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# K<sub>1</sub>-INJECTIVITY FOR PROPERLY INFINITE C\*-ALGEBRAS

ÉTIENNE BLANCHARD

*Dedicated to Alain Connes on the occasion of his 60th birthday.*

## 1. INTRODUCTION

One of the main tools to classify C\*-algebras is the study of its projections and its unitaries. It was proved by J. Cuntz in [Cun81] that if  $A$  is a *purely infinite* simple C\*-algebra, then the kernel of the natural map for the unitary group  $\mathcal{U}(A)$  to the  $K$ -theory group  $K_1(A)$  is reduced to the connected component  $\mathcal{U}^0(A)$ , *i.e.*  $A$  is  $K_1$ -*injective* (see §3). We study in this note a finitely generated C\*-algebra, the  $K_1$ -injectivity of which would imply the  $K_1$ -injectivity of all unital *properly infinite* C\*-algebras.

Note that such a question was already considered in [Blac07], [BRR08].

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## 2. PRELIMINARIES

Let us first review briefly the theory introduced by J. Cuntz ([Cun78]) of comparison of positive elements in a C\*-algebra.

**Definition 2.1.** ([Cun78], [Rør92]) Given two positive elements  $a, b$  in a C\*-algebra  $A$ , one says that:

- $a$  is *dominated* by  $b$  (written  $a \precsim b$ ) if and only if there is a sequence  $\{d_k; k \in \mathbb{N}\}$  in  $A$  such that  $\|d_k^* b d_k - a\| \rightarrow 0$  when  $k \rightarrow \infty$ ,
- $a$  is *properly infinite* if  $a \neq 0$  and  $a \oplus a \precsim a \oplus 0$  in the C\*-algebra  $M_2(A) := M_2(\mathbb{C}) \otimes A$ .

This leads to the following notions of infiniteness for C\*-algebras.

**Definition 2.2.** ([Cun78], [Cun81], [KR00]) A unital C\*-algebra  $A$  is said to be:

- *properly infinite* if its unit  $1_A$  is properly infinite in  $A$ ,
- *purely infinite* if all the non zero positive elements in  $A$  are properly infinite in  $A$ .

**Remark 2.3.** E. Kirchberg and M. Rørdam have proved in [KR00, Theorem 4.16] that a C\*-algebra  $A$  is purely infinite (in the above sense) if and only if there is no character on the C\*-algebra  $A$  and any positive element  $a$  in  $A$  which lies in the closed two-sided ideal generated by another positive element  $b$  in  $A$  satisfies  $a \precsim b$ .

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The first examples of such C\*-algebras were given by J. Cuntz in [Cun81]: For any integer  $n \geq 2$ , the C\*-algebra  $\mathcal{T}_n$  is the universal unital C\*-algebra generated by  $n$  isometries  $s_1, \dots, s_n$  satisfying the relation

$$s_1 s_1^* + \dots + s_n s_n^* \leq 1 \quad (2.1)$$

Then, the closed two sided ideal in  $\mathcal{T}_n$  generated by the *minimal* projection  $p_{n+1} := 1 - s_1 s_1^* - \dots - s_n s_n^*$  is isomorphic to the C\*-algebra  $\mathcal{K}$  of compact operators on an infinite dimension separable Hilbert space and one has an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T}_n \xrightarrow{\pi} \mathcal{O}_n \rightarrow 0, \quad (2.2)$$

where the quotient  $\mathcal{O}_n$  is a purely infinite *simple* unital nuclear C\*-algebra ([Cun81]).

**Remark 2.4.** A unital C\*-algebra  $A$  is properly infinite if and only if there exists a unital \*-homomorphism  $\mathcal{T}_2 \rightarrow A$ .

### 3. $K_1$ -INJECTIVITY OF $\mathcal{T}_n$

Given a unital C\*-algebra  $A$  with unitary group  $\mathcal{U}(A)$ , denote by  $\mathcal{U}^0(A)$  the connected component of  $1_A$  in  $\mathcal{U}(A)$ . For each strictly positive integer  $k \geq 1$ , the upper diagonal embedding  $u \in \mathcal{U}(M_k(A)) \mapsto (u \oplus 1_A) \in \mathcal{U}(M_{k+1}(A))$  sends the connected component  $\mathcal{U}^0(M_k(A))$  into  $\mathcal{U}^0(M_{k+1}(A))$ , whence a canonical homomorphism  $\Theta_A$  from  $\mathcal{U}(A)/\mathcal{U}^0(A)$  to  $K_1(A) := \varinjlim_{k \rightarrow \infty} \mathcal{U}(M_k(A))/\mathcal{U}^0(M_k(A))$ . As noticed by B. Blackadar in [Blac07], this map is (1) neither injective, (2) nor surjective in general:

- (1) If  $\mathfrak{U}_2$  denotes the compact unitary group of the matrix C\*-algebra  $M_2(\mathbb{C})$ ,  $A := C(\mathfrak{U}_2 \times \mathfrak{U}_2, M_2(\mathbb{C}))$  and  $u, v \in \mathcal{U}(A)$  are the two unitaries  $u(x, y) = x$  and  $v(x, y) = y$ , then  $z := uvu^*v^*$  is not unitarily homotopic to  $1_A$  in  $\mathcal{U}(A)$  but the unitary  $z \oplus 1_A$  belongs to  $\mathcal{U}^0(M_2(A))$  ([AJT60]).
- (2) If  $A = C(\mathbb{T}^3)$ , then  $\mathcal{U}(A)/\mathcal{U}^0(A) \cong \mathbb{Z}^3$  but  $K_1(A) \cong \mathbb{Z}^4$ .

**Definition 3.1.** The unital C\*-algebra  $A$  is said to be  $K_1$ -*injective* if the map  $\Theta_A$  is injective.

J. Cuntz proved in [Cun81] that  $\Theta_A$  is surjective as soon as the C\*-algebra  $A$  is properly infinite and that it is also injective if the C\*-algebra  $A$  is simple and purely infinite. Now, the  $K$ -theoretical six-term cyclic exact sequence associated to the exact sequence (2.2) implies that  $K_1(\mathcal{T}_n) = 0$  since  $K_1(\mathcal{K}) = K_1(\mathcal{O}_n) = 0$ . Thus, the map  $\Theta_{\mathcal{T}_n}$  is zero.

**Proposition 3.2.** *For all  $n \geq 2$ , the C\*-algebra  $\mathcal{T}_n$  is  $K_1$ -injective, i.e. any unitary  $u \in \mathcal{U}(\mathcal{T}_n)$  is unitarily homotopic to  $1_{\mathcal{T}_n}$  in  $\mathcal{U}(\mathcal{T}_n)$  (written  $u \sim_h 1_{\mathcal{T}_n}$ ).*

*Proof.* The C\*-algebras  $\mathcal{T}_n$  have real rank zero by Proposition 2.3 of [Zha90]. And Lin proved that any unital C\*-algebra of real rank zero is  $K_1$ -injective ([Lin01, Corollary 4.2.10]).  $\square$

**Corollary 3.3.** *If  $\alpha : \mathcal{T}_3 \rightarrow \mathcal{T}_3$  is a unital \*-endomorphism, then its restriction to the unital copy of  $\mathcal{T}_2$  generated by the two isometries  $s_1, s_2$  is unitarily homotopic to  $id_{\mathcal{T}_2}$  among all unital \*-homomorphisms  $\mathcal{T}_2 \rightarrow \mathcal{T}_3$ .*

*Proof.* The isometry  $\sum_{k=1,2} \alpha(s_k) s_k^*$  extends to a unitary  $u \in \mathcal{U}(\mathcal{T}_3)$  such that  $\alpha(s_k) = u s_k$  for  $k = 1, 2$  ([BRR08, Lemma 2.4]). But Proposition 3.2 yields that  $\mathcal{U}(\mathcal{T}_3) = \mathcal{U}^0(\mathcal{T}_3)$ , whence a homotopy  $u \sim_h 1$  in  $\mathcal{U}(\mathcal{T}_3)$ , and so the desired result holds.  $\square$

**Remark 3.4.** The unital map  $\iota : \mathbb{C} \rightarrow \mathcal{T}_2$  induces an isomorphism in  $K$ -theory. Indeed,  $[1_{\mathcal{T}_2}] = [s_1 s_1^*] + [s_2 s_2^*] + [p_3] = 2[1_{\mathcal{T}_2}] + [p_3]$  and so  $[1_{\mathcal{T}_2}] = -[p_3]$  is invertible in  $K_0(\mathcal{T}_2)$ .

#### 4. $K_1$ -INJECTIVITY OF PROPERLY INFINITE $C^*$ -ALGEBRAS

Denote by  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  the universal unital free product with amalgamation over  $\mathbb{C}$  (in the sequel called full unital free product) of two copies of  $\mathcal{T}_2$  amalgamated over  $\mathbb{C}$  and let  $j_0, j_1$  be the two canonical unital inclusions of  $\mathcal{T}_2$  in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ . We show in this section that the  $K_1$ -injectivity of  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  is equivalent to the  $K_1$ -injectivity of all the unital properly infinite  $C^*$ -algebras. The proof is similar to that of the universality of the full unital free product  $\mathcal{O}_\infty *_\mathbb{C} \mathcal{O}_\infty$  (see Theorem 5.5 of [BRR08]).

**Definition 4.1.** ([Blan09], [BRR08, §2]) If  $X$  is a compact Hausdorff space, a unital  $C(X)$ -algebra is a unital  $C^*$ -algebra  $A$  endowed with a unital  $*$ -homomorphism from the  $C^*$ -algebra  $C(X)$  of continuous functions on  $X$  to the centre of  $A$ .

For any non-empty closed subset  $Y$  of  $X$ , we denote by  $\pi_Y^A$  (or simply by  $\pi_Y$  if no confusion is possible) the quotient map from  $A$  to the quotient  $A_Y$  of  $A$  by the (closed) ideal  $C_0(X \setminus Y) \cdot A$ . For any point  $x \in X$ , we also denote by  $A_x$  the quotient  $A_{\{x\}}$  and by  $\pi_x$  the quotient map  $\pi_{\{x\}}$ .

**Proposition 4.2.** *The following assertions are equivalent.*

- (i)  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  is  $K_1$ -injective.
- (ii)  $\mathcal{D} := \{f \in C([0, 1], \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) ; f(0) \in j_0(\mathcal{T}_2) \text{ and } f(1) \in j_1(\mathcal{T}_2)\}$  is properly infinite.
- (iii) There exists a unital  $*$ -homomorphism  $\theta : \mathcal{T}_2 \rightarrow \mathcal{D}$ .
- (iv) There exists a projection  $q \in \mathcal{D}$  with  $\pi_0(q) = j_0(s_1 s_1^*)$  and  $\pi_1(q) = j_1(s_1 s_1^*)$ .
- (v) Any unital properly infinite  $C^*$ -algebra  $A$  is  $K_1$ -injective.

*Proof.* (i) $\Rightarrow$ (ii) We have a pull-back diagram

$$\begin{array}{ccc}
 & \mathcal{D} & \\
 \swarrow & & \searrow \\
 \mathcal{D}_{[0, \frac{1}{2}]} & & \mathcal{D}_{[\frac{1}{2}, 1]} \\
 \searrow \pi_{\frac{1}{2}} & & \swarrow \pi_{\frac{1}{2}} \\
 & \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2 &
 \end{array}$$

and the two  $C^*$ -algebras  $\mathcal{D}_{[0, \frac{1}{2}]}$  and  $\mathcal{D}_{[\frac{1}{2}, 1]}$  are properly infinite (Remark 2.4). Hence, the implication follows from [BRR08, Proposition 2.7].

(ii) $\Rightarrow$ (iii) is Remark 2.4 applied to the  $C^*$ -algebra  $\mathcal{D}$ .

(iii) $\Rightarrow$ (iv) The two full, properly infinite projections  $j_0(s_1 s_1^*)$  and  $\pi_0 \circ \theta(s_1 s_1^*)$  are unitarily equivalent in  $j_0(\mathcal{T}_2)$  by [LLR00, Lemma 2.2.2] and [BRR08, Proposition 2.3].

Thus, they are homotopic among the projections in the  $C^*$ -algebra  $j_0(\mathcal{T}_2)$  (written  $j_0(s_1 s_1^*) \sim_h \pi_0 \circ \theta(s_1 s_1^*)$ ) by Proposition 3.2. Similarly,  $\pi_1 \circ \theta(s_1 s_1^*) \sim_h j_1(s_1 s_1^*)$  in  $j_1(\mathcal{T}_2)$ . Further,  $\pi_0 \circ \theta(s_1 s_1^*) \sim_h \pi_1 \circ \theta(s_1 s_1^*)$  in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  by hypothesis, whence the result by composition.

(iv) $\Rightarrow$ (v) By [BRR08, Proposition 5.1], it is equivalent to prove that if  $p$  and  $p'$  are two properly infinite full projections in  $A$ , then there exist full properly infinite projections  $p_0$ , and  $p'_0$  in  $A$  such that  $p_0 \leq p$ ,  $p'_0 \leq p'$  and  $p_0 \sim_h p'_0$ .

Fix two such projections  $p$  and  $p'$  in  $A$ . Then, there exist unital  $*$ -homomorphisms  $\sigma : \mathcal{T}_2 \rightarrow pAp$ ,  $\sigma' : \mathcal{T}_2 \rightarrow p'Ap'$  and isometries  $t, t' \in A$  such that  $1_A = t^* p t = (t')^* p' t'$ . Now, one thoroughly defines unital  $*$ -homomorphisms  $\alpha_0 : \mathcal{T}_2 \rightarrow A$  and  $\alpha_1 : \mathcal{T}_2 \rightarrow A$  by

$$\alpha_0(s_k) := \sigma(s_k) \cdot t \quad \text{and} \quad \alpha_1(s_k) := \sigma'(s_k) \cdot t' \quad \text{for } k = 1, 2,$$

whence a unital  $*$ -homomorphism  $\alpha := \alpha_0 * \alpha_1 : \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2 \rightarrow A$  such that  $\alpha \circ j_0 = \alpha_0$  and  $\alpha \circ j_1 = \alpha_1$ .

The two full properly infinite projections  $p_0 = \alpha_0(s_1 s_1^*)$  and  $p'_0 = \alpha_1(s_1 s_1^*)$  satisfy  $p_0 \leq p$  and  $p'_0 \leq p'$ . Further, the projection  $(id \otimes \alpha)(q)$  gives a continuous path of projections in  $A$  from  $p_0$  to  $p'_0$ .  $\square$

**Remark 4.3.** The  $C^*$ -algebra  $M_2(\mathcal{D})$  is properly infinite by [BRR08, Proposition 2.7].

**Lemma 4.4.**  $K_0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) = \mathbb{Z}$  and  $K_1(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) = 0$

*Proof.* The commutative diagram 
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\iota_1} & \mathcal{T}_2 \\ \iota_0 \downarrow & & \downarrow j_1 \\ \mathcal{T}_2 & \xrightarrow{j_0} & \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2 \end{array}$$
 yields by [Ger97, Theorem 2.2]

a six-term cyclic exact sequence

$$\begin{array}{ccccc} K_0(\mathbb{C}) = \mathbb{Z} & \xrightarrow{(\iota_0 \oplus \iota_1)^*} & K_0(\mathcal{T}_2 \oplus \mathcal{T}_2) = \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(j_0)^* - (j_1)^*} & K_0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) & \longleftarrow & K_1(\mathcal{T}_2 \oplus \mathcal{T}_2) = 0 \oplus 0 & \longleftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

Now, Remark 3.4 implies that the map  $(\iota_0 \oplus \iota_1)^*$  is injective, whence the equalities.  $\square$

**Remark 4.5.** G. Skandalis noticed that the  $C^*$ -algebra  $\mathcal{T}_2$  is  $KK$ -equivalent to  $\mathbb{C}$  and so  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  is  $KK$ -equivalent to  $\mathbb{C} *_\mathbb{C} \mathbb{C} = \mathbb{C}$ .

This Lemma entails that the  $K_1$ -injectivity question for unital properly infinite  $C^*$ -algebras boils down to knowing whether  $\mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) = \mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ . Note that Proposition 3.2 already yields that  $\mathcal{U}(\mathcal{T}_2) *_\mathbb{T} \mathcal{U}(\mathcal{T}_2) \subset \mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ .

But the following holds.

**Proposition 4.6.** Set  $p_3 = 1 - s_1 s_1^* - s_2 s_2^*$  in the Cuntz algebra  $\mathcal{T}_2$  and let  $u$  be the canonical unitary generating  $C^*(\mathbb{Z})$ .

(i) The relations  $j_0(s_k) \mapsto s_k$  and  $j_1(s_k) \mapsto u s_k$  ( $k = 1, 2$ ) uniquely define a unital  $*$ -homomorphism  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2 \rightarrow \mathcal{T}_2 *_\mathbb{C} C^*(\mathbb{Z})$  which is injective but not  $K_1$ -surjective.

- (ii) The two projections  $j_0(p_3)$  and  $j_1(p_3)$  satisfy  $j_1(p_3) \not\sim j_0(p_3)$  in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ .  
(iii) There is no  $v \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  such that  $j_1(s_1 s_1^* + s_2 s_2^*) = v j_0(s_1 s_1^* + s_2 s_2^*) v^*$ .  
(iv) There is a unitary  $v \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  such that  $j_1(s_1 s_1^*) = v j_0(s_1 s_1^*) v^*$ .

*Proof.* (i) The unital  $C^*$ -subalgebra of  $\mathcal{O}_3$  generated by the two isometries  $s_1$  and  $s_2$  is isomorphic to  $\mathcal{T}_2$ , whence a unital  $C^*$ -embedding  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2 \subset \mathcal{O}_3 *_\mathbb{C} \mathcal{O}_3$  ([ADEL04]). Let  $\Phi$  be the  $*$ -homomorphism from  $\mathcal{O}_3 *_\mathbb{C} \mathcal{O}_3$  to the free product  $\mathcal{O}_3 *_\mathbb{C} C^*(\mathbb{Z}) = C^*(s_1, s_2, s_3, u)$  fixed by the relations

$$\Phi(j_0(s_k)) = s_k \quad \text{and} \quad \Phi(j_1(s_k)) = u s_k \quad \text{for } k = 1, 2, 3$$

and let  $\Psi : \mathcal{O}_3 *_\mathbb{C} C^*(\mathbb{Z}) \rightarrow \mathcal{O}_3 *_\mathbb{C} \mathcal{O}_3$  be the only  $*$ -homomorphism such that

$$\Psi(u) = \sum_{l=1}^3 j_1(s_l) j_0(s_l)^* \quad \text{and} \quad \Psi(s_k) = j_0(s_k) \quad \text{for } k = 1, 2, 3.$$

For all  $k = 1, 2, 3$ , we have the equalities:

$$\begin{aligned} - \Psi \circ \Phi(j_0(s_k)) &= \Psi(s_k) = j_0(s_k), \\ - \Psi \circ \Phi(j_1(s_k)) &= \Psi(u s_k) = j_1(s_k), \\ - \Phi \circ \Psi(s_k) &= \Phi(j_0(s_k)) = s_k. \end{aligned}$$

Also,  $\Psi(u)^* \Psi(u) = \sum_{l,l'} j_0(s_{l'}) j_1(s_l)^* j_1(s_l) j_0(s_l)^* = 1_{\mathcal{O}_3 *_\mathbb{C} \mathcal{O}_3} = \Psi(u) \Psi(u)^*$ , i.e.  $\Psi(u)$  is a unitary in  $\mathcal{O}_3 *_\mathbb{C} \mathcal{O}_3$  which satisfies:

$$- \Phi \circ \Psi(u) = \sum_{l=1,2,3} \Phi(j_1(s_l)) \Phi(j_0(s_l)^*) = u.$$

Thus,  $\Phi$  is an invertible unital  $*$ -homomorphism with inverse  $\Psi$  ([Blac07]), and the restriction of  $\Phi$  to the  $C^*$ -subalgebra  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  takes values in  $\mathcal{T}_2 *_\mathbb{C} C^*(\mathbb{Z}) \subset \mathcal{O}_3 *_\mathbb{C} C^*(\mathbb{Z})$ .

Now, there is (see [Ger97]) a six-term cyclic exact sequence

$$\begin{array}{ccccc} K_0(\mathbb{C}) = \mathbb{Z} & \hookrightarrow & K_0(\mathcal{T}_2 \oplus C^*(\mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z} & \rightarrow & K_0(\mathcal{T}_2 *_\mathbb{C} C^*(\mathbb{Z})) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{T}_2 *_\mathbb{C} C^*(\mathbb{Z})) & \leftarrow & K_1(\mathcal{T}_2 \oplus C^*(\mathbb{Z})) = 0 \oplus \mathbb{Z} & \leftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

and so,  $K_1(\mathcal{T}_2 *_\mathbb{C} C^*(\mathbb{Z})) = \mathbb{Z}$ , whereas  $K_1(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) = 0$  by Lemma 4.4.

(ii) Let  $\pi_0 : \mathcal{T}_2 \rightarrow L(H)$  be a unital  $*$ -representation on a separable Hilbert space  $H$  such that  $\pi_0(p_3)$  is a rank one projection, let  $\pi_1 : \mathcal{T}_2 \rightarrow L(H)$  be a unital  $*$ -representation such that  $\pi_1(p_3)$  is a rank two projection and consider the induced unital  $*$ -representation  $\pi = \pi_0 * \pi_1$  of the unital free product  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ .

Then the two projections  $\pi[j_0(p_3)] = \pi_0(p_3)$  and  $\pi[j_1(p_3)] = \pi_1(p_3)$  have distinct ranks and so cannot be equivalent in  $L(H)$ . Hence,  $j_0(p_3) \not\sim j_1(p_3)$  in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ .

(iii) This is just a rewriting of the previous assertion since  $s_1 s_1^* + s_2 s_2^* = 1 - p_3$ . Indeed, the partial isometry  $b = j_1(s_1) j_0(s_1)^* + j_1(s_2) j_0(s_2)^*$  defines a Murray-von Neumann equivalence in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  between the projections  $j_0(s_1 s_1^* + s_2 s_2^*) = 1 - j_0(p_3)$  and  $j_1(s_1 s_1^* + s_2 s_2^*) = 1 - j_1(p_3)$ . Thus, they are unitarily equivalent in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  if and only if  $j_0(p_3) \sim j_1(p_3)$  in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  ([LLR00, Proposition 2.2.2]).

(iv) There exists a unitary  $v \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  (which is necessarily  $K_1$ -trivial by Lemma 4.4) such that  $j_1(s_1 s_1^*) = v j_0(s_1 s_1^*) v^*$ . Indeed, we have the inequalities

$$1 > s_2 s_2^* + p_3 > s_2 s_2^* > s_2 s_1 (s_2 s_2^* + p_3) s_1^* s_2^* + s_2 s_2 (s_2 s_2^* + p_3) s_2^* s_2^* \quad \text{in } \mathcal{T}_2.$$

Thus, if we set  $w := j_1(s_1) j_0(s_1)^*$ , then  $1 - w^* w = j_0(s_2 s_2^* + p_3)$  and  $1 - w w^* = j_1(s_2 s_2^* + p_3)$  are two properly infinite and full  $K_0$ -equivalent projections in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ . Thus, there is a partial isometry  $a \in \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  with  $a^* a = 1 - w^* w$  and  $a a^* = 1 - w w^*$  ([Cun81]). The sum  $v = a + w$  has the required properties ([BRR08, Lemma 2.4]).  $\square$

**Remarks 4.7.** (i) The equivalence (iv) $\Leftrightarrow$ (v) in Proposition 4.2 implies that all unital properly infinite  $C^*$ -algebras are  $K_1$ -injective if and only if the unitary  $v \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  constructed in Proposition 4.6.(iv) belongs to the connected component  $\mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ .

Note that  $v \oplus 1 \sim_h 1 \oplus 1$  in  $\mathcal{U}(M_2(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2))$  by [LLR00, Exercice 8.11].

(ii) Let  $\sigma \in \mathcal{U}(\mathcal{T}_2)$  be the symmetry  $\sigma = s_1 s_2^* + s_2 s_1^* + p_3$ , let  $v \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  be a unitary such that  $j_1(s_1 s_1^*) = v j_0(s_1 s_1^*) v^*$  (Proposition 4.6.(iv)) and set  $z := v^* j_1(\sigma) v j_0(\sigma)$ .

Then,  $q_1 = j_0(s_1 s_1^*)$ ,  $q_2 = j_0(s_2 s_2^*)$  and  $q_3 = z j_0(s_2 s_2^*) z^*$  are three properly infinite full projections in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  which satisfy:

- $q_1 q_3 = j_0(s_1 s_1^*) v^* j_1(s_2 s_2^*) v = v^* j_1(s_1 s_1^*) j_1(s_2 s_2^*) v = 0 = q_1 q_2$ ,
- $q_2 \sim_h q_1 \sim_h q_3$  in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  since  $\sigma \in \mathcal{U}^0(\mathcal{T}_2)$  and so  $z \sim_h v^* v = 1$  in  $\mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ ,
- $q_1 + q_3 = v^* j_1(s_1 s_1^* + s_2 s_2^*) v \not\sim_u j_0(s_1 s_1^* + s_2 s_2^*) = q_1 + q_2$  in  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  by Proposition 4.6.(iii).

## Addendum

(iii) Let  $\alpha = \alpha_0 * \alpha_1$  be the unital  $*$ -endomorphism of the free product  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  defined by  $\alpha_0(s_k) = j_0(s_k)$  and  $\alpha_1(s_k) = v^* j_1(s_k)$  for  $k = 1, 2$ . Then  $\alpha_0(s_2 s_2^* + p_3) = 1 - \alpha_0(s_1 s_1^*) = 1 - \alpha_1(s_1 s_1^*) = \alpha_1(s_2 s_2^* + p_3)$  and  $\alpha_0(s_2 s_2^*) \sim_h \alpha_0(s_1 s_1^*) = \alpha_1(s_1 s_1^*) \sim_h \alpha_1(s_2 s_2^*)$  among the projections in  $\alpha(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ . But  $\alpha_0(p_3) \not\sim \alpha_1(p_3)$  in  $\alpha(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ .

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