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A FEW REMARKS ON EXACT C(X)-ALGEBRAS

Etienne Blanchard

We extend in this paper several results of E. Kirchberg, S. Wassermann and the author dealing with continuous fields of C*-algebras to the semi-continuous case. We provide a new characterisation of separable lower semi-continuity C*-bundles and we present exactness criteria for $C(X)$-algebras and unital bisimplifiable Hopf C*-algebras.

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0. INTRODUCTION

In a paper on continuous fields of C*-algebras ([9]), E. Kirchberg and S. Wassermann have introduced several characterisations of the exactness of a continuous field of C*-algebras A over a Hausdorff compact space X through continuity conditions and fibrewise properties of the tensor product with another continuous field. Notice that it has been proved later on by E. Kirchberg that all those conditions are also equivalent in the separable case to the existence of a $C(X)$-linear isomorphism between A and a subfield of the trivial continuous field $\mathcal{O}_2 \otimes C(X)$, where $\mathcal{O}_2$ is the Cuntz C*-algebra ([4, Theorem A.1]).

Our purpose in this article is to analyse these continuity properties and to make the difference between those of lower semi-continuity and those of upper semi-continuity. After a few preliminaries on $C(X)$-algebras, we are led in theorem 2.2 to characterise, up to a $C(X)$-linear isomorphism, every separable lower semi-continuous C*-bundle as a separable $C(X)$-subalgebra of the C*-algebra $\mathcal{L}(\mathcal{E})$ of bounded $C(X)$-linear operators acting on a Hilbert $C(X)$-module $\mathcal{E}$ which admit an adjoint. This enables us to give an answer to a question of Rieffel on the characterisation of lower semi-continuous C*-bundles in the separable case ([11, page 633]) through the generalisation of a previous study of the continuous case ([3, théorème 3.3]). We then present several criteria of exactness for a $C(X)$-algebra which are analogous to those of [9] and we construct an explicit example of an algebraic tensor product of a continuous field of C*-algebras with a C*-algebra which cannot be endowed with any continuous structure. We eventually look at the two notions of exactness for a compact quantum group introduced by E. Kirchberg and S. Wassermann, the language of multiplicative unitaries allowing us to provide a direct proof of the equivalence between these two definitions.

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1. PRELIMINARIES

Let us recall in this section the basic definitions related to the theory of C*-bundles.

Definition 1.1. ([7]) Given a Hausdorff compact space X, a $C(X)$-algebra is a C*-algebra A endowed with a unital morphism from the C*-algebra $C(X)$ of continuous functions on the space X into the centre of the multiplier algebra $\mathcal{M}(A)$ of A.

We associate to such an algebra A the unital $C(X)$-algebra $\mathcal{A}$ generated by A and $u[C(X)]$ in $\mathcal{M}[A \oplus C(X)]$ where $u(g)(a \oplus f) = ga \oplus gf$ for $a \in A$ and $f, g \in C(X)$.

For $x \in X$, let $C_x(X)$ be the kernel of the evaluation map $ev_x : C(X) \to \mathbb{C}$ at x. Denote by A_x the quotient of a $C(X)$-algebra A by the closed ideal $C_x(X)A$ and by a_x the image of an element $a \in A$ in the fibre A_x. Then the function

$$x \mapsto \|a_x\| = \inf\{\|[1 - f + f(x)]a\|, f \in C(X)\}$$

is upper semi-continuous for all $a \in A$ and the $C(X)$-algebra A is said to be a continuous field of C*-algebras over X if the function $x \mapsto \|a_x\|$ is actually continuous for every $a \in A$ ([5]).

Examples. 1. If A is a $C(X)$-algebra and D is a C*-algebra, the spatial tensor product $B = A \otimes D$ is naturally endowed with a structure of $C(X)$-algebra through the map $f \in C(X) \mapsto f \otimes 1_{M(D)} \in \mathcal{M}(A \otimes D)$. In particular, if $A = C(X)$, the tensor product B is a trivial continuous field over X with constant fibre D.
2. If the C*-algebra $C(X)$ is a von Neumann algebra, then any $C(X)$-algebra A is continuous since the lower bound of the continuous functions $x \mapsto \|[1 - f + f(x)]a\|$, $f \in C(X)$, belongs to $C(X)$ for all $a \in A$ in that case. As a consequence, if D is a C*-algebra, the spatial tensor product $A \otimes B$ is a continuous field over X whose fibre at $x \in X$ is not isomorphic to the tensor product $A_x \otimes D$ in general.
3. Given two $C(X)$-algebras A_1 and A_2, the Hausdorff completion $A_1 \overset{m}{\otimes}_{C(X)} A_2$ of the algebraic tensor product $A_1 \odot A_2$ for the semi-norm $\|\alpha\|_m = \sup\{\|\sigma_x(\alpha)\|, x \in X\}$, where for every $x \in X$, σ_x is the map $A_1 \otimes A_2 \to (A_1)_x \otimes (A_2)_x$ which takes $a \otimes b$ to $a_x \otimes b_x$, is also a $C(X)$-algebra which satisfies certain minimal properties ([2, proposition 2.9]).

Definition 1.2. ([3,2]) Given a compact Hausdorff space X, a $C(X)$-representation of a $C(X)$-algebra D in a continuous field of C*-algebras A over X is a $C(X)$-linear morphism π from D into the multiplier algebra $\mathcal{M}(A)$ of A, *i.e.* such that for each $x \in X$, the induced representation π_x in $\mathcal{M}(A_x)$ factorizes through the fibre D_x.

If the $C(X)$-algebra D admits a $C(X)$-representations π in the continuous field A over X, the function

$$x \mapsto \|\pi_x(d)\| = \sup\{\|(\pi(d)a)_x\|, a \in A \text{ such that } \|a\| \leq 1\}$$

is lower semi-continuous for all $d \in D$.

Remark. If the induced representation π_x of the fibre D_x is faithful for each $x \in X$, then the function $x \mapsto \|d_x\| = \|\pi_x(d)\|$ is continuous for all $d \in D$ and the $C(X)$-algebra D is therefore continuous.

In particular, a separable $C(X)$-algebra D is continuous if and only if there exists a Hilbert $C(X)$-module $\mathcal{E}$ and a $C(X)$-representation π of D in the continuous field $\mathcal{K}(\mathcal{E})$ of compact operators acting on $\mathcal{E}$, such that for all $x \in X$, the induced representation of the fibre D_x in the C*-algebra $\mathcal{M}(\mathcal{K}(\mathcal{E})_x) = \mathcal{L}(\mathcal{E}_x)$ (where $\mathcal{E}_x$ is the Hilbert space $\mathcal{E}_x = \mathcal{E} \otimes_{ev_x} \mathbb{C}$) is faithful ([3, théorème 3.3]).

2. THE LOWER SEMI-CONTINUITY

Let us focus on the property of lower semi-continuity associated to any $C(X)$-representation of a $C(X)$-algebra.

Recall first that a faithful family of representations of a C*-algebra A is a family of stellar representations $\{\sigma_\lambda,\ \lambda \in \Lambda\}$ of A such that for any $a \in A$, there exists an index $\lambda \in \Lambda$ satisfying $\sigma_\lambda(a) \neq 0$.

Definition 2.1. Given a Hausdorff compact space X, a lower semi-continuous C*-bundle over X is a pair $(A, \{\sigma_x\})$ where

a) A is a $C(X)$-algebra;

b) $\{\sigma_x,\ x \in X\}$ is a faithful family of representations of the C*-algebra A such that for every $x \in X$, the representation σ_x factorizes through the fibre A_x;

c) for each $a \in A$, the function $x \mapsto \|\sigma_x(a)\|$ is lower semi-continuous.

Remarks. **1.** The faithful $C([0,2])$-representation σ of the $C([0,2])$-algebra $A = C([0,1]) \oplus C([1,2])$ in the Hilbert $C([0,2])$-module $\mathcal{E} = C_0([0,2]\backslash\{1\})$ gives us a typical example of a separable lower semi-continuous C*-bundle $(A, \{\sigma_x\})$ which is not continuous (we shall come back to this example at the end of the section).

2. If $(A, \{\sigma_x\})$ is a lower semi-continuous C*-bundle and $\mathcal{A}$ is the associated unital $C(X)$-algebra from definition 1.1, the pair $(\mathcal{A}, \{\widetilde{\sigma}_x\})$ (where for $x \in X$, $\widetilde{\sigma}_x$ is the unital extension of the map σ_x into the multiplier algebra $\mathcal{M}[\sigma_x(A) \oplus \mathbb{C}]$) is also a unital lower semi-continuous C*-bundle. Indeed, if $\alpha \in \mathcal{A}_+$ is positive, there exist a self-adjoint element $b \in A$ and a positive function $f \in C(X)_+$ such that $\alpha = b + u[f]$. For $x \in X$, one has the equality $Sp(\widetilde{\sigma}_x(\alpha)) \cup \{0\} = [Sp(\sigma_x(b)) + f(x)] \cup \{0\}$. As a consequence, if one writes the decomposition $b = b_+ - b_-$ where the two positive elements b_+ and b_- satisfy the relation $b_+ b_- = 0$, the map $x \mapsto \|\widetilde{\sigma}_x(\alpha)\| = \|\sigma_x(b_+)\| + f(x)$ is a lower semi-continuous function.

We can state the following theorem which generalises a previous characterisation of separable continuous fields of C*-algebras (*cf.* the last remark of the previous section).

THEOREM 2.2. *Given a separable lower semi-continuous C*-bundle $(A, \{\sigma_x\})$ over the Hausdorff compact metrisable space X, there exist a Hilbert $C(X)$-module $\mathcal{E}$ and a*

$C(X)$-representation π of A in the continuous field $\mathcal{K}(\mathcal{E})$ such that for every $x \in X$, one has the isomorphism

$$\pi_x(A) \simeq \sigma_x(A)\,.$$

As in the case of continuous fields, the proof of this theorem relies on the following lemma.

LEMMA 2.3. *Let $(A, \{\sigma_x\})$ be a unital separable lower semi-continuous C*-bundle over the Hausdorff compact space X (whose fibres are all assumed to be non zero). Endow the subspace $\widetilde{\mathcal{S}}(A) \subset S(A)$ of states $\varphi \in S(A)$ on A whose restriction to the unital C*-subalgebra $C(X)$ is the evaluation map at some point $p(\varphi) \in X$ with the restricted weak topology.*

Then the continuous map $p : \widetilde{\mathcal{S}}(A) \to X$ is open.

Remark. The restriction of a state $\varphi \in \widetilde{\mathcal{S}}(A)$ to the ideal $\ker(\sigma_x)$, where $x = p(\varphi) \in X$, is always zero.

Proof. Let Ω be a non empty open set of $\widetilde{\mathcal{S}}(A)$ and consider a point $x \in p(\Omega)$. As the C*-algebra $\sigma_x(A)$ is separable, one can find a faithful state $\varphi \in S(\sigma_x(A))$, positive norm 1 elements $a_2, \cdots a_m$ in A and a constant $\varepsilon \in]0, 1/2[$ such that

a) the state $\varphi \circ \sigma_x$ on A belongs to Ω;

b) the open set $V = \{\psi \in \widetilde{\mathcal{S}}(A), |(\psi - \varphi \circ \sigma_x)(a_k)| < \varepsilon$ for every $2 \leq k \leq m\}$ is an elementary neighbourhood of the state $\varphi \circ \sigma_x$ in Ω.

If we set $a_1 = 1 \in A$, then, replacing the indices $2, \ldots, m$ by a permutation if necessary, there exists an integer $n \leq m$ such that the family $\{\sigma_x(a_1), \cdots, \sigma_x(a_n)\}$ is a linearly independent family of maximal order. Moreover, there exist for every index $j > n$ real numbers λ_k^j, $1 \leq k \leq n$, such that the element $b_j = a_j - \sum_{k=1}^n \lambda_k^j a_k$ satisfies the equality $\sigma_x(b_j) = 0$. Define the positive sum $a_0 = \sum_{j=n+1}^m |b_j| \in \ker \sigma_x$ and the constants $R = \sup\{|\lambda_k^j|\}$, $\varepsilon_1 = \varepsilon/(nR+1) < \varepsilon$. Then the open set

$$V' = \{\psi \in \widetilde{\mathcal{S}}(A), |(\psi - \varphi \circ \sigma_x)(a_k)| < \varepsilon_1 \text{ for all } 0 \leq k \leq n\}$$

is an open neighbourhood of $\varphi \circ \sigma_x$ contained in V. Indeed, if $\psi \in V'$ and $j > n$, then on has $|(\psi - \varphi \circ \sigma_x)(a_j)| \leq \sum_{k=1}^n |\lambda_k^j|\varepsilon_1 + \psi(|b_j|) < \varepsilon$.

Consider for $y \in X$ the linear map $\alpha_y : \mathbb{C}^n \to \sigma_y(A)$ defined by $\lambda = (\lambda_k) \mapsto \sum \lambda_k \sigma_y(a_k)$ and let $\mathbb{S} \subset \mathbb{R}^n$ be the the self-adjoint part of the unit sphere of $\mathbb{C}^n$ for the norm max. By hypothesis, the constant $r = \inf\{\|\alpha_x(\lambda)\|, \lambda \in \mathbb{S}\}$ is strictly positive and the complement U_1 of the projection on X of the closed set

$$F_1 = \{(y, \lambda) \in X \times \mathbb{S}, \|\alpha_y(\lambda)\| \leq r/2\}$$

is an open neighbourhood of x in X on which the $\sigma_y(a_k)$, $1 \leq k \leq n$, are still linearly independent. As the set

$$\begin{aligned} F_2 &= \{(y, \lambda) \in U_1 \times \mathbb{S},\ \|\alpha_y(\lambda) - n\| \leq n \text{ and } \varphi \circ \alpha_x(\lambda) \leq 0\} \\ &= \{(y, \lambda) \in U_1 \times \mathbb{S},\ \alpha_y(\lambda) \geq 0 \text{ and } \varphi \circ \alpha_x(\lambda) \leq 0\} \end{aligned}$$

is also closed, we construct in the same way an open neighbourhood $U_2 \subset U_1$ of x on which the self-adjoint linear form $\varphi_y : \alpha_y(\mathbb{C}^n) \to \mathbb{C}$ defined by the formula $\alpha_y(\lambda) \mapsto \varphi \circ \alpha_x(\lambda)$ is a positive form of norm 1 on the operator system $\alpha_y(\mathbb{C}^n)$. Let ε_2 be the constant $\varepsilon_2 = \varepsilon_1/4$ and U_3 be the open complement of the projection on U_2 of the closed set

$$F_3 = \{(y, \lambda, \mu) \in U_2 \times \mathbb{S} \times [0, 2n/\varepsilon_2],\ \|\alpha_y(\lambda) - \mu\sigma_y(a_0)\| \leq (1 - \varepsilon_2)\|\alpha_x(\lambda)\|\}.$$

We are going to construct for each $y \in U_3$ a state $\Phi_y \in S(\sigma_y(A))$ such that $\Phi_y \circ \sigma_y \in V'$, a result which will end the proof of the lemma.

—> If $\|\sigma_y(a_0)\| \leq \varepsilon_2$, the linear form φ_y extends by Hahn-Banach to a self-adjoint linear form Φ_y of norm $\varphi_y(1) = 1$ on the C*-algebra $\sigma_y(A)$, whence a state $\Phi_y \circ \sigma_y \in V'$.

—> If $\|\sigma_y(a_0)\| > \varepsilon_2$, let us first notice that $\sigma_y(a_0) \notin \alpha_y(\mathbb{R}^n)$: indeed, if there existed a couple $(\lambda, \mu) \in \mathbb{S} \times \mathbb{R}_+^*$ verifying the equality $\sigma_y(a_0) = \mu^{-1}\alpha_y(\lambda)$, we would then have the inequality $|\mu| > 2n/\varepsilon_2$ (since $y \in U_3$), whence the contradiction $\|\sigma_y(a_0)\| = \mu^{-1}\|\alpha_y(\lambda)\| < \varepsilon_2/2$. As a consequence, one can extend the linear form φ_y to a self-adjoint linear form φ' on the operator system $E = \alpha_y(\mathbb{C}^n) + \mathbb{C}a_0$ by setting $\varphi'_y(a_0) = 0$. This unital self-adjoint form satisfies the inequality $\|\varphi'_y\| \leq (1-\varepsilon_2)^{-1}$, inequality which one only needs to check on the self-adjoint part of the unit sphere of E.

a) If $\lambda \in \mathbb{S}$ and $\mu \in [0, 1]$, one has the relations $\|\alpha_y(\lambda) - \mu\sigma_y(a_0)\| \geq (1-\varepsilon_2)\|\alpha_x(\lambda)\| \geq (1 - \varepsilon_2)|\varphi'_y[\alpha_y(\lambda) - \mu\sigma_y(a_0)]\|$ because $y \in U_3$;

b) if $\lambda \in \mathbb{S}$ and $\nu \in [\varepsilon_2/2n, 1]$, then $\|\nu\alpha_y(\lambda) - \sigma_y(a_0)\| = \nu\|\alpha_y(\lambda) - \nu^{-1}\sigma_y(a_0)\| \geq (1 - \varepsilon_2)|\varphi'_y[\nu\alpha_y(\lambda) - \sigma_y(a_0)]|$ thanks to a);

c) if $\lambda \in \mathbb{S}$ and $\nu \in [0, \varepsilon_2/2n[$, the hypothesis $\|\nu\alpha_y(\lambda) - \sigma_y(a_0)\| < \|\nu\alpha_y(\lambda)\|$ implies the inequality $\|\sigma_y(a_0)\| < 2\nu\|\alpha_y(\lambda)\| < \varepsilon_2$, which is absurd.

Choose a self-adjoint extension ψ_y of same norm of the form φ'_y to the C*-algebra $\sigma_y(A)$ and write the polar decomposition $\psi_y = (\psi_y)_+ - (\psi_y)_-$. One has by construction the relations $(\psi_y)_+(1) + (\psi_y)_-(1) = \|\varphi'_y\| \leq (1 - \varepsilon_2)^{-1}$ and $(\psi_y)_+(1) - (\psi_y)_-(1) = \psi_y(1) = 1$, so that $\|(\psi_y)_+ - \psi_y\| < \varepsilon_2$ because $(\psi_y)_-(1) \leq 1/2\dfrac{\varepsilon_2}{1 - \varepsilon_2} \leq \varepsilon_2$ (since $\varepsilon_2 \leq \varepsilon < 1/2$). As $\|(\psi_y)_+\| \leq \|\psi_y\| \leq (1 - \varepsilon_2)^{-1} < 2$, the state $\Phi_y = (\|(\psi_y)_+\|)^{-1}(\psi_y)_+$ satisfies $\|\Phi_y - \psi_y\| \leq \|\Phi_y - \psi_y\| + \varepsilon_2 = \|\psi_y\|(1 - \|\psi_y\|^{-1}) + \varepsilon \leq 3\varepsilon_2$ and so the state $\Phi_y \circ \sigma_y$ belongs to the neighbourhood V'. ∎

Proof of Theorem 2.2. Let us first replace the lower semi-continuous C*-bundle $(A, \{\sigma_x\})$ by the unital lower semi-continuous C*-bundle $(\mathcal{A}, \{\widetilde{\sigma}_x\})$ defined in the remark following the definition of lower semi-continuous C*-bundles, in order to be able to apply the previous lemma.

Given a positive element $a \in \mathcal{A}_+$, a point $x \in X$ and a state $\varphi \in S(\sigma_x(\mathcal{A}))$, corollary 3.7 of [3] enables us to construct a continuous section Φ of the open map $p : \widetilde{\mathcal{S}}(\mathcal{A}) \to X$ satisfying the equality $\Phi_x = \varphi \circ \sigma_x$, whence a $C(X)$-representation θ of A on a Hilbert $C(X)$-module $\mathcal{F}$ such that $\varphi \circ \sigma_x(a) \leq \|\theta_x(a)\| \leq \|\sigma_x(a)\|$ thanks to proposition 2.13 of

[3]. Now, an appropriate sum of such $C(X)$-representations allows us to construct the desired $C(X)$-representation. ∎

One derives from this theorem the following corollary, thanks to proposition 4.1 of [2] in particular.

COROLLARY 2.4. *If $(A, \{\theta_x\})$ (resp. $(B, \{\sigma_y\})$) is a lower semi-continuous C*-bundle over the Hausdorff compact space X (resp. Y), the family of representations $\{\theta_x \otimes \sigma_y\}$ defines a structure of lower semi-continuous C*-bundle with fibres $\{\theta_x(A) \otimes \sigma_y(B)\}$ on the $C(X \times Y)$-algebra $A \otimes B$.*

Furthermore, if the representation θ_x of the fibre A_x is faithful for all $x \in X$ and the two topological spaces X and Y coincide, then the pair $(A \overset{m}{\otimes}_{C(X)} B, \{\theta_x \otimes \sigma_x\})$ is a lower semi-continuous C-bundle.*

Remark. In general, the family $\{\theta_x \otimes \sigma_x; x \in X\}$ is not a faithful family of representations of the C*-algebra $A \overset{m}{\otimes}_{C(X)} B$ (*cf.* example 3.3.2 of [3]).

3. EXACTNESS CRITERIA FOR A C(X)-ALGEBRA

In this section, we reformulate the characterisation of exact continuous fields obtained by E. Kirchberg and S. Wassermann ([9, theorem 4.6]) in the framework of $C(X)$-algebras.

PROPOSITION 3.1. *Given a Hausdorff compact space X and a $C(X)$-algebra A, the following assertions are equivalent:*

1. *the C*-algebra A is exact;*
2. *for any Hausdorff compact space Y and any $C(Y)$-algebra B, one has for every couple of points $(x, y) \in X \times Y$ the isomorphism*

$$(A \otimes B)_{(x,y)} \simeq A_x \otimes B_y \ ;$$

3. *each fibre A_x is exact and for every C*-algebra B, the sequence*

$$0 \to C_x(X)A \otimes B \to A \otimes B \to A_x \otimes B \to 0$$

is exact for any point $x \in X$.

Furthermore, if the space X is metrisable and perfect, the previous assertions are equivalent to the following one :

2'. for every $C(X)$-algebra B, we have for each point $x \in X$ the isomorphism

$$(A \otimes B)_{(x,x)} \simeq A_x \otimes B_x$$

Proof. 1⇒2 Consider a $C(Y)$-algebra B and two points $x \in X$, $y \in Y$. As the C*-algebra A is exact, it satisfies property C of Archbold and Batty ([12, definition 5.3] or [6]). The lines and the columns of the following commutative diagram are therefore exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_x(X)A \otimes C_y(Y)B & \to & C_x(X)A \otimes B & \to & C_x(X)A \otimes B_y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A \otimes C_y(Y)B & \to & A \otimes B & \to & A \otimes B_y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A_x \otimes C_y(Y)B & \to & A_x \otimes B & \to & A_x \otimes B_y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Consequently, the density of the linear space $lin\,\{C_x(X) \otimes C(Y) + C(X) \otimes C_y(Y)\}$ in the ideal $C_{(x,y)}(X \times Y)$ provides us with the exact sequence

$$0 \to C_{(x,y)}(X \times Y)[A \otimes B] \to A \otimes B \to A_x \otimes B_y \to 0\ .$$

2⇒3 Given a $C(Y)$-algebra B and two points $x \in X$, $y \in Y$, we have the canonical sequence of epimorphisms $(A \otimes B)_{(x,y)} \to (A_x \otimes B)_y \to A_x \otimes B_y$, whence the exact sequence

$$0 \to A_x \otimes C_y(Y)B \to A_x \otimes B \to A_x \otimes B_y \to 0\ .$$

In particular, if $Y = \widetilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of $\mathbb{N}$ and B is the $C(\widetilde{\mathbb{N}})$-algebra $B = \prod_{n=1}^{\infty} M_n(\mathbb{C})$, then proposition 4.3 of [9] implies the exactness of the fibre A_x. Now, if the $C(Y)$-algebra B is a trivial continuous field $B = C(Y) \otimes D$, the sequence of epimorphisms $(A \otimes B)_{(x,y)} \to (A \otimes D)_x \to A_x \otimes D$ provides us with the exact sequence $0 \to C_x(X)A \otimes D \to A \otimes D \to A_x \otimes D \to 0$.

Notice that if there exists a sequence of points $x_n \in X \backslash \{x\}$ converging toward the point x, then the topological space $\{x_n\}_n \cup \{x\} \subset X$ is isomorphic to $\widetilde{\mathbb{N}}$, whence the implication *2'⇒3*.

3⇒1 Let B be a C*-algebra, K a two sided closed ideal in B and assume that the operator $d \in A \otimes B$ belongs to the kernel of the quotient map $A \otimes B \to A \otimes (B/K)$. For all $x \in X$, one then has the relation

$$\begin{aligned}
d_x \in \ker\{(A \otimes B)_x \to [A \otimes (B/K)]_x\} &= \ker\{A_x \otimes B \to A_x \otimes (B/K)\} \\
&= A_x \otimes K \quad \text{(since } A_x \text{ is exact)} \\
&= (A \otimes K)_x
\end{aligned}$$

and so $d \in A \otimes K$ since the map $A \otimes B \to \prod_{x \in X}(A \otimes B)_x$ is a monomorphism. This means that the sequence $0 \to A \otimes K \to A \otimes B \to A \otimes (B/K) \to 0$ is exact. ∎

A counter-example

Following ideas of [9] and [3], let us construct explicitly a separable continuous field of C*-algebras A on the Hausdorff compactification $\widetilde{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ of $\mathbb{N}$ and a C*-algebra B such that there is no C*-norm on the algebraic tensor product $A \odot B$ which endows this $C(\widetilde{\mathbb{N}})$-module with a structure of continuous field.

Assume that Γ is an infinite countable residually finite group satisfying property T (for instance $\Gamma = SL_3(\mathbb{Z})$). Consider the countable infinite set of classes of finite dimensional irreducible representations $\{\pi_n\}_{n\in\mathbb{N}}$ of Γ and make the hypothesis that π_0 is the trivial representation. As explained in [9, lemma 4.1], one can then find a strictly growing sequence of integers k_n, $n \in \mathbb{N}$, with $k_0 = 0$, $k_1 = 1$, such that if on sets $\sigma_n = \oplus_{i=k_n}^{k_{n+1}-1} \pi_i$, then the limit $\lim_{n\to\infty} \|\sigma_n(a)\|$ exists for all $a \in A = (\oplus \pi_n)(C^*(\Gamma))$, whence a structure of continuous field over $\widetilde{\mathbb{N}}$ for the C*-algebra A whose fibre at $n \in \mathbb{N}$ is the finite dimensional C*-algebra $A_n = \sigma_n(C^*(\Gamma))$. Define also the C*-algebra $B = (\oplus_{n\in 2\mathbb{N}} \overline{\sigma}_n)(C^*(\Gamma))$, where $\overline{\sigma}_n$ denotes the contragredient representation of the representation σ_n of Γ.

If $p \in A$ is the unique projection satisfying the relation $\|p_n\| = 1$ if and only if $n = 0$, the coproduct $\delta : C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma)$ enables us to construct (as in [3, example 3.3.1]) the image q of the projection $\delta(p)$ in the $C(\widetilde{\mathbb{N}})$-algebra $A \otimes B$. For n finite, this projection satisfies the equalities $\|q_n\| = 1$ if n is even and $\|q_n\| = 0$ if n is odd. Consequently, the sequence $n \mapsto \|q_n\|$ admits no limits as n goes to ∞.

4. UNITAL EXACT HOPF C*-ALGEBRAS

We study in this section a quantised presentation of the equivalence between the exactness of the reduced C*-algebra of a discrete group Γ and the exactness of the group Γ obtained by E. Kirchberg and S. Wassermann. All the basic definitions and notations of the theory of multiplicative unitaries which we shall use may be found in [1].

Let (S, δ) be a unital bisimplifiable Hopf C*-algebra, *i.e.* a unital C*-algebra S endowed with a unital coassociative morphism $\delta : S \to S \otimes S$, called coproduct, such that the two linear subspaces generated by $\delta(S)(1 \otimes S)$ and $\delta(S)(S \otimes 1)$ are both dense in the spatial tensor product $S \otimes S$ (for instance, the reduced C*-algebra $S = C_r^*(\Gamma)$ of a discrete group Γ). Then there exists a Haar measure on S ([14, 13]), *i.e.* a state φ on S satisfying the equalities

$$\forall a \in S, \quad (\varphi \otimes id) \circ \delta(a) = (id \otimes \varphi) \circ \delta(a) = \varphi(a)1 \in S.$$

Let $(\mathcal{H}_\varphi, L, e)$ be the G.N.S. construction associated to this state φ and define the multiplicative unitary $\widehat{V} \in L(S) \otimes \mathcal{L}(\mathcal{H}_\varphi) \subset \mathcal{L}(\mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$, by the formula

$$\forall a, b \in S, \quad \widehat{V}^*(L(a)e \otimes L(b)e) = (L \otimes L)(\delta(b)(a \otimes 1))(e \otimes e).$$

The coproduct of the Hopf C*-algebra $L(S) = \overline{lin}\,\{L(\omega) = (id \otimes \omega)(\widehat{V}), \omega \in \mathcal{L}(\mathcal{H}_\varphi)_*\}$ satisfies the equality $(L \otimes L)\delta(a) = \widehat{V}^*(1 \otimes L(a))\widehat{V}$ for all $a \in S$. For $a \in S$, the operator $\lambda(\varphi a) = (\varphi \otimes id)((a \otimes 1)\widehat{V}) \in \mathcal{L}(\mathcal{H}_\varphi)$ satisfies the relation $1 \otimes \lambda(\varphi a) = (\varphi \otimes L \otimes id)(\delta \otimes id)((a \otimes 1)\widehat{V}) = (\varphi \otimes L \otimes id)(\delta(a)_{12}\widehat{V}_{13})\widehat{V}$ since φ is a Haar state

on S. As the closed linear span $\lambda(\widehat{S}) = \{\lambda(\varphi a), a \in S\}$ defines a non degenerate C*-algebra, the right simplifiability of the Hopf C*-algebra (S, δ) implies that the linear span $lin\,(1 \otimes \lambda(\widehat{S}))\widehat{V}^*(L(S) \otimes 1)$ is dense in $L(S) \otimes \lambda(\widehat{S})$ and so the multiplicative unitary $\widehat{V}$ is regular ([14]). Thus, the C*-algebra $\lambda(\widehat{S})$, which is also the closed linear span of the elements $\lambda(\omega) = (\omega \otimes id)(\widehat{V})$, $\omega \in \mathcal{L}(\mathcal{H}_\varphi)_*$, is endowed with a structure of bisimplifiable Hopf C*-algebra for the coproduct $(\lambda \otimes \lambda)\widehat{\delta}(d) = \widehat{V}(\lambda(d) \otimes 1)\widehat{V}^*$. The couple $(\widehat{S}, \widehat{\delta})$ will be called the dual Hopf C*-algebra of the Hopf C*-algebra $(L(S), (L \otimes L) \circ \delta)$ ([1]).

If a C*-algebra A is endowed with a non-degenerate coaction $\delta_A : A \to \mathcal{M}(A \otimes \widehat{S})$ of the Hopf C*-algebra $\widehat{S}$, a covariant representation of the pair (A, δ_A) is a couple (π, X) where π is a representation of A on a Hilbert space $\mathcal{H}$ and X is a representation of the Hopf C*-algebra $\widehat{S}$ on the same Hilbert space $\mathcal{H}$, *i.e.* a unitary satisfying the relation $X_{12}X_{13}\widehat{V}_{23} = \widehat{V}_{23}X_{12}$ in $\mathcal{L}(\mathcal{H} \otimes \mathcal{H}_\varphi \otimes \mathcal{H}_\varphi)$ and such that for all $a \in A$, one has the equality $(\pi \otimes id)\delta_A(a) = X(\pi(a) \otimes 1)X^*$ in $\mathcal{L}(\mathcal{H} \otimes \widehat{S})$ (*cf.* [3, section 5.2]). The crossed product $A \rtimes S$ is then the closed vector space generated by the products $\pi(a)(id \otimes \omega)(X) \in \mathcal{L}(\mathcal{H})$, $a \in A$ and $\omega \in \mathcal{L}(\mathcal{H}_\varphi)_*$; it is endowed with two canonical non-degenerate morphisms of C*-algebras $\pi : A \to A \rtimes S$ and $L_X : S_f \to \mathcal{M}(A \rtimes S)$, where S_f denotes the full Hopf C*-algebra associated to S.

If one keeps these notations, then one can state the following proposition.

PROPOSITION 4.1. *Given a unital bisimplifiable Hopf C*-algebra (S, δ) whose dual is the Hopf C*-algebra $(\widehat{S}, \widehat{\delta})$, the following assertions are equivalent :*

1. *the reduced C*-algebra $L(S)$ is exact;*
2. *for every $\widehat{S}$-equivariant sequence $0 \to J \to A \to A/J \to 0$, the sequence of reduced crossed products*

$$0 \to J \rtimes_r S \to A \rtimes_r S \to (A/J) \rtimes_r S \to 0$$

is also exact.

Proof. As the implication 2) $\Rightarrow$ 1) is clear (it suffices to look at trivial coactions), the main point of the proposition is contained in the converse implication.

Consider a $\widehat{S}$-equivariant sequence $0 \to J \to A \to A/J \to 0$ and denote by $A \rtimes_f S$ (resp $J \rtimes_f S$, $(A/J) \rtimes_f S$) the full crossed product of A (resp. J, A/J) by S. Let π be the canonical representation of A in $A \rtimes_f S$ and let $X \in \mathcal{L}(A \rtimes_p S \otimes \mathcal{H}_\varphi)$ be the unitary defining the representation of S_f in $\mathcal{M}(A \rtimes_f S)$. The quotient $D = (A \rtimes_f S)/(J \rtimes_f S)$ is generated by the products of images of elements of A and S in D. Furthermore, the image of J in $\mathcal{M}(D)$ is zero by construction and so one has an isomorphism $D \simeq (A/J) \rtimes_f S$.

Notice that the reduced crossed product $A \rtimes_r S$ is linearly generated by the products $\delta_A(a)(id \otimes id \otimes \omega)(\widehat{V}_{23}) \in \mathcal{L}(A \otimes \mathcal{H}_\varphi)$, $a \in A$ and $\omega \in \mathcal{L}(\mathcal{H}_\varphi)_*$ ([1]) and that the unitary $\widehat{V}$ belongs to the C*-algebra $S \otimes \mathcal{L}(\mathcal{H}_\varphi)$. As a consequence, one defines a monomorphism $\Phi_A : A \rtimes_r S \to A \rtimes_f S \otimes S$ through the formula

$$\begin{aligned}\Phi_A\left[\delta_A(a)(id \otimes id \otimes \omega)(\widehat{V}_{23})\right] &= X^*(\pi \otimes id)\left[\delta_A(a)(id \otimes id \otimes \omega)(\widehat{V}_{23})\right]X \\ &= (\pi(a) \otimes 1)(id \otimes id \otimes \omega)(X_{13}\widehat{V}_{23}) \in A \rtimes_p S \otimes S\,.\end{aligned}$$

One constructs in the same way monomorphisms $\Phi_J : J \rtimes_r S \to J \rtimes_f S \otimes S$ and $\Phi_{A/J} : (A/J) \rtimes_r S \to (A/J) \rtimes_f S \otimes S$.

As the C*-algebra $L(S)$ is by assumption exact, one has the following commutative diagram whose lower line is exact.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & J \rtimes_r S & \to & A \rtimes_r S & \to & (A/J) \rtimes_r S & \to & 0 \\
 & & \downarrow{\scriptstyle \Phi_J} & & \downarrow{\scriptstyle \Phi_A} & & \downarrow{\scriptstyle \Phi_{A/J}} & & \\
0 & \to & J \rtimes_f S \otimes L(S) & \to & A \rtimes_f S \otimes L(S) & \to & (A/J) \rtimes_f S \otimes L(S) & \to & 0
\end{array}$$

Assume that $T \in A \rtimes_r S$ belongs to the kernel $\ker(A \rtimes_r S \to (A/J) \rtimes_r S)$ and let $(u_\lambda)_\lambda$ be an approximate identity of the ideal J in A. Then the net $\|\Phi_A(T)[1-(\pi(u_\lambda)\otimes 1)]\| = \|\Phi_A(T[1-\delta_A(u_\lambda)])\|$ converges to zero and so $T \in J \rtimes_r S$, whence the expected exactness of the upper line. ∎

As noticed in the latest remark of [9], one deduces from this proposition the following corollary which generalises corollary 5.10 of [3], where the amenable case was considered.

COROLLARY 4.2. *If S is a unital bisimplifiable Hopf C*-algebra whose reduced C*-algebra $L(S)$ is exact and A is a $C(X)$-algebra endowed with a non-degenerate $C(X)$-linear coaction $\delta_A : A \to \mathcal{M}(A \otimes \widehat{S})$ of the (discrete) quantum dual $\widehat{S}$ of S, then the fibre at $x \in X$ of the reduced crossed product $A \rtimes_r S$ is $(A \rtimes_r S)_x = A_x \rtimes_r S$.*

In particular, if the $C(X)$-algebra A is continuous, the $C(X)$-algebra $A \rtimes_r S$ is a continuous field of C-algebras with fibres $A_x \rtimes_r S$.*

Remark. It would be interesting to know whether proposition 4.1 generalises to the framework of Hopf C*-algebras S_V associated to a regular multiplicative unitary $V \in \mathcal{M}(\widehat{S}_V \otimes S_V)$, *i.e.* given a $\widehat{S}_V$-equivariant sequence $0 \to J \to A \to A/J \to 0$, does the exactness of the C*-algebra S_V imply the exactness of the sequence of reduced crossed products $0 \to J \rtimes_r S_V \to A \rtimes_r S_V \to (A/J) \rtimes_r S_V \to 0$?

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