{K}'_\iota \otimes 1 + 1 \otimes \mathbf{K}_\iota)V + V(\mathbf{K}'_\iota \otimes 1 + 1 \otimes \mathbf{K}_\iota)\Big](w_t \otimes v_t)$. As a consequence, if one sets $Z_1 = v_1$ and $Z'_1 = w_1$, the Taylor formula gives us the inequality

$$\begin{aligned} \|V' - (Z'_1 \otimes Z_1)^*V(Z'_1 \otimes Z_1)\| &\leq \|V' - g(0) - g'(0)\| + \sup\{\|g''(t)\|, 0 \leq t \leq 1\} \\ &\leq 6\,\|R\|^2 + \|R\|^3 + 16\,\Big(\max\{\|\mathbf{K}_\iota\|\,,\,\|\mathbf{K}'_\iota\|\}\Big)^2 \\ &\leq (22 + \|R\|)\,.\,\|R\|^2\,. \end{aligned} \tag{4}$$

Step 3. Let π be the map $T \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H}) \mapsto N^{1/2}(id \otimes \omega_{\hat{e},e})(\Sigma\, T) \in \mathcal{L}(\mathcal{H})$. Then

$$\pi(V) = \pi(V') = \pi(\Sigma\widehat{V}\Sigma) = \pi(\Sigma\widehat{V}\Sigma) = 1 \in \mathcal{L}(\mathcal{H})$$

by lemma 3 since e (resp. $\hat{e}$) is a fix (resp. cofix) vector for each of these multiplicity 1 multiplicative unitaries. Moreover, as the algebra $\lambda(\widehat{S}) \otimes R(S)$ commutes with $V \in \rho(\widehat{S}) \otimes L(S)$ (*cf.* **2.3**), one has the relation

$$\begin{aligned} \pi\,[V' - (Z'_1 \otimes Z_1)^*V(Z'_1 \otimes Z_1)] &= 1 - N^{1/2}\, Z_1^*(id \otimes \omega_{Z'_1\hat{e},Z_1e})(\Sigma\, V)Z'_1 \\ &= 1 - Z_1^*.\mathbf{a}.1_{\mathcal{L}(\mathcal{H})}.\mathbf{b}^*.Z'_1 \end{aligned} \tag{5}$$

where the elements $\mathbf{a} \in R(S)$ and $\mathbf{b} \in \lambda(\widehat{S})$ are uniquely determined by the equalities

i) $Z_1e = \pi(\Sigma\widetilde{V}\Sigma)Z_1e = N^{1/2}(id \otimes \omega_{\hat{e},Z_1e})(\widetilde{V})e = \mathbf{a}e$ and

ii) $Z'_1\hat{e} = \pi(\Sigma\widehat{V}\Sigma)Z'_1\hat{e} = N^{1/2}(\omega_{e,Z'_1\hat{e}} \otimes id)(\widehat{V})\hat{e} = \mathbf{b}\hat{e}$

since the vector e (resp. $\hat{e}$) is separating for the algebra $R(S)$ (resp. $\lambda(\widehat{S})$). Thus equations (3) and (5) give us the inequality $\|1 - Z_1^*\mathbf{a}\mathbf{b}^*Z'_1\| \leq N^{1/2}\,(22 + \|R\|)\,\|R\|^2$.

Set $\mathbf{t} = 23\,N^{1/2}\,\|R\|^2$ and suppose from now on that $\mathbf{t} < 1/2$, an hypothesis which implies in particular that $\|R\| < 1$ and $\|1 - \mathbf{a}\mathbf{b}Z'_1\,Z_1^*\| < 1/2$, so that the operator $\mathbf{a}$ (and similarly $\mathbf{b}$) is invertible. One can then define in $\mathcal{L}(\mathcal{H})$ the two unitaries

$$Z_2 = u^*\,Z_1 \text{ and } Z'_2 = v^*\,Z'_1 \quad \text{ where } \mathbf{a} = u|\mathbf{a}| \text{ and } \mathbf{b} = v|\mathbf{b}|.$$

The previous inequality $\|1 - Z_1^*\mathbf{a}\mathbf{b}^*Z'_1\| = \|Z_2(Z'_2)^* - |\mathbf{a}|.|\mathbf{b}|\,\| \leq 23\,N^{1/2}\,\|R\|^2 < \sin(\frac{\pi}{4})$ implies that the spectrum of the unitary operator $Z_2(Z'_2)^*$ is contained in the arc $\{exp(ix)\,;\,x \in [-\theta,\theta]\}$ where $\theta \in [0,\frac{\pi}{4}]$ is determined by the formula $\sin(\theta) = \mathbf{t}$, since

the spectrum of the invertible element $|\mathbf{a}|.|\mathbf{b}|$ is the same as the one of the positive element $|\mathbf{b}|^{1/2}(|\mathbf{a}|)|\mathbf{b}|^{1/2}$ and is therefore real. This entails

$$\|Z_2 - Z_2'\|^2 \leq |1 - e^{i\theta}|^2 = 2(1 - \sqrt{1 - \mathbf{t}^2}) \leq (1 - \mathbf{t}^2)^{-1/2}\mathbf{t}^2 \leq \frac{2}{\sqrt{3}}\,\mathbf{t}^2 < \frac{49}{36}\,\mathbf{t}^2 \qquad (6)$$

and so $\|V' - (Z_2 \otimes Z_2)^* V (Z_2 \otimes Z_2)\| \leq 23\,\|R\|^2 + 2\,\|Z_2 - Z_2'\| \leq 23\,(1 + (7/3)N^{1/2})\|R\|^2$.

Step 4. The last step consists in finding a unitary Z_3 close to Z_2, which admits the vectors e and $\hat{e}$ as fixed points. By construction $\mathbf{K}_\iota \hat{e} = 0$ and so $Z_1\hat{e} = \hat{e}$, whence $Z_2\hat{e} = |\mathbf{a}|^{-1}\mathbf{a}^*\hat{e} = \hat{e}$ since $\mathbf{a}\hat{e} = N^{1/2}\langle \hat{e}, Z_1 e\rangle \hat{e} = |\mathbf{a}|\hat{e}$. Similarly $Z_2' e = e$ whereas $Z_2^* e = \cos(\theta')e + \sin(\theta')\zeta$ where $\zeta \in \mathcal{H}$ is a unit vector orthogonal to e which satisfies $\langle \hat{e}, \zeta\rangle = \langle \hat{e}, Z_2^* e\rangle = \langle \hat{e}, e\rangle = N^{-1/2}$.

One can therefore find a unitary $\tilde{Z} \in \mathcal{L}(\mathcal{H})$ such that $\tilde{Z}\hat{e} = \hat{e}$, $\tilde{Z}e = Z_2^* e$ and

$$\|\tilde{Z} - 1\| = \|1 - \begin{pmatrix} e^{i\theta'} & \\ & e^{-i\theta'} \end{pmatrix}\| = \|Z_2' e - Z_2 e\| < (7/6) \times 23\, N^{1/2}\,\|R\|^2 \text{ by (6).}$$

Thus the vector e (resp. $\hat{e}$) is a fix (resp. cofix) vector for the multiplicative unitary $(Z_3 \otimes Z_3)^* V (Z_3 \otimes Z_3)$ where $Z_3 = Z_2\tilde{Z}$. Besides one has the inequality

$$\begin{aligned} \|V' - (Z_3 \otimes Z_3)^* V (Z_3 \otimes Z_3)\| &\leq \|V' - (Z_2 \otimes Z_2)^* V (Z_2 \otimes Z_2)\| + 4\,\|\tilde{Z} - 1\| \\ &\leq k\,\|V' - V\|^2 \quad \text{with } k = 23(1 + 7N^{1/2}) \end{aligned}$$

as soon as $23\, N^{1/2}\,\|R\|^2 < 1/2$, a condition which is always fulfilled if $\|R\| < 1/k$.

b) One can find by induction thanks to assertion *a)* unitaries $Z_n \in \mathcal{L}(\mathcal{H})$, $n \geq 3$, such that $\|V' - (Z_{n+1} \otimes Z_{n+1})^* V (Z_{n+1} \otimes Z_{n+1})\| \leq k\,\|V' - (Z_n \otimes Z_n)^* V (Z_n \otimes Z_n)\|^2 \leq (k\,\|V' - V\|)^{n-2}\,\|V' - V\|$ and a weak limit $\mathbf{Z} \in \mathcal{L}(\mathcal{H})$ of the sequence $\{\, Z_n \,\}_{n \geq 3}$ has the desired property. □

One clearly derives from proposition 5.*b)* and [1, théorème 4.7] the following theorem.

Theorem 6. *Given a Hilbert space $\mathcal{H}$ of finite dimension N, there exists only finitely many equivalence classes of multiplicative unitaries $V \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$.*

3.2 In order to refine the upper estimate of the number of possible N-dimensional Hopf C*–algebras, fix an N-dimensional C*–algebra S and let $(\mathcal{H}, L, e)$ be the GNS construction for the trace state φ on S introduced in **2.1**. Suppose also that there is at least one Hopf C*–algebra structure on S, so that one can fix a unit vector $\hat{e} \in \mathcal{H}$ satisfying the relation $\langle \hat{e}, e\rangle = N^{-1/2}$ and such that this vector $\hat{e}$ is a cofix vector (up to equivalence) for any multiplicity 1 multiplicative unitary $V \in \mathcal{L}(\mathcal{H}) \otimes L(S)$.

Assume that the coproducts $\delta\,,\,\delta' :\, S \to S \otimes S$ define two structures of Hopf C*–algebra on S. Then φ defines the Haar state for both (S, δ) and (S, δ') and one can suppose, up to equivalence, that e (resp. $\hat{e}$) is a fix (resp. cofix) vector for both

associated multiplicity 1 multiplicative unitaries V , $V' \in \mathcal{L}(\mathcal{H}) \otimes L(S) \subset \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ (*cf.* **2.2** remark *b*).

Define the antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ by $xe \mapsto x^*e$ for $x \in L(S)$ (cf *e.g.* [2, 1.6]). With the notations of above proposition 5, one has $Z_1 \in L(S)$ since $K = (\tau \otimes id)\,[\,(V - V')V^*] \in L(S)$. The operator $\mathbf{a} \in R(S)$ defined through the equality $\mathbf{a}e = Z_1 e$ therefore satisfies $\mathbf{a} = JZ_1^*J' \in L(S)'$ and the unitary $\tilde{\mathbf{Z}}_{\mathbf{2}} = JZ_1JZ_1$ verifies :

a) $\tilde{\mathbf{Z}}_{\mathbf{2}}e = e$, $\tilde{\mathbf{Z}}_{\mathbf{2}}\hat{e} = \hat{e}$ and

b) $\|V' - (\tilde{\mathbf{Z}}_{\mathbf{2}} \otimes \tilde{\mathbf{Z}}_{\mathbf{2}})^* V (\tilde{\mathbf{Z}}_{\mathbf{2}} \otimes \tilde{\mathbf{Z}}_{\mathbf{2}})\| = \|V' - (\tilde{\mathbf{Z}}_{\mathbf{2}} \otimes Z_1)^* V (\tilde{\mathbf{Z}}_{\mathbf{2}} \otimes Z_1)\| \leq \underline{\mathbf{k}}\,\|R\|^2$ with $\underline{\mathbf{k}} = 23(1 + (7/3)N^{1/2})$.

As any multiplicity 1 multiplicative unitary $V \in \mathcal{L}(\mathcal{H}) \otimes L(S)$ is a rank N operator in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, one gets the pretty rough upper-bound

$$\left[(2\,\underline{\mathbf{k}}^2)^{N^2}\right]^{N-1} = \left[46(1 + (7/3)N^{1/2})\right]^{2N^2(N-1)}$$

for the number of possible Hopf C*–algebra structures on the N-dimensional C*–algebra S.

References

[1] S. Baaj et G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de C*–algèbres*, Ann. scient. Ec. Norm. Sup., 4e série, t. **26** (1993), 425–488.

[2] S. Baaj, E. Blanchard et G. Skandalis, *Unitaires multiplicatifs en dimension finie et leurs sous-objets*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 4, 1305–1344.

[3] M. Beattie, S. Dăscălescu and L. Grünenfelder, *On the number of types of finite dimensional Hopf Algebras*, Inventiones Mathematicae **136** (1999), 1–7.

[4] D. Ştefan, *The set of types of n-dimensional semisimple and cosemisimple Hopf algebras is finite*, J. Algebra **193** (1997), no. 2, 571–580.

[5] A. Van Daele, *The Haar measure on finite quantum groups*, Proc. Amer. Math. Soc. **125** (1997), no. 12, 3489–3500.

[6] S.L. Woronowicz, *Compact Quantum Groups*, in 'Symétries quantiques' (Les Houches, 1995), North–Holland, Amsterdam, 1998, 845-884.

4 Appendix

The given proof of proposition 2 is incorrect as pointed out by A. Wassermann. The correct one actually goes as follows.

Suppose that $(A, \Delta, \varepsilon, \varphi)$ is a finite dimensional Hopf C*-algebra and that the $*$-linear map $T : A \to A \otimes A$ satisfies for all x, y in A (i) $T(xy) = T(x)\Delta(y) + \Delta(x)T(y)$ and (ii) $(id \otimes \Delta)T(x) + (id \otimes T)\Delta(x) = (\Delta \otimes id)T(x) + (T \otimes id)\Delta(x)$ (*cf* proposition 2).

Take Sweedler notations $\Delta(a) = \sum a_1 \otimes a_2 \in A \otimes A$ for $a \in A$ and consider the linear map

$$a \in A \mapsto \alpha(a) = \sum (\varphi \otimes id)\big[(S(a_1) \otimes 1)T(a_2)\big] \in A\,.$$

We have the equality $\sum S(a_1)a_2 = \varepsilon(a)$ and so by (i), $\alpha(ab) = \sum(\varphi \otimes id)\big[(S(b_1)S(a_1) \otimes 1)T(a_2)\Delta(b_2)\big] + \sum(\varphi \otimes id)\big[(S(a_1) \otimes 1)\Delta(a_2)T(b_2)(S(b_2) \otimes 1)\big] = \alpha(a)\,b + a\,\alpha(b)$ for all $a, b \in A$, *i.e.* α is a derivation. There exists therefore $h \in A$ with $\alpha(a) = ah - ha$.

Recall that $(\varphi \otimes id)\Big[(b \otimes 1)\Delta(c)\Big] = (id \otimes \varphi)\Big[\Delta(S(b))(1 \otimes S(c))\Big]$ for all $b, c \in A$ (*cf [6]*). Thus $\Delta\alpha(a) = -T(a) + (\alpha \otimes id)\Delta(a) + \sum(\varphi \otimes id \otimes id)\Big[(S(a_1) \otimes 1 \otimes 1)(\Delta \otimes id)T(a_2)\Big] = -T(a) + (\alpha \otimes id)\Delta(a) + (id \otimes \alpha)\Delta(a)$ thanks to above formula (ii). Set $\alpha_t(a) = \exp(t\,h)a\exp(-t\,h)$ and $\Delta_t(a) = (\alpha_t \otimes \alpha_t)\Delta\alpha_{-t}(a)$ for all $t \in \mathbf{R}$. Then $T(a) = [\Delta(a), 1 \otimes h + h \otimes 1 - \Delta(h)] = \frac{d}{dt}\Delta_t(a)\Big|_{t=0}$ is inner and there are therefore only finitely many possible Hopf C*-algebras structures on the C*-algebra A.

Inst. Math. Luminy, CNRS–Luminy case 907, F–13288 Marseille CEDEX 9
E-mail: E.Blanchard@iml.univ-mrs.fr