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► **To cite this version:**

Alexandre Jollivet. On inverse scattering at high energies for the multidimensional relativistic Newton equation in a long range electromagnetic field. 2013. hal-00922825

HAL Id: hal-00922825

<https://hal.science/hal-00922825>

Preprint submitted on 30 Dec 2013

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On inverse scattering at high energies for the multidimensional relativistic Newton equation in a long range electromagnetic field

Alexandre Jollivet*

December 30, 2013

Abstract

We define scattering data for the relativistic Newton equation in an electric field $-\nabla V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $n \geq 2$, and in a magnetic field $B \in C^1(\mathbb{R}^n, A_n(\mathbb{R}))$ that decay at infinity like $r^{-\alpha-1}$ for some $\alpha \in (0, 1]$, where $A_n(\mathbb{R})$ is the space of $n \times n$ antisymmetric matrices. We provide estimates on the scattering solutions and on the scattering data and we prove, in particular, that the scattering data at high energies uniquely determine the short range part of $(\nabla V, B)$ up to the knowledge of the long range tail of $(\nabla V, B)$. The Born approximation at fixed energy of the scattering data is also considered. We then change the definition of the scattering data to study their behavior in other asymptotic regimes. This work generalizes [Jollivet, 2007] where a short range electromagnetic field was considered.

1 Introduction

Consider the multidimensional relativistic Newton equation in an external and static electromagnetic field:

$$\begin{aligned} \dot{p}(t) &= F(x(t), \dot{x}(t)) := -\nabla V(x(t)) + \frac{1}{c} B(x(t)) \dot{x}(t), & (1.1) \\ p(t) &= \frac{\dot{x}(t)}{\sqrt{1 - \frac{|\dot{x}(t)|^2}{c^2}}}, \quad x(t) \in \mathbb{R}^n, \quad t \in \mathbb{R}, \quad n \geq 2, \end{aligned}$$

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where $\dot{x}(t) = \frac{d}{dt}x(t)$, and where $V \in C^2(\mathbb{R}^n, \mathbb{R})$, $B(x)$ is the $n \times n$ real antisymmetric matrix with elements $B_{i,k}$, $1 \leq i, k \leq n$, and where B satisfies the closure condition

$$\frac{\partial}{\partial x_i} B_{k,m}(x) + \frac{\partial}{\partial x_m} B_{i,k}(x) + \frac{\partial}{\partial x_k} B_{m,i}(x) = 0, \quad (1.2)$$

for $x \in \mathbb{R}^n$ and for $i, k, m = 1 \dots n$. The constant c is positive, and for $\sigma \in (0, +\infty)$ we will denote by $\mathcal{B}(0, \sigma)$ (resp. $\overline{\mathcal{B}}(0, \sigma)$) the open (resp. closed) Euclidean Ball of center 0 and radius σ .

When $n = 3$ the equation (1.1) is the equation of motion of a relativistic particle of mass $m = 1$ and charge $e = 1$ in an external electromagnetic field described by (V, B) (see [2] and, for example, [9, Section 17]). In this equation, x , \dot{x} , p denote the position, the velocity and the impulse of the particle respectively, and t is the time, and c is the speed of light.

We also assume throughout this paper that F satisfies the following conditions

$$F = F^l + F^s, \quad (1.3)$$

where $F^l(x, v) := -\nabla V^l(x) + \frac{1}{c} B^l(x)v$, $F^s(x, v) = -\nabla V^s(x) + \frac{1}{c} B^s(x)v$ and $(V^l, V^s) \in (C^2(\mathbb{R}^n, \mathbb{R}))^2$, $(B^l, B^s) \in (C^1(\mathbb{R}^n, A_n(\mathbb{R})))^2$, and where

$$|\partial_x^{j_1} V^l(x)| \leq \beta_{|j_1|}^l (1 + |x|)^{-(\alpha + |j_1|)}, \quad |\partial_x^{j_2} B_{i,k}^l(x)| \leq \beta_{|j_2|+1}^l (1 + |x|)^{-(\alpha + |j_2|+1)}, \quad (1.4)$$

$$|\partial_x^{j_1} V^s(x)| \leq \beta_{|j_1|+1}^s (1 + |x|)^{-(\alpha + 1 + |j_1|)}, \quad |\partial_x^{j_2} B_{i,k}^s(x)| \leq \beta_{|j_2|+2}^s (1 + |x|)^{-(\alpha + |j_2|+2)}, \quad (1.5)$$

for $x \in \mathbb{R}^n$, $|j_1| \leq 2$ and $|j_2| \leq 1$ and for some $\alpha \in (0, 1]$ (here j is the multiindex $j = (j^1, \dots, j^n) \in (\mathbb{N} \cup \{0\})^n$, $|j| = \sum_{m=1}^n j^m$, and β_m^l and $\beta_{m'}^s$ are positive real constants for $m = 0, 1, 2$ and for $m' = 1, 2, 3$). Note that the assumption $0 < \alpha \leq 1$ includes the decay rate of a Coulombian potential at infinity. Indeed for a Coulombian potential $V^l(x) = \frac{1}{|x|}$ (and no magnetic field $B^l \equiv 0$), estimates (1.4) are satisfied uniformly for $|x| > \varepsilon$ and $\alpha = 1$ for any $\varepsilon > 0$. Although our electromagnetic fields are assumed to be smooth on the entire space, our study may provide interesting results even in presence of singularities.

For equation (1.1) the energy

$$E = c^2 \sqrt{1 + \frac{|p(t)|^2}{c^2}} + V(x(t)) \quad (1.6)$$

is an integral of motion.

For $\sigma \in [0, +\infty)$ set

$$\mu(\sigma) = \sqrt{\frac{2\sigma}{\frac{\sigma}{c^2} + \sqrt{\frac{\sigma^2}{c^4} + 4}}} \quad (1.7)$$

and $\mu^l = \mu(2^8 \alpha^{-1} n^2 \max(\beta_1^l, \beta_2^l))$. Then under conditions (1.4) the following is valid (see Lemma 2.1 given in the next Section): for any $v \in \mathcal{B}(0, c)$, $|v| \geq \mu^l$, there exists a unique solution $z_{\pm}(v, \cdot)$ of the equation

$$\begin{aligned} \dot{p}(t) &= F^l(z(t), \dot{z}(t)), \\ p(t) &= \frac{\dot{z}(t)}{\sqrt{1 - \frac{|\dot{z}(t)|^2}{c^2}}}, \quad t \in \mathbb{R}, \end{aligned} \quad (1.8)$$

so that

$$\dot{z}_{\pm}(v, t) - v = o(1), \text{ as } t \rightarrow \pm\infty, \quad z_{\pm}(v, 0) = 0,$$

and

$$|z_{\pm}(v, t) - tv| \leq \frac{2^{\frac{9}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v|} |t| \text{ for } t \in \mathbb{R}.$$

When $F^l \equiv 0$ then β_1^l, β_2^l and μ^l can be arbitrary close to 0, and we have $z_{\pm}(v, t) = tv$ for $(t, v) \in \mathbb{R} \times \mathcal{B}(0, c)$, $v \neq 0$.

Then under conditions (1.4) and (1.5), the following is valid: for any $(v_-, x_-) \in \mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l) \times \mathbb{R}^n$, the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = z_-(v_-, t) + x_- + y_-(t), \quad (1.9)$$

where $|\dot{y}_-(t)| + |y_-(t)| \rightarrow 0$, as $t \rightarrow -\infty$; in addition for almost any $(v_-, x_-) \in \mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l) \times \mathbb{R}^n$,

$$x(t) = z_+(v_+, t) + x_+ + y_+(t), \quad (1.10)$$

for a unique $(v_+, x_+) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, where $|v_+| = |v_-| \geq \mu^l$ by conservation of the energy (1.6), and where $v_+ =: a(v_-, x_-)$, $x_+ =: b(v_-, x_-)$, and $|\dot{y}_+(t)| + |y_+(t)| \rightarrow 0$, as $t \rightarrow +\infty$. A solution x of (1.1) that satisfies (1.9) and (1.10) for some (v_-, x_-) , $v_- \neq 0$, is called a scattering solution.

We call the map $S : (\mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l)) \times \mathbb{R}^n \rightarrow (\mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l)) \times \mathbb{R}^n$ given by the formulas

$$v_+ = a(v_-, x_-), \quad x_+ = b(v_-, x_-), \quad (1.11)$$

the scattering map for the equation (1.1). In addition, $a(v_-, x_-)$, $b(v_-, x_-)$ are called the scattering data for the equation (1.1), and we define

$$a_{sc}(v_-, x_-) = a(v_-, x_-) - v_-, \quad b_{sc}(v_-, x_-) = b(v_-, x_-) - x_-. \quad (1.12)$$

Our definition of the scattering map is derived from constructions given in [4, 1]. We refer the reader to [4, 1] and references therein for the forward classical scattering theory.

By $\mathcal{D}(S)$ we denote the set of definition of S . Under the conditions (1.4) and (1.5) the map $S : \mathcal{D}(S) \rightarrow (\mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l)) \times \mathbb{R}^n$ is continuous, and $\text{Mes}((\mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l)) \times \mathbb{R}^n \setminus \mathcal{D}(S)) = 0$ for the Lebesgue measure on $\mathcal{B}(0, c) \times \mathbb{R}^n$. In addition the map S is uniquely determined by its restriction to $\mathcal{M}(S) = \mathcal{D}(S) \cap \mathcal{M}$ and by F^l , where $\mathcal{M} = \{(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n \mid v_- \neq 0, v_- \cdot x_- = 0\}$. (Indeed if $x(t)$ is a solution of (1.1) then $x(t + t_0)$ is also a solution of (1.1) for any $t_0 \in \mathbb{R}$.)

One can imagine the following experimental setting that allows to measure the scattering data without knowing the electromagnetic field (V, B) inside a (a priori bounded) region of interest. First find an electromagnetic field (V^l, B^l) that generates the same long range effects as (V, B) does. Then compute the solutions $z_{\pm}(v, \cdot)$ of equation (1.8). Then for a fixed $(v_-, x_-) \in (\mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l)) \times \mathbb{R}^n$ send a particle far away from the region of interest with a trajectory asymptotic to $x_- + z_-(v_-, \cdot)$ at large and negative times. When the particle escapes any bounded region of the space at finite time, then detect the particle and find $S(v_-, x_-) = (v_+, x_+)$ so that the trajectory of the particle is asymptotic to $x_+ + z_+(v_+, \cdot)$ at large and positive times far away from the bounded region of interest.

In this paper we consider the following inverse scattering problem for equation (1.1):

$$\text{Given } S \text{ and given the long range tail } F^l \text{ of the force } F, \text{ find } F^s. \quad (1.13)$$

The main results of the present work consist in estimates and asymptotics for the scattering data (a_{sc}, b_{sc}) and scattering solutions for the equation (1.1) and in application of these asymptotics and estimates to the inverse scattering problem (1.13) at high energies. Our main results include, in particular, Theorem 1.1 given below that provides the high energies asymptotics of the scattering data and the Born approximation of the scattering data at fixed energy.

Consider

$$T\mathbb{S}^{n-1} := \{(\theta, x) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \mid \theta \cdot x = 0\},$$

and for any $m \in \mathbb{N}$ consider the x-ray transform P defined by

$$Pf(\theta, x) := \int_{-\infty}^{+\infty} f(t\theta + x) dt$$

for any function $f \in C(\mathbb{R}^n, \mathbb{R}^m)$ so that $|f(x)| = O(|x|^{-\tilde{\beta}})$ as $|x| \rightarrow +\infty$ for some $\tilde{\beta} > 1$. For $(\sigma, \tilde{\beta}, r, \tilde{\alpha}) \in (0, +\infty)^2 \times (0, \min(1, 2^{-\frac{3}{2}}c)) \times (0, 1]$, let $\rho_0 = \rho_0(\sigma, r, \tilde{\beta}, \tilde{\alpha})$ be defined as the root of the equation

$$1 = \frac{32n^2 \sqrt{1 - \frac{\rho_0^2}{c^2}} \tilde{\beta} (1 + \sigma + \frac{1}{c}) (1 + \frac{1}{\frac{\rho_0}{2\sqrt{2}} - r})}{\tilde{\alpha} (\frac{\rho_0}{2\sqrt{2}} - r) r (1 - r)^{\tilde{\alpha} + 2}}, \quad \rho_0 \in (2^{\frac{3}{2}}r, c). \quad (1.14)$$

Set

$$\begin{aligned} W(v, x) &:= \int_{-\infty}^0 \left(g(g^{-1}(v) + \int_{-\infty}^{\sigma} F^l(z_-(v, \tau) + x, \dot{z}_-(v, \tau)) d\tau) \right. \\ &\quad \left. - g(g^{-1}(v) + \int_{-\infty}^{\sigma} F^l(z_-(v, \tau), \dot{z}_-(v, \tau)) d\tau) \right) d\sigma \\ &\quad + \int_0^{+\infty} \left(g(g^{-1}(a(v, x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v, x), \tau) + x, \dot{z}_+(a(v, x), \tau)) d\tau) \right. \\ &\quad \left. - g(g^{-1}(a(v, x)) - \int_{\sigma}^{+\infty} F^l(z_+(a(v, x), \tau), \dot{z}_+(a(v, x), \tau)) d\tau) \right) d\sigma, \end{aligned} \quad (1.15)$$

for $(v, x) \in \mathcal{D}(S)$, where

$$g(x) := \frac{x}{\sqrt{1 + \frac{|x|^2}{c^2}}} \text{ for } x \in \mathbb{R}^n, \text{ and } g^{-1}(x') := \frac{x'}{\sqrt{1 - \frac{|x'|^2}{c^2}}} \text{ for } x' \in \mathcal{B}(0, c). \quad (1.16)$$

Then we have the following results.

Theorem 1.1. *Let $(\theta, x) \in T\mathbb{S}^{n-1}$. Under conditions (1.4) and (1.5) the following limits are valid*

$$\lim_{\substack{\rho \rightarrow c \\ \rho < c}} \frac{\rho}{\sqrt{1 - \frac{\rho^2}{c^2}}} a_{sc}(\rho\theta, x) = \int_{-\infty}^{+\infty} F(\tau\theta + x, c\theta) d\tau, \quad (1.17)$$

$$\begin{aligned} \lim_{\substack{\rho \rightarrow c \\ \rho < c}} \frac{\rho^2}{\sqrt{1 - \frac{\rho^2}{c^2}}} (b_{sc}(\rho\theta, x) - W(\rho\theta, x)) &= \int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\tau\theta + x, c\theta) d\tau d\sigma \\ &\quad - \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\tau\theta + x, c\theta) d\tau d\sigma + PV^s(\theta, x)\theta. \end{aligned} \quad (1.18)$$

In addition

$$\begin{aligned} & \left| \frac{\rho}{\sqrt{1 - \frac{\rho^2}{c^2}}} a_{sc}(\rho\theta, x) - \int_{-\infty}^{+\infty} F(\tau\theta + x, \rho\theta) d\tau \right| \\ & \leq \beta^2 \sqrt{1 - \frac{\rho^2}{c^2}} \frac{640n^4 \rho \left(\frac{r}{c} + |x| + 1\right) \left(\frac{1}{c} + 1\right) \left(1 + \frac{1}{\frac{\rho}{2\sqrt{2}} - r}\right)^2}{\alpha^2 \left(\frac{\rho}{2\sqrt{2}} - r\right)^2 (1 - r)^{2\alpha + 3}}, \end{aligned} \quad (1.19)$$

$$\begin{aligned}
& \left| \frac{\rho^2}{\sqrt{1 - \frac{\rho^2}{c^2}}} (b_{sc}(\rho\theta, x) - W(\rho\theta, x)) - \int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\tau\theta + x, \rho\theta) d\tau d\sigma \right. \\
& \quad \left. + \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\tau\theta + x, \rho\theta) d\tau d\sigma - \int_{-\infty}^{+\infty} V^s(\sigma\theta + x) d\sigma \frac{\rho^2\theta}{c^2} \right| \\
& \leq \beta^2 \sqrt{1 - \frac{\rho^2}{c^2}} \frac{464n^4 \rho^2 \left(\frac{r}{c} + |x| + 1\right) \left(\frac{1}{c} + 1\right) \left(1 + \frac{1}{\frac{\rho}{2\sqrt{2}} - r}\right)^2}{\alpha^2(\alpha + 1) \left(\frac{\rho}{2\sqrt{2}} - r\right)^3 (1 - r)^{2\alpha+2}}, \quad (1.20)
\end{aligned}$$

for $r \in (0, \min(1, 2^{-\frac{3}{2}}c))$ and for $\rho \in (\rho_0(|x|, r, \beta, \alpha), c)$, where $\beta = \max(\beta_1^l, \beta_2^l, \beta_2^s, \beta_3^s)$.

The vector W defined by (1.15) is known from the scattering data and from F^l . Then from (1.17) and [5, Proposition 1.1] and inversion of the x-ray transform (see [12, 3, 10, 11]) it follows that F^s can be reconstructed from a_{sc} . From (1.18) one can prove the following statements (see [5, Proposition 1.2] and subsequent comments therein): The potential V^s is uniquely determined up to its radial part by b_{sc} ; The magnetic field B^s can be reconstructed from b_{sc} when $n \geq 3$, and up to its radial part when $n = 2$.

The estimates (1.19) and (1.20) also give the asymptotics of a_{sc} , b_{sc} , when the parameters α , n , ρ , θ and x are fixed and β decreases to 0. In that regime the leading term of $a_{sc}(\rho\theta, x)$ and $b_{sc}(\rho\theta, x) - W(\rho\theta, x)$ for $(\theta, x) \in T\mathbb{S}^{n-1}$ and for $\rho \in (\rho_0(|x|, r, \beta, \alpha), c)$ are given by

$$\begin{aligned}
& \frac{\sqrt{1 - \frac{\rho^2}{c^2}}}{\rho} \int_{-\infty}^{+\infty} F(\tau\theta + x, \rho\theta) d\tau, \quad (1.21) \\
& \frac{\sqrt{1 - \frac{\rho^2}{c^2}}}{\rho^2} \left(\int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\tau\theta + x, \rho\theta) d\tau d\sigma \right. \\
& \quad \left. - \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\tau\theta + x, \rho\theta) d\tau d\sigma + \int_{-\infty}^{+\infty} V^s(\sigma\theta + x) d\sigma \frac{\rho^2\theta}{c^2} \right), \quad (1.22)
\end{aligned}$$

respectively. Therefore Theorem 1.1 gives the Born approximation for the scattering data at fixed energy when the electromagnetic field is sufficiently weak, and one can prove the following statements (see [5, Remark 1.1]): The force F^s can be reconstructed from the Born approximation (1.21) of a_{sc} at fixed energy; V^s can be reconstructed from the Born approximation (1.22) of b_{sc} at fixed energy; B^s can be reconstructed from (1.22) when $n \geq 3$, and up to its radial part when $n = 2$.

Theorem 1.1 is a generalization of [5, formulas (1.7a), (1.7b), (1.8a) and (1.8b)] where inverse scattering for the relativistic multidimensional Newton equation was studied in the short range case ($F^l \equiv 0$). The formulas

[5, (1.7b) and (1.8b)] also provide the approximation of the scattering data $(a_{sc}(v_-, x_-), b_{sc}(v_-, x_-))$ for the short range case ($F^l \equiv 0$) when the parameters α , n , v_- and β are fixed and $|x_-| \rightarrow +\infty$. Such an asymptotic regime is not covered by Theorem 1.1. Therefore we shall modify in Section 3 the definition of the scattering map to study these modified scattering data in the following three asymptotic regimes: at high energies, Born approximation at fixed energy, and when the parameters α , n , v_- and β are fixed and $|x_-| \rightarrow +\infty$.

Inverse scattering at high energies for the nonrelativistic multidimensional Newton equation in a short range potential V was first studied by [11]. Then inverse scattering at high energies for this latter equation in a long range potential V was studied by [6]. We develop the approach of [11, 6] to obtain our results.

For inverse scattering at fixed energy for the multidimensional Newton equation, see for example [7] and references therein. For the inverse scattering problem in quantum mechanics, see references given in [5].

Our paper is organized as follows. In Section 2 we transform the differential equation (1.1) with initial conditions (1.9) in an integral equation which takes the form $y_- = A(y_-)$. Then we study the operator A on a suitable space (Lemma 2.2) and we give estimates for the deflection $y_-(t)$ in (1.9) and for the scattering data $a_{sc}(v_-, x_-), b_{sc}(v_-, x_-)$ (Theorem 2.4). We prove Theorem 1.1. Note that we work with small angle scattering compared to the dynamics generated by F^l through the “free” solutions $z_-(v_-, t)$: In particular, the angle between the vectors $\dot{x}(t) = \dot{z}_-(v_-, t) + \dot{y}_-(t)$ and $\dot{z}_-(v_-, t)$ goes to zero when the parameters β , α , n , $v_-/|v_-|$, x_- are fixed and $|v_-|$ increases. In Section 3 we change the definition of the scattering map so that one can obtain for the modified scattering data $(\tilde{a}_{sc}(v_-, x_-), \tilde{b}_{sc}(v_-, x_-))$ their approximation at high energies, or their Born approximation at fixed energy, or their approximation when the parameters α , n , v_- and β are fixed and $|x_-| \rightarrow +\infty$ (Theorem 3.3, Corollary 3.4). Sections 4, 5, 6 and 7 are devoted to proofs of our Theorems and Lemmas.

2 Scattering solutions

2.1 Integral equation

First we need the following Lemma 2.1 that generalizes the statements given in the Introduction on the existence of peculiar solutions z_{\pm} of the equation (1.8).

Lemma 2.1. *Assume conditions (1.4). Let $v \in \mathcal{B}(0, c)$, $v \neq 0$, and $x \in \mathbb{R}^n$ so that $v \cdot x = 0$. Let $(w, q) \in \mathcal{B}(0, c) \times \mathcal{B}(0, 1)$ so that*

$$|v| = |w|, \text{ and } |w - v| \leq \frac{|v|}{2^{\frac{5}{2}}}. \quad (2.1)$$

Assume that

$$\frac{2^8 n^2 \max(\beta_1^l, \beta_2^l) \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v|^2 (1 + \frac{|x|}{\sqrt{2}} - |q|)^\alpha} \leq 1. \quad (2.2)$$

Then there exists a unique solution $z_\pm(w, x + q, \cdot)$ of the equation (1.8) so that

$$\dot{z}_\pm(w, x + q, t) - w = o(1) \text{ as } t \rightarrow \pm\infty, \quad z_\pm(w, x + q, 0) = x + q, \quad (2.3)$$

and

$$\sup_{\mathbb{R}} |\dot{z}_\pm(w, x + q, \cdot) - w| \leq \frac{2^{\frac{9}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v| (1 + \frac{|x|}{\sqrt{2}} - |q|)^\alpha}. \quad (2.4)$$

A proof of Lemma 2.1 is given in Section 4. For the rest of this Section we set

$$\mu^l := \mu(2^8 \alpha^{-1} n^2 \max(\beta_1^l, \beta_2^l)), \quad (2.5)$$

$$z_\pm(v, t) := z_\pm(v, 0, t) \text{ for } t \in \mathbb{R}, \text{ when } |v| \geq \mu^l, \quad (2.6)$$

$$\beta_2 := \max(\beta_2^l, \beta_2^s), \quad (2.7)$$

where the function μ is defined by (1.7).

For the rest of the text $H(f(\tau), \dot{f}(\tau))$ is shortened to $H(f)(\tau)$ for any $(f, \tau) \in C^1(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}$, where H stands for F , F^s or F^l .

Let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$ and $|v_-| \geq \mu^l$. Then the function y_- in (1.9) satisfies the integral equation $y_- = A(y_-)$ where

$$A(f)(t) = \int_{-\infty}^t \dot{A}(f)(\sigma) d\sigma, \quad (2.8)$$

$$\begin{aligned} \dot{A}(f)(t) &= g(g^{-1}(v_-) + \int_{-\infty}^t F(z_-(v_-, \cdot) + x_- + f)(\tau) d\tau) \\ &\quad - g(g^{-1}(v_-) + \int_{-\infty}^t F^l(z_-(v_-, \cdot))(\tau) d\tau), \end{aligned} \quad (2.9)$$

for $t \in \mathbb{R}$ and for $f \in C^1(\mathbb{R}, \mathbb{R}^n)$, $\sup_{(-\infty, 0]} (|f| + |\dot{f}|) < \infty$. We have $A(f) \in C^2(\mathbb{R}, \mathbb{R}^n)$ for $f \in C^1(\mathbb{R}, \mathbb{R}^n)$ so that $\sup_{(-\infty, 0]} (|f| + |\dot{f}|) < \infty$ (see (4.2), (4.6) and (4.7)).

For $r \in (0, 1)$ and for $|v_-| \geq \mu^l$, $|v_-| \geq 2^{\frac{3}{2}}r$, we introduce the following complete metric space M_{r,v_-} endowed with the following norm $\|\cdot\|$

$$M_{r,v_-} = \{f \in C^1(\mathbb{R}, \mathbb{R}^n) \mid \sup_{\mathbb{R}} |\dot{z}_-(v_-, \cdot) + \dot{f}| \leq c, \|f\| \leq r\}, \quad (2.10)$$

$$\|f\| = \max \left(\sup_{t \in (-\infty, 0)} \max \left(1, \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right) |\dot{f}(t)| \right), \sup_{(0, +\infty)} |\dot{f}|, \sup_{(-\infty, 0)} |f| \right). \quad (2.11)$$

The space M_{r,v_-} is a convex subset of $C^1(\mathbb{R}, \mathbb{R}^n)$. Then we have the following estimate and contraction estimate for the map A restricted to M_{r,v_-} .

Lemma 2.2. *Let $(v_-, x_-) \in (\mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l)) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$, and let $r \in (0, \min(\frac{|v_-|}{2\sqrt{2}}, 1))$. When*

$$\frac{2^{\frac{5}{2}}n \max(\beta_1^l, \beta_2) \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 (1-r)^{\alpha+1}} \leq 1, \quad (2.12)$$

then the following estimates are valid

$$\begin{aligned} \|A(f)\| &\leq \lambda_1(n, \alpha, \beta_1^l, \beta_2, |x_-|, |v_-|, r) \quad (2.13) \\ &:= \frac{4n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(|x_-| + r) + 1)\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) (1-r)^{\alpha+1}}, \end{aligned}$$

and

$$\begin{aligned} \|A(f_1) - A(f_2)\| &\leq \lambda_2(n, \alpha, \beta_2, \beta_3^s, |v_-|, r) \|f_1 - f_2\|, \quad (2.14) \\ \lambda_2(n, \alpha, \beta_2, \beta_3^s, |v_-|, r) &:= \frac{4n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\frac{\beta_1^l + \beta_2}{c} + 2n^{\frac{1}{2}}(\beta_2 + \beta_3^s)\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) (1-r)^{\alpha+2}}, \end{aligned}$$

for $(f, f_1, f_2) \in M_{r,v_-}^3$.

A proof of Lemma 2.2 is given in Section 5.

We also need the following result.

Lemma 2.3. *Let $(v_-, x_-) \in (\mathcal{B}(0, c) \setminus \mathcal{B}(0, \mu^l)) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$, and let $r \in (0, \min(\frac{|v_-|}{2\sqrt{2}}, 1))$. When $y_- \in M_{r,v_-}$ is a fixed point for the map A then $x := z_-(v_-, \cdot) + x_- + y_-$ is a scattering solution for equation (1.1) and*

$$x(t) = z_+(a(v_-, x_-), t) + b(v_-, x_-) + y_+(t), \quad (2.15)$$

for $t \geq 0$, where

$$a(v_-, x_-) := g\left(g^{-1}(v_-) + \int_{-\infty}^{+\infty} F(x)(\tau) d\tau\right), \quad (2.16)$$

$$b(v_-, x_-) := x_- + A(y_-)(0) - y_+(0), \quad (2.17)$$

$$\begin{aligned} y_+(t) = & - \int_t^{+\infty} \left(g(g^{-1}(a(v_-, x_-))) - \int_{\sigma}^{+\infty} F(x)(\tau) d\tau \right. \\ & \left. - g(g^{-1}(a(v_-, x_-))) - \int_{\sigma}^{+\infty} F^l(z_+(a(v_-, x_-), \cdot))(\tau) d\tau \right) d\sigma, \end{aligned} \quad (2.18)$$

for $t \geq 0$.

Lemma 2.3 is proved in Section 4.

2.2 Estimates on the scattering solutions

In this Section our main results consist in estimates and asymptotics for the scattering data (a_{sc}, b_{sc}) and scattering solutions for the equation (1.1).

Theorem 2.4. *Under the assumptions of Lemma 2.3 and when*

$$\frac{24n^2 \sqrt{1 - \frac{|v_-|^2}{c^2}} \max(\beta_1^l, \beta_2) \left(1 + \frac{1}{c}\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) (1-r)^{\alpha+1}} \leq 1, \quad (2.19)$$

then the following estimates are valid:

$$|\dot{y}_-(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(|x_-| + r) + 1)\right)}{(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r - \left(\frac{|v_-|}{2\sqrt{2}} - r\right)t\right)^{\alpha+1}}, \quad (2.20)$$

for $t \leq 0$. In addition

$$|a_{sc}(v_-, x_-)| \leq \frac{8n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)^\alpha} \left(\frac{\beta_1^l}{\alpha} + \frac{\beta_2}{(\alpha + 1) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)}\right), \quad (2.21)$$

$$|b_{sc}(v_-, x_-)| \leq \frac{4n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\frac{r\beta_1^l}{c} + \beta_2(n^{\frac{1}{2}}(4|x_-| + r) + 4)\right)}{\alpha(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 (1-r)^\alpha}, \quad (2.22)$$

$$|\dot{y}_+(t)| \leq \frac{2n^{\frac{3}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(3|x_-| + r) + 3) \right) \sqrt{1 - \frac{|v_-|^2}{c^2}}}{(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r \right) (1 - r + t \left(\frac{|v_-|}{2\sqrt{2}} - r \right))^{\alpha+1}}, \quad (2.23)$$

for $t \geq 0$, and

$$\begin{aligned} & |a_{sc}(v_-, x_-) - \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_{-\infty}^{+\infty} F(\tau v_- + x_-, v_-) d\tau| \\ & \leq \frac{520n^4 \beta^2 \left(1 - \frac{|v_-|^2}{c^2}\right) \left(\frac{r}{c} + |x_-| + 1\right) \left(\frac{1}{c} + 1\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)^2}{\alpha^2 \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 (1 - r)^{2\alpha+3}}, \end{aligned} \quad (2.24)$$

$$\begin{aligned} & \left| b_{sc}(v_-, x_-) - W(v_-, x_-) - \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\tau v_- + x_-, v_-) d\tau d\sigma \right. \right. \\ & \left. \left. - \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\tau v_- + x_-, v_-) d\tau d\sigma + \int_{-\infty}^{+\infty} V^s(\sigma v_- + x_-) d\sigma \frac{v_-}{c^2} \right) \right| \\ & \leq \frac{468n^4 \beta^2 \left(1 - \frac{|v_-|^2}{c^2}\right) \left(\frac{r}{c} + |x_-| + 1\right) \left(\frac{1}{c} + 1\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)^2}{\alpha^2 (\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^3 (1 - r)^{2\alpha+2}}, \end{aligned} \quad (2.25)$$

where $\beta = \max(\beta_1^l, \beta_2, \beta_3^s)$.

Theorem 2.4 is proved in Section 6.

Proof of Theorem 1.1. Let $(\theta, x) \in TS^{n-1}$ and let $(r, \rho) \in (0, \min(1, 2^{-\frac{3}{2}}c)) \times (0, +\infty)$, $\rho > \rho_0(|x|, r, \beta, \alpha)$, where ρ_0 is defined in (1.14). Set $(v_-, x_-) = (\rho\theta, x)$. Then note that

$$\max\left(\lambda_0, \frac{\lambda_1}{r}, \lambda_2, \lambda_3\right) \leq \frac{32n^2 \sqrt{1 - \frac{|v_-|^2}{c^2}} \beta \left(1 + |x_-| + \frac{1}{c}\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) r (1 - r)^{\alpha+2}} < 1, \quad (2.26)$$

where λ_1 and λ_2 are defined in (2.13) and (2.14) respectively, and where

$\lambda_0 := \frac{2^8 n^2 \max(\beta_1^l, \beta_2) \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha |v_-|^2}$ and λ_3 is the left-hand side of (2.19) (we also used (1.14)). From estimate (2.26) and Lemma 2.2 we obtain: $|v_-| \geq \mu^l$ (see (2.5)), A is a contraction in M_{r, v_-} , and Theorem 2.4 holds for the unique fixed point $y_- \in M_{r, v_-}$ of A . The estimate (2.24) and (2.25) hold and they provide the estimates (1.19) and (1.20), which proves Theorem 1.1. \square

2.3 Motivations for changing the definition of the scattering map

For a solution x at a nonzero energy for equation (1.1) we say that it is a scattering solution when there exists $\varepsilon > 0$ so that $1 + |x(t)| \geq \varepsilon(1 + |t|)$ for $t \in \mathbb{R}$ (see [1]). In the Introduction and in the previous subsections we choose to parametrize the scattering solutions of equation (1.1) by the solutions $z_{\pm}(v, \cdot)$ of the equation (1.8) (see the asymptotic behaviors (1.9) and (1.10)), and then to formulate the inverse scattering problem (1.13) using this parametrization. We obtain the estimates (1.19) and (1.20) that provide the high energies asymptotics and the Born approximation at fixed energy of the scattering data. However these estimates do not provide the asymptotics of the scattering data (a_{sc}, b_{sc}) when the parameters α , n , ρ and β are fixed and $|x| \rightarrow +\infty$. Motivated by this disadvantage, in the next section we modify the definition of the scattering map given in the Introduction so that one can obtain a result on this asymptotic regime.

3 A modified scattering map

3.1 Changing the parametrization of the scattering solutions

We set

$$\mu_{\sigma}^l := \mu(2^8 \alpha^{-1} n^2 \max(\beta_1^l, \beta_2^l) (1 + \frac{\sigma}{\sqrt{2}})^{-\alpha}), \quad (3.1)$$

for $\sigma \geq 0$, where the function μ is defined by (1.7).

Under conditions (1.4) and (1.5), the following is valid: for any $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$ so that $|v_-| > \mu_{|x_-|}^l$, then the equation (1.1) has a unique solution $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ such that

$$x(t) = z_-(v_-, x_-, t) + y_-(t), \quad (3.2)$$

where $\dot{y}_-(t) \rightarrow 0$, $y_-(t) \rightarrow 0$, as $t \rightarrow -\infty$;

In addition the function y_- in (3.2) satisfies the integral equation $y_- = \mathcal{A}(y_-)$ where

$$\mathcal{A}(f)(t) = \int_{-\infty}^t \dot{\mathcal{A}}(f)(\sigma) d\sigma, \quad (3.3)$$

$$\begin{aligned} \dot{\mathcal{A}}(f)(t) &= g(g^{-1}(v_-) + \int_{-\infty}^t F(z_-(v_-, x_-, \cdot) + f)(\tau) d\tau) \\ &\quad - g(g^{-1}(v_-) + \int_{-\infty}^t F^l(z_-(v_-, x_-, \cdot))(\tau) d\tau), \end{aligned} \quad (3.4)$$

for $t \in \mathbb{R}$ and for $f \in C^1(\mathbb{R}, \mathbb{R}^n)$, $\sup_{(-\infty, 0]}(|f| + |\dot{f}|) < \infty$. We remind that $H(f(\tau), \dot{f}(\tau))$ is shortened to $H(f)(\tau)$ for any $(f, \tau) \in C^1(\mathbb{R}, \mathbb{R}^n) \times \mathbb{R}$ above and in the rest of the text, where H stands for F , F^s or F^l .

We have $\mathcal{A}(f) \in C^2(\mathbb{R}, \mathbb{R}^n)$ for $f \in C^1(\mathbb{R}, \mathbb{R}^n)$ so that $\sup_{(-\infty, 0]}(|f| + |\dot{f}|) < \infty$ (see (4.2), (4.6) and (4.7)).

For $r \in (0, \min(1, \frac{|v_-|}{2^{\frac{3}{2}}}))$, we introduce the following metric space M_{r, v_-, x_-} endowed with the following norm $\|\cdot\|_*$

$$M_{r, v_-, x_-} = \{f \in C^1(\mathbb{R}, \mathbb{R}^n) \mid \sup_{\mathbb{R}} |\dot{z}_-(v_-, x_-, \cdot) + \dot{f}| \leq c, \|f\|_* \leq r\}, \quad (3.5)$$

$$\|f\|_* = \max\left(\sup_{t \in (-\infty, 0)} \max\left(1, \left(1-r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)|t|\right)|\dot{f}(t)|, \sup_{(0, +\infty)} |\dot{f}|, \sup_{(-\infty, 0)} |f|\right).\right) \quad (3.6)$$

The space M_{r, v_-, x_-} is a convex subset of $C^1(\mathbb{R}, \mathbb{R}^n)$. We study the map \mathcal{A} defined by (3.3) and (3.4) on the metric space M_{r, v_-, x_-} . Set

$$\tilde{k}(v_-, x_-, f) := g(g^{-1}(v_-) + \int_{-\infty}^{+\infty} F(z_-(v_-, x_-, \cdot) + f)(t) dt), \quad (3.7)$$

for $f \in M_{r, v_-, x_-}$. For the rest of the section we also set $\beta_2 = \max(\beta_2^l, \beta_2^s)$.

The following Lemma 3.1 is the analog of Lemma 2.2.

Lemma 3.1. *Let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$, $|v_-| > \mu_{|x_-|}^l$, and let $r \in (0, \min(1, \frac{|v_-|}{2^{\frac{3}{2}}}))$. When*

$$\frac{2^{\frac{3}{2}} n \max(\beta_1^l, \beta_2) \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(1 + \frac{1}{1 + \frac{|x_-|}{\sqrt{2}} - r}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)^\alpha} \leq 1, \quad (3.8)$$

then the following estimates are valid

$$\|\mathcal{A}(f)\|_* \leq \tilde{\lambda}_1(n, \alpha, \beta_1^l, \beta_2, |x_-|, |v_-|, r), \quad (3.9)$$

$$\tilde{\lambda}_1 := \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{\beta_1^l r}{c} + 4\beta_2(n^{\frac{1}{2}} r + 1)\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r} + \frac{1}{1 - r + \frac{|x_-|}{\sqrt{2}}}\right),$$

and

$$\|\mathcal{A}(f_1) - \mathcal{A}(f_2)\|_* \leq \tilde{\lambda}_2(n, \alpha, \beta_2, \beta_3^s, |x_-|, |v_-|, r) \|f_1 - f_2\|_*, \quad (3.10)$$

$$\tilde{\lambda}_2 := \frac{4n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{\beta_1^l}{c} + 2\beta_2 n^{\frac{1}{2}} + \frac{\beta_2 + 2\beta_3 n^{\frac{1}{2}}}{1-r + \frac{|x_-|}{\sqrt{2}}}\right)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r} + \frac{1}{1 - r + \frac{|x_-|}{\sqrt{2}}}\right).$$

for $(f, f_1, f_2) \in M_{r, v_-, x_-}^3$. In addition we have

$$|\tilde{k}(v_-, x_-, f) - v_-| \leq \frac{8n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)^\alpha} \left(\frac{\beta_1^l}{\alpha} + \frac{\beta_2}{(\alpha + 1) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)}\right), \quad (3.11)$$

for $f \in M_{r, v_-, x_-}$.

Lemma 3.1 is proved in Section 5.

Let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$ and let $r \in (0, \min(\frac{1}{2}, \frac{|v_-|}{2^{\frac{3}{2}}}))$.

Under the following condition

$$\frac{48n^2(c^{-1} + 1) \max(\beta_1^l, \beta_2) \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) \left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \left(1 + \frac{1}{\frac{|v_-|}{2^{\frac{3}{2}}} - r}\right) \leq 1, \quad (3.12)$$

then condition (2.2) is satisfied for any $q \in \overline{\mathcal{B}(0, \frac{1}{2})}$, and condition (3.8) is also satisfied. In particular, $|v_-| \geq \mu_{|x_-|}^l$, and when $y_- \in M_{r, v_-, x_-}$ is a fixed point of the operator \mathcal{A} then $x := z_-(v_-, x_-, \cdot) + y_-$ is a scattering solution of (1.1) (in the sense given in Section 2.3). We set

$$\tilde{a}(v_-, x_-) := \tilde{k}(v_-, x_-, y_-). \quad (3.13)$$

By conservation of energy $|\tilde{a}(v_-, x_-)| = \lim_{t \rightarrow \infty} |\dot{x}(t)| = |v_-|$. From (3.11) and condition (3.12) it follows that $|\tilde{a}(v_-, x_-) - v_-| \leq 2^{-\frac{5}{2}}|v_-|$, and we can consider the free solution $z_+(\tilde{a}(v_-, x_-), x_- + q, \cdot)$ for any $q \in \overline{\mathcal{B}(0, \frac{1}{2})}$ (see Lemma 2.1). Furthermore, with appropriate changes in the proof of Lemma 2.3, one can prove that the following decomposition holds

$$x(t) = z_+(\tilde{a}(v_-, x_-), x_- + q, t) + \mathcal{G}_{v_-, x_-}(q) - q + h(v_-, x_-, q, t), \quad (3.14)$$

where

$$\mathcal{G}_{v_-, x_-}(q) := \mathcal{A}(y_-)(0) - h(v_-, x_-, q, 0), \quad (3.15)$$

and

$$h(v_-, x_-, q, t) := - \int_t^{+\infty} \left(g(g^{-1}(\tilde{a}(v_-, x_-))) - \int_\sigma^{+\infty} F(x)(\tau) d\tau \right. \\ \left. - g(g^{-1}(\tilde{a}(v_-, x_-))) - \int_\sigma^{+\infty} F^l(z_+(\tilde{a}(v_-, x_-), x_- + q, \cdot))(\tau) d\tau \right) d\sigma, \quad (3.16)$$

for $t \geq 0$ and for $q \in \overline{\mathcal{B}(0, \frac{1}{2})}$. We need the following Lemma.

Lemma 3.2. *Let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$, and let $r \in (0, \min(\frac{1}{2}, \frac{|v_-|}{2^{\frac{3}{2}}}))$. Under condition (3.12) and when $y_- \in M_{r, v_-, x_-}$ is a fixed point of \mathcal{A} , then*

$$|\mathcal{G}_{v_-, x_-}(q)| \leq \frac{4n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} (\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2 + 4\beta_2)}{\alpha(\alpha + 1) (\frac{|v_-|}{2^{\frac{3}{2}}} - r)^2 (\frac{1}{2} + \frac{|x_-|}{\sqrt{2}})^\alpha} \leq \frac{1}{2}, \quad (3.17)$$

for $|q| \leq \frac{1}{2}$, and

$$|\mathcal{G}_{v_-, x_-}(q) - \mathcal{G}_{v_-, x_-}(q')| \leq \frac{4n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} (\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2^l) |q - q'|}{\alpha(\alpha + 1) (\frac{|v_-|}{2^{\frac{3}{2}}})^2 (\frac{1}{2} + \frac{|x_-|}{\sqrt{2}})^\alpha} \leq \frac{|q - q'|}{6}, \quad (3.18)$$

for $(q, q') \in \overline{\mathcal{B}(0, \frac{1}{2})}^2$.

Lemma 3.2 is proved in Section 7.

Under the assumptions of Lemma 3.2 the map \mathcal{G}_{v_-, x_-} is a $\frac{1}{6}$ -contraction map from $\overline{\mathcal{B}(0, \frac{1}{2})}$ to $\overline{\mathcal{B}(0, \frac{1}{2})}$. We denote by $\tilde{b}_{sc}(v_-, x_-)$ its unique fixed point in $\overline{\mathcal{B}(0, \frac{1}{2})}$, and we set $\tilde{b}(v_-, x_-) := x_- + \tilde{b}_{sc}(v_-, x_-)$ and $\tilde{a}_{sc}(v_-, x_-) := \tilde{a}(v_-, x_-) - v_-$. The decomposition (3.14) becomes

$$z_-(v_-, x_-, t) + y_-(t) = z_+(\tilde{a}(v_-, x_-), \tilde{b}(v_-, x_-), t) + y_+(t), \quad (3.19)$$

$$y_+(t) = h(v_-, x_-, \tilde{b}_{sc}(v_-, x_-), t), \quad (3.20)$$

for $t \geq 0$. The map $(\tilde{a}_{sc}, \tilde{b}_{sc})$ are our modified scattering data. The inverse scattering problem for equation (1.1) can now be formulated as follows

$$\text{Given } (\tilde{a}_{sc}, \tilde{b}_{sc}) \text{ and } F^l, \text{ find } F^s. \quad (3.21)$$

3.2 Estimates and asymptotics of the modified scattering data

Let $r \in (0, \min(\frac{1}{2}, 2^{-\frac{3}{2}}c))$ and let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$ so that $|v_-| > \tilde{\rho}_0(|x_-|, r, \beta, \alpha)$, where $\beta = \max(\beta_1^l, \beta_2, \beta_3^s)$ and $\tilde{\rho}_0$ is defined as the root of the following equation

$$1 = \frac{72n^2(c^{-1} + 1)\beta \sqrt{1 - \frac{\tilde{\rho}_0^2}{c^2}} (1 + \frac{1}{\frac{\tilde{\rho}_0}{2^{\frac{3}{2}}} - r})}{\alpha r (\frac{\tilde{\rho}_0}{2^{\frac{3}{2}}} - r) (\frac{1}{2} + \frac{|x_-|}{\sqrt{2}})^\alpha}, \quad \tilde{\rho}_0 \in (2^{\frac{3}{2}}r, c). \quad (3.22)$$

Then note that

$$\max\left(\frac{\tilde{\lambda}_1}{r}, \tilde{\lambda}_2, \tilde{\lambda}_3\right) \leq \frac{72n^2(c^{-1} + 1)\beta\sqrt{1 - \frac{\rho^2}{c^2}\left(1 + \frac{1}{\frac{\rho}{2^{\frac{3}{2}}}-r}\right)}}{\alpha r\left(\frac{\rho}{2^{\frac{3}{2}}}-r\right)\left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^\alpha}, \quad \rho \in (2^{\frac{3}{2}}r, c), \quad (3.23)$$

where $\tilde{\lambda}_1(n, \alpha, \beta_1^l, \beta_2, |x_-|, \rho, r)$ and $\tilde{\lambda}_2(n, \alpha, \beta_2, \beta_3^s, |x_-|, \rho, r)$ are defined in (3.9) and (3.10) respectively, and where $\tilde{\lambda}_3$ is the left-hand side of (3.12) (for $|v_-|$ replaced by ρ in (3.12)). From estimate (3.23), (3.22) and Lemma 3.1 we obtain: $|v_-| \geq \mu_{|x_-|}^l$ (see (3.1)), \mathcal{A} is a contraction in M_{r, v_-, x_-} , and \mathcal{A} has a unique fixed point $y_- \in M_{r, v_-, x_-}$ of \mathcal{A} .

Then set

$$\begin{aligned} \tilde{W}(v_-, x_-) &= g(g^{-1}(v_-) + \int_{-\infty}^0 F^l(z_-(v_-, x_-, \cdot))(\tau) d\tau) \\ &\quad + \int_0^{+\infty} F^l(z_+(\tilde{a}(v_-, x_-), x_-, \cdot))(\tau) d\tau - v_-. \end{aligned} \quad (3.24)$$

Note that \tilde{W} is known from the modified scattering data and from F^l . We obtain the following analog of Theorem 2.4.

Theorem 3.3. *Let $r \in (0, \min(\frac{1}{2}, 2^{-\frac{3}{2}}c))$ and let $(v_-, x_-) \in \mathcal{B}(0, c) \times \mathbb{R}^n$, $v_- \cdot x_- = 0$ so that $|v_-| \geq \tilde{\rho}_0(|x_-|, r, \beta, \alpha)$ where $\beta = \max(\beta_1^l, \beta_2, \beta_3^s)$ and $\tilde{\rho}_0$ is defined by (3.22). Under conditions (1.4), (1.5) the following estimates are valid:*

$$|\dot{y}_-(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}r + 1)\right)}{(\alpha + 1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)\left(1 - r + \frac{|x_-|}{\sqrt{2}} - \left(\frac{|v_-|}{2\sqrt{2}} - r\right)t\right)^{\alpha+1}}, \quad (3.25)$$

for $t \leq 0$; and

$$|\tilde{a}_{sc}(v_-, x_-)| \leq \frac{24n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \max(\beta_1^l, \beta_2)}{\alpha\left(\frac{|v_-|}{2\sqrt{2}} - r\right)\left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^\alpha}, \quad (3.26)$$

$$|\tilde{b}_{sc}(v_-, x_-)| \leq \frac{24n^2 \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \max(\beta_1^l, \beta_2)\left(\frac{1}{c} + 1\right)}{\alpha(\alpha + 1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2\left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^\alpha}, \quad (3.27)$$

$$|\dot{y}_+(t)| \leq \frac{20n^4 \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \max(\beta_1^l, \beta_2)\left(\frac{1}{c} + 1\right)}{(\alpha + 1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)\left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}} + t\frac{|v_-|}{2\sqrt{2}}\right)^{\alpha+1}}. \quad (3.28)$$

for $t \geq 0$. In addition

$$\begin{aligned}
& |\tilde{a}_{sc}(v_-, x_-) - \tilde{W}(v_-, x_-) - \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_{-\infty}^{+\infty} F^s(\tau v_- + x_-, v_-) d\tau| \\
& 944n^4 \beta^2 \left(1 - \frac{|v_-|^2}{c^2}\right) \left(\frac{r}{c} + 1\right) \left(1 + \frac{1}{c}\right) \left(1 + \frac{1}{\left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)}\right)^2 \\
\leq & \frac{\quad}{\alpha^2 \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)^2 \left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^{2\alpha+1}}. \tag{3.29}
\end{aligned}$$

$$\begin{aligned}
& \left| \tilde{b}_{sc}(v_-, x_-) - \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\tau v_- + x_-, v_-) d\tau d\sigma \right. \right. \\
& \left. \left. - \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\tau v_- + x_-, v_-) d\tau d\sigma + \int_{-\infty}^{+\infty} V^s(\tau v_- + x_-) d\tau \frac{v_-}{c^2} \right) \right| \\
& 808n^4 \beta^2 \left(1 - \frac{|v_-|^2}{c^2}\right) \left(\frac{1}{c} + 1\right)^2 \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)^2 \\
\leq & \frac{\quad}{\alpha^2 (\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^3 \left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^{2\alpha}}. \tag{3.30}
\end{aligned}$$

Theorem 3.3 is proved in Section 7.

Estimates (3.29) and (3.30) provide the high energies asymptotics of the modified scattering data. The analog of formulas (1.17) and (1.18) are given in the following Corollary.

Corollary 3.4. *Let $(\theta, x) \in TS^{n-1}$. We have*

$$\lim_{\rho \rightarrow +\infty} \frac{\rho}{\sqrt{1 - \frac{\rho^2}{c^2}}} (\tilde{a}_{sc} - \tilde{W})(\rho\theta, x) = \int_{-\infty}^{+\infty} F(\tau\theta + x, c\theta) d\tau, \tag{3.31}$$

$$\begin{aligned}
\lim_{s \rightarrow +\infty} \frac{\rho^2}{\sqrt{1 - \frac{\rho^2}{c^2}}} \tilde{b}_{sc}(s\theta, x) &= \int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\tau\theta + x, c\theta) d\tau d\sigma \\
& - \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\tau\theta + x, c\theta) d\tau d\sigma + PV^s(\theta, x)\theta. \tag{3.32}
\end{aligned}$$

Then F^s can be reconstructed from the high energies asymptotics of \tilde{a}_{sc} , and from (3.32) one can prove the following statements: The potential V^s is uniquely determined up to its radial part by \tilde{b}_{sc} ; The magnetic field B^s can be reconstructed from \tilde{b}_{sc} when $n \geq 3$, and up to its radial part when $n = 2$.

The estimates (3.29) and (3.30) also gives the Born approximation for the modified scattering data at fixed energy when the electromagnetic field is sufficiently weak. The Born approximation at fixed energy of $\tilde{a}_{sc}(\rho\theta, x) - \tilde{W}(\rho\theta, x)$ and $\tilde{b}_{sc}(\rho\theta, x)$ for $(\theta, x) \in TS^{n-1}$ and for $\tilde{\rho} \in (\tilde{\rho}_0(|x|, r, \beta, \alpha), c)$ are

given by (1.21) and (1.22) respectively, and we have: The force F^s can be reconstructed from the Born approximation (1.21) of \tilde{a}_{sc} at fixed energy; V^s can be reconstructed from the Born approximation (1.22) of \tilde{b}_{sc} at fixed energy; B^s can be reconstructed from (1.22) when $n \geq 3$, and up to its radial part when $n = 2$.

Estimates (3.29) and (3.30) also provide the first leading term in the asymptotics of the modified scattering data $(\tilde{a}_{sc}(\rho\theta, x), \tilde{b}_{sc}(\rho\theta, x))$ when the parameters α, n, s, θ and β are fixed and $|x|$ increases to $+\infty$.

4 Preliminary estimates and proof of Lemmas 2.1 and 2.3

For the rest of the text we use the following properties of the function $g : \mathbb{R}^n \rightarrow \mathcal{B}(0, c)$ defined by (1.16) (see [8, Section 2.5]):

$$|\nabla g_i(x)|^2 \leq \frac{1}{1 + \frac{|x|^2}{c^2}}, \quad (4.1)$$

$$|g(x) - g(y)| \leq \sqrt{n}|x - y| \sup_{\varepsilon \in (0,1)} \frac{1}{\sqrt{1 + \frac{|\varepsilon x + (1-\varepsilon)y|^2}{c^2}}}, \quad (4.2)$$

$$|\nabla g_i(x) - \nabla g_i(y)| \leq \frac{3\sqrt{n}}{c}|x - y| \sup_{\varepsilon \in (0,1)} \frac{1}{1 + \frac{|\varepsilon x + (1-\varepsilon)y|^2}{c^2}}, \quad (4.3)$$

$$\nabla g_i(x) = \frac{1}{(1 + \frac{|x|^2}{c^2})^{\frac{1}{2}}} e_i - \frac{x_i x}{c^2 (1 + \frac{|x|^2}{c^2})^{\frac{3}{2}}}, \quad (4.4)$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, and for $i = 1 \dots n$ where $g = (g_1, \dots, g_n)$, and where the i^{th} component of the vector e_i is equal to 1 and all others components of e_i are equal to zero.

We also used the following properties of the forces (F^l, F^s) (see [8, Section 2.5]):

$$|F^l(x, v)| \leq 2n\beta_1^l (1 + |x|)^{-\alpha-1}, \quad (4.5)$$

$$|F^s(x, v)| \leq 2n\beta_2^s (1 + |x|)^{-\alpha-2}, \quad (4.6)$$

$$\begin{aligned} |F^l(x, v) - F^l(x', v')| &\leq \frac{n\beta_1^l}{c} |v - v'| \sup_{\varepsilon \in (0,1)} (1 + |(1-\varepsilon)x + \varepsilon x'|)^{-\alpha-1} \\ &\quad + 2n^{\frac{3}{2}} \beta_2^l |x - x'| \sup_{\varepsilon \in (0,1)} (1 + |(1-\varepsilon)x + \varepsilon x'|)^{-\alpha-2}, \end{aligned} \quad (4.7)$$

$$\begin{aligned}
|F^s(x, v) - F^s(x', v')| &\leq \frac{n\beta_2^s}{c}|v - v'| \sup_{\varepsilon \in (0,1)} (1 + |(1 - \varepsilon)x + \varepsilon x'|)^{-\alpha-2} \quad (4.8) \\
&\quad + 2n^{\frac{3}{2}}\beta_3^s|x - x'| \sup_{\varepsilon \in (0,1)} (1 + |(1 - \varepsilon)x + \varepsilon x'|)^{-\alpha-3},
\end{aligned}$$

for $(x, x', v, v') \in (\mathbb{R}^n)^4$, $\max(|v|, |v'|) \leq c$.

Proof of Lemma 2.1. We prove the existence and uniqueness of the solution z_+ (similarly one can prove the existence and uniqueness of z_-). Set $C = \frac{2^{\frac{9}{2}}n^{\frac{3}{2}}\beta_1^l\sqrt{1-\frac{|v|^2}{c^2}}}{\alpha|v|(1+\frac{|x|}{\sqrt{2}}-|q|)^\alpha}$. Let $\mathcal{V}_{w,C}$ be the complete metric space defined by

$$\mathcal{V}_{w,C} := \{f \in C^1(\mathbb{R}, \mathbb{R}^n) \mid f(0) = 0 \text{ and } \sup_{\mathbb{R}} |\dot{f}| \leq C, \sup_{\mathbb{R}} |w + f| \leq c\},$$

endowed with the following norm $\|f\|_{\mathcal{V}} := \sup_{\mathbb{R}} |\dot{f}|$. Note that $\mathcal{V}_{w,C}$ is a convex subset of $C^1(\mathbb{R}, \mathbb{R}^n)$. We consider the integral equations

$$G_w(f)(t) := \int_0^t \dot{G}_w(f)(s) ds, \quad (4.9)$$

$$\dot{G}_w(f)(t) := g\left(g^{-1}(w) - \int_t^{+\infty} F^l(x + q + w + f)(\tau) d\tau\right) - w, \quad (4.10)$$

for $f \in \mathcal{V}_{w,C}$ and for $t \in \mathbb{R}$. Then we have (see also (1.16))

$$G_w(f)(0) = 0, \quad |w + \dot{G}_w(f)(t)| < c \text{ for } t \in \mathbb{R}. \quad (4.11)$$

We use the following estimate (4.12)

$$\begin{aligned}
|x + q + \tau w + f(\tau)| &\geq |x + \tau v| - |f(\tau)| - \tau|w - v| - |q| \\
&\geq \frac{|x|}{\sqrt{2}} - |q| + \left(\frac{|v|}{\sqrt{2}} - C - |w - v|\right)|\tau| \geq \frac{|x|}{\sqrt{2}} - |q| + \frac{|v|}{2\sqrt{2}}|\tau|, \quad (4.12)
\end{aligned}$$

for $\tau \in \mathbb{R}$ and $f \in \mathcal{V}_{w,C}$ (we used the estimate $|f(t)| \leq C|t|$ for $t \in \mathbb{R}$ and for $f \in \mathcal{V}_{w,C}$, and we used that $x \cdot v = 0$, and we used (2.1) and (2.2)). Using (4.5) and (4.12) we obtain that

$$\begin{aligned}
\int_t^{+\infty} |F^l(\cdot w + x + q + f)(\tau)| d\tau &\leq 2n\beta_1^l \int_t^{+\infty} \left(1 + \frac{|x|}{\sqrt{2}} - |q| + \frac{|v|}{2\sqrt{2}}|\tau|\right)^{-\alpha-1} d\tau \\
&\leq \frac{2^{\frac{7}{2}}n\beta_1^l}{\alpha|v|(1+\frac{|x|}{\sqrt{2}}-|q|)^\alpha}, \quad (4.13)
\end{aligned}$$

for $t \in \mathbb{R}$ and for $f \in \mathcal{V}_{w,C}$. We used the integral value $\int_{-\infty}^{+\infty} (a + b\tau)^{-\alpha-1} d\tau = \frac{2}{b\alpha a^\alpha}$ for $a > 0$ and $b > 0$. Then we also use the identity $|g^{-1}(v)| = |g^{-1}(w)|$ (see (2.1)) and we use (2.2), and we obtain

$$\begin{aligned} & \left| g^{-1}(w) - \int_t^{+\infty} (\varepsilon F^l(.w + x + q + f_1)(\tau) d\tau + \eta F^l(.w + x + q + f_2)(\tau)) d\tau \right| \\ & \geq |g^{-1}(v)| - \frac{2^{\frac{7}{2}} n \beta_1^l}{\alpha |v| (1 + \frac{|x|}{\sqrt{2}} - |q|)^\alpha} \geq \frac{|v|}{2\sqrt{1 - \frac{|v|^2}{c^2}}}, \end{aligned} \quad (4.14)$$

for $(f_1, f_2) \in \mathcal{V}_{w,C}^2$, and for $(\varepsilon, \eta, t) \in (0, 1)^2 \times \mathbb{R}$, $\varepsilon + \eta \leq 1$. Therefore combining (4.10), (4.13), (4.14) and (4.2) we obtain

$$\begin{aligned} |\dot{G}_w(f)(t)| & \leq \frac{2^{\frac{7}{2}} n^{\frac{3}{2}} \beta_1^l}{\alpha |v| (1 + \frac{|x|}{\sqrt{2}} - |q|)^\alpha} \sup_{\varepsilon \in (0,1)} \frac{1}{\sqrt{1 + \frac{|g^{-1}(w) - \varepsilon \int_t^{+\infty} F^l(.w + x + q + f)(\tau) d\tau|^2}{c^2}}} \\ & \leq \frac{2^{\frac{9}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v| (1 + \frac{|x|}{\sqrt{2}} - |q|)^\alpha}. \end{aligned} \quad (4.15)$$

for $t \in \mathbb{R}$ and $f \in \mathcal{V}_{w,C}$.

Now let $(f_1, f_2) \in \mathcal{V}_{w,C}^2$. Then using (4.2), (4.10) and (4.14) we have

$$\begin{aligned} |\dot{G}_w(f_1)(t) - \dot{G}_w(f_2)(t)| & \leq 2n^{\frac{1}{2}} \sqrt{1 - \frac{|v|^2}{c^2}} \int_t^{+\infty} |F^l(.w + x + q + f_1) \\ & \quad - F^l(.w + x + q + f_2)|(\tau) d\tau \end{aligned} \quad (4.16)$$

for $t \in \mathbb{R}$. Using (4.7) and (4.12) we have

$$\begin{aligned} & |F^l(.w + x + q + f_1) - F^l(.w + x + q + f_2)|(\tau) \\ & \leq \frac{n\beta_1^l}{c} \left(1 + \frac{|x|}{\sqrt{2}} - |q| + \frac{|v|}{2^{\frac{3}{2}}} |\tau|\right)^{-\alpha-1} \sup_{(0, +\infty)} |f_1 - f_2| \\ & \quad + 2n^{\frac{3}{2}} \beta_2^l \left(1 + \frac{|x|}{\sqrt{2}} - |q| + \frac{|v|}{2^{\frac{3}{2}}} |\tau|\right)^{-\alpha-2} |\tau| \sup_{s \in \mathbb{R} \setminus \{0\}} \frac{|(f_1 - f_2)(s)|}{|s|} \\ & \leq \left(\frac{n\beta_1^l}{c} + \frac{2^{\frac{5}{2}} n^{\frac{3}{2}}}{|v|} \beta_2^l\right) \left(1 + \frac{|x|}{\sqrt{2}} - |q| + \frac{|v|}{2^{\frac{3}{2}}} |\tau|\right)^{-\alpha-1} \|f_1 - f_2\|_{\mathcal{V}} \end{aligned} \quad (4.17)$$

for $\tau \in \mathbb{R}$. Therefore we obtain

$$|\dot{G}_w(f_1)(t) - \dot{G}_w(f_2)(t)| \leq \frac{n^{\frac{3}{2}} 2^{\frac{7}{2}} \sqrt{1 - \frac{|v|^2}{c^2}} \left(\frac{\beta_1^l}{c} + \frac{2^{\frac{5}{2}} n^{\frac{1}{2}}}{|v|} \beta_2^l\right)}{\alpha |v| (1 + \frac{|x|}{\sqrt{2}} - |q|)^\alpha} \|f_1 - f_2\|_{\mathcal{V}}, \quad (4.18)$$

for $t \in \mathbb{R}$.

From (4.11), (4.15), (4.18) and (2.2) it follows that the operator G_w is a $\frac{1}{2}$ -contraction map from $\mathcal{V}_{w,C}$ to $\mathcal{V}_{w,C}$. Set $z_+(w, x+q, t) = x+q+tw+f_{w,x+q}(t)$ for $t \in \mathbb{R}$, where $f_{w,x+q}$ denotes the unique fixed point of G_w in $\mathcal{V}_{w,C}$. Then $z_+(w, x+q, \cdot)$ satisfies (1.8), (2.3), (2.4). \square

Before proving Lemma 2.3 we recall the following standard result. For sake of consistency we provide a proof of Lemma 4.1 at the end of this Section.

Lemma 4.1. *Let $x \in C^2(\mathbb{R}, \mathbb{R}^n)$ satisfy equation (1.1) and let $z \in C^2(\mathbb{R}, \mathbb{R}^n)$ satisfy equation (1.8). Assume that there exists $v \in \mathcal{B}(0, c)$, $v \neq 0$, so that $\dot{x}(t) \rightarrow v$ and $\dot{z}(t) \rightarrow v$ as $t \rightarrow +\infty$. Then*

$$\sup_{(0, +\infty)} |x - z| < \infty \text{ and } \sup_{t \in (0, +\infty)} (1+t)^{-1} |\dot{x} - \dot{z}|(t) < \infty. \quad (4.19)$$

Proof of Lemma 2.3. We need the following preliminary estimate (4.21). From the formula $g(\tau) = g(0) + \int_0^\tau \dot{g}(s)ds$ for $g \in C^1(\mathbb{R}, \mathbb{R}^n)$ it follows that ,

$$|(f_1 - f_2)(\tau)| \leq \sup_{(-\infty, 0)} |f_1 - f_2| + |\tau| \sup_{\mathbb{R}} |\dot{f}_1 - \dot{f}_2| \text{ for } \tau \in \mathbb{R}, \quad (4.20)$$

and for $(f_1, f_2) \in M_{r, v_-}^2$. Hence

$$\begin{aligned} |z_-(v_-, \tau) + x + f(\tau)| &\geq |x + \tau v_-| - |z_-(v_-, \tau) - \tau v_-| - |f(\tau)| \\ &\geq \frac{|x|}{\sqrt{2}} - r + |\tau| \left(\frac{|v_-|}{\sqrt{2}} - \frac{2^{\frac{9}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha |v_-|} - r \right) \\ &\geq \frac{|x|}{\sqrt{2}} - r + |\tau| \left(\frac{|v_-|}{2\sqrt{2}} - r \right), \end{aligned} \quad (4.21)$$

for $(f, \tau) \in M_{r, v_-} \times \mathbb{R}$ and for any $x \in \mathbb{R}^n$ so that $x \cdot v_- = 0$. We used (2.4) (for " $(x, w, v, q) = (0, v_-, v_-, 0)$ "), the inequality $|x + \tau v_-| \geq \frac{|x|}{\sqrt{2}} + |\tau| \frac{|v_-|}{\sqrt{2}}$ ($x \cdot v_- = 0$) and (4.20) (for $(f_1, f_2) = (f, 0)$ and $\|f\| \leq r$) and the condition $|v_-| \geq \mu^l$.

Hence the integral $\int_{-\infty}^{+\infty} F(z_-(v_-, \cdot) + x_- + f)(\tau) d\tau$ is absolutely convergent for any $f \in M_{r, v_-}$. And when $y_- \in M_{r, v_-}$ is a fixed point for A then $z_-(v_-, \cdot) + x_- + y_-$ satisfies equation (1.1) (see (2.8) and (2.9)) and $\dot{z}_-(v_-, t) + \dot{y}_-(t) = g(g^{-1}(v_-) + \int_{-\infty}^t F(z_-(v_-, \cdot) + x_- + y_-)(\tau) d\tau) \rightarrow a(v_-, x_-)$ as $t \rightarrow +\infty$, where $a(v_-, x_-)$ is defined in (2.16). Then from Lemma 4.1 it follows that $\sup_{(0, +\infty)} |z_-(v_-, \cdot) + x_- + y_- - z_+(a(v_-, x_-), \cdot)| < +\infty$ and $\sup_{t \in (0, +\infty)} (1+t)^{-1} |\dot{z}_-(v_-, t) + \dot{y}_-(t) - \dot{z}_+(a(v_-, x_-), t)| < \infty$. Using these latter estimates and $y_- \in M_{r, v_-}$, and using (4.6), (4.7) and (4.21) we obtain

that the integral on the right hand side of (2.18) is absolutely convergent. Then the decomposition (2.15) follows from the equality $A(y_-) = y_-$ and (2.8) and (2.9) and straightforward computations. \square

Proof of Lemma 4.1. Note that from (1.1), (1.8) and the property $\lim_{t \rightarrow +\infty} \dot{x}(t) = \lim_{t \rightarrow +\infty} \dot{z}(t) = v \neq 0$ it follows that

$$\dot{x}(t) = g\left(g^{-1}(v) - \int_t^{+\infty} F(x)(s)ds\right), \quad \dot{z}(t) = g\left(g^{-1}(v) - \int_t^{+\infty} F^l(z)(s)ds\right), \quad (4.22)$$

for $t \geq 0$, and that there exists $\varepsilon > 0$ so that

$$1 + |\eta x(t) + (1 - \eta)z(t)| \geq \varepsilon(1 + t), \quad \text{for } (t, \eta) \in [0, +\infty) \times [0, 1]. \quad (4.23)$$

Using (4.23) for $\eta = 1$ and using (4.6) we obtain

$$\int_t^{+\infty} |F^s(x)(\tau)|d\tau \leq 2n\beta_2\varepsilon^{-\alpha-2} \int_t^{+\infty} (1 + \tau)^{-\alpha-2}d\tau = \frac{2n\beta_2\varepsilon^{-\alpha-2}}{(\alpha + 1)(1 + t)^{\alpha+1}}, \quad (4.24)$$

for $t \geq 0$. Thus using (4.22), (4.2) and (4.24) we have

$$|\dot{x}(t) - \dot{z}(t)| \leq n^{\frac{1}{2}} \int_t^{+\infty} |F^l(x) - F^l(z)|(\tau)ds + \frac{2n^{\frac{3}{2}}\beta_2\varepsilon^{-\alpha-2}}{(\alpha + 1)(1 + t)^{\alpha+1}}, \quad (4.25)$$

for $t \geq 0$. Using (4.23) for $\eta = 0, 1$, and using (4.5) we obtain

$$\int_t^{+\infty} (|F^l(x)| + |F^l(z)|)(\tau)d\tau \leq \frac{4\varepsilon^{-\alpha-1}n\beta_1^l}{\alpha(1 + t)^\alpha}, \quad (4.26)$$

for $t \geq 0$, and using (4.25) we obtain

$$|\dot{x}(t) - \dot{z}(t)| \leq \frac{4n^{\frac{3}{2}}\varepsilon^{-\alpha-1}\beta_1^l + 2n^{\frac{3}{2}}\beta_2\varepsilon^{-\alpha-2}}{\alpha(1 + t)^\alpha}, \quad \text{for } t \geq 0. \quad (4.27)$$

We assume without loss of generality that $\alpha \neq \frac{1}{m}$ for $m \in \mathbb{N}$, $m \neq 0$. If $\alpha = \frac{1}{m}$ for some $m \in \mathbb{N}$, $m \neq 0$, then just replace α by $\alpha' \in (\frac{1}{m+1}, \frac{1}{m})$. We denote by $[x]$ the integer part of any real number x . We prove by induction that for $n = 1 \dots [\frac{1}{\alpha}]$ there exists a positive constant C_n so that

$$|\dot{x}(t) - \dot{z}(t)| \leq C_n(1 + t)^{-n\alpha}, \quad \text{for } t \geq 0. \quad (4.28)$$

The estimate (4.28) is proved for $n = 1$ by (4.27). Let $n = 1 \dots [\frac{1}{\alpha}]$. Integrating (4.28) over $(0, t)$ we have

$$|x(t) - z(t)| \leq C'_n + \frac{C_n}{|1 - n\alpha|}(1 + t)^{1-n\alpha}, \quad \text{for } t \geq 0, \quad (4.29)$$

and for some constant C'_n . Using (4.25), (4.28), (4.29), (4.7) and (4.23) we have

$$\begin{aligned}
|\dot{x}(t) - \dot{z}(t)| &\leq \int_t^{+\infty} \sup_{\eta \in (0,1)} \frac{\frac{n^{\frac{3}{2}}\beta_1^l}{c} |\dot{x}(s) - \dot{z}(s)|}{(1 + |\eta x(s) + (1-\eta)z(s)|)^{\alpha+1}} ds \\
&\quad + \int_t^{+\infty} \sup_{\eta \in (0,1)} \frac{2n^2\beta_2 |x(s) - z(s)|}{(1 + |\eta x(s) + (1-\eta)z(s)|)^{\alpha+2}} ds \quad (4.30) \\
&\quad + \frac{2n^{\frac{3}{2}}\beta_2 \varepsilon^{-\alpha-2}}{(\alpha+1)(1+t)^{\alpha+1}} \\
&\leq n^{\frac{3}{2}}\varepsilon^{-\alpha-2} \left(\frac{\beta_1^l C_n \varepsilon}{c} + 2 \frac{n^{\frac{1}{2}} C_n \beta_2}{|1-n\alpha|} \right) \int_t^{+\infty} (1+s)^{-1-(n+1)\alpha} ds \\
&\quad + 2n^2\beta_2 C'_n \varepsilon^{-\alpha-2} \int_t^{+\infty} (1+s)^{-\alpha-2} ds + \frac{2n^{\frac{3}{2}}\beta_2 \varepsilon^{-\alpha-2}}{(\alpha+1)(1+t)^{\alpha+1}} \\
&\leq \frac{n^{\frac{3}{2}}\varepsilon^{-\alpha-2} \left(\frac{\beta_1^l C_n \varepsilon}{c} + 2 \frac{n^{\frac{1}{2}} C_n \beta_2}{|1-n\alpha|} \right)}{(n+1)\alpha(1+t)^{(n+1)\alpha}} + \frac{2n^{\frac{3}{2}}\varepsilon^{-\alpha-2}\beta_2(1+n^{\frac{1}{2}}C'_n)}{(\alpha+1)(1+t)^{\alpha+1}}, \quad (4.31)
\end{aligned}$$

for $t \geq 0$. This proves (4.28) for " $n+1$ ". The induction step is proved. Then (4.28), (4.29) also hold for $n = \lfloor \frac{1}{\alpha} \rfloor + 1$, and using $(\lfloor \frac{1}{\alpha} \rfloor + 1)\alpha > 1$, we obtain (4.19). \square

5 Proof of Lemmas 2.2 and 3.1

Proof of Lemma 2.2. We shorten $z_-(v_-, \cdot)$ to z_- in this paragraph. We first prove the estimates (5.9), (5.10) and (5.12) given below. Let $f \in M_{r,v_-}$. Using (4.6) and (4.21) we have

$$|F^s(z_- + \varepsilon x_- + f)(\tau)| \leq \frac{2n\beta_2}{\left(1 + \frac{\varepsilon|x_-|}{\sqrt{2}} - r + |\tau| \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\right)^{\alpha+2}}, \quad (5.1)$$

for $\tau \in \mathbb{R}$ and for $\varepsilon \in [0, 1]$. Integrating both sides of (5.1) over $(-\infty, t)$ we obtain

$$\int_{-\infty}^t |F^s(z_- + \varepsilon x_- + f)(\tau)| d\tau \leq \frac{4n\beta_2}{(\alpha+1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 + \frac{\varepsilon|x_-|}{\sqrt{2}} - r\right)^{\alpha+1}}, \quad (5.2)$$

for $t \in \mathbb{R}$ and for $\varepsilon \in [0, 1]$. Similarly using (4.5) instead of (4.6) we have

$$\int_{-\infty}^t |F^l(z_- + \varepsilon x_- + f)(\tau)| d\tau \leq \frac{4n\beta_1^l}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 + \frac{\varepsilon|x_-|}{\sqrt{2}} - r\right)^\alpha}, \quad (5.3)$$

for $t \in \mathbb{R}$ and for $\varepsilon \in [0, 1]$. We combine (5.2) and (5.3), and we use (2.12), and we have

$$\begin{aligned} & |g^{-1}(v_-) + \eta_1 \int_{-\infty}^t (F(z_- + x_- + f_1)(\tau) + (\eta_2 F^l + \eta_2 \mu F^s)(z_- + \varepsilon x_- + f_2)(\tau)) d\tau| \\ & \geq |g^{-1}(v_-)| - \frac{8n \max(\beta_1^l, \beta_2)}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) (1-r)^{\alpha+1}} \geq \frac{1}{2} |g^{-1}(v_-)|. \end{aligned} \quad (5.4)$$

for $(f_1, f_2) \in M_{r, v_-}^2$ and for $(t, \eta_1, \eta_2, \mu, \varepsilon) \in \mathbb{R} \times [0, 1]^4$ so that $\eta_1 + \eta_2 \leq 1$. Then from (2.9), (5.4) and (4.2) it follows that

$$|\dot{A}(f)(t)| \leq 2n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left| \int_{-\infty}^t (F(z_- + x_- + f)(\tau) - F^l(z_-)(\tau)) d\tau \right|, \quad (5.5)$$

for $t \in \mathbb{R}$. Using (5.1) we have

$$\int_{-\infty}^t |F^s(z_- + x_- + f)(\tau)| d\tau \leq \frac{2n\beta_2}{(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} - r - t \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\right)^{\alpha+1}}, \quad (5.6)$$

for $t \leq 0$. Using (4.7) and (4.21) (with " $x'' = (1 - \varepsilon + \varepsilon\mu)x_-$ ", $(\varepsilon, \mu) \in [0, 1]^2$) we obtain

$$\begin{aligned} & |F^l(z_- + x_- + f_1)(\tau) - F^l(z_- + \mu x_- + f_2)(\tau)| \\ & \leq \frac{\frac{n\beta_1^l}{c} |\dot{f}_1 - \dot{f}_2|(\tau)}{\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right) |\tau|\right)^{\alpha+1}} + \frac{2n^{\frac{3}{2}} \beta_2 ((1 - \mu)|x_-| + |f_1 - f_2|(\tau))}{\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right) |\tau|\right)^{\alpha+2}} \end{aligned} \quad (5.7)$$

for $(f_1, f_2) \in M_{r, v_-}^2$ and for $(\tau, \mu) \in \mathbb{R} \times [0, 1]$. We integrate (5.7) over $(-\infty, t)$, and we use the estimates $|f(\tau)| \leq r$ and $|\dot{f}(\tau)| \leq r(1 - r + (\frac{|v_-|}{2\sqrt{2}} - r)|\tau|)^{-1}$ for $\tau \leq 0$, and we obtain

$$\begin{aligned} & \int_{-\infty}^t |F^l(z_- + x_- + f)(\tau) - F^l(z_-)(\tau)| d\tau \\ & \leq \left(\frac{n\beta_1^l r}{c} + 2n^{\frac{3}{2}} \beta_2 (|x_-| + r)\right) \int_{-\infty}^t \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right) |\tau|\right)^{-\alpha-2} d\tau \\ & \leq \frac{\frac{nr\beta_1^l}{c} + 2n^{\frac{3}{2}} \beta_2 (|x_-| + r)}{(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r - \left(\frac{|v_-|}{2\sqrt{2}} - r\right) t\right)^{\alpha+1}}, \end{aligned} \quad (5.8)$$

for $t \leq 0$.

Combining (5.5), (5.8), (5.6) we obtain

$$|\dot{A}(f)(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(|x_-| + r) + 1)\right)}{(\alpha + 1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)\left(1 - r - \left(\frac{|v_-|}{2\sqrt{2}} - r\right)t\right)^{\alpha+1}}, \quad (5.9)$$

$$|A(f)(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(|x_-| + r) + 1)\right)}{\alpha(\alpha + 1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2\left(1 - r - \left(\frac{|v_-|}{2\sqrt{2}} - r\right)t\right)^\alpha}, \quad (5.10)$$

for $t \leq 0$.

Let $t \geq 0$ and $|v_-| > 2\sqrt{2}r$, $r < 1$. Integrating (5.7) over $(0, t)$ and using the estimate $\sup_{(0, +\infty)} |\dot{f}| \leq r$ and (4.20) (for $(f_1, f_2) = (f, 0)$ and $\|f\| \leq r$), we obtain

$$\begin{aligned} & \int_0^t |F^l(z_- + x_- + f)(\tau) - F^l(z_-)(\tau)| d\tau \\ & \leq \frac{nr\beta_1^l}{c} \int_0^t \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|\tau|\right)^{-\alpha-1} d\tau \\ & \quad + 2n^{\frac{3}{2}}\beta_2 \int_0^t \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|\tau|\right)^{-\alpha-2} (|x_-| + r + r|\tau|) d\tau \\ & \leq \frac{n}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right)(1-r)^\alpha} \left(\frac{r\beta_1^l}{c} + \frac{2n^{\frac{1}{2}}\beta_2(|x_-| + r)}{(\alpha+1)(1-r)} + \frac{2n^{\frac{1}{2}}\beta_2r}{\alpha\left(\frac{|v_-|}{2\sqrt{2}} - r\right)}\right). \end{aligned} \quad (5.11)$$

Hence combining (5.5), (5.2), (5.11) and (5.8) (for " $t = 0$ ") we obtain

$$|\dot{A}(f)(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right)(1-r)^\alpha} \left(\frac{r\beta_1^l}{c} + \frac{4\beta_2(n^{\frac{1}{2}}(|x_-| + r) + 1)}{(\alpha+1)(1-r)} + \frac{r\beta_1^l}{c\alpha} + \frac{2n^{\frac{1}{2}}\beta_2r}{\alpha\left(\frac{|v_-|}{2\sqrt{2}} - r\right)}\right). \quad (5.12)$$

Then estimate (2.13) follows from (5.9), (5.10) and (5.12).

Now we prove the estimates (5.18), (5.19) and (5.21) given below. Estimate (2.14) follows from those latter estimates. Let $|v_-| > 2\sqrt{2}r$, $r < 1$, and let (f_1, h_1) and $(f_2, h_2) \in M_{r, v_-}$. From (4.8) and (4.21) it follows that

$$\begin{aligned} & |F^s(z_- + x_- + f_1)(\tau) - F^s(z_- + x_- + f_2)(\tau)| \\ & \leq \frac{\frac{n\beta_2}{c} |\dot{f}_1 - \dot{f}_2|(\tau)}{\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)|\tau|\right)^{\alpha+2}} + \frac{2n^{\frac{3}{2}}\beta_3^s |f_1 - f_2|(\tau)}{\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)|\tau|\right)^{\alpha+3}} \end{aligned} \quad (5.13)$$

for $\tau \in \mathbb{R}$. Note that

$$\begin{aligned} \dot{A}(f_1)(t) - \dot{A}(f_2)(t) &= g\left(g^{-1}(v_-) + \int_{-\infty}^t F(z_- + x_- + f_1)(\tau)d\tau\right) \\ &- g\left(g^{-1}(v_-) + \int_{-\infty}^t F(z_- + x_- + f_2)(\tau)d\tau\right). \end{aligned} \quad (5.14)$$

Hence using (4.2) we obtain

$$|\dot{A}(f_1)(t) - \dot{A}(f_2)(t)| \leq 2n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} J(t), \quad (5.15)$$

$$J(t) := \int_{-\infty}^t |F(z_- + x_- + f_2)(\tau) - F(z_- + x_- + f_1)(\tau)| d\tau, \quad (5.16)$$

for $t \in \mathbb{R}$. We integrate (5.7) and (5.13) over $(-\infty, t)$, and we use the estimates $|(f_1 - f_2)(\tau)| \leq \|f_1 - f_2\|$ and $|(\dot{f}_1 - \dot{f}_2)(\tau)| \leq (1 - r + (\frac{|v_-|}{2\sqrt{2}} - r)|\tau|)^{-1} \|f_1 - f_2\|$ for $\tau \leq 0$, and we have

$$J(t) \leq \|f_1 - f_2\| \int_{-\infty}^t \left(\frac{(\frac{n\beta_1^l}{c} + 2n^{\frac{3}{2}}\beta_2)}{(1 - r + (\frac{|v_-|}{2\sqrt{2}} - r)|\tau|)^{\alpha+2}} + \frac{(\frac{n\beta_2}{c} + 2n^{\frac{3}{2}}\beta_3^s)}{(1 - r + (\frac{|v_-|}{2\sqrt{2}} - r)|s|)^{\alpha+3}} \right) d\tau. \quad (5.17)$$

Let $t \leq 0$. We also use (5.15) and we obtain

$$\begin{aligned} |\dot{A}(f_1)(t) - \dot{A}(f_2)(t)| &\leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \|f_1 - f_2\|}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)^{\alpha+1}} \\ &\times \left(\frac{\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2}{(\alpha+1)} + \frac{\frac{\beta_2}{c} + 2n^{\frac{1}{2}}\beta_3^s}{(\alpha+2) \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)} \right), \end{aligned} \quad (5.18)$$

$$\begin{aligned} |A(f_1)(t) - A(f_2)(t)| &\leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \|f_1 - f_2\|}{(\alpha+1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)^\alpha} \\ &\times \left(\frac{\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2}{\alpha} + \frac{\frac{\beta_2}{c} + 2n^{\frac{1}{2}}\beta_3^s}{(\alpha+2) \left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)} \right). \end{aligned} \quad (5.19)$$

Let $t \geq 0$ and $|v_-| > 2\sqrt{2}r$, $r < 1$, and let $(f_1, f_2) \in M_{r,v_-}^2$. We integrate (5.7) and (5.13) over $(0, t)$, and we use the estimates (4.20) and $|\dot{f}_1(\tau) -$

$\dot{f}_2(\tau) \leq \|f_1 - f_2\|$ for $\tau \in \mathbb{R}$, and we have

$$\begin{aligned}
J(t) - J(0) &= \int_0^t |F(x_- + z_- + f_2)(\tau) - F(x_- + z_- + f_1)(\tau)| d\tau \\
&\leq \int_0^t \frac{\frac{n\beta_1^l}{c} \|f_1 - f_2\| d\tau}{(1-r + (\frac{|v_-|}{2\sqrt{2}} - r)\tau)^{\alpha+1}} + \int_0^t \frac{2n^{\frac{3}{2}}\beta_2(1+\tau) \|f_1 - f_2\| d\tau}{(1-r + (\frac{|v_-|}{2\sqrt{2}} - r)\tau)^{\alpha+2}} \\
&\quad + \int_0^t \frac{\frac{n\beta_2}{c} \|f_1 - f_2\| d\tau}{(1-r + (\frac{|v_-|}{2\sqrt{2}} - r)\tau)^{\alpha+2}} + \int_0^t \frac{2n^{\frac{3}{2}}\beta_3^s(1+\tau) \|f_1 - f_2\| d\tau}{(1-r + (\frac{|v_-|}{2\sqrt{2}} - r)\tau)^{\alpha+3}} \\
&\leq \frac{n\|f_2 - f_1\|}{(1-r)^\alpha(\frac{|v_-|}{2\sqrt{2}} - r)} \left(\frac{\beta_1^l}{c\alpha} + \frac{\beta_2}{c(\alpha+1)(1-r)} + \frac{2n^{\frac{1}{2}}}{(1-r)} \left(\frac{\beta_2}{\alpha+1} \right. \right. \\
&\quad \left. \left. + \frac{\beta_3^s}{(\alpha+2)(1-r)} \right) + \frac{2n^{\frac{1}{2}}}{(\frac{|v_-|}{2\sqrt{2}} - r)} \left(\frac{\beta_2}{\alpha} + \frac{\beta_3^s}{(\alpha+1)(1-r)} \right) \right). \quad (5.20)
\end{aligned}$$

Using (5.15), (5.17) (with “ $t = 0$ ”) and (5.20) we obtain

$$\begin{aligned}
|\dot{A}(f_1)(t) - \dot{A}(f_2)(t)| &\leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \|f_1 - f_2\|}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right)(1-r)^\alpha} \\
&\times \left[\frac{\beta_1^l}{c(\alpha+1)(1-r)} + \frac{\beta_2}{c(\alpha+2)(1-r)^2} \right. \\
&+ \left. \left(\frac{\beta_1^l}{c\alpha} + \frac{\beta_2}{c(\alpha+1)(1-r)} \right) + \frac{4n^{\frac{1}{2}}}{(1-r)} \left(\frac{\beta_2}{\alpha+1} + \frac{\beta_3^s}{(\alpha+2)(1-r)} \right) \right. \\
&+ \left. \frac{2n^{\frac{1}{2}}}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right)} \left(\frac{\beta_2}{\alpha} + \frac{\beta_3^s}{(\alpha+1)(1-r)} \right) \right]. \quad (5.21)
\end{aligned}$$

□

Proof of Lemma 3.1. In this paragraph we shorten $z_-(v_-, x_-, \cdot)$ to z_- .

Similarly to (4.21) we have

$$\begin{aligned}
|z_-(\tau) + f(\tau)| &\geq \frac{|x_-|}{\sqrt{2}} - r + |\tau| \left(\frac{|v_-|}{\sqrt{2}} - \frac{2^{\frac{9}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha |v_-| \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} - r \right) \\
&\geq \frac{|x_-|}{\sqrt{2}} - r + |\tau| \left(\frac{|v_-|}{2\sqrt{2}} - r \right). \quad (5.22)
\end{aligned}$$

for $\tau \in \mathbb{R}$ and for $f \in M_{r,v_-,x_-}$. We used (2.4) (for $(x, w, v, q) = (x_-, v_-, v_-, 0)$) and the condition (2.2). Using (4.6), (4.5) and (5.22) we have

$$|F^s(z_- + f)(\tau)| \leq 2n\beta_2 \left(1 + \frac{|x_-|}{\sqrt{2}} - r + |\tau| \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\right)^{-\alpha-2}, \quad (5.23)$$

$$|F^l(z_- + f)(\tau)| \leq 2n\beta_1^l \left(1 + \frac{|x_-|}{\sqrt{2}} - r + |\tau| \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\right)^{-\alpha-1}, \quad (5.24)$$

for $\tau \in \mathbb{R}$. Hence using also condition (3.8) we obtain

$$\begin{aligned} & |g^{-1}(v_-) + \int_{-\infty}^t (\eta F(z_- + f_1) + (1 - \eta)(F^l + \mu F^s)(z_- + f_2))(\tau) d\tau| \\ & \geq |g^{-1}(v_-)| - \frac{4n}{\left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)^\alpha} \left(\frac{\beta_1^l}{\alpha} + \frac{\beta_2}{(\alpha + 1) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)}\right) \\ & \geq \frac{|g^{-1}(v_-)|}{2}, \end{aligned} \quad (5.25)$$

for $(t, \eta, \mu) \in \mathbb{R} \times [0, 1]^2$ and for $(f_1, f_2) \in M_{r,v_-,x_-}^2$.

Using (5.22), (4.7) and (4.8) we have

$$\begin{aligned} & |F^l(z_- + f_1) - F^l(z_- + f_2)|(\tau) \leq \frac{\frac{n\beta_1^l}{c} |\dot{f}_1 - \dot{f}_2|(\tau)}{\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) |\tau|\right)^{\alpha+1}} \\ & + \frac{2n^{\frac{3}{2}} \beta_2 |f_1 - f_2|(\tau)}{\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) |\tau|\right)^{\alpha+2}}, \end{aligned} \quad (5.26)$$

$$\begin{aligned} & |F^s(z_- + f_1)(\tau) - F^s(z_- + f_2)(\tau)| \leq \frac{\frac{n\beta_2}{c} |\dot{f}_1 - \dot{f}_2|(\tau)}{\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) |\tau|\right)^{\alpha+2}} \\ & + \frac{2n^{\frac{3}{2}} \beta_3^s |f_1 - f_2|(\tau)}{\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) |\tau|\right)^{\alpha+3}}, \end{aligned} \quad (5.27)$$

for $\tau \in \mathbb{R}$.

Then the proof of the following estimates (5.28), (5.29), (5.32), (5.30), (5.31) and (5.33) is similar to the proof of the estimates (5.9), (5.10), (5.12), (5.18), (5.19) and (5.21), and we have

$$|\dot{\mathcal{A}}(f)(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}r + 1)\right)}{(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}} - \left(\frac{|v_-|}{2\sqrt{2}} - r\right)t\right)^{\alpha+1}}, \quad (5.28)$$

$$|\mathcal{A}(f)(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}r + 1)\right)}{\alpha(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 \left(1 - r + \frac{|x_-|}{\sqrt{2}} - \left(\frac{|v_-|}{2\sqrt{2}} - r\right)t\right)^\alpha}, \quad (5.29)$$

$$\begin{aligned}
|\dot{\mathcal{A}}(f_1)(t) - \dot{\mathcal{A}}(f_2)(t)| &\leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \|f_1 - f_2\|_*}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)^{\alpha+1}} \\
&\times \left(\frac{\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2}{(\alpha+1)} + \frac{\frac{\beta_2}{c} + 2n^{\frac{1}{2}}\beta_3^s}{(\alpha+2)\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)} \right), \tag{5.30}
\end{aligned}$$

$$\begin{aligned}
|\mathcal{A}(f_1)(t) - \mathcal{A}(f_2)(t)| &\leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \|f_1 - f_2\|_*}{(\alpha+1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 \left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)^\alpha} \\
&\times \left(\frac{\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2}{\alpha} + \frac{\frac{\beta_2}{c} + 2n^{\frac{1}{2}}\beta_3^s}{(\alpha+2)\left(1 - r + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)|t|\right)} \right), \tag{5.31}
\end{aligned}$$

for $t \leq 0$, and

$$|\dot{\mathcal{A}}(f)(t)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}}}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \left(\frac{r\beta_1^l}{c} + 4\beta_2(n^{\frac{1}{2}}r + 1) \right) + \frac{r\beta_1^l}{c\alpha} + \frac{2n^{\frac{1}{2}}\beta_2r}{\alpha\left(\frac{|v_-|}{2\sqrt{2}} - r\right)}, \tag{5.32}$$

$$\begin{aligned}
|\dot{\mathcal{A}}(f_1)(t) - \dot{\mathcal{A}}(f_2)(t)| &\leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \|f_1 - f_2\|_*}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \\
&\times \left[\frac{\beta_1^l}{c(\alpha+1)\left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)} + \frac{\beta_2}{c(\alpha+2)\left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^2} \right. \\
&+ \left. \left(\frac{\beta_1^l}{c\alpha} + \frac{\beta_2}{c(\alpha+1)\left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)} \right) + \frac{4n^{\frac{1}{2}}}{\left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)} \left(\frac{\beta_2}{\alpha+1} + \frac{\beta_3^s}{(\alpha+2)\left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)} \right) \right. \\
&\left. + \frac{2n^{\frac{1}{2}}}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right)} \left(\frac{\beta_2}{\alpha} + \frac{\beta_3^s}{(\alpha+1)\left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)} \right) \right], \tag{5.33}
\end{aligned}$$

for $t \geq 0$. Then estimate (3.9) follows from (5.28), (5.29) and (5.32), and estimate (3.10) follows from (5.30), (5.31) and (5.33).

Now we prove (3.11). From (3.7), (5.25), (4.2), (5.23) and (5.24) it follows that

$$\begin{aligned}
|\tilde{k}(v_-, x_-, f) - v_-| &\leq 2n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_{-\infty}^{+\infty} |F(z_- + f)(\tau)| d\tau \\
&\leq \frac{8n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)^\alpha} \left(\frac{\beta_1^l}{\alpha} + \frac{\beta_2}{(\alpha+1)\left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)} \right). \tag{5.34}
\end{aligned}$$

□

6 Proof of Theorem 2.4

Estimate (2.20) follows from estimate (5.9). The proof of (2.21) is similar to the proof of (3.11) given at the end of Section 5. In this Section we shorten $z_-(v_-, \cdot)$, $a_{sc}(v_-, x_-)$, $a(v_-, x_-)$ and $b_{sc}(v_-, x_-)$ to z_- , a_{sc} , a and b_{sc} .

We prove (2.24). First note that

$$a_{sc} - \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_{-\infty}^{+\infty} F(.v_- + x_-)(\tau) d\tau = \Delta_{1,1} + \Delta_{1,2}, \quad (6.1)$$

where

$$\Delta_{1,1} := \lim_{t \rightarrow +\infty} (\dot{A}(y_-) - \dot{A}(0))(t), \quad (6.2)$$

$$\begin{aligned} \Delta_{1,2} := & \left(\langle \nabla g_j(g^{-1}(v_-)), \int_{-\infty}^{+\infty} (F(z_- + x_-)(\tau) - F(.v_- + x_-)(\tau)) d\tau \rangle \right)_{j=1\dots n} \\ & + \left(\int_0^1 \langle \nabla g_j(g^{-1}(v_-) + \varepsilon \int_{-\infty}^{+\infty} F(z_- + x_-)(\tau) d\tau) - \nabla g_j(g^{-1}(v_-)), \right. \\ & \left. \int_{-\infty}^{+\infty} F(z_- + x_-)(\tau) d\tau \rangle_{j=1\dots n} > d\varepsilon \right), \end{aligned} \quad (6.3)$$

and where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n . For the decomposition (6.1) we used (4.4) and the identity $\langle v_-, \int_{-\infty}^{+\infty} F(.v_- + x_-)(\tau) d\tau \rangle = 0$ to obtain

$$\left(\langle \nabla g_j(g^{-1}(v_-)), \int_{-\infty}^{+\infty} F(.v_- + x_-)(\tau) d\tau \rangle \right)_{j=1\dots n} = \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_{-\infty}^{+\infty} F(.v_- + x_-)(\tau) d\tau. \quad (6.4)$$

We prove the estimates (6.6) and (6.11) given below that provide a bound for $\Delta_{1,i}$, $i = 1, 2$. Then adding those bounds and using the decomposition (6.1) we obtain (2.24).

We use (5.21), and we have

$$|\Delta_{1,1}| \leq \frac{8n^{\frac{3}{2}} \max(\beta_1^l, \beta_2, \beta_3^s) \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{1}{c} + 2n^{\frac{1}{2}}\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right) \|y_-\|}{\alpha \left(\frac{|v_-|}{2\sqrt{2}} - r\right) (1-r)^{\alpha+2}}. \quad (6.5)$$

Then $\|y_-\| = \|A(y_-)\|$ is bounded by the right hand side of (2.13), and combining this upper bound with (6.5) we obtain

$$|\Delta_{1,1}| \leq \frac{32n^3 \max(\beta_1^l, \beta_2, \beta_3^s)^2 \left(1 - \frac{|v_-|^2}{c^2}\right) \left(\frac{r}{c} + 2n^{\frac{1}{2}}(|x_-| + r) + 2\right) \left(\frac{1}{c} + 2n^{\frac{1}{2}}\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)^2}{\alpha^2 \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 (1-r)^{2\alpha+3}}. \quad (6.6)$$

We use (4.1), (4.3) and (5.4), and we obtain

$$\begin{aligned} |\Delta_{1,2}| &\leq n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_{-\infty}^{+\infty} |F(z_- + x_-)(\tau) - F(v_- + x_-)(\tau)| d\tau \\ &\quad + \frac{6n}{c} \left(1 - \frac{|v_-|^2}{c^2}\right) \left| \int_{-\infty}^{+\infty} F(z_- + x_-)(\tau) d\tau \right|^2. \end{aligned} \quad (6.7)$$

Using (5.2) and (5.3) (with “ (f, r) ” = $(0, 0)$) we obtain

$$\left| \int_{-\infty}^{+\infty} F(z_- + x_-)(\tau) d\tau \right| \leq \frac{2^{\frac{7}{2}} n}{|v_-| \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \left(\frac{\beta_1^l}{\alpha} + \frac{\beta_2}{(\alpha+1) \left(1 + \frac{|x_-|}{\sqrt{2}}\right)} \right). \quad (6.8)$$

Then we use (4.7) and (2.4) (with “ (w, v, x, q) ” = $(v_-, v_-, 0, 0)$), and we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} |F^l(z_- + x_-)(\tau) - F^l(v_- + x_-)(\tau)| d\tau \\ &\leq \int_{-\infty}^{+\infty} \frac{\frac{n\beta_1^l}{c} \sup_{\mathbb{R}} |\dot{z}_- - v_-| d\tau}{\left(1 + \frac{|x_-|}{\sqrt{2}} + |\tau| \frac{|v_-|}{2^{\frac{3}{2}}}\right)^{\alpha+1}} + \int_{-\infty}^{+\infty} \frac{2n^{\frac{3}{2}} \beta_2 |\tau| \sup_{\mathbb{R}} |\dot{z}_- - v_-| d\tau}{\left(1 + \frac{|x_-|}{\sqrt{2}} + |\tau| \frac{|v_-|}{2^{\frac{3}{2}}}\right)^{\alpha+2}} \\ &\leq \frac{2^7 n^{\frac{5}{2}} \beta_1^l \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha^2 |v_-|^2 \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \left(\frac{\beta_1^l}{c} + \frac{2^{\frac{5}{2}} n^{\frac{1}{2}} \beta_2}{|v_-|} \right). \end{aligned} \quad (6.9)$$

Similarly we use (4.8) and (2.4), and we obtain

$$\int_{-\infty}^{+\infty} |F^s(z_- + x_-) - F^s(v_- + x_-)|(\tau) d\tau \leq \frac{2^7 n^{\frac{5}{2}} \beta_1^l \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{\beta_2}{c} + \frac{2^{\frac{5}{2}} n^{\frac{1}{2}} \beta_3^s}{|v_-|}\right)}{\alpha(\alpha+1) |v_-|^2 \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{\alpha+1}}. \quad (6.10)$$

Combining (6.7), (6.8), (6.9) and (6.10), we obtain

$$\begin{aligned} |\Delta_{1,2}| &\leq \frac{2^8 \cdot 3n^3 \left(1 - \frac{|v_-|^2}{c^2}\right)}{c |v_-|^2 \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{2\alpha}} \left(\frac{\beta_1^l}{\alpha} + \frac{\beta_2}{(\alpha+1) \left(1 + \frac{|x_-|}{\sqrt{2}}\right)} \right)^2 \\ &\quad + \frac{2^7 n^3 \beta_1^l \left(1 - \frac{|v_-|^2}{c^2}\right)}{\alpha |v_-|^2 \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} \left(\frac{1}{c} \left(\frac{\beta_1^l}{\alpha} + \frac{\beta_2}{(\alpha+1) \left(1 + \frac{|x_-|}{\sqrt{2}}\right)} \right) \right. \\ &\quad \left. + \frac{2^{\frac{5}{2}} n^{\frac{1}{2}}}{|v_-|} \left(\frac{\beta_2}{\alpha} + \frac{\beta_3^s}{(\alpha+1) \left(1 + \frac{|x_-|}{\sqrt{2}}\right)} \right) \right). \end{aligned} \quad (6.11)$$

Now we prove (2.22) and (2.23). We rewrite y_+ as follows

$$y_+ = h_0 + h_1, \quad (6.12)$$

where

$$h_0(t) := \int_t^{+\infty} \left(< \int_0^1 \nabla g_j(g^{-1}(a) - \varepsilon \int_\sigma^{+\infty} F(z_- + x_- + y_-)(\tau) d\tau \right. \quad (6.13)$$

$$\left. - (1 - \varepsilon) \int_\sigma^{+\infty} F^l(z_+(a, \cdot))(\tau) d\tau \right) d\tau d\varepsilon, \int_\sigma^{+\infty} F^s(z_- + x_- + y_-)(\tau) d\tau > \Big)_{j=1 \dots n} d\sigma,$$

$$h_1(t) := \int_t^{+\infty} \left(< \int_0^1 \nabla g_j(g^{-1}(a) - \varepsilon \int_\sigma^{+\infty} F(z_- + x_- + y_-)(\tau) d\tau \right.$$

$$\left. - (1 - \varepsilon) \int_\sigma^{+\infty} F^l(z_+(a, \cdot))(\tau) d\tau \right) d\tau d\varepsilon,$$

$$\int_\sigma^{+\infty} (F^l(z_- + x_- + y_-) - F^l(z_+(a, \cdot)))(\tau) d\tau > \Big)_{j=1 \dots n} d\sigma, \quad (6.14)$$

for $t \geq 0$. We estimate \dot{h}_0 . We also need the following estimate (6.16). For $\varepsilon, \varepsilon' \in (0, 1)$ and $\tau \geq 0$ we have

$$\begin{aligned} & |(1 - \varepsilon)(z_-(\tau) + y_-(\tau)) + \varepsilon z_+(a, \tau) + \varepsilon' x_-| \\ & \geq |\varepsilon' x_- + \tau v_-| - (1 - \varepsilon)|z_-(\tau) - \tau v_-| - r\tau - r \\ & \quad - \varepsilon|z_+(a, \tau) - \tau a| - \varepsilon|a_{sc}|\tau \\ & \geq \varepsilon' \frac{|x_-|}{\sqrt{2}} - r + \tau \left(\frac{|v_-|}{\sqrt{2}} - r - \frac{2^{\frac{9}{2}} n^{\frac{3}{2}} \beta_1^l \sqrt{1 - \frac{|v_-|^2}{c^2}}}{\alpha |v_-|} \right) - \varepsilon|a_{sc}|\tau. \end{aligned} \quad (6.15)$$

We used (2.4) for “ $(v, w, x, q) = (v_-, v_-, 0, 0)$ ” and for “ $(v, w, x, q) = (a, a, 0, 0)$ ”, and we used the identity $|v_-| = a$ that holds by conservation of energy. Then we use (2.19) and (6.15), and we have for $(\varepsilon, \varepsilon') \in (0, 1)^2$ and $\tau \geq 0$

$$|(1 - \varepsilon)(z_-(\tau) + y_-(\tau)) + \varepsilon z_+(a, \tau) + \varepsilon' x_-| \geq \varepsilon' \frac{|x_-|}{\sqrt{2}} - r + \tau \left(\frac{|v_-|}{2\sqrt{2}} - r \right). \quad (6.16)$$

Then the following estimate that is similar to (5.4) holds

$$\begin{aligned} & |g^{-1}(a) - \eta_1 \int_{-\infty}^{+\infty} F(z_- + x_- + y_-)(\tau) d\tau - \int_\sigma^{+\infty} (\eta_2 F(z_- + x_- + y_-) \\ & + \eta_3 F^l(z_+(a, \cdot)) + \eta_4 F^l(z_+(a, \cdot) + x_-))(\tau) d\tau| \geq \frac{1}{2} |g^{-1}(v_-)|, \end{aligned} \quad (6.17)$$

for $(\sigma, \eta_1, \eta_2, \eta_3, \eta_4) \in [0, +\infty) \times [0, 1]^4$, $\eta_1 + \eta_2 + \eta_3 + \eta_4 \leq 1$.

From (6.17), (4.1), (4.6) and (6.13) it follows that

$$\begin{aligned} |\dot{h}_0(t)| &\leq 2n^{\frac{1}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \int_t^{+\infty} |F^s(z_- + x_- + y_-)|(\tau) d\tau \\ &\leq \frac{4\beta_2 n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}}}{(\alpha + 1) \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} - r + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)t\right)^{\alpha+1}}, \end{aligned} \quad (6.18)$$

for $t \geq 0$. Now set

$$\delta_r := \max \left(\sup_{(0, +\infty)} |b_{sc} + y_+|, \sup_{(0, +\infty)} \left(1 - r + \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)s\right)^{-1} |\dot{y}_+(s)| \right). \quad (6.19)$$

We remind that δ_r is finite by Lemma 4.1 (for “ (x, z) ” = $(z_- + x_- + y_-, z_+(a, \cdot))$). Then we use (4.1), (4.7), (6.17) and (6.16), and we obtain

$$\begin{aligned} |\dot{h}_1(t)| &\leq 2n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_t^{+\infty} \frac{2n^{\frac{3}{2}} \beta_2 |x_-| + \left(\frac{n\beta_1^l}{c} + 2n^{\frac{3}{2}} \beta_2\right) \delta_r}{\left(1 - r + \tau \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\right)^{\alpha+2}} d\tau d\sigma \\ &\leq \frac{2n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(2n^{\frac{1}{2}} \beta_2 |x_-| + \left(\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}} \beta_2\right) \delta_r\right)}{(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + t \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\right)^{\alpha+1}}, \end{aligned} \quad (6.20)$$

for $t \geq 0$. Hence combining (6.12), (6.18) and (6.20) we have

$$|\dot{y}_+(t)| \leq \frac{2n^{\frac{3}{2}} \left(2\beta_2 + 2n^{\frac{1}{2}} \beta_2 |x_-| + \left(\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}} \beta_2\right) \delta_r\right) \sqrt{1 - \frac{|v_-|^2}{c^2}}}{(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 - r + t \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)\right)^{\alpha+1}}, \quad (6.21)$$

for $t \geq 0$. In addition from (2.17) and (5.10) at $t = 0$ it follows that

$$|b_{sc} + y_+(0)| = |A(y_-)(0)| \leq \frac{2n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2 \left(n^{\frac{1}{2}} (|x_-| + r) + 1\right)\right)}{\alpha(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 (1 - r)^\alpha}. \quad (6.22)$$

From (6.19), (6.21) and (6.22) it follows that

$$\delta_r \leq \frac{2n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2 \left(n^{\frac{1}{2}} (2|x_-| + r) + 2\right) + \left(\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}} \beta_2\right) \delta_r\right) \max \left(1, \frac{1}{\alpha \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)}\right)}{(\alpha + 1) \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right) (1 - r)^\alpha}. \quad (6.23)$$

Then we use condition (2.19) and we obtain

$$\frac{\delta_r}{2} \leq \frac{2n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(2|x_-| + r) + 2) \right)}{(\alpha + 1) \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r \right) (1 - r)^\alpha} \max \left(1, \frac{1}{\alpha \left(\frac{|v_-|}{2^{\frac{3}{2}}} - r \right)} \right). \quad (6.24)$$

Estimate (6.24) and condition (2.19) also provide the following estimate

$$\left(\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2 \right) \delta_r \leq \frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(2|x_-| + r) + 2). \quad (6.25)$$

Estimate (2.23) follows from (6.21) and (6.25). Then we integrate over $(0, +\infty)$ both sides of (2.23) and we obtain a bound on $|y_+(0)|$. Then we add this bound with the bound given in (5.10) for $A(y_-)(0)$, and we use (2.17) and we obtain (2.22).

It remains to prove (2.25). From (2.17), (6.12), (6.13) and (6.14) at $t = 0$ it follows by straightforward computations that

$$\begin{aligned} \sum_{j=1}^8 \Delta_{2,j} &= b_{sc}(v_-, x_-) - \int_{-\infty}^0 \left(g(g^{-1}(v_-) + \int_{-\infty}^{\sigma} F^l(z_- + x_-)(\tau) d\tau) \right. \\ &\quad \left. - g(g^{-1}(v_-) + \int_{-\infty}^{\sigma} F^l(z_-)(\tau) d\tau) \right) d\sigma - \int_0^{+\infty} \left(g(g^{-1}(a) - \int_{\sigma}^{+\infty} F^l(z_+(a, \cdot) + x_-)(\tau) d\tau) \right. \\ &\quad \left. - g(g^{-1}(a) - \int_{\sigma}^{+\infty} F^l(z_+(a, \cdot))(\tau) d\tau) \right) d\sigma - \left(\langle \nabla g_j(g^{-1}(v_-)), \right. \\ &\quad \left. \int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\cdot v_- + x_-)(\tau) d\tau d\sigma - \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\cdot v_- + x_-)(\tau) d\tau d\sigma \right)_{j=1 \dots n}, \end{aligned} \quad (6.26)$$

where

$$\Delta_{2,1} := A(y_-)(0) - A(0)(0), \quad (6.27)$$

$$\begin{aligned} \Delta_{2,2} &:= \int_{-\infty}^0 \left(\langle \nabla g_j(g^{-1}(v_-) + \varepsilon \int_{-\infty}^{\sigma} F(z_- + x_-)(\tau) d\tau) \right. \\ &\quad \left. + (1 - \varepsilon) \int_{-\infty}^{\sigma} F^l(z_- + x_-)(\tau) d\tau - \nabla g_j(g^{-1}(v_-)), \int_{-\infty}^{\sigma} F^s(z_- + x_-)(\tau) d\tau \right)_{j=1 \dots n} d\sigma, \end{aligned} \quad (6.28)$$

$$\Delta_{2,3} := \int_{-\infty}^0 \left(\langle \nabla g_j(g^{-1}(v_-)), \int_{-\infty}^{\sigma} (F^s(z_- + x_-) - F^s(\cdot v_- + x_-))(\tau) d\tau \right)_{j=1 \dots n} d\sigma, \quad (6.29)$$

$$\begin{aligned} \Delta_{2,4} := & - \int_0^{+\infty} \left(\int_0^1 (\nabla g_j(g^{-1}(a)) - \varepsilon \int_\sigma^{+\infty} F(z_- + x_- + y_-)(\tau) d\tau \right. \\ & \left. - (1 - \varepsilon) \int_\sigma^{+\infty} F^l(z_+(a, \cdot))(\tau) d\tau - \nabla g_j(g^{-1}(v_-)) \right) d\varepsilon, \\ & \left. \int_\sigma^{+\infty} F^s(z_- + x_- + y_-)(\tau) d\tau > \right)_{j=1\dots n} d\sigma, \end{aligned} \quad (6.30)$$

$$\Delta_{2,5} := - \int_0^{+\infty} \left(\int_0^1 \nabla g_j(g^{-1}(v_-)), \int_\sigma^{+\infty} (F^s(z_- + x_- + y_-) - F^s(z_- + x_-))(\tau) d\tau > \right)_{j=1\dots n} d\sigma, \quad (6.31)$$

$$\Delta_{2,6} := - \int_0^{+\infty} \left(\int_0^1 \nabla g_j(g^{-1}(v_-)), \int_\sigma^{+\infty} (F^s(z_- + x_-) - F^s(v_- + x_-))(\tau) d\tau > \right)_{j=1\dots n} d\sigma, \quad (6.32)$$

$$\begin{aligned} \Delta_{2,7} := & - \int_0^{+\infty} \left(\int_0^1 \nabla g_j(g^{-1}(a)) - \varepsilon \int_\sigma^{+\infty} F(z_- + x_- + y_-)(\tau) d\tau \right. \\ & \left. - (1 - \varepsilon) \int_\sigma^{+\infty} F^l(z_+(a, \cdot))(\tau) d\tau \right) d\varepsilon, \\ & \left. \int_\sigma^{+\infty} (F^l(z_- + x_- + y_-) - F^l(z_+(a, \cdot) + x_-))(\tau) d\tau > \right)_{j=1\dots n} d\sigma, \end{aligned} \quad (6.33)$$

$$\begin{aligned} \Delta_{2,8} := & - \int_0^{+\infty} \left(\int_0^1 (\nabla g_j(g^{-1}(a)) - \varepsilon \int_\sigma^{+\infty} F(z_- + x_- + y_-)(\tau) d\tau \right. \\ & \left. - (1 - \varepsilon) \int_\sigma^{+\infty} F^l(z_+(a, \cdot))(\tau) d\tau \right. \\ & \left. - \nabla g_j(g^{-1}(a)) - \varepsilon \int_\sigma^{+\infty} F^l(z_+(a, \cdot) + x_-)(\tau) d\tau - (1 - \varepsilon) \int_\sigma^{+\infty} F^l(z_+(a, \cdot))(\tau) d\tau \right) d\varepsilon, \\ & \left. \int_\sigma^{+\infty} (F^l(z_+(a, \cdot) + x_-) - F^l(z_+(a, \cdot))) (\tau) d\tau > \right)_{j=1\dots n} d\sigma. \end{aligned} \quad (6.34)$$

Using (5.19) (for $(f_1, f_2) = (y_-, 0)$ at time $t = 0$) and (2.13) we obtain

$$|\Delta_{2,1}| \leq \frac{12n^3 \left(1 - \frac{|v_-|^2}{c^2}\right) \max(\beta_1^l, \beta_2, \beta_3^s)^2 \left(\frac{r}{c} + 2(n^{\frac{1}{2}}(|x_-| + r) + 1)\right) \left(\frac{1}{c} + 2n^{\frac{1}{2}}\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)}{\alpha^2(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^3 (1 - r)^{2\alpha+2}}. \quad (6.35)$$

Using (4.3), (5.4) and then (4.5), (4.6) and (4.21) (with “ (f, r) ” = $(0, 0)$) we

have

$$\begin{aligned}
|\Delta_{2,2}| &\leq \frac{6n}{c} \left(1 - \frac{|v_-|^2}{c^2}\right) \int_{-\infty}^0 \max \left(\int_{-\infty}^{\sigma} |F(z_-(v_-, \cdot) + x_-)(\tau)| d\tau, \right. \\
&\quad \left. \int_{-\infty}^{\sigma} |F^l(z_-(v_-, \cdot) + x_-)(\tau)| d\tau \right) \int_{-\infty}^{\sigma} |F^s(z_-(v_-, \cdot) + x_-)(\tau)| d\tau d\sigma \\
&\leq \int_{-\infty}^0 \left(\frac{2^{\frac{5}{2}} \beta_1^l n}{\alpha |v_-| \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{\alpha}} + \frac{2^{\frac{5}{2}} \beta_2 n}{(\alpha + 1) |v_-| \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{\alpha+1}} \right) \frac{\frac{6n^2}{c} \left(1 - \frac{|v_-|^2}{c^2}\right) 2^{\frac{5}{2}} \beta_2 d\sigma}{(\alpha + 1) |v_-| \left(1 + \frac{|x_-|}{\sqrt{2}} + \frac{|v_-|}{2\sqrt{2}} |\sigma|\right)^{\alpha+1}} \\
&\leq \frac{48n^3 \left(1 - \frac{|v_-|^2}{c^2}\right) \max(\beta_1^l, \beta_2)^2}{c\alpha^2 (\alpha + 1) \left(\frac{|v_-|}{2^{\frac{3}{2}}}\right)^3 \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{2\alpha}}. \tag{6.36}
\end{aligned}$$

We use (4.1), (4.8), (4.21) (with “(f, r)” = (0, 0)) and (2.4), and we obtain

$$\begin{aligned}
\max(|\Delta_{2,3}|, |\Delta_{2,6}|) &\leq \frac{2^{\frac{9}{2}} n^2 \beta_1^l \left(1 - \frac{|v_-|^2}{c^2}\right)}{\alpha |v_-|} \int_0^{+\infty} \int_{\sigma}^{+\infty} \left[\frac{n\beta_2}{c} \left(1 + \frac{|x_-|}{\sqrt{2}} + \frac{|v_-|}{2\sqrt{2}} \tau\right)^{-\alpha-2} \right. \\
&\quad \left. + 2n^{\frac{3}{2}} \beta_3^s \tau \left(1 + \frac{|x_-|}{\sqrt{2}} + \left(\frac{|v_-|}{2\sqrt{2}} \tau\right)^{-\alpha-3}\right) \right] d\tau d\sigma \\
&\leq \frac{8n^3 \max(\beta_1^l, \beta_2, \beta_3^s)^2 \left(1 - \frac{|v_-|^2}{c^2}\right)}{\alpha^2 (\alpha + 1) \left(\frac{|v_-|}{2^{\frac{3}{2}}}\right)^3 \left(1 + \frac{|x_-|}{\sqrt{2}}\right)^{\alpha}} \left(\frac{1}{c} + \frac{2n^{\frac{1}{2}}}{\frac{|v_-|}{2^{\frac{3}{2}}}}\right). \tag{6.37}
\end{aligned}$$

We use (4.3) and (6.17), and then we use (4.5), (4.6) and (6.16), and we have

$$\begin{aligned}
|\Delta_{2,4}| &\leq \frac{6n}{c} \left(1 - \frac{|v_-|^2}{c^2}\right) \int_0^{+\infty} \left(\int_{-\infty}^{+\infty} |F(z_- + x_- + y_-)(\tau)| d\tau \right. \\
&\quad \left. + \int_{\sigma}^{+\infty} |F^l(z_+(a, \cdot))(\tau)| d\tau \right) \int_{\sigma}^{+\infty} |F^s(z_- + x_- + y_-)(s)| ds d\sigma \\
&\leq \frac{6n}{c} \left(1 - \frac{|v_-|^2}{c^2}\right) \int_0^{+\infty} \left(\int_0^{+\infty} \frac{4n\beta_2 d\tau}{\left(1 + \frac{|x_-|}{\sqrt{2}} - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\tau\right)^{\alpha+2}} \right. \\
&\quad \left. + \int_0^{+\infty} \frac{6n\beta_1^l d\tau}{\left(1 - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\tau\right)^{\alpha+1}} \right) \int_{\sigma}^{+\infty} \frac{2n\beta_2 ds d\sigma}{\left(1 + \frac{|x_-|}{\sqrt{2}} - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)s\right)^{\alpha+2}} \\
&\leq \frac{24n^3 \left(1 - \frac{|v_-|^2}{c^2}\right) \max(\beta_1^l, \beta_2)^2}{c\alpha^2 (\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^3 (1 - r)^{2\alpha}} \left(3 + \frac{2}{1 + \frac{|x_-|}{\sqrt{2}} - r}\right). \tag{6.38}
\end{aligned}$$

We use (4.1), (5.13) and (2.13) ($y_- = A(y_-)$), and we obtain

$$\begin{aligned}
|\Delta_{2,5}| &\leq n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_0^{+\infty} \int_{\sigma}^{+\infty} |F^s(z_- + x_- + y_-)(\tau) - F^s(z_- + x_-)(\tau)| d\tau d\sigma \\
&\leq n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \|y_-\| \int_0^{+\infty} \int_{\sigma}^{+\infty} \left(\frac{\frac{n\beta_2}{c}}{\left(1 + \frac{|x_-|}{\sqrt{2}} - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\tau\right)^{\alpha+2}} \right. \\
&\quad \left. + \frac{2n^{\frac{3}{2}}\beta_3^s(1+\tau)}{\left(1 + \frac{|x_-|}{\sqrt{2}} - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\tau\right)^{\alpha+3}} \right) d\tau d\sigma \\
&\leq \frac{4n^3 \left(1 - \frac{|v_-|^2}{c^2}\right) \max(\beta_1^l, \beta_2, \beta_3^s)^2 \left(\frac{r}{c} + 2(n^{\frac{1}{2}}(|x_-| + r) + 1)\right) \left(\frac{1}{c} + \frac{2n^{\frac{1}{2}}}{\frac{|v_-|}{2\sqrt{2}} - r} + n^{\frac{1}{2}}\right) \left(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r}\right)}{\alpha^2(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^3 (1 - r)^{2\alpha+2}}.
\end{aligned} \tag{6.39}$$

From (6.17), (4.1), (4.7), (6.16) and (6.24) it follows that

$$\begin{aligned}
|\Delta_{2,7}| &\leq 2n^{\frac{1}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_0^{+\infty} \int_{\sigma}^{+\infty} \frac{\left(\frac{n\beta_1^l}{c} + 2n^{\frac{3}{2}}\beta_2\right) \delta_r d\tau d\sigma}{\left(1 + \frac{|x_-|}{\sqrt{2}} - r + \left(\frac{|v_-|}{2\sqrt{2}} - r\right)\tau\right)^{\alpha+2}} \\
&\leq \frac{8n^3 \left(1 - \frac{|v_-|^2}{c^2}\right) \left(\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2\right) \left(\frac{r\beta_1^l}{c} + 2\beta_2(n^{\frac{1}{2}}(2|x_-| + r) + 2)\right) \max\left(1, \frac{1}{\alpha\left(\frac{|v_-|}{2\sqrt{2}} - r\right)}\right)}{\alpha(\alpha + 1)^2 \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^3 (1 - r)^{2\alpha}}.
\end{aligned} \tag{6.40}$$

Combining (6.16), (4.3), (6.17), (4.7), (4.5) and (4.6) we obtain

$$\begin{aligned}
|\Delta_{2,8}| &\leq \frac{6n\left(1 - \frac{|v_-|^2}{c^2}\right)}{c} \int_0^{+\infty} \left(\int_{\sigma}^{+\infty} |F(z_- + x_- + y_-)(\tau)| \right. \\
&\quad \left. + |F^l(z_+(a, \cdot) + x_-)(\tau)| d\tau \right) \int_{\sigma}^{+\infty} \frac{2n^{\frac{3}{2}}\beta_2|x_-|}{\left(1 + \frac{|v_-|}{2\sqrt{2}}\tau\right)^{\alpha+2}} d\tau d\sigma \\
&\leq \frac{24n^{\frac{7}{2}}\beta_2|x_-| \left(1 - \frac{|v_-|^2}{c^2}\right) \left(\frac{2\beta_1^l}{\alpha} + \frac{\beta_3^s}{(\alpha+1)\left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)}\right)}{c\alpha(\alpha + 1) \left(\frac{|v_-|}{2\sqrt{2}}\right)^2 \left(\frac{|v_-|}{2\sqrt{2}} - r\right) \left(1 + \frac{|x_-|}{\sqrt{2}} - r\right)^{\alpha}}.
\end{aligned} \tag{6.41}$$

Then we add the bounds on the right-hand sides of (6.35)–(6.41), and we use (6.26), and we obtain (2.25). \square

7 Proof of Lemma 3.2 and Theorem 3.3

7.1 Preliminary Lemma

Lemma 7.1. *Let $(v, w, x, q) \in \mathcal{B}(0, c)^2 \times \mathbb{R}^n \times \mathcal{B}(0, 1)$ so that $|v| = |w| \neq 0$, $|v - w| < \frac{|v|}{2^{\frac{3}{2}}}$ and $v \cdot x = 0$. Assume that*

$$\frac{2^{\frac{7}{2}} n^{\frac{3}{2}} \left(\frac{1}{c} + 2n^{\frac{1}{2}}\right) \max(\beta_1^l, \beta_2^l) \sqrt{1 - \frac{|v|^2}{c^2}}}{\alpha |v| \left(1 + \frac{|x|}{\sqrt{2}} - |q|\right)^\alpha} \left(1 + \frac{2^{\frac{3}{2}}}{|v|}\right) \leq 1. \quad (7.1)$$

Then the following estimates are valid

$$|\eta_1(z_-(v, x, t) + f(t)) + \eta_2 z_+(w, x + q, t) + \eta_3 z_+(w, x + q', t)| \geq \frac{|x|}{\sqrt{2}} - (\eta_2 + \eta_3)|q| - \eta_1 r + \left(\frac{|v|}{2^{\frac{3}{2}}} - \eta_1 r\right)t, \quad (7.2)$$

$$\delta_{+, q, q'} := \max\left(\sup_{(0, +\infty)} |\omega_{+, q, q'}|, \sup_{t \in (0, +\infty)} \left(1 + \frac{|x|}{\sqrt{2}} - |q| + t \frac{|v|}{2^{\frac{3}{2}}}\right) |\dot{\omega}_{+, q, q'}(t)|\right) \leq 2|q - q'|, \quad (7.3)$$

for $(r, \eta_1, \eta_2, \eta_3, t, q') \in (0, \min(\frac{|v|}{2^{\frac{3}{2}}}, 1)) \times [0, 1]^3 \times [0, +\infty) \times \mathbb{R}^n$ and for $f \in M_{r, v, x}$ so that $\eta_1 + \eta_2 + \eta_3 = 1$, $|q'| \leq |q|$, where $\omega_{+, q, q'} = z_+(w, x + q, \cdot) - z_+(w, x + q', \cdot)$.

Proof of Lemma 7.1. We have

$$\begin{aligned} & |\eta_1(z_-(v, x, t) + f(t)) + \eta_2 z_+(w, x + q, t) + \eta_3 z_+(w, x + q', t)| \\ & \geq |x + \eta_2 q + \eta_3 q' + tv| - \eta_1 |z_-(v, x, t) - tv - x| - \eta_1 |f(t)| - (\eta_2 + \eta_3) |v - w| |t| \\ & \quad - \eta_2 |z_+(w, x + q, t) - wt - x - q| - \eta_3 |z_+(w, x + q', t) - wt - x - q'|. \end{aligned} \quad (7.4)$$

Then we use (2.4), (2.2), (5.22), the equality $x \cdot v = 0$ and the estimates $|q'| \leq |q|$ and $|v - w| \leq \frac{|v|}{2^{\frac{3}{2}}}$ to obtain (7.2).

From Lemma 4.1 it follows that $\delta_{+, q, q'}$ is finite. Combining (4.7) and (7.4) we obtain

$$|F^l(z_+(w, x + q, \cdot))(\tau) - F^l(z_+(w, x + q', \cdot))(\tau)| \leq \frac{\left(\frac{n\beta_1^l}{c} + 2n^{\frac{3}{2}}\beta_2^l\right) \delta_{+, q, q'}}{\left(1 + \frac{|x|}{\sqrt{2}} - |q| + \tau \frac{|v|}{2^{\frac{3}{2}}}\right)^{\alpha+2}}, \quad (7.5)$$

for $\tau \geq 0$. Note that

$$\begin{aligned} \dot{\omega}_{+, q, q'}(t) &= g(g^{-1}(w) - \int_t^{+\infty} F^l(z_+(w, x + q, \cdot))(\tau) d\tau) \\ &\quad - g(g^{-1}(w) - \int_t^{+\infty} F^l(z_+(w, x + q', \cdot))(\tau) d\tau), \end{aligned} \quad (7.6)$$

for $t \geq 0$. Combining (7.5), (7.6) and (4.2) and (4.14) we have

$$|\dot{\omega}_{+,q,q'}(t)| \leq \frac{2^{\frac{5}{2}}n^{\frac{3}{2}}\sqrt{1-\frac{|v|^2}{c^2}}\left(\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2^l\right)\delta_{+,q,q'}}{(\alpha+1)|v|\left(1+\frac{|x|}{\sqrt{2}}-|q|+t\frac{|v|}{2^{\frac{3}{2}}}\right)^{\alpha+1}}, \quad (7.7)$$

for $t \geq 0$. Then we use the estimate (7.7) and the estimate $|g(t)| \leq |g(0)| + \int_0^t |\dot{g}(s)|ds$ for $t \geq 0$ and $g = \omega_{+,q,q'}$ ($g(0) = q - q'$), and we obtain a bound on $\omega_{+,q,q'}$, and then we have

$$\delta_{+,q,q'} \leq |q - q'| + \frac{2^{\frac{5}{2}}n^{\frac{3}{2}}\left(\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2^l\right)\max\left(1,\frac{2^{\frac{3}{2}}}{\alpha|v|}\right)\sqrt{1-\frac{|v|^2}{c^2}}\delta_{+,q,q'}}{(\alpha+1)|v|\left(1+\frac{|x|}{\sqrt{2}}-|q|\right)^\alpha}. \quad (7.8)$$

Using (7.1) we obtain $\delta_{+,q,q'} \leq |q - q'| + \frac{\delta_{+,q,q'}}{2}$, which proves (7.3). \square

For the rest of the text we shorten $z_-(v_-, x_-, \cdot)$, $\tilde{a}_{sc}(v_-, x_-)$, $\tilde{a}(v_-, x_-)$ and $\tilde{b}_{sc}(v_-, x_-)$ to z_- , \tilde{a}_{sc} , \tilde{a} and \tilde{b}_{sc} .

7.2 Proof of Lemma 3.2

We first need to estimate

$$\delta := \max\left(\sup_{(0,+\infty)} |\omega|, \sup_{t \in (0,+\infty)} \left(1 + \frac{|x_-|}{\sqrt{2}} - r + t\left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)\right)|\dot{\omega}(t)|\right). \quad (7.9)$$

where $\omega = z_- + y_- - z_+(\tilde{a}, x_-, \cdot)$. Then under condition (3.12) we have

$$|g^{-1}(v_-)| - \int_{-\infty}^{+\infty} |F(z_- + y_-)(\tau)|d\tau - \int_0^{+\infty} |F^l(z_+(\tilde{a}, x_-, \cdot))(\tau)|d\tau \geq \frac{1}{2}|g^{-1}(v_-)|. \quad (7.10)$$

Note also that $\omega(0) = y_-(0)$ and

$$\dot{\omega}(t) = g(g^{-1}(\tilde{a}) - \int_t^{+\infty} F(z_- + y_-)(\tau)d\tau) - g(g^{-1}(\tilde{a}) - \int_t^{+\infty} F^l(z_+(\tilde{a}, x_-, \cdot))(\tau)d\tau), \quad (7.11)$$

for $t \geq 0$. Then similarly to (6.21) we obtain

$$|\dot{\omega}(t)| \leq \frac{2n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}(2\beta_2 + \delta\left(\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2\right))}{(\alpha+1)\left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)\left(1 - r + \frac{|x_-|}{\sqrt{2}} + t\left(\frac{|v_-|}{2^{\frac{3}{2}}} - r\right)\right)^{\alpha+1}}, \quad (7.12)$$

$$\delta \leq |y_-(0)| + \frac{2n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} (2\beta_2 + \delta(\frac{\beta_1^t}{c} + 2n^{\frac{1}{2}}\beta_2)) \max(1, \frac{1}{\alpha(\frac{|v_-|}{2^{\frac{3}{2}}} - r)})}{(\alpha + 1)(\frac{|v_-|}{2^{\frac{3}{2}}} - r)(1 - r + \frac{|x_-|}{\sqrt{2}})^\alpha}, \quad (7.13)$$

for $t \geq 0$. Then from (3.12) and (5.29) ($y_-(0) = (\mathcal{A}(y_-)(0))$) and from the estimate $r \leq \frac{1}{2}$ it follows that

$$\frac{\delta}{2} \leq \frac{2n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} (2\beta_2 + \frac{r\beta_1^t}{c} + (n^{\frac{1}{2}}r + 1)2\beta_2) \max(1, \frac{1}{\alpha(\frac{|v_-|}{2^{\frac{3}{2}}} - r)})}{(\alpha + 1)(\frac{|v_-|}{2^{\frac{3}{2}}} - r)(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}})^\alpha}. \quad (7.14)$$

In addition under condition (3.12) we have $\delta(\frac{\beta_1^t}{c} + 2n^{\frac{1}{2}}\beta_2) \leq (2\beta_2 + \frac{r\beta_1^t}{c} + (n^{\frac{1}{2}}r + 1)2\beta_2)$, and from (7.12) it follows that

$$|\dot{\omega}(t)| \leq \frac{2n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} (\frac{r\beta_1^t}{c} + (n^{\frac{1}{2}}r + 3)2\beta_2)}{(\alpha + 1)(\frac{|v_-|}{2^{\frac{3}{2}}} - r)(1 - r + \frac{|x_-|}{\sqrt{2}} + t(\frac{|v_-|}{2^{\frac{3}{2}}} - r))^{\alpha+1}}, \quad (7.15)$$

Then note that

$$\mathcal{G}_{v_-, x_-}(q) = \mathcal{A}(y_-)(0) + \int_0^{+\infty} (\dot{\omega}(s) - \dot{\omega}_{+, q, 0}(s)) ds, \quad (7.16)$$

where $\omega_{+, q, q'}$ is defined in Lemma 7.1 for “ w ” = \tilde{a} and for any $q' \in \overline{\mathcal{B}(0, \frac{1}{2})}$. Then we use (5.29) at $t = 0$, (7.7) and (7.3) (“ $q' = 0$ ”) and (7.15), and we obtain

$$|\mathcal{G}_{v_-, x_-}(q)| \leq \frac{4n^{\frac{3}{2}} \left(1 - \frac{|v_-|^2}{c^2}\right)^{\frac{1}{2}} \left(\left(\frac{\beta_1^t}{c} + 2n^{\frac{1}{2}}\beta_2\right)r + 4\beta_2\right)}{\alpha(\alpha + 1)\left(\frac{|v_-|}{2\sqrt{2}} - r\right)^2 \left(1 - r + \frac{|x_-|}{\sqrt{2}}\right)^\alpha} + \frac{2^5 n^{\frac{3}{2}} \sqrt{1 - \frac{|v_-|^2}{c^2}} \left(\frac{\beta_1^t}{c} + 2n^{\frac{1}{2}}\beta_2\right) |q|}{\alpha(\alpha + 1) |v_-|^2 \left(1 + \frac{|x_-|}{\sqrt{2}} - |q|\right)^\alpha}, \quad (7.17)$$

for $|q| \leq \frac{1}{2}$. Estimates (3.17) follow from (7.17), condition (3.12) and $\max(r, |q|) \leq \frac{1}{2}$. Note also that for $(q, q') \in \overline{\mathcal{B}(0, \frac{1}{2})}^2$

$$\mathcal{G}_{v_-, x_-}(q) - \mathcal{G}_{v_-, x_-}(q') = \int_0^{+\infty} \dot{\omega}_{+, q, q'}(s) ds. \quad (7.18)$$

Then we use (7.7) and (7.3) and condition (3.12), and we obtain (3.18). \square

7.3 Proof of Theorem 3.3

Estimate (3.25) follows from the identity $\mathcal{A}(y_-) = y_-$ and (5.28) at $t = 0$. Estimate (3.26) follows from (3.11) and $r \leq \frac{1}{2}$. Estimate (3.27) follows from (3.17). We add (7.15) and (7.7) for $(q, q') = (\tilde{b}_{sc}, 0)$, and we use (7.3) and we obtain

$$|\dot{y}_+(t)| \leq \frac{8n^2 \sqrt{1 - \frac{|v_-|^2}{c^2}} \max(\beta_1^l, \beta_2) (\frac{1}{c} + 1) (2 + |\tilde{b}_{sc}|)}{(\alpha + 1) (\frac{|v_-|}{2^{\frac{3}{2}}} - r) (\frac{1}{2} + \frac{|x_-|}{\sqrt{2}} + t (\frac{|v_-|}{2^{\frac{3}{2}}} - r))^{\alpha+1}}. \quad (7.19)$$

Then we use $\tilde{b}_{sc} \leq \frac{1}{2}$, and we obtain (3.28).

Now we prove (3.29). Note that

$$\tilde{a}_{sc}(v_-, x_-) - \tilde{W}(v_-, x_-) - \sqrt{1 - \frac{|v_-|^2}{c^2}} \int_{-\infty}^{+\infty} F^s(\tau v_- + x_-, v_-) d\tau =: \sum_{j=1}^3 \Delta_{fc,j}, \quad (7.20)$$

where

$$\begin{aligned} \Delta_{fc,1} &:= \tilde{a}(v_-, x_-) - g(g^{-1}(v_-)) + \int_{-\infty}^0 F^l(z_-)(\tau) d\tau \\ &\quad + \int_{-\infty}^{+\infty} F^s(z_-)(\tau) d\tau + \int_0^{+\infty} F^l(z_+(\tilde{a}, x_-, \cdot))(\tau) d\tau, \end{aligned} \quad (7.21)$$

$$\begin{aligned} \Delta_{fc,2} &:= \left(\langle \int_0^1 (\nabla g_j(g^{-1}(v_-)) + \int_{-\infty}^0 F^l(z_-)(\tau) d\tau + \int_0^{+\infty} F^l(z_+(\tilde{a}, x_-, \cdot))(\tau) d\tau \right. \\ &\quad \left. + \varepsilon \int_{-\infty}^{+\infty} F^s(z_-)(\tau) d\tau - \nabla g_j(g^{-1}(v_-)) \rangle d\varepsilon, \int_{-\infty}^{+\infty} F^s(z_-)(\tau) d\tau \right)_{j=1 \dots n}, \end{aligned} \quad (7.22)$$

$$\Delta_{fc,3} := \left(\langle \nabla g_j(g^{-1}(v_-)) \rangle, \int_{-\infty}^{+\infty} (F^s(z_-)(\tau) - F^s(\cdot v_- + x_-)(\tau)) d\tau \right)_{j=1 \dots n}, \quad (7.23)$$

and where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^n . Under condition (3.12) it follows that

$$\begin{aligned} &|g^{-1}(v_-)| - \int_{-\infty}^{+\infty} ((|F^l| + |F^s|)(z_- + y_-)(\tau) + (|F^l| + |F^s|)(z_-)(\tau)) d\tau \\ &+ |F^l(z_+(\tilde{a}, x_-, \cdot))(\tau)| d\tau \geq \frac{|g^{-1}(v_-)|}{2}. \end{aligned} \quad (7.24)$$

Then we use (4.2) and we have

$$\begin{aligned}
|\Delta_{fc,1}| &\leq 2n^{\frac{1}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\left(\int_{-\infty}^0|F(z_-+y_-)(\tau)-F(z_-)(\tau)|d\tau\right. \\
&\quad + \int_0^{+\infty}|F^s(z_-+y_-)(\tau)-F^s(z_-)(\tau)|d\tau \\
&\quad \left. + \int_0^{+\infty}|F^l(z_-+y_-)(\tau)-F^l(z_+(\tilde{a},x_-,\cdot))(\tau)|d\tau\right). \quad (7.25)
\end{aligned}$$

Hence from (4.7) and (4.8) it follows that

$$\begin{aligned}
|\Delta_{fc,1}| &\leq 2n^{\frac{3}{2}}\sqrt{1-\frac{|v_-|^2}{c^2}}\left(\frac{(\frac{\beta_1^l}{c}+2n^{\frac{1}{2}}\beta_2)(\|y_-\|_*+\delta)}{(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha+1}}\right. \\
&\quad \left. + \frac{(\frac{2\beta_2}{c}+4n^{\frac{1}{2}}\beta_3^s)\|y_-\|_*}{(\alpha+2)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha+2}} + \frac{2n^{\frac{1}{2}}\beta_3^s\|y_-\|_*}{(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^2(1+\frac{|x_-|}{\sqrt{2}}-r)^{\alpha+1}}\right), \quad (7.26)
\end{aligned}$$

where δ is defined in (7.9). We use the bound (7.14) on δ and the bound (3.9) on $y_- = \mathcal{A}(y_-)$, and we use the estimate $r \leq \frac{1}{2}$, and we obtain

$$|\Delta_{fc,1}| \leq \frac{768n^4\beta^2(1-\frac{|v_-|^2}{c^2})(\frac{r}{c}+1)(1+\frac{1}{c})(1+\frac{1}{(\frac{|v_-|}{2^{\frac{3}{2}}}-r)})^2}{\alpha^2(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^2(\frac{1}{2}+\frac{|x_-|}{\sqrt{2}})^{2\alpha+1}}. \quad (7.27)$$

Then we use (7.24) and (4.3), and we obtain

$$\begin{aligned}
|\Delta_{fc,2}| &\leq \frac{6n(1-\frac{|v_-|^2}{c^2})}{c}\left(\int_{-\infty}^0|F^l(z_-)|(\tau)d\tau + \int_0^{+\infty}|F^l(z_+(\tilde{a},x_-,\cdot))|(\tau)d\tau\right. \\
&\quad \left. + \int_{-\infty}^{+\infty}|F^s(z_-)(\tau)|d\tau\right) \int_{-\infty}^{+\infty}|F^s(z_-)|(\tau)d\tau. \quad (7.28)
\end{aligned}$$

We use (4.5), (4.6) and (7.2), and we have

$$|\Delta_{fc,2}| \leq \frac{144n^2\beta^2(1-\frac{|v_-|^2}{c^2})}{c\alpha(\alpha+1)(\frac{|v_-|}{2^{\frac{3}{2}}}-r)^2(1+\frac{|x_-|}{\sqrt{2}})^{2\alpha+1}}. \quad (7.29)$$

We use (4.1) and an estimate similar to (6.10), and we have

$$|\Delta_{fc,3}| \leq \frac{2^7n^{\frac{7}{2}}\beta_1^l(1-\frac{|v_-|^2}{c^2})}{\alpha(\alpha+1)|v_-|^2(1+\frac{|x_-|}{\sqrt{2}})^{2\alpha+1}}\left(\frac{\beta_2}{c} + \frac{2^{\frac{5}{2}}n^{\frac{1}{2}}\beta_3^s}{|v_-|}\right) \quad (7.30)$$

Then we add the bounds of (7.27), (7.29) and (7.30), and we obtain (3.29).

We prove (3.30). From (3.15) and $\tilde{b}_{sc} = \mathcal{G}_{v_-, x_-}(\tilde{b}_{sc})$ it follows that

$$\begin{aligned} & \tilde{b}_{sc}(v_-, x_-) - \left(\langle \nabla g_j(g^{-1}(v_-)), \int_{-\infty}^0 \int_{-\infty}^{\sigma} F^s(\cdot v_- + x_-)(\tau) d\tau d\sigma \right. \\ & \left. - \int_0^{+\infty} \int_{\sigma}^{+\infty} F^s(\cdot v_- + x_-)(\tau) d\tau d\sigma \rangle \right)_{j=1 \dots n} = \sum_{j=1}^8 \Delta_{sc,j}, \end{aligned} \quad (7.31)$$

where

$$\Delta_{sc,1} := \mathcal{G}_{v_-, x_-}(\tilde{b}_{sc}) - \mathcal{G}_{v_-, x_-}(0), \quad \Delta_{sc,2} := \mathcal{A}(y_-)(0) - \mathcal{A}(0)(0), \quad (7.32)$$

$$\begin{aligned} \Delta_{sc,3} := & \int_{-\infty}^0 \left(\langle \int_0^1 \nabla g_j(g^{-1}(v_-) + \int_{-\infty}^{\sigma} (F^l + \varepsilon F^s)(z_-)(\tau) d\tau) d\varepsilon \right. \\ & \left. - \nabla g_j(g^{-1}(v_-)), \int_{-\infty}^{\sigma} F^s(z_-)(\tau) d\tau \rangle \right)_{j=1 \dots n} d\sigma \end{aligned} \quad (7.33)$$

$$\Delta_{sc,4} := \int_{-\infty}^0 \left(\langle \nabla g_j(g^{-1}(v_-)), \int_{-\infty}^{\sigma} (F^s(z_-)(\tau) - F^s(\cdot v_- + x_-)(\tau)) d\tau \rangle \right)_{j=1 \dots n} d\sigma, \quad (7.34)$$

$$\begin{aligned} \Delta_{sc,5} := & - \int_0^{+\infty} \left(\langle \int_0^1 \left(\nabla g_j(g^{-1}(\tilde{a}) - \int_{\sigma}^{+\infty} (\varepsilon F(z_- + y_-) + (1 - \varepsilon) F^l(z_+(\tilde{a}, x_-, \cdot))) (\tau)) d\tau \right. \right. \\ & \left. \left. - \nabla g_j(g^{-1}(v_-)) \right) d\varepsilon, \int_{\sigma}^{+\infty} F^s(z_- + y_-)(\tau) d\tau \rangle \right)_{j=1 \dots n} d\sigma, \end{aligned} \quad (7.35)$$

$$\Delta_{sc,6} := - \int_0^{+\infty} \left(\langle \nabla g_j(g^{-1}(v_-)), \int_{\sigma}^{+\infty} (F^s(z_- + y_-)(\tau) - F^s(z_-)(\tau)) d\tau \rangle \right)_{j=1 \dots n} d\sigma. \quad (7.36)$$

$$\Delta_{sc,7} := - \int_0^{+\infty} \left(\langle \nabla g_j(g^{-1}(v_-)), \int_{\sigma}^{+\infty} (F^s(z_-)(\tau) - F^s(\cdot v_- + x_-)(\tau)) d\tau \rangle \right)_{j=1 \dots n} d\sigma, \quad (7.37)$$

$$\begin{aligned} \Delta_{sc,8} = & - \int_0^{+\infty} \left(\langle \int_0^1 \nabla g_j(g^{-1}(\tilde{a}) - \int_{\sigma}^{+\infty} (\varepsilon F(z_- + y_-) + (1 - \varepsilon) F^l(z_+(\tilde{a}, x_-, \cdot))) (\tau)) d\tau \right. \\ & \left. \int_{\sigma}^{+\infty} (F^l(z_- + y_-)(\tau) - F^l(z_+(\tilde{a}, x_-, \cdot))(\tau)) d\tau \rangle \right)_{j=1 \dots n} d\sigma. \end{aligned} \quad (7.38)$$

From (3.18) and (3.27) it follows that

$$|\Delta_{sc,1}| \leq \frac{8n^2 \sqrt{1 - \frac{|v_-|^2}{c^2}} \max(\beta_1^l, \beta_2) \left(\frac{1}{c} + 1\right) |\tilde{b}_{sc}|}{\alpha(\alpha + 1) \left(\frac{|v_-|}{2^{\frac{3}{2}}}\right)^2 \left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^{\alpha}} \leq \frac{192n^4 \left(1 - \frac{|v_-|^2}{c^2}\right) \max(\beta_1^l, \beta_2)^2 \left(\frac{1}{c} + 1\right)^2}{\alpha^2(\alpha + 1)^2 \left(\frac{|v_-|}{2\sqrt{2}} - r\right)^4 \left(\frac{1}{2} + \frac{|x_-|}{\sqrt{2}}\right)^{2\alpha}}. \quad (7.39)$$

We use (5.31) (" $(f_1, f_2) = (y_-, 0)$ "), and we use the bound of $\|y_-\|_* = \|\mathcal{A}(y_-)\|_*$ given in (3.9) and $r \leq \frac{1}{2}$, and we obtain

$$|\Delta_{sc,2}| \leq \frac{192n^4(1 - \frac{|v_-|^2}{c^2})(\frac{1}{c} + 1)(\frac{r}{c} + 1) \max(\beta_1^l, \beta_2)(1 + \frac{1}{\frac{|v_-|}{2\sqrt{2}} - r})}{\alpha^2(\alpha + 1)(\frac{|v_-|}{2\sqrt{2}} - r)^3(1 - r + \frac{|x_-|}{\sqrt{2}})^{2\alpha}}. \quad (7.40)$$

The proof of the following estimates (7.41), (7.42), (7.43) and (7.44) given below is similar to the proof of the estimates (6.36), (6.37), (6.38) and (6.39)

$$|\Delta_{sc,3}| \leq \frac{48n^3 \max(\beta_1^l, \beta_2)(1 - \frac{|v_-|^2}{c^2})}{c\alpha^2(\alpha + 1)(\frac{|v_-|}{2^{\frac{3}{2}}})^3(1 + \frac{|x_-|}{\sqrt{2}})^{2\alpha}}, \quad (7.41)$$

$$\max(|\Delta_{sc,4}|, |\Delta_{sc,7}|) \leq \frac{8n^3\beta^2(1 - \frac{|v_-|^2}{c^2})(\frac{1}{c} + \frac{2n^{\frac{1}{2}}}{\frac{|v_-|}{2^{\frac{3}{2}}}})}{\alpha^2(\alpha + 1)(\frac{|v_-|}{2^{\frac{3}{2}}})^3(1 + \frac{|x_-|}{\sqrt{2}})^{2\alpha}}, \quad (7.42)$$

$$|\Delta_{sc,5}| \leq \frac{168n^3(1 - \frac{|v_-|^2}{c^2}) \max(\beta_1^l, \beta_2)^2}{c\alpha^2(\alpha + 1)(\frac{|v_-|}{2\sqrt{2}} - r)^3(1 - r + \frac{|x_-|}{\sqrt{2}})^{2\alpha}}, \quad (7.43)$$

$$|\Delta_{sc,6}| \leq \frac{96n^4\beta^2(1 - \frac{|v_-|^2}{c^2})(1 + \frac{1}{c})^2(1 + \frac{1}{\frac{|v_-|}{2^{\frac{3}{2}} - r}})^2}{\alpha^2(\alpha + 1)(\frac{|v_-|}{2^{\frac{3}{2}}} - r)^3(1 + \frac{|x_-|}{2^{\frac{3}{2}} - r})^{2\alpha}}. \quad (7.44)$$

From (7.24), (4.1), (4.7) and (7.14) it follows that

$$\begin{aligned} |\Delta_{sc,8}| &\leq 2n^{\frac{1}{2}}(1 - \frac{|v_-|^2}{c^2})^{\frac{1}{2}} \int_0^{+\infty} \int_{\sigma}^{+\infty} \frac{(\frac{\beta_1^l}{c} + 2n^{\frac{1}{2}}\beta_2)\delta d\tau d\sigma}{(1 - r + \frac{|x_-|}{\sqrt{2}} + (\frac{|v_-|}{2^{\frac{3}{2}}} - r)\tau)^{\alpha+2}} \\ &\leq \frac{96n^4\beta^2(1 - \frac{|v_-|^2}{c^2})(1 + \frac{1}{c})^2(1 + \frac{1}{\frac{|v_-|}{2^{\frac{3}{2}} - r}})^2}{\alpha^2(\alpha + 1)(\frac{|v_-|}{2^{\frac{3}{2}}} - r)^3(\frac{1}{2} + \frac{|x_-|}{2^{\frac{3}{2}}})^{2\alpha}}. \end{aligned} \quad (7.45)$$

Then we add the bounds on the right-hand sides of (7.39)–(7.45), and we use (7.31) and we obtain (3.30). \square

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