



**HAL**  
open science

## Remarks on the consistency of Upwind Source at Interface schemes on nonuniform grids

Chiara Simeoni

► **To cite this version:**

Chiara Simeoni. Remarks on the consistency of Upwind Source at Interface schemes on nonuniform grids. *Journal of Scientific Computing*, 2011, 48 (1-3), pp.333-338. 10.1007/s10915-010-9442-6 . hal-00922834

**HAL Id: hal-00922834**

**<https://hal.science/hal-00922834>**

Submitted on 30 Dec 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Remarks on the consistency of Upwind Source at Interface schemes on nonuniform grids

Chiara Simeoni <sup>\*†</sup>

**Abstract.** This note presents a preliminary study of the *supra-convergence* of well-balanced (finite volume) schemes for conservation laws with a (non-linear) geometrical source term. In particular, we consider scalar (linear) advection equations in one dimension, for which smooth analytical solutions are available, and upwind interfacial discretizations for the numerical simulation on nonuniform grids (also generated by adaptive procedures). We point out the inconsistent characteristics of the (local) truncation error, and then we illustrate through a simple example the consistency condition formulated by Wendroff and White, which turns out to be effectively compatible with a (strong) convergence theory at optimal rates.

**Keywords.** conservation laws - geometrical source term - upwind interfacial schemes - consistency - supra-convergence

**AMS 2000** 65M10 (65D25, 76M12, 65M50, 65L05)

---

<sup>\*</sup>Dipartimento di Matematica Pura ed Applicata, Università degli Studi dell'Aquila, via Vetoio (snc) - Località Coppito, I-67010 L'Aquila, Italy

<sup>†</sup>e-mail : [chiara.simeoni@dm.univaq.it](mailto:chiara.simeoni@dm.univaq.it)

Hyperbolic conservation laws with source terms arise in many applications, especially as effective mathematical models for geophysical flows (shallow water equations, nozzle flows, debris avalanches), and their numerical simulation leads to specific difficulties. In comparison with the homogeneous case, a significant snag is the accurate computation of nonconstant steady states, occurring for the balance between source term and internal forces (we refer to [15] for an overall discussion of *well-balanced schemes*). Moreover, the utilization of unstructured spatial grids is required for problems incorporating composite geometries, also in view of the recent theoretical advances on adaptive techniques for mesh refinement in the resolution of multi-scale complex flows (see [2], [17] and [16], for instance). Motivated by the debate on the stability mechanisms possibly generated by such strategies, we aim at studying the features of finite volume schemes setting on nonuniform grids, with an emphasis for their consistency properties with respect to the question of convergence. We confine our investigation to elementary transport problems with linear fluxes, in order to prevent classical complications (computation of discontinuous solutions, stability of the interpolation for general nonlinear fluxes, resonance's effects), and we shall perform a formal error analysis attempting to elucidate the influence of the non-uniformity of the mesh mainly on the convergence's rates.

For the sake of simplicity in the notation, although our analysis applies to linear systems as well, we restrict to scalar advection equations, and we focus on the so-called *geometrical source term*, in one space dimension,

$$\partial_t u + a \partial_x u + z'(x)b(u) = 0, \quad t \in \mathbb{R}^+, x \in \mathbb{R}, \quad (1.1)$$

$$u(0, x) = u_0(x) \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 1 \leq p < +\infty, \quad (1.2)$$

with  $a \in \mathbb{R}$ ,  $z' \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$  and  $b \in C^1(\mathbb{R})$ , for which the stationary solutions are described by the (integral) relation

$$D(u(x)) + z(x) = C^{st}, \quad D'(u) = \frac{a}{b(u)} \in L^\infty(\mathbb{R}). \quad (1.3)$$

The assumptions made above ensure the existence and uniqueness of (weak) entropic solutions to the problem (1.1)-(1.2), for any  $T > 0$ , in the functional space  $C(\mathbb{R}^+, L^p(\mathbb{R})) \cap L^\infty((0, T) \times \mathbb{R})$ , according to the *extended Kruřkov's theory* [6]. Besides, assuming  $D$  given in (1.3) to be strictly monotonic, the existence of a unique Lipschitz continuous steady state is earned.

We set up a nonuniform mesh on  $\mathbb{R}$ , and we denote by  $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$  the finite volume (cell) centered at the vertices  $x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$ ,  $i \in \mathbb{Z}$ , where  $x_{i-\frac{1}{2}}$  and  $x_{i+\frac{1}{2}}$  are the cell's interfaces (see Figure 1) and  $\Delta x_i = \text{length}(C_i)$ , so that the characteristic space-step is  $h = \sup_{i \in \mathbb{Z}} \Delta x_i$ . We also introduce a variable time-step  $\Delta t_n = t_{n+1} - t_n$ ,  $n \in \mathbb{N}$ , and we set  $\Delta t = \sup_{n \in \mathbb{N}} \Delta t_n$ ,

therefore we have to consider a *CFL-condition* on the ratio  $\Delta t_n/\Delta x_i$  for the numerical stability. We construct a piecewise constant approximation of the function  $z$  on the mesh, whose coefficients are  $z_i = \frac{1}{\Delta x_i} \int_{C_i} z(x) dx$ , and the discrete unknowns are defined to approximate the cell-averages of the analytical solution (cell-centered conservative quantities), namely

$$v_i^n \approx \frac{1}{\Delta x_i} \int_{C_i} u(t_n, x) dx, \quad n \in \mathbb{N}, i \in \mathbb{Z}. \quad (1.4)$$

A suitable treatment of the geometrical source term (originally formulated by Roe [14], then adapted in [3], [4] and various others) consists in upwind discretizations at the mesh's interfaces, as for the numerical fluxes, that is particularly relevant to properly reproduce the steady states. According to the typical formalism of an explicit (three points) method, the simplest one-sided upwind (and possibly well-balanced) scheme applied to (1.1) reads

$$\frac{v_i^{n+1} - v_i^n}{\Delta t_n} + a \frac{v_i^n - v_{i-1}^n}{\Delta x_i} + b(v_i^n) \frac{z_i - z_{i-1}}{\Delta x_i} = 0, \quad (1.5)$$

for  $a > 0$ , hence the additional constraint  $b(u) > 0$  is adopted for (1.3), and appropriate initial and boundary data must be taken into account.

The (local) truncation error, denoted by  $T_i^n$ , is defined by substituting the analytical solution, expressed in terms of its cell-averages (1.4), into the numerical formulation (1.5), so that a straightforward calculation yields

$$T_i^n = \partial_t u(t_n, x_i) + \frac{\Delta x_{i-1} + \Delta x_i}{2 \Delta x_i} \left[ a \partial_x u(t_n, x_i) + z'(x_i) b(u(t_n, x_i)) \right] + \mathcal{O}(\Delta t, h), \quad (1.6)$$

thus revealing the well-known lack of consistency with the underlying conservation law, as the space-step  $\Delta x_i$  could be very different from the length of an interfacial interval  $|x_i - x_{i-1}| = \frac{\Delta x_{i-1} + \Delta x_i}{2}$  (refer to [10], [19] and [13], for example). In conclusion, unless the spatial grid is quasi-uniform, and  $\Delta x_{i-1} = \Delta x_i + \mathcal{O}(h^2)$ , it seems that convergence as usually stated by the *Lax theorem* cannot be expected, or at least a reduction in the rate occurs.

Nevertheless, the main issue of an error analysis with optimal rates can be pursued, by virtue of the results concerning the *supra-convergence phenomenon* for numerical approximation of hyperbolic conservation laws. In fact, despite a deterioration of the pointwise consistency is observed in consequence of the non-uniformity of the mesh, the formal accuracy is actually maintained as the global error behaves better than the (local) truncation error would indicate. This property of enhancement of the numerical error has been widely explored, starting from [18], [11] and [21]. A comprehensive interpretation of a priori error estimates, based on the original *Kuznecov's*

theory, is given in [7] for flux-splitting monotone schemes applied to (scalar) nonlinear problems with low regularity. Otherwise, the question of (finite volume) upwind schemes for linear equations on two-dimensional triangulations is addressed in [8] and [9], with proof of convergence at optimal rates for smooth solutions. This bibliography is by far not exhaustive, and must be complemented with the references therein the papers quoted above.

Therefore, to elucidate the mechanism of supra-convergence, we imitate the procedure established in [21], then generalized to the multi-dimensional case in [9] and also in [5], and we postulate that (1.4) is replaced by

$$v_i^n \approx \frac{1}{\Delta x_i} \int_{C_i} \left[ u(t_n, x) + \frac{\Delta x_i}{2} \partial_x u(t_n, x) \right] dx, \quad n \in \mathbb{N}, i \in \mathbb{Z}. \quad (1.7)$$

The introduction of a (first order) correction in (1.7) allows to compensate the wrong terms of the truncation error (1.6), as it represents precisely the discrepancy between cell-centered and interfacial averages of the solution, which do not coincide for upwind schemes on nonuniform meshes.

In effects, rewriting the scheme (1.5) for the reconstruction (1.7), analogous calculations to deduce (1.6) lead to the following *modified equation*,

$$\begin{aligned} R_i^n = & \partial_t u(t_n, x_i) + a \partial_x u(t_n, x_i) + z'(x_i) b(u(t_n, x_i)) + \\ & + \frac{\Delta x_{i-1}}{2} \left[ a \partial_{xx} u(t_n, x_i) + z''(x_i) b(u(t_n, x_i)) \right] \frac{\Delta x_{i-1} + \Delta x_i}{2 \Delta x_i} + \\ & + \frac{\Delta x_i}{2} \left[ \partial_{tx} u(t_n, x_i) + z'(x_i) b'(u(t_n, x_i)) \partial_x u(t_n, x_i) \right] + \mathcal{O}(\Delta t, h), \end{aligned} \quad (1.8)$$

that involves the pointwise form of the exact model (1.1), with extra terms depending on an (admissible) improved regularity of the solution, while the higher order remainders are bounded through the usual mesh's condition

$$\exists \alpha, \beta > 0 \quad / \quad \alpha \Delta x_{i+1} \leq \Delta x_i \leq \beta \Delta x_{i+1}, \quad \forall i \in \mathbb{Z}. \quad (1.9)$$

Moreover, basic first order expansions in the right-hand side of (1.7) justify to use  $E_i^n = v_i^n - u(t_n, x_i) - \frac{\Delta x_i}{2} \partial_x u(t_n, x_i)$  as the reference quantity for the error analysis, together with  $S_i^n = z_i - z(x_i) - \frac{\Delta x_i}{2} z'(x_i)$ . Finally, fixing also  $b(u) = b > 0$  to avoid inessential technicalities, we derive from (1.5) and (1.7) the typical *stability equation* for the (fully discrete) numerical scheme,

$$E_i^{n+1} = E_i^n - a \frac{\Delta t_n}{\Delta x_i} (E_i^n - E_{i-1}^n) - b \frac{\Delta t_n}{\Delta x_i} (S_i^n - S_{i-1}^n) - \Delta t_n R_i^n,$$

and, in principle, we can infer (strong) convergence of  $L^p$ -type with optimal rate from (1.8), for nonuniform grids verifying the hypothesis (1.9).

We mention that a proper investigation of (1.8) constitutes an easy way to

identify as well the diffusive and dispersive characteristics of the scheme (1.5).

**Remark 1.** The truncation error (1.6) vanishes for the simulation of steady states (1.3), thus indicating that the *well-balance property* enforces an overall stability of the finite volume method, because fully conservative schemes demonstrate to perform substantially better on unstructured meshes when compared to those not preserving some special structures even for uniform meshes (see [20] for an interesting discussion).

We conclude this note by providing numerical tests<sup>1</sup> for (1.1)-(1.2) with periodic boundary conditions, in the simple case of a source term given by  $z(x) = \sin(\pi x)$ , for  $x \in [0, 1]$ , and  $b(u) = u$ , such that an analytical (smooth) solution is available to make direct comparisons. The nonuniform meshes used for the computation are built through the adaptive procedure in [1], that indeed generates cell's sizes conforming with (1.9), whereas no specific restriction on the *CFL-condition* is required to insure numerical stability.

We report in Table 1 the experimental errors obtained from the simulation at  $T = 0.3$ , with advection  $a = 0.5$ , for the (first order) scheme (1.5) when  $CFL = 0.9$  is arbitrarily chosen to determine the variable time-step.

The Table 2 presents similar computations for a stationary solution (1.3) at  $T = 1.5$ , with advection  $a = 1$ . Because of the smoothness of the source term, and the corresponding numerical solution, an adaptive technique customarily yields quasi-uniform meshes (see Figure 2), hence upwind schemes like (1.5) actually turn to higher order approximations. On the other hand, a *well-balance property* is known to significantly improve the numerical accuracy, and this also constitutes an interpretation for the results in Table 2.

**Remark 2.** The formal analysis presented above directly extends to the general framework in [12], and it should concern many effective schemes developed in the literature (extensively reviewed therein), thus encouraging to perform a rigorous theoretical study in a forthcoming paper.

---

<sup>1</sup>the code for reproducing the numerical tests is available upon request to the author

## References

- [1] Arvanitis, C. (2008). Mesh redistribution strategies and finite element schemes for hyperbolic conservation laws. *J. Sci. Comput.* 34 (1), 1-25.
- [2] Arvanitis, C., and Delis, A.I. (2006). Behavior of finite volume schemes for hyperbolic conservation laws on adaptive redistributed spatial grids. *J. Sci. Comput.* 28 (5), 1927-1956.
- [3] Ben-Artzi, M., and Falcovitz, J. (1986). An upwind second-order scheme for compressible duct flows. *SIAM J. Sci. Statist. Comput.* 7 (3), 744-768.
- [4] Bermudez, A., and Vazquez, M.E. (1994). Upwind methods for hyperbolic conservation laws with source terms. *Computers & Fluids* 23 (8), 1049-1071.
- [5] Bouche, D., Ghidaglia, J.-M., and Pascal, F. (2005). Error estimate and the geometric corrector for the upwind finite volume method applied to the linear advection equation. *SIAM J. Numer. Anal.* 43 (2), 578-603.
- [6] Bouchut, F., and Perthame, B. (1998). Kruřkov's estimates for scalar conservation laws revisited. *Trans. Amer. Math. Soc.* 350 (7), 2847-2870.
- [7] Cockburn, B., and Gremaud, P.-A. (1997). A priori error estimates for numerical methods for scalar conservation laws. II. Flux-splitting monotone schemes on irregular Cartesian grids. *Math. Comp.* 66 (218), 547-572.
- [8] Després, B. (2004). Lax theorem and finite volume schemes. *Math. Comp.* 73 (247), 1203-1234.
- [9] Després, B. (2004). An explicit a priori estimate for a finite volume approximation of linear advection on non-Cartesian grids. *SIAM J. Numer. Anal.* 42 (2), 484-504.
- [10] Hoffman, J.D. (1982). Relationship between the truncation errors of centered finite-difference approximations on uniform and nonuniform meshes. *J. Comput. Phys.* 46 (3), 469-474.
- [11] Kreiss, H.-O., Manteuffel, T.A., Swartz, B., Wendroff, B., and White, A.B. Jr. (1986). Supra-convergent schemes on irregular grids. *Math. Comp.* 47 (176), 537-554.
- [12] Perthame, B., and Simeoni, C. (2003). Convergence of the Upwind Interface Source method for hyperbolic conservation laws. In *Hyperbolic problems : theory, numerics, applications*, Springer, Berlin, pp. 61-78.
- [13] Pike, J. (1987). Grid adaptive algorithms for the solution of the Euler equations on irregular grids. *J. Comput. Phys.* 71 (1), 194-223.
- [14] Roe, P.L. (1987). Upwind differencing schemes for hyperbolic conservation laws with source terms. In Carasso, C., Raviart, P.A., and Serre D. (ed.), *Nonlinear Hyperbolic Problems*, Lecture Notes in Math. 1270, Springer-Verlag, Berlin, pp. 41-51.
- [15] Rogers, B.D., Borthwick, A.G.L., and Taylor, P.H. (2003). Mathematical balancing of flux gradient and source terms prior to using Roe's approximate Riemann solver. *J. Comput. Phys.* 192 (2), 422-451.

- [16] Sfakianakis, N. (2009). Finite Difference schemes on non-uniform meshes for Hyperbolic Conservation Laws, Ph.D. thesis, University of Crete, Heraklion.
- [17] Teng, Z.-H. (2008). Modified equation with adaptive monotone difference schemes and its convergent analysis. *Math. Comp.* 77 (263), 1453-1465.
- [18] Tikhonov, A.N., and Samarsky, A.A. (1963). On the theory of homogeneous difference schemes. *Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963)*, Acad. Sci. USSR Siberian Branch, Moscow, pp. 266-273.
- [19] Turkel, E. (1986). Accuracy of schemes with nonuniform meshes for compressible fluid flows. *Appl. Numer. Math.* 2 (6), 529-550.
- [20] Vasilyev, O.V. (2000). High order finite difference schemes on non-uniform meshes with good conservation properties. *J. Comput. Phys.* 157 (2), 746-761.
- [21] Wendroff, B., and White, A.B. Jr. (1989). A supraconvergent scheme for nonlinear hyperbolic systems. *Comput. Math. Appl.* 18 (8), 761-767.



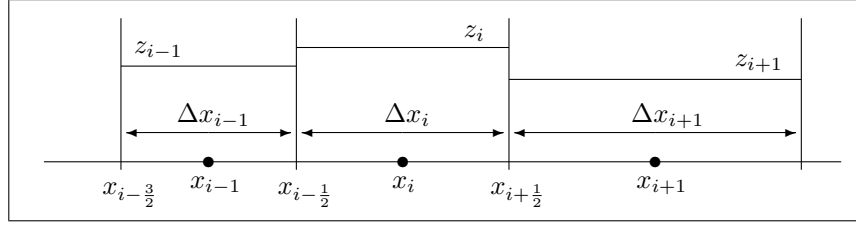


Figure 1: (first order) reconstruction on nonuniform (spatial) grid

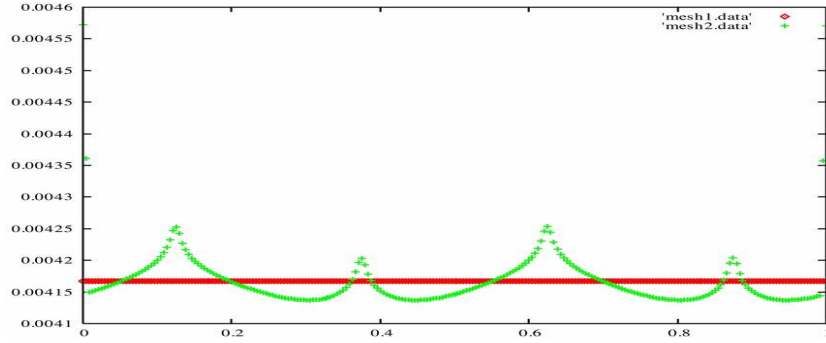


Figure 2: cells' sizes after mesh refinement (green) over uniform mesh

	$\ e(t)\ _{L^1}$		$\ e(t)\ _{L^2}$		$\ e(t)\ _{L^\infty}$	
	Error	Rate	Error	Rate	Error	Rate
30	0.322152E-01		0.354948E-01		0.587403E-01	
60	0.149459E-01	1.116	0.164315E-01	1.119	0.260567E-01	1.181
120	0.715125E-02	1.092	0.787522E-02	1.093	0.119510E-01	1.156
240	0.349139E-02	1.074	0.384303E-02	1.075	0.565209E-02	1.132
480	0.171988E-02	1.062	0.189368E-02	1.062	0.273081E-02	1.112

Table 1:  $\min_{i \in \mathbb{Z}} \Delta x_i = 0.230874E-02$ ,  $\min_{n \in \mathbb{N}} \Delta t_n = 0.373335E-02$ 

	$\ e(t)\ _{L^1}$		$\ e(t)\ _{L^2}$		$\ e(t)\ _{L^\infty}$	
	Error	Rate	Error	Rate	Error	Rate
30	0.195000E-02		0.232276E-02		0.419508E-02	
60	0.569572E-01	1.820	0.679157E-01	1.819	0.123060E-02	1.814
120	0.218606E-01	1.619	0.260709E-01	1.618	0.470566E-01	1.619
240	0.960606E-00	1.484	0.114540E-01	1.484	0.206341E-01	1.485
480	0.450752E-00	1.391	0.537440E-00	1.391	0.967720E-00	1.392

Table 2: experimental well-balance error =  $0.224508E-02$