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# First and second order error estimates for the Upwind Interface Source method

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## Abstract

The Upwind Interface Source method for hyperbolic conservation laws presented in [30] is essentially first order accurate. Under appropriate hypotheses of consistency on the source discretization, we prove  $L^p$ -error estimates, for  $1 \leq p < +\infty$ , in the case of a uniform spatial mesh, for which an optimal result can be obtained. We thus conclude that the same convergence rate holds as in the corresponding homogeneous problem (refer to [8]). To improve the numerical accuracy, we develop two different approaches of treating the source term and we discuss the question to derive second order error estimates. Numerical evidence shows that those techniques produce high resolution schemes compatible with the Upwind Interface Source method.

**Key-words:** scalar conservation laws, source terms, upwind interfacial methods, consistency, error estimates.

# 1 Introduction

We consider the initial values problem for a transport equation with non-linear source term, in one space dimension,

$$\partial_t u + \partial_x u = B(x, u), \quad t \in \mathbb{R}_+, \quad x \in \mathbb{R}, \quad (1.1)$$

$$u(0, x) = u_0(x) \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 1 \leq p < +\infty, \quad (1.2)$$

with  $u(t, x) \in \mathbb{R}$  and the analytical source operator is given by

$$B(x, u) = z'(x) b(u), \quad z' \in L^p(\mathbb{R}), \quad b \in C^1(\mathbb{R}). \quad (1.3)$$

The system (1.1)-(1.3) corresponds to the simplest model of scalar conservation law with a geometrical source term, extensively treated in [30].

The entropy inequalities associated to (1.1) are described by the equation

$$\partial_t S(u) + \partial_x S(u) + S'(u)B(x, u) \leq 0, \quad (1.4)$$

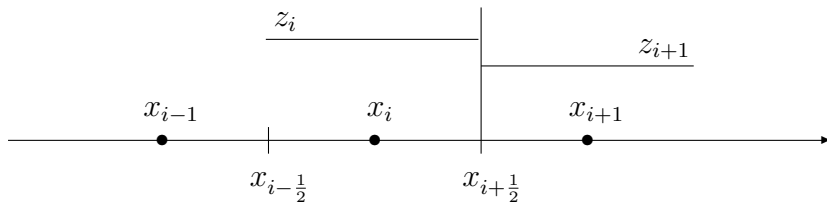
for any convex entropy function  $S$  (see [17] and [18]). Under stronger assumptions on the source term, Kruřkov [17] proved existence and uniqueness of the entropy solution to the problem (1.1)-(1.2), in the functional space  $L^\infty([0, T]; L^p(\mathbb{R}))$ , for all  $T \in \mathbb{R}_+$ . Another approach, based on convergence analysis for special approximations, is presented in [1]. In the case of singular source terms (namely,  $z$  discontinuous), a uniqueness result has recently been proved by Vasseur [35].

## 1.1 Formalism of the Upwind Interface Source method

We set up a uniform mesh on  $\mathbb{R}$ , whose vertices are  $x_i, i \in \mathbb{Z}$  and with characteristic space-step  $h$ . We denote by  $C_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})$  the control volume (cell) centered on  $x_i$ , where  $x_{i+\frac{1}{2}} = \frac{x_i + x_{i+1}}{2}$  are the cell interfaces, so that  $h = \text{length}(C_i)$ . Then we construct a piecewise constant approximation of the function  $z$  on the mesh, for example

$$z^h(x) = \sum_{i \in \mathbb{Z}} z_i \mathbb{1}_{C_i}(x), \quad z_i = \frac{1}{h} \int_{C_i} z(x) dx, \quad (1.5)$$

where  $\mathbb{1}_{C_i}$  is the characteristic function of the cell  $C_i$ .



We also introduce a piecewise constant approximation of the analytical solution to the problem (1.1)-(1.2), defined by

$$u^h(t, x) = \sum_{i \in \mathbb{Z}} u_i(t) \mathbb{1}_{C_i}(x), \quad u_i(t) = \frac{1}{h} \int_{C_i} u(t, x) dx. \quad (1.6)$$

In the above framework, the numerical solution obtained from a finite volume scheme applied to (1.1)-(1.2) is a function  $v^h(t, x)$ , whose cell-averages

$$v_i(t) = \frac{1}{h} \int_{C_i} v^h(t, x) dx, \quad i \in \mathbb{Z}, \quad (1.7)$$

are interpreted as approximations of the cell-averages of the analytical solution,  $v_i(t) \approx u_i(t)$ ,  $i \in \mathbb{Z}$ . The general scheme for (1.1) reads

$$\partial_t v^h + \partial_x v^h = B^N(x, v^h), \quad (1.8)$$

with initial data corresponding to the approximate initial condition

$$v_0^h(x) = \sum_{i \in \mathbb{Z}} u_{0i} \mathbb{1}_{C_i}(x), \quad u_{0i} = \frac{1}{h} \int_{C_i} u_0(x) dx. \quad (1.9)$$

According to the Upwind Interface Source method in [30], appropriate discretizations of the source term in (1.8) are given by

$$B^N(x, v^h) = \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(v_{i-1}, v_i, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(v_i, v_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i}, \quad (1.10)$$

where we set  $\Delta z_{i+\frac{1}{2}} = z_{i+1} - z_i$  (we dropped the time and space dependence in the formula, for simplicity). We assume the following consistency properties for the numerical source operator (1.10), in respect of (1.3), which are fundamental to the convergence analysis,

$$\mathcal{B}^\pm \in C^2, \quad \mathcal{B}^\pm(u, v, 0) = 0, \quad \frac{\partial \mathcal{B}^\pm}{\partial u}(u, v, 0) = \frac{\partial \mathcal{B}^\pm}{\partial v}(u, v, 0) = 0, \quad (1.11a)$$

$$\lim_{\zeta \rightarrow 0} \frac{\mathcal{B}^+(u, u, \zeta) + \mathcal{B}^-(u, u, \zeta)}{\zeta} = b(u). \quad (1.11b)$$

The last limit holds uniformly in  $u$ , as specified by the further assumption

$$\left| \frac{\mathcal{B}^+(u, u, \zeta) + \mathcal{B}^-(u, u, \zeta)}{\zeta} - b(u) \right| \leq K_B \zeta, \quad (1.12)$$

where  $K_B$  is a fixed constant (independent of  $u$ ). Moreover, we denote by  $L_b$  and  $L_B$  any Lipschitz constant associated respectively to the continuous or discrete source operator.

## 1.2 What is a second order scheme for the Upwind Interface Source method?

In order to obtain second order extensions of the discrete solver (1.8)-(1.9), we apply a slope limiter method to the numerical functions: the basic idea is to replace the piecewise constant reconstruction on the mesh of the approximate solution by more accurate reconstructions, namely piecewise linear (refer to [10] and [22] for a survey of high resolution methods).

We associate to the numerical solution (1.7) some coefficients, defined as second order interpolation of the discrete unknowns,

$$\bar{v}_i(t, x) = v_i(t) + (x - x_i)v'_i, \quad i \in \mathbb{Z}, x \in C_i, \quad (1.13)$$

where  $v'_i$  indicates a generic numerical derivative (computed by means of an appropriate *limiter*, as it will be discussed more precisely later on).

From (1.6), analogous definitions are introduced for the analytical solution,

$$\bar{u}_i(t, x) = u_i(t) + (x - x_i)u'_i, \quad i \in \mathbb{Z}, x \in C_i. \quad (1.14)$$

The function  $z$  can also be represented in terms of piecewise linear approximations on the spatial mesh, departing from (1.5), with coefficients

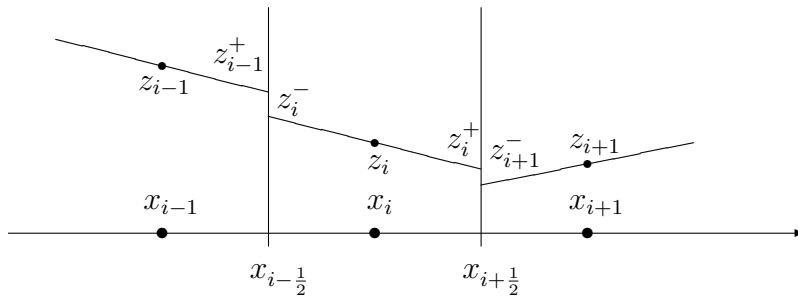
$$\bar{z}_i(x) = z_i + (x - x_i)z'_i, \quad x \in C_i.$$

At the cell interfaces, the values of the numerical functions are given by

$$v_i^- = \bar{v}_i(x_{i-\frac{1}{2}}) = v_i - \frac{h}{2}v'_i, \quad v_i^+ = \bar{v}_i(x_{i+\frac{1}{2}}) = v_i + \frac{h}{2}v'_i, \quad (1.15a)$$

$$z_i^- = \bar{z}_i(x_{i-\frac{1}{2}}) = z_i - \frac{h}{2}z'_i, \quad z_i^+ = \bar{z}_i(x_{i+\frac{1}{2}}) = z_i + \frac{h}{2}z'_i, \quad (1.15b)$$

as represented in the figure below, so that  $\Delta z_{i+\frac{1}{2}} = z_{i+1}^- - z_i^+$  in this case (we drop the time and space dependence when no mistake is possible).



Therefore, it is natural to perform a discretization of the source term (1.3)

by using the interfacial values (1.15a)-(1.15b), as follows,

$$B^N(x, v^h) = \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(v_{i-1}^+, v_i^-, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(v_i^+, v_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i} + \sum_{i \in \mathbb{Z}} z'_i b(v_i) \mathbb{1}_{C_i}, \quad (1.16)$$

with an additional term in comparison to the discrete source operator (1.10), which depends on the cell-averages and is necessary to achieve second order estimates (see Section 4 for details).

An alternative approach to formulating second order extensions of the Upwind Interface Source method is based on improving the consistency properties of the numerical source operator.

We consider a piecewise constant approximation (1.5) of the function  $z$  and piecewise linear reconstructions (1.13) of the numerical solution on the mesh, to define the upwind interfacial discretization

$$B^N(x, v^h) = \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(v_{i-1}^+, v_i^-, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(v_i^+, v_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i}, \quad (1.17)$$

where the numerical functions are computed on the interfacial values (1.15a) and  $\Delta z_{i+\frac{1}{2}} = z_{i+1} - z_i$ . To obtain second order accuracy, we need to assume that (1.11a) holds and the second order definition of consistency

$$\left| \frac{\mathcal{B}^+(u, u, \zeta) + \mathcal{B}^-(u, u, \zeta)}{\zeta} - b(u) \right| \leq K_B \zeta^2. \quad (1.18)$$

This is suggested by the particular form of the source term (1.3), given by the product of functions which exhibit different orders of derivative.

**Remark 1.1.** *In effect, the two discretizations (1.16) and (1.17) are strictly related, as formally verified by means of standard asymptotic expansions on the numerical functions and simple algebraic calculations with the discrete differences of values (1.5) or (1.15b). We also note that many of the second order schemes proposed in the literature do not include the additional term in (1.16), for the sake of simplicity (see [25], [2] and [29], for instance), but that is probably recovered implicitly in the formulation.*

### 1.3 Convergence and error estimates

To deal with the question of deriving error estimates for the approximation (1.8) to the equation (1.1), we introduce the error function

$$e(t, x) = u(t, x) - v^h(t, x), \quad (1.19)$$

which satisfies

$$\begin{aligned}\partial_t e + \partial_x e &= B(x, u) - B^N(x, v^h) \\ &= (B(x, u) - B^N(x, u^h)) + (B^N(x, u^h) - B^N(x, v^h)) \\ &:= \mathcal{C}(u; u^h) + \mathcal{S}(u^h; v^h).\end{aligned}\quad (1.20)$$

From (1.6) and (1.7), we obtain the usual expression for the cell-averages,

$$e_i(t) = \frac{1}{h} \int_{C_i} e(t, x) dx = u_i(t) - v_i(t), \quad i \in \mathbb{Z}. \quad (1.21)$$

The operators  $\mathcal{C}(u; u^h)$  and  $\mathcal{S}(u^h; v^h)$  in the formula (1.20) indicate the consistency and stability error term respectively.

The following result constitute the main stage of the convergence analysis for the Upwind Interface Source method.

**Theorem 1.2.** *We assume  $z \in W^{2,p}$ ,  $1 \leq p < +\infty$ , and we consider the numerical source operator (1.10) in (1.20). Then, for all  $t \in \mathbb{R}_+$ , the error function (1.19) verifies the first order estimate*

$$\|e(t)\|_{L^p} \leq C(t) \left( \|e_0\|_{L^p} + h\|z\|_{W^{2,p}} + h \int_0^t \exp\{-Cs\} \|u(s)\|_{W^{1,p}} ds \right), \quad (1.22)$$

where  $C(t)$  is a constant independent of  $h$ .

The convergence properties of second order schemes are notably affected by the technique used to construct piecewise linear approximations of the numerical functions, namely the choice of the slope *limiter*, as pointed out by several authors (see [27], [36] and [15]). Without appropriate hypotheses on the coefficients of such approximations, the proof of the consistency estimate given in Section 3.2 fails and numerical evidence shows that the discretization (1.16) loses second order accuracy (refer to Section 4 for details).

The following results extend the one which is established in Theorem 1.2 to the discretization (1.16) and (1.17).

**Theorem 1.3.** *We assume  $z \in W^{3,p}$ ,  $1 \leq p < +\infty$ , and we consider the numerical source operator (1.16), with discrete derivatives computed in the restricted class of slope limiters introduced in Section 3. Then, for all  $t \in \mathbb{R}_+$ , the error function (1.19) verifies the second order estimate*

$$\|e(t)\|_{L^p} \leq C(t) \left( \|e_0\|_{L^p} + h^2\|z\|_{W^{3,p}} + h^2 \int_0^t \exp\{-Cs\} \|u(s)\|_{W^{2,p}} ds \right), \quad (1.23)$$

where  $C(t)$  is a constant independent of  $h$ .

**Theorem 1.4.** *We assume  $z \in W^{3,p}$ ,  $1 \leq p < +\infty$ , and we consider the numerical source operator (1.17), with the consistency property (1.18). Then, for all  $t \in \mathbb{R}_+$ , the error function (1.19) verifies the second order estimate*

$$\|e(t)\|_{L^p} \leq C(t) \left( \|e_0\|_{L^p} + h^2 \|z\|_{W^{3,p}} + h^2 \int_0^t \exp\{-Cs\} \|u(s)\|_{W^{2,p}} ds \right), \quad (1.24)$$

where  $C(t)$  is a constant independent of  $h$ .

Because of the definition (1.6) and (1.9), we have  $v_0^h = u_0^h$  and then we deduce from (1.19) that  $e_0(x) = u_0(x) - u_0^h(x)$ ,  $x \in \mathbb{R}$ . Besides, the following statements are classical and not difficult to prove (see [5], for instance),

$$\begin{aligned} \|u_0^h\|_{L^p} &\leq \|u_0\|_{L^p}, \quad 1 \leq p < +\infty, \\ \|e_0\|_{L^p} &\leq Ch \quad \text{if } u_0 \in W^{1,p}, \\ \|e_0\|_{L^p} &\leq Ch^2 \quad \text{if } u_0 \in W^{2,p}. \end{aligned}$$

The convergence of initial data in (1.22), (1.23) and (1.24), as the mesh size tends to zero, is thus guaranteed by the first and second order convergence of piecewise constant approximations.

The detailed proofs of Theorem 1.2, Theorem 1.3 and Theorem 1.4, with the corresponding intermediate stages, are presented in Section 2 and Section 3.

**Remark 1.5.** *The same approach as described above applies to nonlinear scalar conservation laws with a source term, also to define numerical fluxes in semi-discrete methods (refer to [30] for specific notations). Therefore, the arguments developed in this paper might extend to the general case, to derive complete error estimates for the Upwind Interface Source method.*

## 2 Error estimates for first order schemes

Before giving details about the estimates, we introduce some relations on the discrete differences of numerical functions, we will frequently use later on the proofs.

We consider a generic function  $w \in C^1$ , whose cell-averages on the spatial mesh are denoted by  $w_i = \frac{1}{h} \int_{C_i} w(x) dx$ ,  $i \in \mathbb{Z}$ . By performing appropriate



expansions, we obtain

$$w_{i+1} - w_i = \int_{C_i} w'(\xi(x)) dx \quad (2.1a)$$

$$= hw'(x_i) + \int_{C_i} w''(\eta(x))(x - x_i) dx, \quad (2.1b)$$

$$w_{i+1} - 2w_i + w_{i-1} = h \int_{C_i} w''(\vartheta(x)) dx, \quad (2.1c)$$

for some  $\xi(x), \eta(x), \vartheta(x) \in C_i$ . We also recall the classical *Taylor's formula*,

$$w(x) = \sum_{k=0}^n \frac{1}{k!} w^{(k)}(x_i)(x - x_i)^k + \frac{1}{n!} \int_{x_i}^x (x - s)^n w^{(n+1)}(s) ds, \quad (2.2)$$

in the particular form with an integral expression for the remainder.

## 2.1 Stability estimate

We begin by dealing with the stability error term  $\mathcal{S}(u^h; v^h)$  in (1.20), to test the stability of the numerical source operator.

**Lemma 2.1.** *For the assumptions of Theorem 1.2, together with (1.11a), there exists a constant  $C := C(L_B, \|z'\|_{L^\infty})$ , independent of  $h$ , such that*

$$\left| \int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \operatorname{sgn}(e) dx \right| \leq C \|e\|_{L^p}^p. \quad (2.3)$$

*Proof.* From (1.20), we have

$$\int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \operatorname{sgn}(e) dx = \int_{\mathbb{R}} [B^N(x, u^h) - B^N(x, v^h)] |e|^{p-1} \operatorname{sgn}(e) dx.$$

Then, according to (1.10) and (1.6), we deduce

$$\begin{aligned}
& \int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \operatorname{sgn}(e) dx \\
&= \int_{\mathbb{R}} \left\{ \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(u_{i-1}, u_i, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i} \right. \\
&\quad \left. - \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(v_{i-1}, v_i, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(v_i, v_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i} \right\} |e|^{p-1} \operatorname{sgn}(e) dx \\
&= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_{i-1}, u_i, \Delta z_{i-\frac{1}{2}}) - \mathcal{B}^+(v_{i-1}, v_i, \Delta z_{i-\frac{1}{2}}) \right] e_i^{p-1} \\
&\quad + \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^-(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^-(v_i, v_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \\
&= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^+(v_i, v_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_{i+1}^{p-1} \\
&\quad + \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^-(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^-(v_i, v_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} := S_1 + S_2,
\end{aligned}$$

where we set  $e_i^{p-1} = \frac{1}{h} \int_{C_i} |e|^{p-1} \operatorname{sgn}(e) dx$  (as usually, we dropped the time and space dependence in the above formulas for simplicity).

We estimate the terms  $S_1$  and  $S_2$  separately. For  $S_1$ , we have

$$\begin{aligned}
S_1 &= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^+(v_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_{i+1}^{p-1} \\
&\quad + \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(v_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^+(v_i, v_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_{i+1}^{p-1} \\
&= \sum_{i \in \mathbb{Z}} \left( \int_{v_i}^{u_i} \frac{\partial \mathcal{B}^+}{\partial u}(u, u_{i+1}, \Delta z_{i+\frac{1}{2}}) du \right) e_{i+1}^{p-1} \\
&\quad + \sum_{i \in \mathbb{Z}} \left( \int_{v_{i+1}}^{u_{i+1}} \frac{\partial \mathcal{B}^+}{\partial v}(v_i, v, \Delta z_{i+\frac{1}{2}}) dv \right) e_{i+1}^{p-1},
\end{aligned} \tag{2.4}$$

so that, in view of (1.11a), we get

$$\begin{aligned}
S_1 &= \sum_{i \in \mathbb{Z}} \left( \int_{v_i}^{u_i} \left[ \frac{\partial \mathcal{B}^+}{\partial u}(u, u_{i+1}, \Delta z_{i+\frac{1}{2}}) - \frac{\partial \mathcal{B}^+}{\partial u}(u, u_{i+1}, 0) \right] du \right) e_{i+1}^{p-1} \\
&\quad + \sum_{i \in \mathbb{Z}} \left( \int_{v_{i+1}}^{u_{i+1}} \left[ \frac{\partial \mathcal{B}^+}{\partial v}(v_i, v, \Delta z_{i+\frac{1}{2}}) - \frac{\partial \mathcal{B}^+}{\partial v}(v_i, v, 0) \right] dv \right) e_{i+1}^{p-1} \\
&\leq L_B \sum_{i \in \mathbb{Z}} |\Delta z_{i+\frac{1}{2}}| (|u_i - v_i| + |u_{i+1} - v_{i+1}|) |e_{i+1}^{p-1}|.
\end{aligned} \tag{2.5}$$

We proceed in similar way for  $S_2$  and we establish the relations corresponding to (2.4) and (2.5). Therefore, also recalling (1.21), we conclude

$$S_1 \leq L_B \sum_{i \in \mathbb{Z}} |\Delta z_{i+\frac{1}{2}}| (|e_i| + |e_{i+1}|) |e_{i+1}^{p-1}|, \quad (2.6)$$

$$S_2 \leq L_B \sum_{i \in \mathbb{Z}} |\Delta z_{i+\frac{1}{2}}| (|e_i| + |e_{i+1}|) |e_i^{p-1}|. \quad (2.7)$$

Because of  $|\operatorname{sgn}(e)| \leq 1$ , by using the *Hölder's inequality* for  $1 \leq p < +\infty$ , simple computations lead to obtain  $|e_i^{p-1}| \leq |e_i|^{p-1}$ . This implies, after rearranging terms in (2.6) and (2.7), that

$$\begin{aligned} & \left| \int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \operatorname{sgn}(e) dx \right| \\ & \leq L_B \sum_{i \in \mathbb{Z}} |\Delta z_{i+\frac{1}{2}}| (|e_i|^p + |e_i|^{p-1}|e_{i+1}| + |e_i||e_{i+1}|^{p-1} + |e_{i+1}|^p). \end{aligned} \quad (2.8)$$

Now the *Young's inequality*,  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , applied to (2.8) and the immediate property  $|e_i|^p \leq \frac{1}{h} \int_{C_i} |e|^p dx$ ,  $i \in \mathbb{Z}$ , provide

$$\left| \int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \operatorname{sgn}(e) dx \right| \leq 2L_B \sum_{i \in \mathbb{Z}} \frac{|\Delta z_{i+\frac{1}{2}}|}{h} \left( \int_{C_i} |e|^p dx + \int_{C_{i+1}} |e|^p dx \right). \quad (2.9)$$

In the case of (1.5), according to (2.1a), a direct estimate yields the first order approximation  $\frac{|\Delta z_{i+\frac{1}{2}}|}{h} \leq \|z'\|_{L^\infty}$ . The proof of (2.3) is thus completed.  $\square$

## 2.2 Consistency estimate

We turn our attention to the consistency error term  $\mathcal{C}(u; u^h)$  in (1.20), for which an optimal result in terms of the rate of convergence is obtained.

**Lemma 2.2.** *For the assumptions of Theorem 1.2, together with (1.11a), (1.11b) and (1.12), there exists a constant independent of  $h$  such that*

$$\left| \int_{\mathbb{R}} \mathcal{C}(u; u^h) |e|^{p-1} \operatorname{sgn}(e) dx \right| \leq Ch (\|z\|_{W^{2,p}} + \|u\|_{W^{1,p}}) \|e\|_{L^p}^{p-1}. \quad (2.10)$$

*Proof.* From (1.20), we have

$$\int_{\mathbb{R}} \mathcal{C}(u; u^h) |e|^{p-1} \operatorname{sgn}(e) dx = \int_{\mathbb{R}} [B(x, u) - B^N(x, u^h)] |e|^{p-1} \operatorname{sgn}(e) dx.$$

We compute the integral of the discrete source operator,

$$\begin{aligned}
& \int_{\mathbb{R}} B^N(x, u^h) |e|^{p-1} \operatorname{sgn}(e) dx \\
&= \int_{\mathbb{R}} \left\{ \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(u_{i-1}, u_i, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i} \right\} |e|^{p-1} \operatorname{sgn}(e) dx \\
&= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_{i-1}, u_i, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1},
\end{aligned}$$

where  $e_i^{p-1} = \frac{1}{h} \int_{C_i} |e|^{p-1} \operatorname{sgn}(e) dx$ . Then we decompose as follows,

$$\begin{aligned}
& \int_{\mathbb{R}} B^N(x, u^h) |e|^{p-1} \operatorname{sgn}(e) dx \\
&= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) + \mathcal{B}^-(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} + \\
&+ \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_{i-1}, u_i, \Delta z_{i-\frac{1}{2}}) - \mathcal{B}^+(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} := T_1 + T_2.
\end{aligned} \tag{2.11}$$

We estimate each  $T_j$ ,  $j = 1, 2$  separately. Setting  $\mathcal{B} = \mathcal{B}^+ + \mathcal{B}^-$ , we write

$$\begin{aligned}
T_1 &= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}(u_i, u_i, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \\
&+ \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i, u_i, \Delta z_{i+\frac{1}{2}}) + \mathcal{B}^-(u_i, u_i, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} := T_1^1 + T_1^2.
\end{aligned} \tag{2.12}$$

For the remainder  $T_1^1$ , thanks to (1.11a), we get

$$\begin{aligned}
T_1^1 &= \sum_{i \in \mathbb{Z}} \left( \int_{u_i}^{u_{i+1}} \left[ \frac{\partial \mathcal{B}}{\partial v}(u_i, v, \Delta z_{i+\frac{1}{2}}) - \frac{\partial \mathcal{B}}{\partial v}(u_i, v, 0) \right] dv \right) e_i^{p-1} \\
&\leq L_B \sum_{i \in \mathbb{Z}} |\Delta z_{i+\frac{1}{2}}| |u_{i+1} - u_i| |e_i^{p-1}|,
\end{aligned}$$

so that (2.1a) applied to (1.5) and (1.6) leads to conclude

$$T_1^1 \leq L_B \|z'\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \tag{2.13}$$

The second term  $T_1^2$  of (2.12) can be further decomposed into three parts,

$$\begin{aligned}
T_1^2 &= \sum_{i \in \mathbb{Z}} \frac{\mathcal{B}(u_i, u_i, \Delta z_{i+\frac{1}{2}})}{\Delta z_{i+\frac{1}{2}}} \frac{\Delta z_{i+\frac{1}{2}}}{h} h e_i^{p-1} \\
&= \sum_{i \in \mathbb{Z}} \frac{\mathcal{B}(u_i, u_i, \Delta z_{i+\frac{1}{2}})}{\Delta z_{i+\frac{1}{2}}} \left[ \frac{\Delta z_{i+\frac{1}{2}}}{h} - z'(x_i) \right] h e_i^{p-1} \\
&\quad + \sum_{i \in \mathbb{Z}} \left[ \frac{\mathcal{B}(u_i, u_i, \Delta z_{i+\frac{1}{2}})}{\Delta z_{i+\frac{1}{2}}} - b(u_i) \right] z'(x_i) h e_i^{p-1} \\
&\quad + \sum_{i \in \mathbb{Z}} z'(x_i) b(u_i) h e_i^{p-1} := T_1^{2,1} + T_1^{2,2} + T_1^{2,3}.
\end{aligned} \tag{2.14}$$

We give details for each part. By using (1.11a), from (2.1b) we deduce that

$$\begin{aligned}
T_1^{2,1} &\leq \sum_{i \in \mathbb{Z}} \left| \frac{\mathcal{B}(u_i, u_i, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}(u_i, u_i, 0)}{\Delta z_{i+\frac{1}{2}}} \right| \left| \frac{\Delta z_{i+\frac{1}{2}}}{h} - z'(x_i) \right| h |e_i^{p-1}| \\
&\leq L_B \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z''| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right).
\end{aligned} \tag{2.15}$$

Because of the consistency property (1.12) and (2.1a) for (1.5), we derive

$$T_1^{2,2} \leq K_B \|z'\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z'| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \tag{2.16}$$

Finally, the third term in (2.14) is equivalent to the integral of the analytical source operator (1.3). Indeed, by means of Taylor's expansions in (1.6), we obtain the *midpoint formula* (dropping the time dependence, for simplicity)

$$u_i = u(x_i) + R_i, \quad R_i = \frac{1}{h} \int_{C_i} \partial_x u(\xi(x))(x - x_i) dx,$$

for some  $\xi(x) \in C_i$ , and the regularity assumed in (1.3) guarantees that

$$b(u_i) = b(u(x_i)) + b'(\nu_i) R_i, \quad |b'(\nu_i)| \leq L_b, \quad \forall i \in \mathbb{Z}.$$

We thus write

$$T_1^{2,3} = \sum_{i \in \mathbb{Z}} z'(x_i) b(u(x_i)) h e_i^{p-1} + R_1, \tag{2.17}$$

where the remainder satisfies

$$R_1 \leq L_b \|z'\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \tag{2.18}$$

The *Taylor's formula* (2.2), applied to the source term (1.3), yields

$$\int_{\mathbb{R}} B(x, u) |e|^{p-1} \operatorname{sgn}(e) dx = \sum_{i \in \mathbb{Z}} \int_{C_i} z'(x_i) b(u(x_i)) |e|^{p-1} \operatorname{sgn}(e) dx + R_2$$

and we readily check that the remainder satisfies

$$\begin{aligned} R_2 &\leq L_b \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z''| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right) \\ &\quad + L_b \|z'\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \end{aligned} \quad (2.19)$$

Therefore, from (2.17) we have

$$T_1^{2,3} = \int_{\mathbb{R}} B(x, u) |e|^{p-1} \operatorname{sgn}(e) dx + R_1 - R_2, \quad (2.20)$$

with the remainders (2.18) and (2.19), which are conveniently bounded.

Coming back to decomposition (2.11), using (1.11a) we have for the last term

$$\begin{aligned} T_2 &= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_{i-1}, u_i, \Delta z_{i-\frac{1}{2}}) - \mathcal{B}^+(u_i, u_i, \Delta z_{i-\frac{1}{2}}) \right] e_i^{p-1} \\ &\quad + \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i, u_i, \Delta z_{i-\frac{1}{2}}) - \mathcal{B}^+(u_i, u_i, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \\ &\quad + \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i, u_i, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^+(u_i, u_{i+1}, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \\ &= \sum_{i \in \mathbb{Z}} \left( \int_{u_i}^{u_{i-1}} \left[ \frac{\partial \mathcal{B}^+}{\partial u}(u, u_i, \Delta z_{i-\frac{1}{2}}) - \frac{\partial \mathcal{B}^+}{\partial u}(u, u_i, 0) \right] du \right) e_i^{p-1} \\ &\quad + \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i, u_i, \Delta z_{i-\frac{1}{2}}) - \mathcal{B}^+(u_i, u_i, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \\ &\quad + \sum_{i \in \mathbb{Z}} \left( \int_{u_{i+1}}^{u_i} \left[ \frac{\partial \mathcal{B}^+}{\partial v}(u_i, v, \Delta z_{i+\frac{1}{2}}) - \frac{\partial \mathcal{B}^+}{\partial v}(u_i, v, 0) \right] dv \right) e_i^{p-1} \\ &\leq L_B \sum_{i \in \mathbb{Z}} \left( |\Delta z_{i-\frac{1}{2}}| |u_i - u_{i-1}| + |\Delta z_{i+\frac{1}{2}} - \Delta z_{i-\frac{1}{2}}| \right. \\ &\quad \left. + |\Delta z_{i+\frac{1}{2}}| |u_{i+1} - u_i| \right) |e_i^{p-1}|, \end{aligned}$$

so that we conclude by means of (2.1a) and (2.1c),

$$\begin{aligned} T_2 &\leq 2L_B \|z'\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right) \\ &\quad + L_B \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z''| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \end{aligned} \quad (2.21)$$

We put (2.11), (2.12), (2.14) and (2.20) together, with the corresponding estimates stated in (2.13), (2.15), (2.16), (2.18), (2.19) and (2.21). We apply the discrete Hölder's inequality,  $\sum_{i \in \mathbb{Z}} a_i b_i \leq (\sum_{i \in \mathbb{Z}} a_i^p)^{\frac{1}{p}} \cdot (\sum_{i \in \mathbb{Z}} b_i^q)^{\frac{1}{q}}$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , to the products and then the usual continuous inequality to each integral on the mesh cells. This provides a coefficient  $h$  in front of all expressions and the result in (2.10) thus follows, with  $C := C(L_B, K_B, L_b, \|z'\|_{L^\infty})$ .  $\square$

### 2.3 Proof of Theorem 1.2

We multiply equation (1.20) by  $|e|^{p-1} \text{sgn}(e)$  and we integrate as follows,

$$\begin{aligned} & \int_{\mathbb{R}} (\partial_t e + \partial_x e) |e|^{p-1} \text{sgn}(e) dx \\ &= \int_{\mathbb{R}} \mathcal{C}(u; u^h) |e|^{p-1} \text{sgn}(e) dx + \int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \text{sgn}(e) dx. \end{aligned} \quad (2.22)$$

An integration by parts shows  $\int_{\mathbb{R}} |e|^{p-1} \text{sgn}(e) \partial_x e dx = 0$ , then we deduce from (2.22), (2.3) and (2.10) that

$$\frac{1}{p} \partial_t \|e(t)\|_{L^p}^p \leq C \|e(t)\|_{L^p}^p + Ch (\|z\|_{W^{2,p}} + \|u(t)\|_{W^{1,p}}) \|e(t)\|_{L^p}^{p-1}. \quad (2.23)$$

Let  $t^* \in \mathbb{R}_+$  be such that  $\|e(t^*)\|_{L^p} = \max_{t \in \mathbb{R}_+} \|e(t)\|_{L^p}$ . By integrating in time from 0 to  $t^*$ , we get

$$\begin{aligned} \|e(t^*)\|_{L^p}^p &\leq \|e(0)\|_{L^p}^p + Cp \int_0^{t^*} \|e(s)\|_{L^p}^p ds \\ &\quad + Cph \|z\|_{W^{2,p}} \int_0^{t^*} \|e(s)\|_{L^p}^{p-1} ds + Cph \int_0^{t^*} \|u(s)\|_{W^{1,p}} \|e(s)\|_{L^p}^{p-1} ds \\ &\leq \|e(0)\|_{L^p} \|e(t^*)\|_{L^p}^{p-1} + Cp \|e(t^*)\|_{L^p}^{p-1} \int_0^{t^*} \|e(s)\|_{L^p} ds \\ &\quad + Cph t^* \|z\|_{W^{2,p}} \|e(t^*)\|_{L^p}^{p-1} + Cph \|e(t^*)\|_{L^p}^{p-1} \int_0^{t^*} \|u(s)\|_{W^{1,p}} ds, \end{aligned}$$

which implies that

$$\begin{aligned} \|e(t^*)\|_{L^p} &\leq \|e(0)\|_{L^p} + Cp \int_0^{t^*} \|e(s)\|_{L^p} ds \\ &\quad + Cph t^* \|z\|_{W^{2,p}} + Cph \int_0^{t^*} \|u(s)\|_{W^{1,p}} ds. \end{aligned} \quad (2.24)$$

Finally, a straightforward extension of Gronwall's inequality yields the desired result (1.22), where  $C(t) := C(t; p, L_B, K_B, L_b, \|z'\|_{L^\infty})$  is any positive constant depending on time by the factor  $\exp\{-Ct\}$ .

### 3 Error estimates for second order schemes

The convergence properties of the approximation (1.16) are shown by mimicking the proof of the analogous results for (1.10), provided in Section 2.

As in the case of first order approximations, we derive some preliminary estimates on the discrete differences of numerical functions.

For a function  $w \in C^2$ , with cell-averages  $w_i = \frac{1}{h} \int_{C_i} w(x) dx$ ,  $i \in \mathbb{Z}$ , we construct piecewise linear approximations on the spatial mesh by means of the coefficients

$$\bar{w}_i(x) = w_i + (x - x_i)w'_i, \quad i \in \mathbb{Z}, x \in C_i, \quad (3.1)$$

where the numerical derivatives are defined as appropriate interpolations of the discrete increments between neighboring cells,

$$w'_i = \text{lmtr} \left\{ \frac{w_{i+1} - w_i}{h}, \frac{w_i - w_{i-1}}{h} \right\}, \quad i \in \mathbb{Z}. \quad (3.2)$$

We consider a general representation of the slope *limiter* introduced in the above formula, i.e. if  $M = \text{lmtr}\{\alpha, \beta\}$ , then  $M = \kappa\alpha + \lambda\beta$ , with  $\kappa, \lambda \in [0, 1]$  and  $\kappa + \lambda = 1$  or  $\kappa + \lambda = 0$ . In particular, we restrict our analysis to the special class of operators which satisfy the condition  $\kappa_i + \lambda_i = 1$ ,  $\forall i \in \mathbb{Z}$  (that excludes, for instance, the classical *minmod limiter* in the case of nonmonotonic numerical functions). We also assume that the numerical application (3.2) relating the cell-averages  $w_j$ ,  $j = i-1, i, i+1$ , to the discrete derivative  $w'_i$  is Lipschitz continuous on its arguments, with constant  $\frac{C}{h}$ . Several examples of slope *limiter* which satisfies these properties have been formulated in the literature (refer to [12], [13], [14], [26], [32] and [33]).

We deduce from those definitions that

$$w'_i = \kappa_i \frac{w_{i+1} - w_i}{h} + \lambda_i \frac{w_i - w_{i-1}}{h}, \quad i \in \mathbb{Z}. \quad (3.3)$$

The interfacial values of the reconstruction (3.1) are given by

$$w_i^- = \bar{w}_i(x_{i-\frac{1}{2}}) = w_i - \frac{h}{2}w'_i, \quad w_i^+ = \bar{w}_i(x_{i+\frac{1}{2}}) = w_i + \frac{h}{2}w'_i, \quad (3.4)$$

and we are interested in evaluating the jumps at the interfaces, i.e.  $w_{i+1}^- - w_i^+$ . Taking into account (3.4) and (3.3), we have

$$\begin{aligned} w_{i+1}^- - w_i^+ &= w_{i+1} - w_i - \frac{h}{2}(w'_i + w'_{i+1}) \\ &= \left(1 - \frac{\kappa_i}{2} - \frac{\lambda_{i+1}}{2}\right)W_1 - \frac{\lambda_i}{2}W_2 - \frac{\kappa_{i+1}}{2}W_3, \end{aligned} \quad (3.5)$$



where we indicate

$$W_1 = w_{i+1} - w_i, \quad W_2 = w_i - w_{i-1}, \quad W_3 = w_{i+2} - w_{i+1}. \quad (3.6)$$

Consequently, to deal with (3.5) we use the same procedure as (2.1a)-(2.1b), based on high order Taylor expansions applied to the various terms in (3.6). Besides, we observe that

$$w_{i+1} - w_i = \frac{1}{h} \int_{C_i} [w(x+h) - w(x)] dx = \frac{1}{h} \int_{C_i} \int_0^h w'(x+s) ds dx,$$

then the following result holds,

$$|W_j| \leq \|w'\|_{L^1(C_i)}, \quad i \in \mathbb{Z}, \quad j = 1, 2, 3. \quad (3.7)$$

The simplest first order approximation reads

$$\begin{aligned} w_{i+1}^- - w_i^+ &= \left(1 - \frac{\kappa_i}{2} - \frac{\lambda_{i+1}}{2}\right) \int_{C_i} w'(\xi(x)) dx \\ &\quad - \frac{\lambda_i}{2} \int_{C_i} w'(\eta(x)) dx - \frac{\kappa_{i+1}}{2} \int_{C_i} w'(\vartheta(x)) dx, \end{aligned} \quad (3.8)$$

for some  $\xi(x), \eta(x), \vartheta(x) \in C_i$ , so it follows also from (3.7) that

$$|w_{i+1}^- - w_i^+| \leq D_{i+\frac{1}{2}} \|w\|_{W^{1,1}} \quad \text{or} \quad |w_{i+1}^- - w_i^+| \leq D_{i+\frac{1}{2}} h \|w'\|_{L^\infty}. \quad (3.9)$$

Recalling that  $\kappa_i + \lambda_i = 1, \forall i \in \mathbb{Z}$ , we obtain the second order approximation

$$\begin{aligned} w_{i+1}^- - w_i^+ &= \left(1 - \frac{\kappa_i}{2} - \frac{\lambda_{i+1}}{2}\right) \int_{C_i} w''(\xi(x))(x - x_i) dx \\ &\quad + \frac{\lambda_i}{2} \int_{C_i} w''(\eta(x))(x - x_i) dx - \frac{3}{2} \kappa_{i+1} \int_{C_i} w''(\vartheta(x))(x - x_i) dx, \end{aligned}$$

for some  $\xi(x), \eta(x), \vartheta(x) \in C_i$ , and then it follows

$$|w_{i+1}^- - w_i^+| \leq D_{i+\frac{1}{2}} h \|w\|_{W^{2,1}} \quad \text{or} \quad |w_{i+1}^- - w_i^+| \leq D_{i+\frac{1}{2}} h^2 \|w''\|_{L^\infty}. \quad (3.10)$$

**Remark 3.1.** We note that the constant in (3.9) and (3.10) satisfies, uniformly for  $i \in \mathbb{Z}$ , the estimate  $D_{i+\frac{1}{2}} \leq \max \left\{ \left(1 - \frac{\kappa_i}{2} - \frac{\lambda_{i+1}}{2}\right), \frac{\lambda_i}{2}, \frac{3}{4} \kappa_{i+1} \right\} \leq 1$ . Moreover, for any set of values  $(\kappa_i, \lambda_i)_{i \in \mathbb{Z}}$ , the bounds on these quantities are always not degenerate.

Finally, a long but straightforward computation, involving also the third order expansions, leads to conclude that

$$\begin{aligned}
w_{i+1}^- - w_i^+ &= (\lambda_{i+1} - \kappa_i) \frac{h^2}{2} w''(x_i) \\
&+ \left(1 - \frac{\kappa_i}{2} - \frac{\lambda_{i+1}}{2}\right) \int_{C_i} w'''(\xi(x))(x - x_i)^2 dx \\
&- \frac{\lambda_i}{2} \int_{C_i} w'''(\eta(x))(x - x_i)^2 dx - \frac{\kappa_{i+1}}{2} \int_{C_i} w'''(\vartheta(x))(x - x_i)^2 dx,
\end{aligned} \tag{3.11}$$

for some  $\xi(x), \eta(x), \vartheta(x) \in C_i$ .

**Remark 3.2.** According to the piecewise linear reconstruction (3.1), discrete interfacial jumps approximate the second derivative of the numerical functions, as it can be roughly deduced from (3.5).

### 3.1 Stability estimate

The following result corresponds to that presented in Section 2.1 and then we adapt the proof of Lemma 2.1 in the case of the numerical source operator (1.16).

**Lemma 3.3.** For the assumptions of Theorem 1.3, together with (1.11a), there exists a constant  $C := C(L_B, L_b, \|z'\|_{L^\infty}, \|z''\|_{L^\infty})$ , independent of  $h$ , such that

$$\left| \int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \text{sgn}(e) dx \right| \leq C \|e\|_{L^p}^p. \tag{3.12}$$

*Proof.* From (1.20), (1.16) and (1.6), we deduce

$$\begin{aligned}
&\int_{\mathbb{R}} \mathcal{S}(u^h; v^h) |e|^{p-1} \text{sgn}(e) dx \\
&= \int_{\mathbb{R}} \left\{ \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(u_{i-1}^+, u_i^-, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(u_i^+, u_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i} \right. \\
&\quad \left. - \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(v_{i-1}^+, v_i^-, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(v_i^+, v_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i} \right\} |e|^{p-1} \text{sgn}(e) dx \\
&+ \int_{\mathbb{R}} \left\{ \sum_{i \in \mathbb{Z}} z_i' b(u_i) \mathbb{1}_{C_i} - \sum_{i \in \mathbb{Z}} z_i' b(v_i) \mathbb{1}_{C_i} \right\} |e|^{p-1} \text{sgn}(e) dx
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i^+, u_{i+1}^-, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^+(v_i^+, v_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] e_{i+1}^{p-1} \\
&+ \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^-(u_i^+, u_{i+1}^-, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^-(v_i^+, v_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \quad (3.13) \\
&+ \sum_{i \in \mathbb{Z}} z'_i [b(u_i) - b(v_i)] h e_i^{p-1} := S_1 + S_2 + S_3,
\end{aligned}$$

where  $e_i^{p-1} = \frac{1}{h} \int_{C_i} |e|^{p-1} \text{sgn}(e) dx$ .

To deal with  $S_1$  and  $S_2$ , we proceed exactly as in (2.4) and (2.5). Because of the definition (3.3) and (3.4), together with the Lipschitz property of the application (3.2), simple computations lead to verify that

$$|u_i^+ - v_i^+| \leq \max(|u_i - v_i| + |u_{i+1} - v_{i+1}| + |u_{i-1} - v_{i-1}|),$$

and the same relation is satisfied by  $|u_i^- - v_i^-|$ . So, we can establish for the second order methods similar estimates to (2.6) and (2.7).

On the other hand, a direct treatment of the last term in (3.13) yields

$$S_3 \leq L_b \sum_{i \in \mathbb{Z}} |z'_i| |u_i - v_i| h |e_i^{p-1}|. \quad (3.14)$$

We give some details about the estimate of numerical derivatives (3.3), for the particular case of (1.5), we will use later on the proofs.

By performing appropriate expansions, also recalling that  $\kappa_i + \lambda_i = 1, \forall i \in \mathbb{Z}$ , we obtain

$$\begin{aligned}
z'_i &= \frac{\kappa_i}{h} \int_{C_i} z'(\xi(x)) dx + \frac{\lambda_i}{h} \int_{C_i} z'(\eta(x)) dx \\
&= z'(x_i) + (\kappa_i - \lambda_i) \frac{h}{2} z''(x_i) \quad (3.15) \\
&+ \frac{\kappa_i}{3} h \int_{C_i} z'''(\vartheta(x)) dx + \frac{\lambda_i}{3} h \int_{C_i} z'''(\varrho(x)) dx,
\end{aligned}$$

for some  $\xi(x), \eta(x), \vartheta(x), \varrho(x) \in C_i$ , which implies that  $|z'_i| \leq \|z'\|_{L^\infty}$  in (3.14). Thanks to the arguments used for passing to (2.8) and (2.9), with the first order approximation (3.9) applied to (1.15b), we conclude (3.12).  $\square$

## 3.2 Consistency estimate

The proof of the following result is also an extension of that of Lemma 2.2.

**Lemma 3.4.** *For the assumptions of Theorem 1.3, together with (1.11a), (1.11b) and (1.12), there exists a constant independent of  $h$  such that*

$$\left| \int_{\mathbb{R}} \mathcal{C}(u; u^h) |e|^{p-1} \operatorname{sgn}(e) dx \right| \leq Ch^2 (\|z\|_{W^{3,p}} + \|u\|_{W^{2,p}}) \|e\|_{L^p}^{p-1}. \quad (3.16)$$

*Proof.* We consider the integral of the source operator (1.16), computed on the approximation (1.6) of the analytical solution,

$$\begin{aligned} & \int_{\mathbb{R}} B^N(x, u^h) |e|^{p-1} \operatorname{sgn}(e) dx \\ &= \int_{\mathbb{R}} \left\{ \sum_{i \in \mathbb{Z}} \frac{1}{h} \left[ \mathcal{B}^+(u_{i-1}^+, u_i^-, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(u_i^+, u_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] \mathbb{1}_{C_i} \right\} |e|^{p-1} \operatorname{sgn}(e) dx \\ &+ \int_{\mathbb{R}} \left\{ \sum_{i \in \mathbb{Z}} z'_i b(u_i) \mathbb{1}_{C_i} \right\} |e|^{p-1} \operatorname{sgn}(e) dx \\ &= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_{i-1}^+, u_i^-, \Delta z_{i-\frac{1}{2}}) + \mathcal{B}^-(u_i^+, u_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} + \sum_{i \in \mathbb{Z}} z'_i b(u_i) h e_i^{p-1}, \end{aligned}$$

where we set  $e_i^{p-1} = \frac{1}{h} \int_{C_i} |e|^{p-1} \operatorname{sgn}(e) dx$ . In the sequel, we neglect the dependence on time of the numerical functions to simplify the notation.

We decompose the first part of the above formula, similarly to (2.11), into two terms  $T_j$ ,  $j=1, 2$  treated separately. The remainder can be rewritten as

$$\begin{aligned} T_2 &= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_{i-1}^+, u_i^-, \Delta z_{i-\frac{1}{2}}) - \mathcal{B}^+(u_i^+, u_i^-, \Delta z_{i-\frac{1}{2}}) \right] e_i^{p-1} \\ &+ \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i^+, u_i^-, \Delta z_{i-\frac{1}{2}}) - \mathcal{B}^+(u_i^+, u_i^-, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \\ &+ \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}^+(u_i^+, u_i^-, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}^+(u_i^+, u_{i+1}^-, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \end{aligned}$$

and the usual procedures for the differences, by using (1.11a), leads to deduce

$$\begin{aligned} T_2 &\leq L_B \sum_{i \in \mathbb{Z}} \left( |\Delta z_{i-\frac{1}{2}}| |u_i^+ - u_{i-1}^+| + |\Delta z_{i+\frac{1}{2}} - \Delta z_{i-\frac{1}{2}}| \right. \\ &\quad \left. + |\Delta z_{i+\frac{1}{2}}| |u_{i+1}^- - u_i^-| \right) |e_i^{p-1}|. \end{aligned} \quad (3.17)$$

According to the definition (3.3) and (3.4), concerning (1.14), we easily obtain

$$|u_i^+ - u_{i-1}^+| = u_i - u_{i-1} + \frac{h}{2} (u'_i - u'_{i-1}) \leq \int_{C_i} |\partial_x u| dx,$$

with an analogous estimate for  $|u_{i+1}^- - u_i^-|$ , while a second order approximation is needed for the central term in (3.17), that is

$$|\Delta z_{i+\frac{1}{2}} - \Delta z_{i-\frac{1}{2}}| = (z_{i+1} - 2z_i + z_{i-1}) - \frac{h}{2} (z'_{i+1} - z'_{i-1}) \leq h^2 \int_{C_i} |z'''| dx.$$

These estimates and (3.10) for (1.15b) provide an analogous inequality to (2.21),

$$\begin{aligned} T_2 &\leq 2L_B h \|z''\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right) \\ &\quad + L_B h \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z'''| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \end{aligned} \quad (3.18)$$

For the term corresponding to (2.12), setting  $\mathcal{B} = \mathcal{B}^+ + \mathcal{B}^-$ , we thus have

$$\begin{aligned} T_1 &= \sum_{i \in \mathbb{Z}} \left[ \mathcal{B}(u_i^+, u_{i+1}^-, \Delta z_{i+\frac{1}{2}}) - \mathcal{B}(u_i^+, u_i^+, \Delta z_{i+\frac{1}{2}}) \right] e_i^{p-1} \\ &\quad + \sum_{i \in \mathbb{Z}} \mathcal{B}(u_i^+, u_i^+, \Delta z_{i+\frac{1}{2}}) e_i^{p-1} := T_1^1 + T_1^2. \end{aligned} \quad (3.19)$$

We use again the property (1.11a) and we deduce

$$T_1^1 \leq L_B \sum_{i \in \mathbb{Z}} |\Delta z_{i+\frac{1}{2}}| |u_{i+1}^- - u_i^+| e_i^{p-1},$$

to conclude from (3.8) and (3.10) that

$$T_1^1 \leq L_B h \|z''\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \quad (3.20)$$

The second term of (3.19) is further decomposed, also thanks to (3.11),

$$\begin{aligned} T_1^2 &= \sum_{i \in \mathbb{Z}} \frac{\mathcal{B}(u_i^+, u_i^+, \Delta z_{i+\frac{1}{2}})}{\Delta z_{i+\frac{1}{2}}} \left[ \frac{\Delta z_{i+\frac{1}{2}}}{h} - Q_{i+\frac{1}{2}} \frac{h}{2} z''(x_i) \right] h e_i^{p-1} \\ &\quad + \sum_{i \in \mathbb{Z}} \left[ \frac{\mathcal{B}(u_i^+, u_i^+, \Delta z_{i+\frac{1}{2}})}{\Delta z_{i+\frac{1}{2}}} - b(u_i^+) \right] Q_{i+\frac{1}{2}} \frac{h}{2} z''(x_i) h e_i^{p-1} \\ &\quad + \sum_{i \in \mathbb{Z}} Q_{i+\frac{1}{2}} \frac{h}{2} z''(x_i) b(u_i^+) h e_i^{p-1} := T_1^{2,1} + T_1^{2,2} + T_1^{2,3}, \end{aligned} \quad (3.21)$$

where  $Q_{i+\frac{1}{2}} = \lambda_{i+1} - \kappa_i \leq 1$ ,  $\forall i \in \mathbb{Z}$ , for the properties of coefficients in (3.3). We give a few details of the estimate for each part. We proceed as in (2.15), by means of (1.11a) and (3.11), to obtain

$$T_1^{2,1} \leq L_B h \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z'''| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \quad (3.22)$$

From the consistency bound (1.12), together with the approximation (3.8), we derive that

$$T_1^{2,2} \leq K_B h \|z''\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z'| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \quad (3.23)$$

Then we pass to the crucial point of the proof, to show the convergence towards the integral of the analytical source operator (1.3). On the one hand, by applying to that function classical Taylor's expansions, we have

$$\begin{aligned} \int_{\mathbb{R}} B(x, u) |e|^{p-1} \operatorname{sgn}(e) dx &= \sum_{i \in \mathbb{Z}} \int_{C_i} z'(x_i) b(u(x_i)) |e|^{p-1} \operatorname{sgn}(e) dx \\ &\quad + \sum_{i \in \mathbb{Z}} \int_{C_i} (z' b(u))'(\xi(x_i)) (x - x_i) |e|^{p-1} \operatorname{sgn}(e) dx, \end{aligned} \quad (3.24)$$

for some  $\xi(x_i) \in C_i$ . On the other hand, recalling the definition of interfacial values (1.15a) and by the regularity assumed in (1.3), we can write

$$b(u_i^+) = b(u_i) + b'(\nu_i) \frac{h}{2} u'_i, \quad |b'(\nu_i)| \leq L_b, \quad \forall i \in \mathbb{Z},$$

so that from (3.21) we deduce

$$T_1^{2,3} = \sum_{i \in \mathbb{Z}} Q_{i+\frac{1}{2}} \frac{h}{2} z''(x_i) b(u_i) h e_i^{p-1} + R_1 \quad (3.25)$$

and we use analogous approximations to (3.15) for the numerical derivatives of the analytical solution to obtain

$$R_1 \leq L_b h \|z''\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \quad (3.26)$$

To conclude the announced result, we need to take into account the contribution of the additional term in the numerical source operator, neglected in the first part of the proof. The second order approximation of cell-averages,

$$u_i = u(x_i) + R_i, \quad R_i = \frac{1}{h} \int_{C_i} \partial_{xx} u(\xi(x)) \frac{(x - x_i)^2}{2} dx, \quad (3.27)$$

for some  $\xi(x) \in C_i$ , together with the usual Taylor's expansion

$$b(u_i) = b(u(x_i)) + b'(\nu_i) R_i, \quad |b'(\nu_i)| \leq L_b, \quad \forall i \in \mathbb{Z}, \quad (3.28)$$

after proceeding according to (3.15), setting  $P_{i+\frac{1}{2}} = \kappa_i - \lambda_i$ ,  $i \in \mathbb{Z}$ , leads to

$$\begin{aligned} \sum_{i \in \mathbb{Z}} z'_i b(u_i) h e_i^{p-1} &= \sum_{i \in \mathbb{Z}} z'(x_i) b(u(x_i)) h e_i^{p-1} + R_2 \\ &+ \sum_{i \in \mathbb{Z}} P_{i+\frac{1}{2}} \frac{h}{2} z''(x_i) b(u_i) h e_i^{p-1} + R_3, \end{aligned} \quad (3.29)$$

with the following estimates for the remainders,

$$R_2 \leq L_b h \|z'\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_{xx} u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right), \quad (3.30)$$

$$R_3 \leq L_b h \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z'''| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right). \quad (3.31)$$

Therefore, up to the bounded remainders (3.26), (3.30) and (3.31), by combining (3.24), (3.25) and (3.29), we have

$$\begin{aligned} &\int_{\mathbb{R}} B(x, u) |e|^{p-1} \operatorname{sgn}(e) dx - T_1^{2,3} - \sum_{i \in \mathbb{Z}} z'_i b(u_i) h e_i^{p-1} \\ &= \sum_{i \in \mathbb{Z}} \int_{C_i} z''(\xi(x_i)) b(u(\xi(x_i))) (x - x_i) |e|^{p-1} \operatorname{sgn}(e) dx \\ &+ \sum_{i \in \mathbb{Z}} \int_{C_i} z'(\xi(x_i)) b'(u(\xi(x_i))) u'(\xi(x_i)) (x - x_i) |e|^{p-1} \operatorname{sgn}(e) dx \quad (3.32) \\ &- \sum_{i \in \mathbb{Z}} (P_{i+\frac{1}{2}} + Q_{i+\frac{1}{2}}) \frac{h}{2} z''(x_i) b(u(x_i)) h e_i^{p-1} \\ &- \sum_{i \in \mathbb{Z}} (P_{i+\frac{1}{2}} + Q_{i+\frac{1}{2}}) \frac{h}{2} z''(x_i) b'(u_i) R_i h e_i^{p-1}, \end{aligned}$$

where again we used (3.27)-(3.28) and  $P_{i+\frac{1}{2}} + Q_{i+\frac{1}{2}} = \lambda_{i+1} - \lambda_i$ ,  $i \in \mathbb{Z}$ .

We introduce an appropriate hypothesis on the slope *limiter* (3.2)-(3.3), as discussed in Section 1.3, namely an additional property for its coefficients,

$$\exists \Lambda_0 > 0 \quad \text{such that} \quad \lambda_{i+1} - \lambda_i \geq \Lambda_0, \quad \forall i \in \mathbb{Z}. \quad (3.33)$$

This condition and general properties of the numerical functions allow us to rewrite the difference between first and third term in the right-hand side of (3.32) in integral form, to conclude that

$$\begin{aligned} &\sum_{i \in \mathbb{Z}} \int_{C_i} z''(\xi(x_i)) b(u(\xi(x_i))) (x - x_i) |e|^{p-1} \operatorname{sgn}(e) dx \\ &- \sum_{i \in \mathbb{Z}} (P_{i+\frac{1}{2}} + Q_{i+\frac{1}{2}}) \frac{h}{2} z''(x_i) b(u(x_i)) h e_i^{p-1} \end{aligned}$$

$$\begin{aligned}
&\leq h \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |(z''b(u))'| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right) \\
&\leq L_b h \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |z'''| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right) \\
&+ L_b h \|z''\|_{L^\infty} \sum_{i \in \mathbb{Z}} \left( \int_{C_i} |\partial_x u| dx \right) \left( \int_{C_i} |e|^{p-1} dx \right).
\end{aligned}$$

An analogous estimate is proved for the difference between second and fourth term in the right-hand side of (3.32), which also involves the second derivative of the analytical solution, similarly to the remainder (3.30).

We apply to the previous computations the arguments in the conclusion of Lemma 2.2, to obtain (3.16) with  $C = C(L_B, K_B, L_b, \|z'\|_{L^\infty}, \|z''\|_{L^\infty})$ .  $\square$

### 3.3 Proof of Theorem 1.3 and Theorem 1.4

With the stability estimate (3.12) and the consistency estimate (3.16), we proceed as in Section 2.3 to conclude the second order error estimate (1.23).

The proof of Theorem 1.4 is obtained by using the main tools introduced for proving the results of Section 2 and Section 3.

Because of the consistency hypotheses (1.11a)-(1.18), the same techniques as in Lemma 2.2 extend to the numerical source operator (1.17), while we apply the arguments formulated in Lemma 3.3 to deduce stability estimates.

## 4 Remarks and numerical evidence

The principal issue in the proofs of Theorem 1.2 and Theorem 1.3 is to establish the consistency estimates (2.10) and (3.16) respectively, in particular to show the convergence of the numerical source operators towards the analytical source term (1.3) from the relations (2.17) and (3.25).

We note that, due to the introduction of piecewise linear reconstructions of the function  $z$ , the differences of interfacial values approximate the second order derivative and the upwind part of the discretization (1.16) “overtakes” the desired result; an additional term is thus needed to recover the first order derivatives in the Taylor’s expansion of the source term. Moreover, some restrictions on the definition of the slope *limiter* are also required, to guarantee the occurrence of suitable error estimates (refer also to [7], [23] and [24]). Without these assumptions, only suboptimal results are derived (see [19] and [20], for instance).

These considerations can also be justified numerically: the tables above reproduce the convergence rates observed when the Upwind Interface Source



	$\ e(t)\ _{L^1}$		$\ e(t)\ _{L^2}$		$\ e(t)\ _{L^\infty}$	
	Error	Rate	Error	Rate	Error	Rate
50	0.323743E-02		0.772868E-02		0.368811E-01	
100	0.816610E-03	1.987	0.270992E-02	1.512	0.184893E-01	0.996
200	0.207343E-03	1.982	0.951254E-03	1.511	0.921217E-02	1.001
400	0.516765E-04	1.990	0.336009E-03	1.508	0.461648E-02	0.999
800	0.128919E-04	1.993	0.118745E-03	1.506	0.231075E-02	0.999
1600	0.321149E-05	1.995	0.419888E-04	1.505	0.115659E-02	0.999

Table 1

	$\ e(t)\ _{L^1}$		$\ e(t)\ _{L^2}$		$\ e(t)\ _{L^\infty}$	
	Error	Rate	Error	Rate	Error	Rate
50	0.129563E-02		0.148671E-02		0.324608E-02	
100	0.326417E-03	1.989	0.368537E-03	2.012	0.814699E-03	1.994
200	0.819300E-04	1.992	0.917495E-04	2.009	0.203996E-03	1.996
400	0.205217E-04	1.993	0.228873E-04	2.007	0.509962E-04	1.997
800	0.513540E-05	1.995	0.571567E-05	2.006	0.127479E-04	1.998
1600	0.128447E-05	1.996	0.142815E-05	2.005	0.318655E-05	1.998

Table 2

method illustrated in this paper is applied to the simplified problem

$$\partial_t u = z'(x), \quad u(0, x) = u_0(x),$$

with  $z(x) = \sin(\pi * x)$ ,  $x \in [0, 1]$ , for which an analytical solution is available to make direct comparisons,  $u(t, x) = u_0(x) + z'(x) t$ .

The results plotted correspond to the discretization (1.16), for the standard *VanLeer limiter*, with a simple TVD reconstruction (see [34]) in Table 1 and with an appropriate ENO reconstruction (see [12]) in Table 2.

The problems just discussed do not arise in the case of discretization (1.17), for which stronger consistency hypotheses are made, to compensate reduced regularity of the reconstructions.

Some classical convergence results for numerical approximations of hyperbolic conservation laws are presented in [28], [37], [11], [6] and its references. Further applications of these methods to different situations are proposed in [3] and [4].

Although the question of preserving stationary states at the discrete level is only handled rigorously for discretizations of the first order (refer to [30]), the numerical results obtained for the Saint-Venant system indicate that the

discretization (1.16) exactly simulates simple equilibria (refer to [16]). As far as we know, similar issues are only addressed in [9] and [21].

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