

# CONTINUOUS FIELDS WITH FIBRES $\mathcal{O}_\infty$

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ABSTRACT. We study in this article a class of unital continuous  $C^*$ -bundles the fibres of which are all isomorphic to the Cuntz  $C^*$ -algebra  $\mathcal{O}_\infty$ . This enables us to give several equivalent reformulations for the triviality of all these  $C^*$ -bundles.

## 1. INTRODUCTION

A programme of classification for separable nuclear  $C^*$ -algebras through K-theoretical invariants has been launched by Elliott ([Ell94]). Quite a number of results have already been obtained for simple  $C^*$ -algebras (see e.g. [ET08], [BPT08] for an account of them).

One of the central  $C^*$ -algebras for this programme is the simple unital nuclear Cuntz  $C^*$ -algebra  $\mathcal{O}_\infty$  generated by a countable family of isometries  $\{s_k; k \in \mathbb{N}\}$  with pairwise orthogonal ranges ([Cun77]). This  $C^*$ -algebra belongs to the category of unital strongly self-absorbing  $C^*$ -algebras systematically studied by Toms and Winter:

**Definition 1.1** ([TW07]). *Let  $A, B$  and  $D_0$  be separable unital  $C^*$ -algebras distinct from  $\mathbb{C}$ .*

a) *Two unital completely positive (u.c.p.) maps  $\theta_1$  and  $\theta_2$  from  $A$  to  $B$  are said to be approximately unitarily equivalent (written  $\theta_1 \approx_{a.u.} \theta_2$ ) if there is a sequence  $\{v_m\}_m$  of unitaries in  $B$  such that  $\|\theta_2(a) - v_m \theta_1(a) v_m^*\| \xrightarrow{m \rightarrow \infty} 0$  for all  $a \in A$ .*

b) *The  $C^*$ -algebra  $D_0$  is said to be strongly self-absorbing if there is an isomorphism  $\pi : D_0 \rightarrow D_0 \otimes D_0$  such that  $\pi \approx_{a.u.} \iota_{D_0} \otimes 1_{D_0}$ .*

*Remark 1.2.* All separable unital strongly self-absorbing  $C^*$ -algebras are simple and nuclear ([TW07]). Besides, these  $C^*$ -algebras are  $K_1$ -injective (see e.g. Definition 2.9 for a definition and [Win09] for a proof of it).

This property of strong self-absorption is pretty invariant under continuous deformation. Indeed, any separable unital continuous  $C(X)$ -algebra  $D$  the fibres of which are all isomorphic to a given strongly self-absorbing  $C^*$ -algebra  $D_0$  satisfies an isomorphism of  $C(X)$ -algebra

$$D \cong D_0 \otimes C(X)$$

provided the second countable compact metric space  $X$  is of finite topological dimension ([BK04a], [HRW07], [DW08], [Win09]).

This is not anymore always the case when the compact Hausdorff space  $X$  has infinite topological dimension. Indeed, Hirshberg, Rørdam and Winter constructed non-trivial

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2010 *Mathematics Subject Classification.* Primary: 46L80; Secondary: 46L06, 46L35.

*Key words and phrases.*  $C^*$ -algebra, Classification, Proper Infiniteness.

unital continuous fields of algebras over the infinite dimensional compact space  $Y = \prod_{n=1}^{\infty} S^2$  with constant strongly self-absorbing fibre  $D_0$  in case that  $C^*$ -algebra  $D_0$  is a UHF algebra of infinite type or the Jiang-Su algebra ([HRW07, Examples 4.7-4.8]). Then, Dădărlat has constructed in [Dad09b] unital continuous fields of  $C^*$ -algebras  $A$  over the contractible Hilbert cube  $\mathfrak{X}$  of infinite topological dimension such that  $K_*(A)$  is non-trivial and each fibre  $A_x$  is isomorphic to the Cuntz  $C^*$ -algebra  $\mathcal{O}_2$ , so that

$$K_0(A) \oplus K_1(A) \neq K_*(C(\mathfrak{X}; \mathcal{O}_2)) = 0 \oplus 0.$$

We analyse in this present paper the case  $D_0 = \mathcal{O}_{\infty}$  through explicit examples of Cuntz-Pimsner  $C(\mathfrak{X})$ -algebra associated with Hilbert  $C(\mathfrak{X})$ -module with infinite dimensional fibres, so that all the fibres of these Cuntz-Pimsner  $C(\mathfrak{X})$ -algebras are isomorphic to the  $C^*$ -algebra  $\mathcal{O}_{\infty}$ . We were not able to prove here whether these continuous  $C(\mathfrak{X})$ -algebras are always trivial. But we describe a number of equivalent formulations of this problem (see *e.g.* Proposition 5.1) and we especially look at the case of the Cuntz-Pimsner  $C(\mathfrak{X})$ -algebra for the non-trivial continuous field of Hilbert spaces  $\mathcal{E}$  defined by Dixmier and Douady in [DD63].

More precisely, we first recall in section §2 the construction of the Cuntz-Pimsner  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  for any continuous field  $E$  of infinite dimensional Hilbert spaces over a compact Hausdorff space  $X$  ([DD63]). We show in section §3 that this  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is always locally purely infinite. We observe that basic  $K$ -theoretical arguments are not enough to decide whether such a continuous  $C(X)$ -algebra with fibres isomorphic to  $\mathcal{O}_{\infty}$  is or is not trivial unlike the case where all the fibres are isomorphic to the Cuntz  $C^*$ -algebra  $\mathcal{O}_2$  ([Dad09b]). We study in the following section whether this  $C(X)$ -algebra is at least purely infinite, *i.e.* whether  $\mathcal{T}_{C(X)}(E) \cong \mathcal{T}_{C(X)}(E) \otimes \mathcal{O}_{\infty}$ , before we analyse in section §6 whether  $\mathcal{T}_{C(X)}(E)$  is always properly infinite, *i.e.* whether there is always a unital embedding of  $C(X)$ -algebra  $\mathcal{O}_{\infty} \otimes C(X) \hookrightarrow \mathcal{T}_{C(X)}(E)$ .

The author especially thanks Wuhan University for its relaxing atmosphere which enabled him to order the different steps of the presented constructions. He also thanks A. Freslon, E. Kirchberg, M. Rørdam and the referee for inspiring questions and remarks.

## 2. PRELIMINARIES

We fix in this section a few notations which will be used all along this paper.

We shall denote by  $\mathbb{N} := \{0, 1, 2, \dots\}$  the set of *positive* integers and by  $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$  the subset of *strictly positive* integer. We can then define the two separable Hilbert spaces

$$\begin{aligned} - \ell^2(\mathbb{N}^*) &:= \{x = (x_m) \in \mathbb{C}^{\mathbb{N}^*} ; \sum_{m \geq 1} |x_m|^2 < +\infty\}, \\ - \ell^2(\mathbb{N}) &:= \mathbb{C} \oplus \ell^2(\mathbb{N}^*). \end{aligned} \tag{2.1}$$

Let now  $X$  be a non-zero compact Hausdorff space and denote by  $C(X)$  the  $C^*$ -algebra of continuous functions on  $X$  with values in the complex field  $\mathbb{C}$ . For all points

$x$  in the compact space  $X$ , one denotes by  $C_0(X \setminus \{x\}) \subset C(X)$  the closed two-sided ideal of continuous functions on  $X$  which are zero at  $x$ .

**Definition 2.1** ([DD63]). *Assume that  $X$  is a compact Hausdorff space.*

a) *A Banach  $C(X)$ -module is a Banach space  $E$  endowed with a structure of  $C(X)$ -module such that  $1_{C(X)} \cdot e = e$  and  $\|f \cdot e\| \leq \|f\| \cdot \|e\|$  for all  $e$  in  $E$  and  $f$  in  $C(X)$ .*

b) *A Hilbert  $C(X)$ -module is a Banach  $C(X)$ -module  $E$  with a  $C(X)$ -valued sesquilinear inner product  $\langle \cdot, \cdot \rangle : E \times E \rightarrow C(X)$  such that*

- $f \cdot e = e \cdot f \in E$  for all pairs  $(e, f)$  in  $E \times C(X)$ ,
- $\langle e_1, e_2 \cdot f \rangle = \langle e_1, e_2 \rangle \cdot f$ ,  $\langle e_2, e_1 \rangle = \langle e_1, e_2 \rangle^*$ ,  $\langle e_1, e_1 \rangle \geq 0$  and  $\langle e_1, e_1 \rangle = 0$  if and only if  $e_1 = 0$  for all triples  $(e_1, e_2, f)$  in  $E \times E \times C(X)$ ,
- the Banach space  $E$  is complete for the norm  $\|e\| := \|\langle e, e \rangle\|^{1/2}$ .

For all Banach  $C(X)$ -modules  $E$  and all points  $x \in X$ , the set  $C_0(X \setminus \{x\}) \cdot E$  is closed in  $E$  (by Cohen's factorization theorem). The quotient  $E_x := E/C_0(X \setminus \{x\}) \cdot E$  is a Hilbert space which is called the *fibre* at  $x$  of the  $C(X)$ -module  $E$ . One will denote in the sequel by  $e_x$  the image of a *section*  $e \in E$  in the fibre  $E_x$ .

The map  $x \mapsto \|e_x\| = \inf\{\|[1 - f + f(x)] \cdot e\|; f \in C(X)\}$  is always upper semi-continuous. If that map is actually continuous for all sections  $e \in E$ , then the  $C(X)$ -module  $E$  is said to be *continuous*.

*Remark 2.2.* The continuous fields of Hilbert spaces over a compact Hausdorff space  $X$  studied by Dixmier and Douady in [DD63] are now called *continuous Hilbert  $C(X)$ -modules*.

*Examples 2.3.* a) If  $H$  is a Hilbert space, then the tensor product  $E := H \otimes C(X)$  is a *trivial* Hilbert  $C(X)$ -module the fibres of which are all isomorphic to  $H$ .

b) If  $E$  is a Hilbert  $C(X)$ -module, one defines inductively for all  $m \in \mathbb{N}$  the Hilbert  $C(X)$ -modules  $E^{(\otimes_{C(X)})^m}$  by  $E^{(\otimes_{C(X)})^0} = C(X)$  and  $E^{(\otimes_{C(X)})^{m+1}} = E^{(\otimes_{C(X)})^m} \otimes_{C(X)} E$ . Then, the *full Fock Hilbert  $C(X)$ -module*  $\mathcal{F}(E)$  of  $E$  is the sum  $\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(X)})^m}$ .

Note that many of these full Fock Hilbert  $C(X)$ -modules are trivial Hilbert  $C(X)$ -modules, as the referee noticed through the following lemma.

**Lemma 2.4.** *Assume that  $X$  is a compact Hausdorff space and  $E$  is countably generated Hilbert  $C(X)$ -module.*

*Then the two Hilbert  $C(X)$ -modules  $\mathcal{F}(E)$  and  $\ell^2(\mathbb{N}) \otimes C(X)$  are isomorphic if and only if the Hilbert  $C(X)$ -module  $E$  is full, i.e. all the fibres of  $E$  are non-zero.*

*Proof.* One has  $\mathcal{F}(E)_x \cong \mathcal{F}(E_x) = \mathbb{C} \oplus E_x \oplus (E_x \otimes E_x) \oplus \dots$  for all  $x \in X$ . And so, the existence of an isomorphism  $\mathcal{F}(E) \cong \ell^2(\mathbb{N}) \otimes C(X)$  implies that each fibre  $E_x$  is non-zero, i.e.  $E$  is full.

Conversely, if one assumes that the Hilbert  $C(X)$ -module  $E$  is full, then for all points  $x$  in  $X$ , there exists a section  $\zeta$  in  $E$  with

$$\langle \zeta, \zeta \rangle(x) = \langle \zeta(x), \zeta(x) \rangle = 1.$$

Moreover, this implies by continuity that there are a closed neighbourhood  $F(x) \subset X$  of that point  $x$  and a norm 1 section  $\zeta'$  in  $C(X) \cdot \xi \subset E$  such that

$$\langle \zeta', \zeta' \rangle(y) = 1 \quad \text{for all } y \in F(x).$$

The fullness of the  $C(X)$ -module  $E$ , the continuity of the map  $x \in X \mapsto \langle \zeta, \zeta \rangle(x)$  for any section  $\zeta \in E$  and the compactness of the space  $X$  imply that there are:

- a finite covering  $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$  by the interiors of closed subsets  $F_1, \dots, F_n$  in  $X$ ,
- norm 1 sections  $\zeta_1, \zeta_2, \dots, \zeta_n$  in  $E$  such that:

$$\langle \zeta_k, \zeta_k \rangle(y) = 1 \quad \text{for all } k \in \{1, \dots, n\} \text{ and } y \in F_k. \quad (2.2)$$

Fix a partition of unity  $1_{C(X)} = \sum_{1 \leq k \leq n} \phi_k$ , where each positive contraction  $\phi_k \in C(X)$  has support in the closed subset  $F_k$  ( $1 \leq k \leq n$ ). Denote by  $\xi^{\otimes j}$  the tensor product  $\xi \otimes \dots \otimes \xi$  in  $(E_x)^{\otimes j}$  for all pairs  $(x, j)$  in  $X \times \mathbb{N}^*$  and all  $\xi$  in the Hilbert space  $E_x$ .

For all  $m \in \mathbb{N}$ , let  $E(m)$  be the Hilbert  $C(X)$ -module  $E(m) := \bigoplus_{k=1}^n E^{\otimes_{C(X)}(mn+k)}$  and let  $\xi_m \in E(m)$  be the section given by the formula

$$x \in X \mapsto \xi_m(x) := \sum_{k=1}^n \phi_k(x)^{1/2} \cdot \zeta_k(x)^{\otimes(mn+k)}. \quad (2.3)$$

It satisfies  $\langle \xi_m, \xi_m \rangle(x) = 1$  for all  $x \in X$ , so that we have an Hilbert  $C(X)$ -module isomorphism  $E(m) \cong C(X) \oplus F(m)$ , where  $F(m)$  is the Hilbert  $C(X)$ -module

$$F(m) := \{ \zeta \in E(m); \langle \xi_m, \zeta \rangle = 0 \}.$$

Thus, the Dixmier-Douady stabilization theorem for separable Hilbert  $C(X)$ -modules ([DD63, Theorem 4]) implies that

$$\begin{aligned} \mathcal{F}(E) &= C(X) \oplus \bigoplus_{m \in \mathbb{N}^*} E^{\otimes_{C(X)} m} = C(X) \oplus \bigoplus_{m \in \mathbb{N}} E(m) \\ &\cong \left[ \ell^2(\mathbb{N}) \otimes C(X) \right] \oplus C(X) \oplus \bigoplus_{m \in \mathbb{N}} F(m) \\ &\cong \ell^2(\mathbb{N}) \otimes C(X). \end{aligned}$$

□

c) A non-trivial full Hilbert  $C(X)$ -module with infinite dimensional fibres is the following construction by Dixmier and Douady:

**Definition 2.5** ([DD63]). Let  $\mathfrak{X} := \{x = (x_m) \in \ell^2(\mathbb{N}^*); \|x\|^2 = \sum_{m \geq 1} |x_m|^2 \leq 1\}$  be the unit ball of the Hilbert space  $\ell^2(\mathbb{N}^*)$ : It is called the compact Hilbert cube when it is endowed with the distance  $d(x, y) = (\sum_{m \geq 1} 2^{-m} |x_m - y_m|^2)^{1/2}$ .

Let also  $\eta \in \ell^\infty(\mathfrak{X}; \ell^2(\mathbb{N})) = \ell^\infty(\mathfrak{X}; \mathbb{C} \oplus \ell^2(\mathbb{N}^*))$  be the normalized section  $x \mapsto (\sqrt{1 - \|x\|^2}, x)$  and let  $\theta_{\eta, \eta}$  be the projection  $\zeta \mapsto \eta \cdot \langle \eta, \zeta \rangle$ .

Then the norm closure  $\mathcal{E} := \overline{(1 - \theta_{\eta, \eta}) C(\mathfrak{X}; 0 \oplus \ell^2(\mathbb{N}^*))} \subset \ell^\infty(\mathfrak{X}; \ell^2(\mathbb{N})) \cap \eta^\perp$  is a Hilbert  $C(\mathfrak{X})$ -module with infinite dimensional fibres (called the Dixmier-Douady Hilbert  $C(\mathfrak{X})$ -module), where for all sections  $\zeta \in C(\mathfrak{X}; \ell^2(\mathbb{N}^*))$  and all points  $x \in \mathfrak{X}$ :

$$(1 - \theta_{\eta, \eta}) \zeta(x) = \left( -\langle x, \zeta(x) \rangle \cdot \sqrt{1 - \|x\|^2}, \zeta(x) - \langle x, \zeta(x) \rangle \cdot x \right) \in \ell^2(\mathbb{N}).$$

*Remarks 2.6.* a) There is a canonical isomorphism of Hilbert  $C(\mathfrak{X})$ -module:

$$\begin{aligned} C(\mathfrak{X}) \oplus \mathcal{E} &\cong C(\mathfrak{X}) \cdot \eta + \mathcal{E} \subset \ell^\infty(\mathfrak{X}; \ell^2(\mathbb{N})) \\ (f, \xi) &\mapsto f \cdot \eta + \xi \end{aligned} \quad (2.4)$$

b) The constant  $n$  given in formula (2.2) is greater than 2 for the Dixmier-Douady Hilbert  $C(\mathfrak{X})$ -module  $\mathcal{E}$  since any section  $\zeta$  in that Hilbert  $C(\mathfrak{X})$ -module satisfies  $\zeta(x) = 0$  for at least one point  $x \in \mathfrak{X}$  (see [DD63], or [BK04a, Proposition 3.6]).

**Definition 2.7.** *If  $X$  is a non-empty compact Hausdorff space, a  $C(X)$ -algebra is a  $C^*$ -algebra  $A$  endowed with a unital  $*$ -homomorphism from  $C(X)$  to the centre of the multiplier  $C^*$ -algebra  $\mathcal{M}(A)$  of  $A$ .*

*For all closed subsets  $F \subset X$ , the two-sided ideal  $C_0(X \setminus F) \cdot A$  is closed in  $A$  (by Cohen's factorization theorem). One denotes by  $A|_F$  the quotient  $A/C_0(X \setminus F) \cdot A$ .*

*If  $F$  is reduced to a single point  $x$ , the quotient  $A|_{\{x\}}$  is also called the fibre  $A_x$  at  $x$  of the  $C(X)$ -algebra  $A$  and one denotes by  $a_x$  the image of a section  $a \in A$  in  $A_x$ .*

*The  $C(X)$ -algebra  $A$  is said to be a continuous  $C(X)$ -algebra if the map  $x \mapsto \|a_x\|$  is continuous for all  $a \in A$ .*

We shall mainly study here the following Cuntz-Pimsner  $C(X)$ -algebras.

**Definition 2.8** ([DD63]). *If  $E$  is a full separable Hilbert  $C(X)$ -module and  $\hat{1}_{C(X)}$  is a unit vector generating the first direct summand of the full Fock Hilbert  $C(X)$ -module  $\mathcal{F}(E)$ , then one defines for all  $\zeta \in E$  the creation operator  $\ell(\zeta) \in \mathcal{L}_{C(X)}(\mathcal{F}(E))$  through the formulae:*

$$\begin{aligned} - \ell(\zeta)(f \cdot \hat{1}_{C(X)}) &= f \cdot \zeta = \zeta \cdot f && \text{for } f \in C(X) \quad \text{and} \\ - \ell(\zeta)(\zeta_1 \otimes \dots \otimes \zeta_k) &= \zeta \otimes \zeta_1 \otimes \dots \otimes \zeta_k && \text{for } \zeta_1, \dots, \zeta_k \in E \quad \text{if } k \geq 1. \end{aligned} \quad (2.5)$$

*The Cuntz-Pimsner  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  of the Hilbert  $C(X)$ -module  $E$  is the  $C(X)$ -algebra generated in  $\mathcal{L}_{C(X)}(\mathcal{F}(E))$  by all the creation operators  $\ell(\zeta)$ ,  $\zeta \in E$ .*

Let us recall the definition of a  $K_1$ -injective  $C^*$ -algebra.

**Definition 2.9.** *Let  $\mathcal{U}(A)$  be the group of unitaries in a unital  $C^*$ -algebra  $A$  and let  $\mathcal{U}^0(A)$  be the normal connected component of the unit  $1_A$  in  $\mathcal{U}(A)$ .*

*Then the tensor product  $M_m(A) := M_m(\mathbb{C}) \otimes A$  is a unital  $C^*$ -algebra for all  $m \in \mathbb{N}^*$  and the group  $\mathcal{U}(M_m(A))$  embeds in  $\mathcal{U}(M_{m+1}(A))$  by  $u \mapsto u \oplus 1$ .*

*The  $C^*$ -algebra  $A$  is said to be  $K_1$ -injective if the canonical map  $\mathcal{U}(A)/\mathcal{U}^0(A) \rightarrow K_1(A) = \lim_{m \rightarrow \infty} \mathcal{U}(M_m(A))/\mathcal{U}^0(M_m(A))$  is injective.*

### 3. A QUESTION OF LOCAL PURELY INFINITENESS

We show in this section that the Cuntz-Pimsner  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  of any Hilbert  $C(X)$ -module  $E$  with infinite-dimensional fibres is locally purely infinite ([BK04b, Definition 1.3]).

**Proposition 3.1.** *Let  $E$  be a separable Hilbert  $C(X)$ -module the fibres of which are all infinite dimensional Hilbert spaces. Then, the Cuntz-Pimsner  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is a locally purely infinite continuous  $C(X)$ -algebra the fibres of which are all isomorphic to the Cuntz  $C^*$ -algebra  $\mathcal{O}_\infty$ .*

*Proof.* All the fibres of the  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  are isomorphic to the  $C^*$ -algebra  $\mathcal{O}_\infty$  by universality ([Cun81]).

As the  $C^*$ -algebra  $\mathcal{O}_\infty$  is simple (and non-zero), the separable unital  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is continuous ([Dad09a, Lemma 2.3] or [Blan09, Lemma 4.5]) and the  $C^*$ -representation of  $\mathcal{T}_{C(X)}(E)$  on the Hilbert  $C(X)$ -module  $\mathcal{F}(E)$  is a continuous field of faithful representations ([Blan97]).

Besides, this  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is locally purely infinite (l.p.i.) since all its fibres are simple and purely infinite (p.i.) ([BK04b, Proposition 5.1]).  $\square$

#### 4. A QUESTION OF TRIVIALITY

We observe in this section that basic K-theory arguments are not enough to decide whether the Cuntz-Pimsner  $C(\mathfrak{X})$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(E)$  associated to a Hilbert  $C(\mathfrak{X})$ -module  $E$  with infinite-dimensional fibres is always a trivial continuous  $C(\mathfrak{X})$ -algebra, *i.e.*, whether  $\mathcal{T}_{C(\mathfrak{X})}(E) \cong C(\mathfrak{X}; \mathcal{O}_\infty)$ .

Let  $\{\mathfrak{X}_k\}_{k \in \mathbb{N}^*}$  be an increasing sequence of closed subspaces with finite (covering) dimension in the Hilbert cube  $\mathfrak{X}$  (Definition 2.5) given by

$$\mathfrak{X}_k := \{(x_m)_m \in \mathfrak{X}; x_m = 0 \text{ for all } m > k\}.$$

Then, Dădărlat has constructed in [Dad09b] non  $K_*$ -trivial unital continuous  $C(\mathfrak{X})$ -algebras the restriction of which to each compact subset  $\mathfrak{X}_k$  is isomorphic to the K-trivial  $C(\mathfrak{X}_k)$ -algebra  $\mathcal{O}_2 \otimes C(\mathfrak{X}_k)$  ([DW08, Theorem 1.1]).

Now, this K-theory proof fails to decide whether the Cuntz-Pimsner  $C(\mathfrak{X})$ -algebra associated to a Hilbert  $C(\mathfrak{X})$ -module  $E$  with infinite dimensional fibres is trivial or not. Indeed, we only know that the restriction  $\mathcal{T}_{C(\mathfrak{X})}(E)|_{\mathfrak{X}_k}$  is a unital  $C(\mathfrak{X}_k)$ -algebra the fibres of which are all isomorphic to the  $K_1$ -injective ([Roh09]) strongly self-absorbing ([Win09])  $C^*$ -algebra  $\mathcal{O}_\infty$ , so that there is an isomorphism of  $C(\mathfrak{X}_k)$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(E)|_{\mathfrak{X}_k} \cong C(\mathfrak{X}_k; \mathcal{O}_\infty)$  for each  $k \geq 1$  ([DW08, Theorem 1.1]).

*Remarks 4.1.* a) For all  $k \in \mathbb{N}$ , let  $j_k : C(\mathfrak{X}_k; \mathcal{O}_\infty) \hookrightarrow C(\mathfrak{X}_{k+1}; \mathcal{O}_\infty)$  be the embedding

$$\forall (f, x) \in C(\mathfrak{X}_k; \mathcal{O}_\infty) \times \mathfrak{X}_{k+1}, j_k(f)(x_0, \dots, x_k, x_{k+1}, 0, \dots) = f(x_0, \dots, x_k, 0, \dots) \quad (4.1)$$

Then the inductive limit satisfies  $\lim_{k \in \mathbb{N}^*} (C(\mathfrak{X}_k; \mathcal{O}_\infty), j_k) = C(\mathfrak{X}; \mathcal{O}_\infty)$ . But the diagramme

$$\begin{array}{ccc} C(\mathfrak{X}_k; \mathcal{O}_\infty) & \xrightarrow{\sim} & \mathcal{T}_{C(\mathfrak{X})}(E)|_{\mathfrak{X}_k} \\ j_k \downarrow & & \downarrow \\ C(\mathfrak{X}_{k+1}; \mathcal{O}_\infty) & \xrightarrow{\sim} & \mathcal{T}_{C(\mathfrak{X})}(E)|_{\mathfrak{X}_{k+1}} \end{array}$$

is not asked to be commutative for every  $k \in \mathbb{N}$ .

If this is the case, then  $\mathcal{T}_{C(\mathfrak{X})}(E) \cong \mathcal{T}_{C(\mathfrak{X})}(E) \otimes \mathcal{O}_\infty$  because  $\mathcal{O}_\infty \cong \mathcal{O}_\infty \otimes \mathcal{O}_\infty$ .

b) If  $X$  is a second countable compact Hausdorff space, a more general question would be to know when two separable Hilbert  $C(X)$ -modules  $E_1$  and  $E_2$  have isomorphic Cuntz-Pimsner  $C(X)$ -algebras.

Dădărlat has characterized in [Dad10] when  $\mathcal{T}_{C(X)}(E_1)$  and  $\mathcal{T}_{C(X)}(E_2)$  have isomorphic quotient Cuntz  $C(X)$ -algebras (with simple fibres) if all the fibres of  $E_1$  and  $E_2$  have the same finite dimension.

## 5. A QUESTION OF PURE INFINITENESS

We show in this section that the  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  associated to any full separable Hilbert  $C(X)$ -module  $E$  is purely infinite (in the sense of [KR00, definition 4.1]) if and only if it tensorially absorbs  $\mathcal{O}_\infty$  ([TW07]), a property which is local.

**Proposition 5.1.** *Given a second countable compact Hausdorff space  $X$  and a full separable Hilbert  $C(X)$ -module  $E$ , the following assertions are equivalent:*

- a) *The  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is purely infinite (abbreviated p.i.),*
- b) *The  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is strongly purely infinite,*
- c) *One has  $\mathcal{T}_{C(X)}(E) \cong \mathcal{T}_{C(X)}(E) \otimes \mathcal{O}_\infty$ ,*
- d) *There exists a unital  $*$ -homomorphism*

$$\rho : \mathcal{O}_\infty \rightarrow (c_b(\mathbb{N}; \mathcal{T}_{C(X)}(E)) / c_0(\mathbb{N}; \mathcal{T}_{C(X)}(E))) \cap \iota(\mathcal{T}_{C(X)}(E))',$$

- e) *There are unital  $*$ -homomorphisms  $\rho_m : \mathcal{O}_\infty \rightarrow \mathcal{T}_{C(X)}(E)$  ( $m \in \mathbb{N}^*$ ) such that:*

$$\|[\rho_m(s_k), b]\| \xrightarrow{m \rightarrow \infty} 0 \quad \text{for all pairs } (k, b) \text{ in } \mathbb{N} \times \mathcal{T}_{C(X)}(E),$$

- f) *Any point  $x \in X$  has a closed neighbourhood  $F(x) \subset X$  such that  $\mathcal{T}_{C(X)}(E)|_{F(x)} \cong \mathcal{T}_{C(X)}(E)|_{F(x)} \otimes \mathcal{O}_\infty$ ,*
- g) *Any point  $x \in X$  has a closed neighbourhood  $F(x) \subset X$  such that the quotient  $\mathcal{T}_{C(X)}(E)|_{F(x)}$  is p.i.*

Let us first prove the following technical lemma:

**Lemma 5.2.** *Let  $A_1, A_2, B, D$  be separable unital  $C^*$ -algebras such that:*

- *$D$  is strongly self-absorbing ([TW07, Definition 1.3]), hence nuclear,*
- *$A_1$  and  $A_2$  are  $D$ -stable, i.e.  $A_i \cong A_i \otimes D$  for  $i = 1, 2$ ,*
- *there are  $*$ -epimorphisms  $\pi_1 : A_1 \rightarrow B$  and  $\pi_2 : A_2 \rightarrow B$ .*

*Then the amalgamated sum  $A_1 \oplus_B A_2 = \{(a_1, a_2) \in A_1 \oplus A_2; \pi_1(a_1) = \pi_2(a_2)\}$  is also  $D$ -stable.*

*Proof.* The closed two sided ideal  $I := \ker \pi_2 \triangleleft A_2$  absorbs  $D$  tensorially ([TW07, Corollary 3.3]) and the sequence  $0 \rightarrow I \rightarrow A_1 \oplus_B A_2 \rightarrow A_1 \rightarrow 0$  is exact. Hence,  $A_1 \oplus_B A_2$  is  $D$ -stable ([TW07, Corollary 4.3]).  $\square$

*Remarks 5.3.* a) Corollary 4.3 in [TW07] answers Question 9.8 in [KR02], i.e. if  $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$  is an exact sequence of separable  $C^*$ -algebras, then the  $C^*$ -algebra  $A$  is  $\mathcal{O}_\infty$ -stable if and only if the ideal  $I$  and the quotient  $B$  are  $\mathcal{O}_\infty$ -stable.

b) Winter noticed that Lemma 5.2 does not hold if the  $C^*$ -morphisms  $\pi_1$  and  $\pi_2$  are not surjective. Indeed, if  $A_1 = A_2 = D$  is a strongly self-absorbing  $C^*$ -algebra,  $B = D \otimes D$  and  $\pi_1, \pi_2$  are the first and second factor canonical unital embeddings, then the amalgamated sum  $A_1 \oplus_B A_2$  is isomorphic to  $\mathbb{C}$ , and so it is not  $D$ -stable.

*Proof of Proposition 5.1.* The equivalences a) $\Leftrightarrow$ b) $\Leftrightarrow$ c) $\Leftrightarrow$ d) $\Leftrightarrow$ e) are proved respectively in [BK04b, Theorem 5.8], [KR02, Theorem 8.5], [TW07, Theorem. 2.2], and [DW08, Proposition 3.7].

The implication c) $\Rightarrow$ f) is contained in [TW07, Corollary 3.3] while the converse implication f) $\Rightarrow$ c) follows from Lemma 5.2.

At last, the proof of f) $\Leftrightarrow$ g) is the same as the one of a) $\Leftrightarrow$ c).  $\square$

*Remarks 5.4.* a) If the  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is purely infinite, then  $\mathcal{T}_{C(X)}(E)|_F \cong \mathcal{T}_{C(X)}(E)|_F \otimes \mathcal{O}_\infty$  for all closed subsets  $F \subset X$  (Proposition 5.1) and so all the quotient  $C^*$ -algebras  $\mathcal{T}_{C(X)}(E)|_F$  are  $K_1$ -injective ([Phil00], [BRR08, Proposition 6.1]).

b) Given a separable Hilbert  $C(X)$ -module with infinite dimensional fibres, the locally purely infinite  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is isomorphic to  $C(X; \mathcal{O}_\infty)$  if and only if it is at the same time purely infinite and  $\text{KK}_{C(X)}$ -equivalent to  $C(X) \otimes \mathcal{O}_\infty$ . Indeed,  
– the pure infiniteness implies that  $\mathcal{T}_{C(X)}(E) \cong \mathcal{T}_{C(X)}(E) \otimes \mathcal{O}_\infty$  (Proposition 5.1),  
– the  $\text{KK}_{C(X)}$ -equivalence to  $C(X) \otimes \mathcal{O}_\infty$  implies that  $\mathcal{K}(\ell^2(\mathbb{N}^*)) \otimes \mathcal{T}_{C(X)}(E) \cong \mathcal{K}(\ell^2(\mathbb{N}^*)) \otimes \mathcal{O}_\infty \otimes C(X)$  ([Kir99], [Dad09b]),  
– the proper infiniteness of the two full  $K_0$ -equivalent projections  $1_{\mathcal{T}_{C(X)}(E)}$  and  $1_{C(X; \mathcal{O}_\infty)}$  then implies that  $\mathcal{T}_{C(X)}(E) \cong C(X; \mathcal{O}_\infty)$  ([BRR08, Lemma 2.3]).

## 6. A QUESTION OF PROPER INFINITENESS

Given a compact metric space  $X$  and a full separable Hilbert  $C(X)$ -module  $E$ , we study in this section whether the Cuntz-Pimsner  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(E)$  is always properly infinite, *i.e.*, if there exists a unital  $*$ -homomorphism  $\mathcal{O}_\infty \rightarrow \mathcal{T}_{C(X)}(E)$ . We also make the link between that question and several problems raised in [BRR08].

1) Let us first state the following general lemma.

**Lemma 6.1.** *Let  $E$  be a full separable Hilbert  $C(X)$ -module. Then the Cuntz-Pimsner  $C(X)$ -algebra  $\mathcal{T}_{C(X)}(C(X) \oplus E)$  is a unital properly infinite  $C(X)$ -algebra.*

*Proof.* The separable Hilbert  $C(X)$ -module  $E$  is full by assumption. Thus, there exists by Lemma 2.4

- a finite integer  $n \in \mathbb{N}^*$ ,
- positive functions  $\phi_1, \dots, \phi_n$  in  $C(X)$  such that  $1_{C(X)} = \phi_1 + \dots + \phi_n$ ,
- contractive sections  $\zeta_1, \dots, \zeta_n$  in  $E$  such that  $\|\zeta_k(x)\| = 1$  for all  $(k, x)$  in  $\{1, \dots, n\} \times X$  with  $\phi_k(x) > 0$ .

For all integers  $k, k'$  in  $\{1, \dots, n\}$  and all integers  $m, m'$  in  $\mathbb{N}$ , a direct computation gives the equality

$$\ell(1_{C(X)} \oplus 0)^* \cdot (\ell(0 \oplus \zeta_k)^{mn+k})^* \cdot \ell(0 \oplus \zeta_{k'})^{m'n+k'} \cdot \ell(1_{C(X)} \oplus 0) = \delta_{k=k'} \cdot \delta_{m=m'} \cdot \langle \zeta_k, \zeta_k \rangle^{mn+k}.$$

Hence, the sequence  $\{(\sum_{k=1}^n (\phi_k)^{1/2} \cdot \ell(0 \oplus \zeta_k)^{mn+k}) \cdot \ell(1_{C(X)} \oplus 0)\}_{m \in \mathbb{N}}$  is by linearity a countable family of isometries in  $\mathcal{T}_{C(X)}(C(X) \oplus E)$  with pairwise orthogonal ranges. In other words, these isometries define by universality (see [Pim95, Theorem 3.4]) a unital morphism of  $C(X)$ -algebra

$$\mathcal{O}_\infty \otimes C(X) \rightarrow \mathcal{T}_{C(X)}(C(X) \oplus E). \quad (6.1)$$

□

2) On the other hand, if  $\mathcal{E}$  is the Dixmier-Douady Hilbert  $C(\mathfrak{X})$ -module (Definition 2.5),  $\tilde{\mathcal{E}}$  is the Hilbert  $C(\mathfrak{X})$ -module  $\tilde{\mathcal{E}} := C(\mathfrak{X}) \oplus \mathcal{E}$  and  $\alpha : \mathcal{T}_{C(\mathfrak{X})}(\tilde{\mathcal{E}}) \rightarrow \mathcal{T}_{C(\mathfrak{X})}(\mathcal{E}) \otimes C(\mathbb{T})$  is



the *only* coaction (by [Pim95]) of the circle group  $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$  on the Cuntz-Pimsner  $C(\mathfrak{X})$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(\tilde{\mathcal{E}})$  such that

$$\alpha(\ell(\zeta)) = \ell(\zeta) \otimes z \quad \text{for all } \zeta \in \tilde{\mathcal{E}}, \quad (6.2)$$

then the fixed point  $C(\mathfrak{X})$ -subalgebra  $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha$  of elements  $a \in \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})$  with  $\alpha(a) = a \otimes 1$  is the closed linear span of  $C(\mathfrak{X}) \cdot 1$  and the words of the form  $\ell(\zeta_1) \dots \ell(\zeta_k) \ell(\zeta_k)^* \dots \ell(\zeta_1)^*$  for some integer  $k \geq 1$  and some sections  $\zeta_1, \dots, \zeta_k$  in  $0 \oplus \mathcal{E}$ . This unital  $C^*$ -subalgebra is not properly infinite since:

- There is a unital epimorphism of  $C(\mathfrak{X})$ -algebra

$$\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha \rightarrow C(\mathfrak{X}) \cdot 1_{\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})} + \mathcal{K}_{C(\mathfrak{X})}(\mathcal{E}).$$

- The  $C^*$ -algebra  $\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})$  is not properly infinite ([BK04a, Corollary 3.7]).

Thus, any unital  $*$ -homomorphism  $\mathcal{O}_\infty \rightarrow \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha$  would give by composition a unital  $*$ -homomorphism from  $\mathcal{O}_\infty$  to the unital  $C^*$ -subalgebra  $C(\mathfrak{X}) \cdot 1_{\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})} + \mathcal{K}_{C(\mathfrak{X})}(\mathcal{E})$  of  $\mathcal{L}_{C(\mathfrak{X})}(\mathcal{E})$ , something which cannot be.

**Question 6.2.** *Let  $D := [\ell(0 \oplus \mathcal{E}) \cdot \ell(0 \oplus \mathcal{E})^*] \subset \mathcal{L}_{C(\mathfrak{X})}(\mathcal{F}(\tilde{\mathcal{E}})) = \mathcal{L}_{C(\mathfrak{X})}(\mathcal{F}(C(\mathfrak{X}) \oplus \mathcal{E}))$ . Then the  $C^*$ -algebra  $D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha \cdot D = [D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \cdot D]^\alpha$  is a non-stable  $C^*$ -algebra since:*

- $C(\mathfrak{X})$  is a quotient of the fixed point  $C(\mathfrak{X})$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha$ . And so  $D$  is a quotient  $C(\mathfrak{X})$ -algebra of  $[D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \cdot D]^\alpha$ .
- Any non-trivial quotient  $C^*$ -algebra of a stable  $C^*$ -algebra is stable ([Ror04]). But the  $C^*$ -algebra  $D$  is not stable.

*Is the  $C^*$ -algebra  $D \cdot \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \cdot D$  also not stable?*

*Kirchberg noticed that this would imply that the locally purely infinite  $C^*$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})$  is not purely infinite (see question 3.8(iii) in [BK04a]).*

**3)** There is (by [DS01, claim 3.4] or [Blan09, Lemma 4.5]) a sequence of unital inclusions of  $C(\mathfrak{X})$ -algebras:

$$\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})^\alpha \subset \mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E}) \subset \mathcal{T}_{C(\mathfrak{X})}(C(\mathfrak{X}) \oplus \mathcal{E}) \cong \mathcal{T}_{C(\mathfrak{X})}(C(\mathfrak{X}) \oplus \mathcal{E})^\alpha \rtimes \mathbb{N}.$$

Now, we can reformulate the question of whether the intermediate  $C(\mathfrak{X})$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(0 \oplus \mathcal{E})$  is properly infinite or not through the following result of gluing.

**Proposition 6.3.** *Let  $\mathfrak{X}$  be the compact Hilbert cube and let  $\mathcal{T}_{C(\mathfrak{X})}(E)$  be the Cuntz-Pimsner  $C(\mathfrak{X})$ -algebra of a separable Hilbert  $C(\mathfrak{X})$ -module  $E$  with infinite dimensional fibres. Then there exist:*

- a finite covering  $\mathfrak{X} = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$  by the interiors of closed contractible subsets
- unital  $*$ -homomorphisms  $\sigma_k : \mathcal{O}_\infty \rightarrow \mathcal{T}_{C(\mathfrak{X})}(E)|_{F_k}$  ( $1 \leq k \leq n$ )
- unitaries  $u_{i,j} \in \mathcal{U}(\mathcal{T}_{C(\mathfrak{X})}(E)|_{F_i \cap F_j})$  ( $1 \leq i, j \leq n$ )

*such that for all triples  $1 \leq i, j, k \leq n$  with  $F_i \cap F_j \cap F_k \neq \emptyset$ :*

$$(1) \pi_{F_i \cap F_j} \left( \sigma_i(s_m) \right) = u_{i,j} \cdot \pi_{F_j \cap F_i} \left( \sigma_j(s_m) \right) \quad \text{for all } m \in \mathbb{N}$$

$$(2) u_{i,j} \oplus 1 \sim_h 1 \oplus 1 \quad \text{in } \mathcal{U} \left( M_2(\mathbb{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(E)|_{F_i \cap F_j} \right)$$

$$(3) \quad \pi_{F_i \cap F_j \cap F_k}(u_{i,j}) \cdot \pi_{F_i \cap F_j \cap F_k}(u_{j,k}) \cdot \pi_{F_i \cap F_j \cap F_k}(u_{k,i}) = 1 \quad \text{in } \mathcal{U}(\mathcal{T}_{C(\mathfrak{X})}(E)|_{F_i \cap F_j \cap F_k}).$$

*Proof.* For all points  $x \in \mathfrak{X}$ , the fibre  $[\mathcal{T}_{C(\mathfrak{X})}(E)]_x \cong \mathcal{T}_C(E_x)$  of the continuous  $C(\mathfrak{X})$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(E)$  is isomorphic to the semiprojective  $C^*$ -algebra  $\mathcal{O}_\infty$  ([Blac04, Theorem 3.2]). And so, there exists a closed ball  $F(x) \subset \mathfrak{X}$  of strictly positive radius around the point  $x$  and a unital  $*$ -monomorphism from  $\mathcal{O}_\infty$  to the quotient  $\mathcal{T}_{C(\mathfrak{X})}(E)|_{F(x)} := \mathcal{T}_{C(\mathfrak{X})}(E) / C_0(\mathfrak{X} \setminus F(x)) \cdot \mathcal{T}_{C(\mathfrak{X})}(E)$  lifting the unital  $*$ -isomorphism  $\mathcal{O}_\infty \xrightarrow{\sim} \mathcal{T}_C(E_x)$ .

The compactness of the convex metric space  $\mathfrak{X}$  implies that there are

- a finite covering  $\mathfrak{X} = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$  by the interiors of closed contractible subsets  $F_1, \dots, F_n$  of  $\mathfrak{X}$  and
- unital  $*$ -homomorphisms  $\sigma_k : \mathcal{O}_\infty \rightarrow \mathcal{T}_{C(\mathfrak{X})}(E)|_{F_k}$  ( $1 \leq k \leq n$ ).

If two closed subsets  $F_i, F_j \subset \mathfrak{X}$  of this finite covering have a non-zero intersection, the partial isometry

$$u_{i,j} := \sum_{m \in \mathbb{N}} \pi_{F_i \cap F_j}(\sigma_i(s_m)) \cdot \pi_{F_i \cap F_j}(\sigma_j(s_m)^*) \in \mathcal{L}_{C(F_i \cap F_j)}(\mathcal{F}(E)|_{F_i \cap F_j}) \quad (6.3)$$

satisfies

$$u_{i,j}^* u_{i,j} = \sum_{m \in \mathbb{N}} \pi_{F_i \cap F_j}(\sigma_i(s_m s_m^*)) = \pi_{F_i \cap F_j}(\sigma_i(1_{\mathcal{O}_\infty})) = 1_{\mathcal{T}_{C(\mathfrak{X})}(E)|_{F_i \cap F_j}} \quad (6.4)$$

since the only projection in  $\mathcal{O}_\infty$  which dominates all the pairwise orthogonal projections  $s_m s_m^*$  ( $m \in \mathbb{N}$ ) is  $1_{\mathcal{O}_\infty}$ . Similarly,  $u_{i,j} u_{i,j}^* = 1_{\mathcal{T}_{C(\mathfrak{X})}(E)|_{F_i \cap F_j}}$  and so  $u_{i,j}$  is a unitary in  $\mathcal{L}_{C(F_i \cap F_j)}(\mathcal{F}(E)|_{F_i \cap F_j})$  which satisfies the relation (1).

As the  $C(\mathfrak{X})$ -linear  $*$ -representation  $\pi : \mathcal{T}_{C(\mathfrak{X})}(E) \rightarrow \mathcal{L}_{C(\mathfrak{X})}(\mathcal{F}(E))$  is a continuous field of faithful representations and each  $\pi_x(u_{i,j})$  belongs to  $\mathcal{U}(\mathcal{T}(E_x)) = \mathcal{U}^0(\mathcal{T}(E_x))$  (for all  $x \in F_i \cap F_j$ ), the unitary  $u_{i,j}$  actually belongs to the unital  $C^*$ -subalgebra  $\mathcal{T}_{C(\mathfrak{X})}(E)|_{F_i \cap F_j} \subset \mathcal{L}_{C(F_i \cap F_j)}(\mathcal{F}(E)|_{F_i \cap F_j})$ .

This unitary  $u_{i,j}$  also satisfies the relation (2) by [LLR00, Exercise 8.11].

At last, if  $\pi_{i,j,k}$  denotes the quotient map  $\pi_{F_i \cap F_j \cap F_k}$  onto  $\mathcal{L}_{C(F_i \cap F_j \cap F_k)}(\mathcal{F}(E)|_{F_i \cap F_j \cap F_k})$ , then

$$\pi_{i,j,k}(u_{i,j}) \cdot \pi_{i,j,k}(u_{j,k}) \cdot \pi_{i,j,k}(u_{k,i}) = \sum_{m \in \mathbb{N}} \pi_{i,j,k}(\sigma_i(s_m s_m^*)) = 1_{C(F_i \cap F_j \cap F_k)} \quad (6.5)$$

□

*Remarks 6.4.* a) One has  $u_{i,i} = 1_{C(F_i)}$  and  $u_{i,j} = (u_{j,i})^{-1}$  for all  $1 \leq i, j \leq n$ .

b) The contractibility of two closed subsets  $F_1, F_2$  of a metric space  $(X, d)$  does not imply the contractibility of their union  $F_1 \cup F_2$  or their intersection  $F_1 \cap F_2$  (take *e.g.*  $X = \mathbb{T}$ ,  $F_1 = \{z \in \mathbb{T}; z + z^* \geq 0\}$  and  $F_2 = \{z \in \mathbb{T}; z + z^* \leq 0\}$ ).

c) For all indices  $1 \leq i, j \leq n$  and all points  $x \in F_i \cap F_j$ , the unitary  $\pi_x(u_{i,j})$  belongs to the connected component  $\mathcal{U}^0(\mathcal{T}(E_x)) \cong \mathcal{U}(\mathcal{O}_\infty)$ . Nonetheless this does not necessarily imply that the unitary  $u_{i,j}$  belongs to the connected component  $\mathcal{U}^0(\mathcal{T}_{C(\mathfrak{X})}(E)|_{F_i \cap F_j})$  as is shown in the following:

**Lemma 6.5.** *Let  $B$  be a unital  $C^*$ -algebra which is the pull-back of two unital  $C^*$ -algebras  $B_1$  and  $B_2$  along the  $*$ -epimorphisms  $\pi_1 : B_1 \rightarrow B_{12}$  and  $\pi_2 : B_2 \rightarrow B_{12}$ :*

$$\begin{array}{ccc}
 & B & \\
 \sigma_1 \swarrow & & \searrow \sigma_2 \\
 B_1 & & B_2 \\
 \pi_1 \searrow & & \swarrow \pi_2 \\
 & B_{12} &
 \end{array}$$

If  $\mathcal{U}(B) := \{u \in \mathcal{U}(B); \sigma_1(u) \in \mathcal{U}^0(B_1) \text{ and } \sigma_2(u) \in \mathcal{U}^0(B_2)\}$ , one has the sequence of inclusions of normal subgroups:  $\mathcal{U}^0(B) \triangleleft \mathcal{U}(B) \triangleleft \mathcal{U}(B)$ .

Yet the subgroup  $\mathcal{U}^0(B)$  is distinct from  $\mathcal{U}(B)$  in general, even if the unital  $C^*$ -algebra  $B$  is  $K_1$ -injective.

*Proof.* One has  $\mathcal{U}(B) = \{g \in \mathcal{U}(B); \sigma_1(g) \in \mathcal{U}^0(B_1)\} \cap \{g \in \mathcal{U}(B); \sigma_2(g) \in \mathcal{U}^0(B_2)\}$ , whence the expected sequence of inclusions of normal subgroups.

Now, let  $B_1 = B_2 = C([0, 1]; \mathcal{O}_\infty)$ ,  $B_{12} = \mathbb{C}^2 \otimes \mathcal{O}_\infty$ . Set:

$$\pi_1(f) = (f(0), f(1)) \text{ and } \pi_2(f) = (f(1), f(0)) \quad \text{for all } f \in C([0, 1]; \mathcal{O}_\infty).$$

Then, there is a  $C^*$ -isomorphism  $\alpha : B = B_1 \oplus_{B_{12}} B_2 \cong C(\mathbb{T}, \mathcal{O}_\infty)$  given by:

$$\alpha(f_1, f_2)(e^{i\pi t}) = \begin{cases} f_1(t) & \text{if } 0 \leq t \leq 1 \\ f_2(1-t) & \text{if } 1 \leq t \leq 2 \end{cases} \quad \text{for all } (f_1, f_2) \in A.$$

Thus, the  $C^*$ -algebra  $B$  satisfies  $B \cong B \otimes \mathcal{O}_\infty$  and so is  $K_1$ -injective ([Roh09]). If  $v_1(t) = e^{2i\pi t} \cdot 1_{\mathcal{O}_\infty}$  and  $v_2(t) = 1_{\mathcal{O}_\infty}$  for  $t \in [0, 1]$ , then the pair  $(v_1, v_2)$  belongs to  $\mathcal{U}(B) \setminus \mathcal{U}^0(B)$  since  $\alpha(v_1, v_2) \sim_h (z \mapsto z \cdot 1_{\mathcal{O}_\infty})$  in  $\mathcal{U}(B)$  and  $[z] \neq [1]$  in  $K_1(B) \cong \mathbb{Z}$ .  $\square$

4) If  $\mathbb{E}$  denotes the trivial Hilbert  $C(\mathfrak{X})$ -module  $\mathbb{E} := \ell^2(\mathbb{N}) \otimes C(\mathfrak{X})$ , then Theorem 2.11 of [BRR08] and Equation (2) of the above Proposition 6.3 imply by finite induction the sequence of unital inclusions of  $C(\mathfrak{X})$ -algebras:

$$\mathcal{O}_\infty \otimes C(\mathfrak{X}) = \mathcal{T}_{C(\mathfrak{X})}(\mathbb{E}) \hookrightarrow M_{2^{n-1}}(\mathbb{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(E) \subset M_{2^{n-1}}(\mathbb{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(\mathbb{E}) \quad (6.6)$$

In particular, the tensor product  $M_{2^{n-1}}(\mathbb{C}) \otimes \mathcal{T}_{C(\mathfrak{X})}(E)$  is properly infinite (thus answering question 3.8(ii) in [BK04a]). Note that this does not *a priori* imply that the  $C^*$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(E)$  itself is properly infinite (see [Ror97, Theorem 5.3], [Ror04]).

Let us eventually recall the following link between the proper infiniteness of the  $C(\mathfrak{X})$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(E)$  and the  $K_1$ -injectivity of its properly infinite quotient  $C^*$ -algebras.

**Lemma 6.6** ([BRR08]). *Suppose that:*

- $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$  is a finite covering of a compact metric space  $X$  by the interiors of closed subsets  $F_k \subset X$ ,
- $B$  is a unital continuous  $C(X)$ -algebra such that all the quotients  $C^*$ -algebras  $B|_{F_1}, \dots, B|_{F_n}$  are properly infinite.

Then the  $C^*$ -algebra  $B$  is a properly infinite  $C^*$ -algebra as soon as all the properly infinite quotients  $B|_{F_k \cap (F_1 \cup \dots \cup F_{k-1})}$  are  $K_1$ -injective ( $2 \leq k \leq n$ ).

This result easily derives from Proposition 2.7 in [BRR08]. Nevertheless we sketch a self-contained proof for the completeness of this paper.

*Proof.* We can construct inductively for all  $k \in \{1, \dots, n\}$  a unitary  $d_k \in \mathcal{U}^0(B_{|F_k})$  and a unital  $*$ -homomorphism  $\sigma'_k$  from the  $C^*$ -subalgebra  $\mathcal{T}_{n+2-k} := C^*(s_1, \dots, s_{n+2-k}) \subset \mathcal{O}_\infty$  to the quotient  $B_{|F_1 \cup \dots \cup F_k}$  such that:

$$\pi_{F_j}(\sigma'_k(s_m)) = d_j \cdot \sigma_j(s_m) \in B_{|F_k} \quad (6.7)$$

for all  $j \in \{1, \dots, k\}$  and all  $m \in \{1, \dots, n+2-k\}$  in the following way.

Set first  $d_1 := 1_{|F_1}$  and let  $\sigma'_1 : \mathcal{T}_{n+1} \rightarrow B_{|F_1}$  be the only unital  $*$ -homomorphism such that  $\sigma'_1(s_m) := \sigma_1(s_m)$  for all integers  $1 \leq m \leq n+1$ .

Take now an integer  $k$  in the finite set  $\{2, \dots, n\}$  and suppose already constructed the first  $k-1$  unitaries  $d_1, \dots, d_{k-1}$  and a unital homomorphism of  $C^*$ -algebra  $\sigma'_{k-1} : \mathcal{T}_{n+3-k} \rightarrow B_{|F_1 \cup \dots \cup F_{k-1}}$  so that:

$$\begin{aligned} - \pi_{F_j}(\sigma'_{k-1}(s_m)) &= d_j \cdot \sigma_j(s_m) \quad \text{for all } 1 \leq j \leq k-1 \text{ and } 1 \leq m \leq n+3-k, \\ - u_{i,j} &= \pi_{F_i \cap F_j}(d_i)^* \cdot \pi_{F_i \cap F_j}(d_j) \quad \text{for all } 1 \leq i, j \leq k-1. \end{aligned}$$

As the projection  $1 - \sum_{j=1}^{n+2-k} s_j s_j^*$  is properly infinite and full in the  $C^*$ -algebra  $\mathcal{T}_{n+3-k}$ ,

Lemma 2.4 of [BRR08] implies that there exists a unitary  $z_k \in \mathcal{U}(B_{|F_k \cap (F_1 \cup \dots \cup F_{k-1})})$  such that:

$$\begin{aligned} - [z_k] &= [1] && \text{in } K_1(B_{|F_k \cap (F_1 \cup \dots \cup F_{k-1})}) \\ - \pi_{F_k \cap (F_1 \cup \dots \cup F_{k-1})}(\sigma'_{k-1}(s_m)) &= z_k \cdot \pi_{F_k \cap (F_1 \cup \dots \cup F_{k-1})}(\sigma_k(s_m)) && \text{for all } 1 \leq m \leq n+2-k. \end{aligned}$$

The assumed  $K_1$ -injectivity of the quotient  $C^*$ -algebra  $B_{|F_k \cap (F_1 \cup \dots \cup F_{k-1})}$  implies that the  $K_1$ -trivial unitary  $z_k$  belongs to the connected component  $\mathcal{U}^0(B_{|F_k \cap (F_1 \cup \dots \cup F_{k-1})})$ . Hence, it admits a lifting  $d_k$  in  $\mathcal{U}^0(B_{|F_k})$  ([BRR08, Proposition 2.7]) and there exists one and only one unital  $*$ -homomorphism  $\sigma'_k : \mathcal{T}_{n+2-k} \rightarrow B_{|F_1 \cup \dots \cup F_k}$  such that for all  $m \in \{1, \dots, n+2-k\}$ :

$$\begin{aligned} - \pi_{F_1 \cup \dots \cup F_{k-1}} \circ \sigma'_k(s_m) &= \sigma'_{k-1}(s_m) \\ - \pi_{F_k} \circ \sigma'_k(s_m) &= d_k \cdot \sigma_k(s_m). \end{aligned}$$

The composition of a unital  $*$ -homomorphism  $\mathcal{O}_\infty \hookrightarrow \mathcal{T}_2$  with the above constructed  $*$ -homomorphism  $\sigma'_n$  gives a convenient unital  $*$ -homomorphism  $\mathcal{O}_\infty \rightarrow B$ .  $\square$

*Remarks 6.7.* a) If  $E$  is a separable Hilbert  $C(\mathfrak{X})$ -module with infinite dimensional fibres, then the non proper infiniteness of the  $C^*$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(E)$  implies by Proposition 6.3 and Lemma 6.6 that one of the quotients  $\mathcal{T}_{C(\mathfrak{X})}(E)|_{F_k \cap (F_1 \cup \dots \cup F_{k-1})}$  is a unital properly infinite  $C^*$ -algebra which is not  $K_1$ -injective (see [BRR08, Theorem 5.5]).

b) Is the  $C^*$ -algebra  $\mathcal{T}_{C(\mathfrak{X})}(E)$  weakly purely infinite for any separable full Hilbert  $C(\mathfrak{X})$ -module  $E$ ? (see [BK04a])

c) Is a quotient of a unital  $K_1$ -injective properly infinite  $C^*$ -algebra always  $K_1$ -injective? (see [BRR08, Theorem 5.5])

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