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# AMALGAMATED PRODUCTS OF $C^*$ -BUNDLES

ÉTIENNE BLANCHARD

ABSTRACT. We describe which classical amalgamated products of continuous  $C^*$ -bundles are continuous  $C^*$ -bundles and we analyse the involved extension problems for continuous  $C^*$ -bundles.

## INTRODUCTION

Different (fibrewise) amalgamated products of continuous  $C^*$ -bundles have been studied over the last years ([1], [8], [6], [4]), one of the main questions being to know when these amalgamated products are still continuous  $C^*$ -bundles.

In order to gather these different approaches in a joint survey, we first recall a few definitions from the theory of deformations of  $C^*$ -algebras and we fix several notations which will be used in the sequel.

Then we present a few possible extension properties for continuous  $C^*$ -bundles. More precisely, given a compact Hausdorff space  $X$  which is perfect, *i.e.* without any isolated point, we first recall in §2 that there is no general  $C(X)$ -linear version of the Hahn-Banach extension theorem for continuous  $C(X)$ -algebra. But we describe in §3 a Tietze extension property for continuous  $C(X)$ -algebras which will enable us to characterize in the following sections:

- when the canonical fiberwise amalgamated tensor products of a given continuous  $C(X)$ -algebra  $A$  with any other continuous  $C(X)$ -algebra  $B$  is a continuous  $C(X)$ -algebra ([6, Theorem 1.1 and Theorem 1.2]),
- when the canonical fiberwise amalgamated free products of a given continuous  $C(X)$ -algebra  $A$  with any other continuous  $C(X)$ -algebra  $B$  is a continuous  $C(X)$ -algebra ([4, Theorem 3.7 and Corollary 4.8]).

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## 1. $C(X)$ -ALGEBRAS

We recall first a few definitions from deformation theory for  $C^*$ -algebras and we fix the notations which will be used in the sequel.

Let  $X$  be a compact Hausdorff space and  $C(X)$  the  $C^*$ -algebra of continuous functions on  $X$  with values in the complex field  $\mathbb{C}$ .

**Definition 1.1.** A  $C(X)$ -algebra is a  $C^*$ -algebra  $A$  endowed with a unital  $*$ -homomorphism from  $C(X)$  to the centre of the multiplier  $C^*$ -algebra  $\mathcal{M}(A)$  of  $A$ .

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Given a closed subset  $Y \subset X$ , we denote by  $C_0(X \setminus Y)$  the closed ideal of continuous functions on  $X$  that vanish of  $Y$ . If  $A$  is a  $C(X)$ -algebra, then the subset  $C_0(X \setminus Y).A$  is a *closed* ideal in  $A$  (by Cohen factorisation Theorem) and we denote by  $\pi_Y^X$  the quotient map  $A \rightarrow A/C_0(X \setminus Y).A$ .

If the closed subset  $Y$  is reduced to a point  $x$  and the element  $a$  belongs to the  $C(X)$ -algebra  $A$ , we usually write  $\pi_x$ ,  $A_x$  and  $a_x$  for  $\pi_{\{x\}}^X$ ,  $\pi_{\{x\}}^X(A)$  and  $\pi_{\{x\}}^X(a)$ .

Note that the function

$$(1.1) \quad x \mapsto \|a_x\| = \inf\{\| [1 - f + f(x)]a \|; f \in C(X)\}$$

is always upper semi-continuous by construction. And the  $C(X)$ -algebra  $A$  is said to be **continuous** (or to be a *continuous  $C^*$ -bundle over  $X$* ) if the function  $x \mapsto \|a_x\|$  is actually continuous for all  $a$  in  $A$ .

**Definition 1.2.** A *continuous field of states* on a unital  $C(X)$ -algebra  $A$  is a unital positive  $C(X)$ -linear map  $\varphi : A \rightarrow C(X)$ .

**Remark 1.3.** A (unital) separable  $C(X)$ -algebra  $A$  is continuous if and only if (iff) there exists a continuous field of states  $\varphi : A \rightarrow C(X)$  such that for all  $x \in X$ , the induced state  $\varphi_x : a_x \in A_x \mapsto \varphi(a)(x)$  is faithful on  $A_x$  ([2]).

## 2. HAHN-BANACH EXTENSION PROPERTIES

Given a compact Hausdorff space  $X$ , a continuous unital  $C(X)$ -algebra  $A$ , a unital  $C(X)$ -subalgebra  $B \subset A$  and a continuous field of states  $\phi : B \rightarrow C(X)$ , there does not exist in general a  $C(X)$ -linear positive unital map  $\varphi : A \rightarrow C(X)$  extending  $\phi$ , *i.e.* a continuous field of states  $\varphi$  on  $A$  making the following diagram commutative:

$$\begin{array}{ccc} B & \xrightarrow{\phi} & C(X) \\ & & \parallel \\ \cap & & \\ & & \\ A & \xrightarrow{\varphi} & C(X) \end{array}$$

The problem happens as soon as the interior of  $X$  is non empty. Indeed, consider:

- the compact space  $X := \{0\} \cup \{\frac{1}{n}; n \in \mathbb{N}^*\}$ ,
- the unital continuous  $C(X)$ -algebra  $A := C(X) \oplus C(X)$  and
- the  $C(X)$ -subalgebra  $B := C(X).1_A + \left( C_0(X \setminus \{0\}) \oplus C_0(X \setminus \{0\}) \right) \subset A$

And let  $\phi : B \rightarrow C(X)$  be the continuous field of states on  $B$  fixed by the formulae

$$\phi((b_1, b_2))\left(\frac{1}{n}\right) = \begin{cases} b_1\left(\frac{1}{n}\right) & \text{if } n \text{ is odd} \\ b_2\left(\frac{1}{n}\right) & \text{otherwise} \end{cases} \quad \text{for } (b_1, b_2) \in C_0(X \setminus \{0\}) \oplus C_0(Y \setminus \{0\})$$

Then, there cannot be any continuous field of states  $\varphi : A \rightarrow C(X)$  such that  $\varphi(b) = \phi(b)$  for all  $b \in B$ . Indeed, if  $a = 1 \oplus 0 \in A$ , one has that:

- (1)  $\varphi(a)(\frac{1}{n}) = 1$  if  $n$  is odd and
- (2)  $\varphi(a)(\frac{1}{n}) = 0$  otherwise.

Hence, the function  $x \mapsto \varphi(a)(x)$  cannot be continuous at  $x = 0$ .

On the other hand, if  $Z$  is a second countable compact Hausdorff space  $Z$  and  $X \subset Z$  is a non empty closed subspace, then any continuous field of states  $\phi : \pi_X^Z(A) \rightarrow C(X)$  on the restriction  $\pi_X^Z(A)$  can be extended to a continuous field of states  $\varphi : A \rightarrow C(Z)$  by Michael continuous selection theorem (see e.g. [2, Proposition 3.13]), *i.e.* such that the following diagramme commutes:

$$\begin{array}{ccc}
 \pi_X^Z(A) & \xrightarrow{\phi} & C(X) \\
 \uparrow & & \uparrow \\
 A & \xrightarrow{\varphi} & C(Z)
 \end{array}$$

### 3. TIETZE EXTENSION PROPERTIES

Given a second countable compact Hausdorff space  $X$  and a closed non empty subspace  $Y \subset X$ , we describe in this section when a continuous  $C(Y)$ -algebra  $A$  can be extended to  $X$ , *i.e.* when there exists a continuous  $C(X)$ -algebra  $D$  with a  $C(Y)$ -algebra isomorphic  $\pi_Y^X(D) \cong A$ .

If the  $C^*$ -algebra  $A$  is an *exact* separable  $C^*$ -algebra, then there exists a unital embedding of the  $C(Y)$ -algebra  $A$  into the trivial  $C(Y)$ -algebra  $C(Y; \mathcal{O}_2) \cong C(Y) \otimes \mathcal{O}_2$ , where  $\mathcal{O}_2$  is the unital Cuntz  $C^*$ -algebra generated by two isometries  $s_1, s_2$  satisfying the relation  $1_{\mathcal{O}_2} = s_1(s_1)^* + s_2(s_2)^*$  ([3]). Hence, the continuous  $C(X)$ -algebra  $D := \{f \in C(X, \mathcal{O}_2); \pi_Y^X(f) \in A\}$  answers the question in that case.

But there are continuous  $C(Y)$ -algebras which are not exact  $C^*$ -algebras. Thus, in order to study extensions in the general case, let us define in  $X \times Y \times [0, 1]$ :

- the open subspace  $U = \{(x, y, t) \in X \times Y \times [0, 1]; 0 < t\}$  and
  - the closed subspace  $Z = \{(x, y, t) \in X \times Y \times [0, 1]; 0 \leq t.d(x, Y) \leq 2d(x, Y) - d(x, y)\}$ .
- And let  $\bar{d}$  be the metric on  $Z$  given by  $\bar{d}((x, y, t), (x', y', t')) = d(x, x') + d(y, y') + |t - t'|$ .

**Proposition 3.1.** ([6]) *The coordinate map  $p_1 : (x, y, t) \mapsto x$  gives a structure of  $C(X)$ -algebra on  $C(Z)$  and the ideal  $C_0(U \cap Z)$  is a continuous  $C(X)$ -algebra such that  $C_0(U \cap Z)|_Y \cong C_0(Y \times (0, 1])$ , *i.e.* the map  $(x, y, t) \in U \cap Z \mapsto x \in X$  is open.*

*Proof.* Given a function  $f$  in  $C_0(U \cap Z)$ , let us prove the continuity of the function

$$x \in X \mapsto \|\pi_x^X(f)\| = \sup\{|f(z)|; z \in p_1^{-1}(\{x\})\}$$

This map is already upper semi-continuous (u. s. c.) by construction. Hence, it only remains to show that for any point  $x_0 \in X$  and any constant  $\varepsilon > 0$ , one has  $\|\pi_x^X(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon$  for all points  $x$  in a neighbourhood of  $x_0$  in  $X$ .

The uniform continuity of the function  $f$  implies that there exists  $\delta > 0$  such that  $|f(z) - f(z')| < \varepsilon$  for all  $z, z'$  in  $Z$  with  $\bar{d}(z, z') < \delta$ . Now three cases can appear:

1) If  $x_0 \in Y$  and  $x \in Y$  satisfies  $d(x_0, x) < \delta/2$ , then  $|f(x, x, t) - f(x_0, x_0, t)| < \varepsilon$  for all  $t \in [0, 1]$ . And so  $\|\pi_x^X(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon$ .

2) If  $x_0 \in Y$  and  $x \in X \setminus Y$  satisfies  $d(x_0, x) < \delta/4$ , then for all  $y \in Y$ , the relation  $d(x, y) \leq 2d(x, Y)$  implies that  $d(y, x_0) \leq d(y, x) + d(x, x_0) \leq 2d(x, Y) + d(x, x_0) \leq \frac{3}{4}\delta$  and so  $|f(x, y, t) - f(x_0, x_0, t)| < \varepsilon$  for all  $t \in [0, 2 - \frac{d(x, y)}{d(x, Y)}]$ . Whence the inequality  $\|\pi_x^X(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon$ .

3) If  $x_0 \notin Y$  and the triple  $(x_0, y_0, t_0) \in U \cap Z$  satisfies  $|f(x_0, y_0, t_0)| = \|\pi_{x_0}^X(f)\| \neq 0$ , then  $d(x_0, y_0) < 2d(x_0, Y)$ . Thus, there exists by continuity a constant  $\alpha(x_0) \in ]0, \delta/2[$  such that all  $x \in X$  in the ball of radius  $\alpha(x_0)$  around  $x_0$  satisfy:

$$\text{a) } d(x, Y) > 0, \quad \text{b) } d(x, y_0) < 2d(x, Y), \quad \text{c) } t_0 < 2 - \frac{d(x, y_0)}{d(x, Y)} + \delta/2.$$

And so  $\|\pi_x^X(f)\| \geq \left| f(x, y_0, \inf\{t_0, 2 - \frac{d(x, y_0)}{d(x, Y)}\}) \right| > \|\pi_{x_0}^X(f)\| - \varepsilon. \quad \square$

**Remark 3.2.** S. Wassermann pointed out that if  $Y = \{0, 1\} \subset X = [0, 1]$ , then  $Z = \{(x, 0, t) \in [0, 1] \times \{0\} \times [0, 1]; t \leq \frac{2-3x}{1-x}\} \cup \{(x, 1, t) \in [0, 1] \times \{1\} \times [0, 1]; t \leq \frac{3x-1}{x}\}$ . Hence, the  $C(X)$ -algebra  $C(Z)$  is not continuous at  $x = \frac{1}{3}$  and  $x = \frac{2}{3}$ .

But the essential ideal  $C_0(Z \cap (0, 1] \times \{0, 1\} \times (0, 1])$  is a continuous  $C(X)$ -algebra.

The following Corollary will be essential in the proof of Proposition 4.2.

**Corollary 3.3.** *Let  $X$  be a second countable compact space,  $Y \subset X$  a non zero closed subset and  $A$  a continuous  $C(Y)$ -algebra.*

- a)  $B := C(X) \otimes A \otimes C([0, 1])$  is a continuous  $C(X \times Y \times [0, 1])$ -algebra.
- b)  $D := [C_0(U).B]_{|Z} = C_0(U).B / C_0(U \setminus U \cap Z).B$  is a continuous  $C(X)$ -algebra.
- c) There is an isomorphism of  $C(Y)$ -algebras  $D|_Y \cong A \otimes C_0((0, 1])$ .

*Proof.* a) holds because the  $C^*$ -algebras  $C(X)$  and  $C_0((0, 1])$  are nuclear.

b) Let  $b \in D$ . Then for all  $x \in X$ , we have

$$\|\pi_x^X(b)\| = \|b + C_0(X \setminus \{x\})D\| = \sup\{\|\pi_z^Z(b)\|; z \in p_1^{-1}(\{x\})\},$$

whence the continuity of the map  $x \mapsto \|\pi_x^X(b)\|$  by a) and Proposition 3.1.

c) One has  $D_Y \cong [C_0(U).B]_Y \cong A \otimes C_0((0, 1])$  by Proposition 3.1.  $\square$

#### 4. AMALGAMATED TENSOR PRODUCTS OF CONTINUOUS $C(X)$ -ALGEBRAS

Given a fixed compact Hausdorff space  $X$ , we study in this section the continuity properties of the different tensor products amalgamated over  $C(X)$  of two given continuous  $C(X)$ -algebras  $A$  and  $B$ .

More precisely, let  $A \odot B$  denote the algebraic tensor product (over  $\mathbb{C}$ ) of  $A$  and  $B$ , let  $\mathcal{I}_X(A, B)$  be the ideal in  $A \odot B$  generated by the differences  $af \otimes b - a \otimes fb$  ( $a \in A$ ,  $b \in B$ ,  $f \in C(X)$ ) and let  $A \underset{C(X)}{\odot} B$  denote the quotient of  $A \odot B$  by  $\mathcal{I}_X(A, B)$ .

If  $C_\Delta(X \times X) \subset C(X \times X)$  is the ideal of continuous function of  $X \times X$  which are

zero on the diagonal and  $A \overset{m}{\otimes} B$  (resp.  $A \overset{M}{\otimes} B$ ) is the minimal (resp. maximal) tensor product over  $\mathbb{C}$  of the two *continuous*  $C(X)$ -algebras  $A$  and  $B$ , then the quotient  $A \overset{m}{\otimes}_{C(X)} B := A \overset{m}{\otimes} B / C_{\Delta}(X \times X) A \overset{m}{\otimes} B$  (resp.  $A \overset{M}{\otimes}_{C(X)} B := A \overset{M}{\otimes} B / C_{\Delta}(X \times X) A \overset{M}{\otimes} B$ ) is the minimal (resp. maximal) completion of the algebraic amalgamated tensor product  $A \odot_{C(X)} B$ . Further, the  $*$ -algebra  $A \odot_{C(X)} B$  embeds in the  $C(X)$ -algebra  $A \overset{m}{\otimes}_{C(X)} B$  ([1]) and we have

$$(4.1) \quad \forall x \in X, \quad (A \overset{m}{\otimes}_{C(X)} B)_x \cong A_x \overset{m}{\otimes} B_x \quad \text{and} \quad (A \overset{M}{\otimes}_{C(X)} B)_x \cong A_x \overset{M}{\otimes} B_x.$$

Let us also recall a characterisation of exactness given by Kirchberg and Wassermann.

**Proposition 4.1.** ([8, Theorem 4.5]) *Let  $Y = \mathbb{N} \cup \{\infty\}$  be the one point compactification of  $\mathbb{N}$  and let  $D$  be a  $C^*$ -algebra. Then the following assertions are equivalent.*

- i) *The  $C^*$ -algebra  $A$  is exact.*
- ii) *For all continuous  $C(Y)$ -algebra  $B$ , the minimal tensor product  $A \overset{m}{\otimes} B$  is a continuous  $C(Y)$ -algebra with fibres  $A \overset{m}{\otimes} B_y$  ( $y \in Y$ ).*

It induces the following results for fibrewise tensor products of continuous  $C(X)$ -algebras.

**Proposition 4.2.** ([6], [4]) *Let  $X$  be a second countable compact Hausdorff space and  $A$  a separable unital continuous  $C(X)$ -algebra.*

*If the topological space  $X$  is perfect (i.e. without isolated point), then the following assertions  $\alpha_e$ ) and  $\beta_e$ ) (resp.  $\alpha_n$ ) and  $\beta_n$ ) are equivalent.*

$\alpha_e$ ) *The  $C^*$ -algebra  $A$  is exact.*

$\beta_e$ ) *For all continuous  $C(X)$ -algebra  $B$ , the amalgamated tensor product  $A \overset{m}{\otimes}_{C(X)} B$  is a continuous  $C(X)$ -algebra with fibres  $A_x \overset{m}{\otimes} B_x$  ( $x \in X$ ).*

$\alpha_n$ ) *The  $C^*$ -algebra  $A$  is nuclear.*

$\beta_n$ ) *For all continuous  $C(X)$ -algebra  $B$ , the amalgamated tensor product  $A \overset{M}{\otimes}_{C(X)} B$  is a continuous  $C(X)$ -algebra with fibres  $A_x \overset{M}{\otimes} B_x$  ( $x \in X$ ).*

*Proof.*  $\alpha_e$ )  $\Rightarrow$   $\beta_e$ ) If the  $C^*$ -algebra  $A$  is exact, then the spatial tensor product  $A \overset{m}{\otimes} D$  is a continuous  $C(X \times X)$ -algebra with fibres  $A_x \overset{m}{\otimes} D_{x'}$  ( $x, x' \in X$ ) ([8]). Hence, its restriction to the diagonal is as desired.

$\beta_e$ )  $\Rightarrow$   $\alpha_e$ ) Suppose conversely that the  $C(X)$ -algebra  $A$  satisfies  $\beta_e$ ). And let us prove step by step that the  $C^*$ -algebra  $A$  is exact.

*Step a)* All the fibres  $A_x$  are exact ( $x \in X$ ). Indeed, given a point  $x$  in  $X$ , take a sequence of points  $x_n$  in  $X$  converging to  $x$  such that there is a topological isomorphism

$Y := \{x_n; n \in \mathbb{N}\} \cup \{x\} \cong \mathbb{N} \cup \{\infty\}$ . Then, for any separable continuous  $C(Y)$ -algebra  $B$ , there is a continuous  $C(X)$ -algebra  $\mathcal{B}$  such that  $\mathcal{B}|_Y = B \otimes C_0((0, 1])$  (Corollary 3.3).

Now, the continuity of the  $C(X)$ -algebra  $\mathcal{B} \otimes_{C(X)}^m A$  given by  $\beta_e$ ) implies that of its

restriction  $\left(\mathcal{B} \otimes_{C(X)}^m A\right)|_Y \cong (C_0((0, 1]) \otimes B) \otimes_{C(Y)}^m A|_Y$ , whence that of the  $C(Y)$ -algebra

$B \otimes_{C(Y)}^m A|_Y$  since there is an isometric  $C(Y)$ -linear embedding  $B \hookrightarrow \mathcal{B}|_Y$ . And this

implies the exactness of the  $C^*$ -algebra  $A_x$  by Proposition 4.1.

*Step b)* If  $B$  is a  $C^*$ -algebra and  $\mathcal{B}$  is the constant  $C(X)$ -algebra  $C(X; B)$ , then for all  $x \in X$ , we have the exact sequence

$$0 \rightarrow C_x(X)A \otimes_{C(X)}^m B \rightarrow (A \otimes_{C(X)}^m \mathcal{B})_x = A \otimes^m B \rightarrow A_x \otimes^m B \rightarrow 0.$$

*Step c)* If  $B$  is a  $C(X)$ -algebra, then for all  $x \in X$ , we have the sequence of epimor-

$$\left(A \otimes_{C(X)}^m B\right)_x \twoheadrightarrow \left(A_x \otimes^m B\right)_x \twoheadrightarrow A_x \otimes^m B_x$$

*Step d)* Now, let  $B$  be a  $C^*$ -algebra,  $K \triangleleft B$  a closed two sided ideal in  $B$  and take an element  $d \in \ker\{A \otimes^m B \rightarrow A \otimes^m B/K\}$ . Then for all  $x \in X$ , we have

$$\begin{aligned} d_x &\in \ker\{(A \otimes^m B)_x \rightarrow (A \otimes^m B/K)_x\} \\ &= \ker\{A_x \otimes^m B \rightarrow A_x \otimes^m B/K\} && \text{by } b) \\ &= A_x \otimes^m K && \text{by } a) \\ &= (A \otimes^m K)_x && \text{by } c) \end{aligned}$$

Thus,  $d \in A \otimes^m K$ . And so, the  $C^*$ -algebra  $A$  is exact.

$\alpha_n) \Rightarrow \beta_n)$  has a similar proof to that of  $\alpha_e) \Rightarrow \beta_e)$ .

$\beta_n) \Rightarrow \alpha_n)$  If a  $C^*$ -algebra  $A$  satisfies  $\beta_n)$ , then all the fibres  $A_x$  ( $x \in X$ ) are nuclear by [8, Theorem 3.2] and so the  $C^*$ -algebra  $A$  itself is nuclear (see e.g. [2, Proposition 3.23]).  $\square$

**Remark 4.3.** These characterisations do not hold anymore if the compact space  $X$  is not perfect. Indeed, if the space  $X$  is reduced to a point, then both the amalgamated tensor products  $A \otimes_{C(X)}^m B$  and  $A \otimes_{C(X)}^M B$  are constant, hence continuous.

Proposition 4.2 implies the following characterisation of exact continuous  $C(X)$ -algebras in the framework of  $C(X)$ -algebras.

**Corollary 4.4.** *Let  $X$  be a perfect compact metric space and  $A$  be a separable continuous  $C(X)$ -algebra. Then the following are equivalent*

- (1) *The  $C^*$ -algebra  $A$  is exact.*
- (2) *For all exact sequence of continuous  $C(X)$ -algebras  $0 \rightarrow J \rightarrow B \rightarrow D \rightarrow 0$ , the sequence  $0 \rightarrow A \otimes_{C(X)}^m J \rightarrow A \otimes_{C(X)}^m B \rightarrow A \otimes_{C(X)}^m D \rightarrow 0$  is exact.*

*Proof.* (2) $\Rightarrow$ (1) If the unital continuous  $C(X)$ -algebra  $A$  satisfies (2) and the sequence  $0 \rightarrow J_0 \rightarrow B_0 \rightarrow D_0 \rightarrow 0$  is an exact sequence of  $C^*$ -algebras, then the sequence  $0 \rightarrow J := C(X) \otimes J_0 \rightarrow B := C(X) \otimes B_0 \rightarrow D := C(X) \otimes D_0 \rightarrow 0$  is an exact sequence of  $C(X)$ -algebras. And the condition (2) implies the exactness of the sequence

$$0 \rightarrow A \underset{C(X)}{\overset{m}{\otimes}} J = A \overset{m}{\otimes} J_0 \rightarrow A \underset{C(X)}{\overset{m}{\otimes}} B = A \overset{m}{\otimes} B_0 \rightarrow A \underset{C(X)}{\overset{m}{\otimes}} D = A \overset{m}{\otimes} D_0 \rightarrow 0.$$

whence the exactness of  $A$ .

(1) $\Rightarrow$ (2) If the  $C(X)$ -algebra  $A$  is an exact  $C^*$ -algebra and  $0 \rightarrow J \rightarrow B \rightarrow D \rightarrow 0$  is an exact sequence of  $C(X)$ -algebras, then the two first lines of the following diagram are exact by assumption (1)

$$\begin{array}{ccccc} C_\Delta(X \times X)A \overset{m}{\otimes} J & \rightarrow & C_\Delta(X \times X)A \overset{m}{\otimes} B & \rightarrow & C_\Delta(X \times X)A \overset{m}{\otimes} D \\ \downarrow \scriptstyle m & & \downarrow \scriptstyle m & & \downarrow \scriptstyle m \\ A \otimes J & \rightarrow & A \otimes B & \rightarrow & A \otimes D \\ \downarrow & & \downarrow & & \downarrow \\ A \underset{C(X)}{\overset{m}{\otimes}} J & \dashrightarrow & A \underset{C(X)}{\overset{m}{\otimes}} B & \dashrightarrow & A \underset{C(X)}{\overset{m}{\otimes}} D \end{array}$$

Besides, all the columns of the diagram are exact by definition, whence the exactness of the last line by a diagram chasing.  $\square$

## 5. AMALGAMATED FREE PRODUCTS OF CONTINUOUS $C(X)$ -ALGEBRAS

In this section, we describe the continuity properties of different free products amalgamated over  $C(X)$  of two given unital continuous  $C(X)$ -algebras  $A$  and  $B$ .

**Proposition 5.1.** ([4]) *Let  $X$  be a second countable perfect compact Hausdorff space and  $A$  a separable unital continuous  $C(X)$ -algebra.*

*Then the following assertions are equivalent.*

$\alpha_e$ ) *The  $C^*$ -algebra  $A$  is exact.*

$\gamma_e$ ) *For all separable unital continuous  $C(X)$ -algebra  $B$  and all continuous fields of faithful states  $\phi : A \rightarrow C(X)$ ,  $\psi : B \rightarrow C(X)$ , the reduced amalgamated free product  $(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$  is a continuous  $C(X)$ -algebra with fibres  $(C_x, \phi_x * \psi_x) = (A_x, \phi_x) * (B_x, \psi_x)$ .*

*Proof.*  $\gamma_e \Rightarrow \alpha_e$ ) Let  $B$  be a unital separable continuous  $C(X)$ -algebra and let  $\psi$  be a continuous field of faithful states  $\psi$  on  $B$ . Set  $D = A \underset{C(X)}{\overset{m}{\otimes}} B$  and let  $E$  be the Hilbert  $D$ -bimodule  $E = L^2(D, \phi \otimes \psi) \underset{C(X)}{\otimes} D$ .

Then, the following assertions are equivalent ([4, Lemma 4.5]).

- $D$  is a continuous  $C(X)$ -algebra with fibres  $D_x \cong A_x \overset{m}{\otimes} B_x$  ( $x \in X$ ).
- The Pimsner  $C^*$ -algebra  $\mathcal{T}_D(E \oplus D)$  of the full Hilbert  $D$ -bimodule  $E \oplus D$  is a continuous  $C(X)$ -algebra with fibres  $\mathcal{T}_{D_x}(E_x \oplus D_x)$  ( $x \in X$ ).

But there is a  $C(X)$ -linear isomorphism  $\mathcal{T}_D(E \oplus D) \cong C \rtimes \mathbb{N}$ . And so, these two assertions are equivalent to the continuity of the reduced amalgamated free product  $(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$  since the group  $\mathbb{Z}$  is amenable ([2, Corollaire 5.10]).

$\alpha_e) \Rightarrow \gamma_e)$  Conversely, if  $A$  is an exact  $C^*$ -algebra and  $B$  is a unital separable continuous  $C(X)$ , the amalgamated tensor product  $D = A \underset{C(X)}{\overset{m}{\otimes}} B$  is a continuous  $C(X)$ -algebra

with fibres  $D_x \cong A_x \overset{m}{\otimes} B_x$  for  $x \in X$  ([1]). Hence, the reduced amalgamated free product  $(C, \phi * \psi) = (A, \phi) \underset{C(X)}{*} (B, \psi)$  is a continuous  $C(X)$ -algebra ([4, Theorem 4.1]).  $\square$

**Remark 5.2.** There is no similar result for full amalgamated free product. Indeed, the full amalgamated free product  $A \underset{C(X)}{\overset{f}{*}} B$  of two unital continuous  $C(X)$ -algebras  $A$  and  $B$  is always a continuous  $C(X)$ -algebra with fibres  $A_x \overset{f}{*} B_x$  ( $x \in X$ ) ([4, Theorem 3.7]).

**Sketch of proof.** The algebraic amalgamated free product  $A \underset{C(X)}{\overset{\circledast}{\otimes}} B$  is a dense  $C(X)$ -submodule of the amalgamated Haagerup tensor product  $A \underset{C(X)}{\overset{h}{\otimes}} B$ , which itself is contained in the full amalgamated free product  $A \underset{C(X)}{\overset{f}{*}} B$  ([9]). And for all  $d \in A \underset{C(X)}{\overset{\circledast}{\otimes}} B$ , one has

$$\begin{aligned} \|d_x\|_{A_x \overset{f}{*} B_x} &= \|d_x\|_{A_x \overset{h}{\otimes} B_x} = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \cdot \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} ; d_x = \sum_i a_i \otimes b_i \right\} \\ &= \sup \left\{ \left| \left\langle \xi, \sum_i \pi(a_i) \cdot \sigma(b_i) \eta \right\rangle \right| ; \xi, \eta \text{ unit vectors in the Hilbert space } \ell^2(\mathbb{N}) \right. \\ &\quad \left. ; \pi, \sigma \text{ unital } * \text{-rep. of } A_x, B_x \text{ on } \ell^2(\mathbb{N}) \right\} \end{aligned}$$

Hence, the map  $x \mapsto \|d_x\|$  is both upper and lower semi-continuous if  $d \in A \underset{C(X)}{\overset{\circledast}{\otimes}} B$ .

The proof for the continuity of the map  $x \mapsto \|d_x\|$  for elements  $d$  in the algebraic amalgamated free product  $A \underset{C(X)}{\overset{\circledast}{\otimes}} B$  is similar ([4]).

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