

ON UNITAL $C(X)$ -ALGEBRAS AND $C(X)$ -VALUED CONDITIONAL EXPECTATIONS OF FINITE INDEX

ETIENNE BLANCHARD AND ILJA GOGIĆ

ABSTRACT. Let X be a compact Hausdorff space and let A be a unital $C(X)$ -algebra, where $C(X)$ is embedded as a unital C^* -subalgebra of the centre of A . We consider the problem of characterizing the existence of a conditional expectation $E : A \rightarrow C(X)$ of finite index in terms of the underlying C^* -bundle of A over X . More precisely, we show that if A admits a $C(X)$ -valued conditional expectation of finite index, then A is necessarily a continuous $C(X)$ -algebra, and there exists a positive integer N such that every fibre A_x of A is finite-dimensional, with $\dim A_x \leq N$. We also give some sufficient conditions on A that ensure the existence of a $C(X)$ -valued conditional expectation of finite index.

1. INTRODUCTION

Let $B \subseteq A$ be two unital C^* -algebras with the same unit element. A *conditional expectation* (abbreviated by C.E.) from A to B is a completely positive contraction $E : A \rightarrow B$ such that $E(b) = b$ for all $b \in B$, and which is B -bilinear, i.e.

$$E(b_1 a b_2) = b_1 E(a) b_2$$

for all $a \in A$ and $b_1, b_2 \in B$. By a result of Y. Tomiyama (see [22, Theorem 1] or [4, Theorem II.6.10.2]), a map $E : A \rightarrow B$ is a C.E. if and only if E is a projection of norm one.

If $E(a^* a) = 0$ ($a \in A$) implies $a = 0$, E is said to be *faithful*. Every faithful conditional expectation $E : A \rightarrow B$ introduces a pre-Hilbert B -module structure on A , whose inner product is defined by

$$(1.1) \quad \langle a_1, a_2 \rangle_E := E(a_1^* a_2) \quad (a_1, a_2 \in A).$$

The notion of finite index was introduced by V. F. R. Jones [14] in order to classify the subfactors of a type II_1 factor. Soon afterwards H. Kosaki [16] extended the Jones index theory to arbitrary factors. In order to generalize the results of [14, 16], M. Pimsner and S. Popa introduced in [19, 20] a definition for conditional expectations of finite index in the context of W^* -algebras: There must exist a constant $K \geq 1$ such that the map $K \cdot E - \text{id}_A$ is positive on A . Then, following the idea of M. Baillel, Y. Denizeau and J.-F. Havet (see [3]), the index of E can be defined in the following way: Since the map $K \cdot E - \text{id}_A$ is positive, E defines a (complete) Hilbert B -module structure on A , with respect to the inner product

Date: October 24, 2013.

2010 Mathematics Subject Classification. Primary 46L65; Secondary 46L05, 55R05.

Key words and phrases. $C(X)$ -algebra, C^* -bundle, conditional expectation of finite index, non-commutative branched covering.

(1.1). If $\{x_i\}$ is a quasi-orthonormal basis in A , the *index* of E is the sum $\sum_{i=1}^{\infty} x_i^* x_i$, with respect to the ultraweak topology.

Y. Watatani also considered C.E. of (algebraically) finite index, when the original C^* -algebra A is a finitely generated Hilbert C^* -module over B (see [23]).

The results of M. Baillel, Y. Denizeau and J.-F. Havet in [3] also indicated that there might occur some difficulties in order to extend the notion of "finite index" for conditional expectations of C^* -algebras with arbitrary centres. However, this problem was solved by M. Frank and E. Kirchberg in [11]. The main result of their paper is [11, Theorem 1]:

Theorem 1.1 (M. Frank and E. Kirchberg). *For a C.E. $E : A \rightarrow B$, where $B \subseteq A$ are unital C^* -algebras with the same unit element, the following conditions are equivalent:*

- (i) *There exists a constant $K \geq 1$ such that the map $K \cdot E - \text{id}_A$ is positive.*
- (ii) *There exists a constant $L \geq 1$ such that the map $L \cdot E - \text{id}_A$ is completely positive.*
- (iii) *A becomes a (complete) Hilbert B -module when equipped with the inner product (1.1).*

Moreover, if

$$K(E) := \inf\{K \geq 1 : K \cdot E - \text{id}_A \text{ is positive}\},$$

$$L(E) := \inf\{L \geq 1 : L \cdot E - \text{id}_A \text{ is completely positive}\},$$

with $K(E) = \infty$ or $L(E) = \infty$ if no such number K or L exists, then

$$K(E) \leq L(E) \leq \lfloor K(E) \rfloor K(E),$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number.

The importance of this result is that it gives the right general definition for conditional expectations on C^* -algebras to be of finite index:

Definition 1.2. If $B \subseteq A$ are two unital C^* -algebras with the same unit element, then a C.E. $E : A \rightarrow B$ is said to be of *finite index* (abbreviated C.E.F.I.) if E satisfies one of the equivalent conditions of Theorem 1.1.

In this case the index value of E can be calculated in the enveloping von Neumann algebra A^{**} (see [11, Definition 3.1]).

For a unital inclusion $A \subseteq B$ of unital C^* -algebras we introduce the following constant

$$K(A, B) := \inf\{K(E) : E : A \rightarrow B \text{ is C.E.F.I.}\},$$

with $K(A, B) = \infty$, if no such C.E.F.I. exists. This constant will play an important role in this paper.

More recently, A. Pavlov and E. Troitsky considered in [17] the problem of existence of a C.E.F.I. $E : C(Y) \rightarrow C(X)$ for a unital inclusion $\varphi : C(X) \hookrightarrow C(Y)$ of unital commutative C^* -algebras. The main result of their paper is [17, Theorem 1.1], which shows that such a C.E.F.I. exists if and only if the transpose map $\varphi_* : Y \rightarrow X$ is a *branched covering*. This means that φ_* is an open map with uniformly bounded number of pre-images (i.e. $\sup_{x \in X} |\varphi_*^{-1}(x)| < \infty$). This result motivated A. Pavlov and E. Troitsky to define the *noncommutative branched coverings*, as unital inclusion $B \subseteq A$ of unital C^* -algebras such that there exists a C.E.F.I. from A to B (see [17, Definition 1.2]).

Using the above inclusion $\varphi : C(X) \hookrightarrow C(Y)$ we may consider $C(Y)$ as a $C(X)$ -algebra. Then the map φ_* is open if and only if $C(Y)$ is a continuous $C(X)$ -algebra, and φ_* has uniformly bounded number of pre-images if and only if $C(Y)$ is subhomogeneous $C(X)$ -algebra. This means that there exists a positive integer N such that every fibre $C(Y)_x$ of $C(Y)$ is finite-dimensional with $\dim C(Y)_x \leq N$ (see Section 2). Therefore, we can restate [17, Theorem 1.1] in terms of $C(X)$ -algebras as follows:

Theorem 1.3 (A. Pavlov and E. Troitsky). *Let A be a unital commutative $C(X)$ -algebra, where $C(X)$ is embedded as a unital C^* -subalgebra of A . Then A admits a $C(X)$ -valued C.E.F.I. if and only if A is a continuous subhomogeneous $C(X)$ -algebra.*

The purpose of the present paper is to consider a possible extension of Theorem 1.3 to the case when A is an arbitrary (not necessarily commutative) unital $C(X)$ -algebra. The necessary condition for the existence of a $C(X)$ -valued C.E.F.I. appears to be identical to the one of Theorem 1.3:

Theorem 1.4. *Let A be a unital $C(X)$ -algebra, where $C(X)$ is embedded as a unital C^* -subalgebra of the centre of A . If A admits a $C(X)$ -valued C.E.F.I., then A is a continuous subhomogeneous $C(X)$ -algebra. Moreover, in this case the following inequality holds:*

$$K(A, C(X)) \geq r(A),$$

where $r(A)$ is the rank of A , i.e.

$$r(A) = \max \left\{ \sum_{[\pi_x] \in \widehat{A}_x} \dim \pi_x : x \in X \right\}.$$

We shall prove Theorem 1.4 in Section 3. At the moment we do not know if the converse of Theorem 1.4 also holds. However, if all the fibres of a continuous unital $C(X)$ -algebra A are $*$ -isomorphic to the same finite-dimensional C^* -algebra (i.e. A is a homogeneous $C(X)$ -algebra), then there exists a unique C.E. $E : A \rightarrow C(X)$ such that the map $r(A) \cdot E - \text{id}_A$ is positive (Proposition 3.4). In particular, we have the equality $K(A, C(X)) = r(A)$ in this case. Also, a direct consequence of this fact is that any unital $C(X)$ -algebra A which can be embedded as a $C(X)$ -subalgebra of some continuous homogeneous unital $C(X)$ -algebra also admits a $C(X)$ -valued C.E.F.I.. However, this embedding condition is not necessary for the existence of such C.E.F.I.. Indeed, there exists a continuous unital $C(X)$ -algebra A over a second-countable compact Hausdorff space X with fibres $M_2(\mathbb{C})$ or \mathbb{C} which admits a $C(X)$ -valued C.E.F.I., but which cannot be embedded as a $C(X)$ -subalgebra into any continuous homogeneous unital $C(X)$ -algebra (Example 3.6). At the end of this paper we also show that any continuous unital $C(X)$ -algebra A of rank 2 admits a C.E. $E : A \rightarrow C(X)$ such that the map $2 \cdot E - \text{id}_A$ is positive (Proposition 3.7). In particular, the equality $K(A, C(X)) = r(A)$ also holds in this class of $C(X)$ -algebras.

2. NOTATION AND PRELIMINARIES

Throughout this paper A will be a C^* -algebra. We denote by A_{sa} and A_+ the self-adjoint and the positive parts of A . The centre of A is denoted by $Z(A)$. By

\hat{A} and $\text{Prim}(A)$ we respectively denote the *spectrum* of A (i.e. the set of all classes of irreducible representations of A) and the *primitive spectrum* of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology. By a *dimension* of $[\pi] \in \hat{A}$, which is denoted by $\dim \pi$, we mean the dimension of the underlying Hilbert space of some representative of $[\pi]$.

Let X be a compact Hausdorff space. For each point $x \in X$ let

$$C_x(X) := \{f \in C(X) : f(x) = 0\}$$

be the corresponding maximal ideal of $C(X)$.

Definition 2.1. A $C(X)$ -algebra is a C^* -algebra A endowed with a unital $*$ -homomorphism ψ_A from $C(X)$ to the centre of the multiplier algebra of A .

Remark 2.2. Given $f \in C(X)$ and $a \in A$, we write fa for the product $\psi_A(f) \cdot a$ if no confusion is possible.

There is a natural connection between $C(X)$ -algebras and upper semicontinuous C^* -bundles over X . We first give a formal definition of such bundles:

Definition 2.3. Following [24] by an *upper semicontinuous C^* -bundle* we mean a triple $\mathfrak{A} = (p, \mathcal{A}, X)$ where \mathcal{A} is a topological space with a continuous open surjection $p : \mathcal{A} \rightarrow X$, together with operations and norms making each *fibre* $\mathcal{A}_x := p^{-1}(x)$ into a C^* -algebra, such that the following conditions are satisfied:

- (A1) The maps $\mathbb{C} \times \mathcal{A} \rightarrow \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \rightarrow \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \rightarrow \mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{A}$ given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous ($\mathcal{A} \times_X \mathcal{A}$ denotes the Whitney sum over X).
- (A2) The map $\mathcal{A} \rightarrow \mathbb{R}$, defined by norm on each fibre, is upper semicontinuous.
- (A3) If $x \in X$ and if (a_α) is a net in \mathcal{A} such that $\|a_\alpha\| \rightarrow 0$ and $p(a_\alpha) \rightarrow x$ in X , then $a_\alpha \rightarrow 0_x$ in \mathcal{A} (0_x denotes the zero-element of \mathcal{A}_x).

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that \mathfrak{A} is a *continuous C^* -bundle*.

By a *section* of an upper semicontinuous C^* -bundle \mathfrak{A} we mean a map $s : X \rightarrow \mathcal{A}$ such that $p(s(x)) = x$ for all $x \in X$. We denote by $\Gamma(\mathfrak{A})$ the set of all continuous sections of \mathfrak{A} . Then $\Gamma(\mathfrak{A})$ becomes a $C(X)$ -algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a $C(X)$ -algebra A , one can always associate an upper semicontinuous C^* -bundle \mathfrak{A} over X such that $A \cong \Gamma(\mathfrak{A})$, as follows. Set $J_x := C_x(X) \cdot A$ and note that J_x is a closed two-sided ideal in A (by Cohen factorization theorem [7], [6, Theorem A.6.2]). The quotient $A_x := A/J_x$ is called the *fibre* at the point x , and we denote by a_x the image in A_x of an element $a \in A$. Let

$$\mathcal{A} := \bigsqcup_{x \in X} A_x,$$

and let $p : \mathcal{A} \rightarrow X$ be the canonical associated projection. For $a \in A$ we define the map $\hat{a} : X \rightarrow \mathcal{A}$ by $\hat{a}(x) := a_x$, and let $\Omega := \{\hat{a} : a \in A\}$. Since for each $a \in A$ we have

$$\|a_x\| = \inf\{\| [1 - f + f(x)] \cdot a \| : f \in C(X)\},$$

the norm function $x \mapsto \|a_x\|$ is upper semicontinuous on X . Hence, by Fell's theorem [24, Theorem C.25] there exists a unique topology on \mathcal{A} for which $\mathfrak{A} := (p, \mathcal{A}, X)$

becomes an upper semicontinuous C^* -bundle such that $\Omega \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee's theorem [24, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the *generalized Gelfand transform* $\mathcal{G} : a \in A \mapsto \hat{a} \in \Gamma(\mathfrak{A})$, is an isomorphism of $C(X)$ -algebras, from A onto $\Gamma(\mathfrak{A})$.

Definition 2.4. Let A be a $C(X)$ -algebra. If all the norm functions $x \mapsto \|a_x\|$ ($a \in A$) are continuous on X , we say that A is a *continuous $C(X)$ -algebra*.

Note that the $C(X)$ -algebra A is continuous if and only if \mathfrak{A} is continuous as a C^* -bundle.

The C^* -algebra A is said to be

- (n -)homogeneous ($n \in \mathbb{N}$), if $\dim \pi = n$ for all $[\pi] \in \widehat{A}$,
- (n -)subhomogeneous ($n \in \mathbb{N}$), if $\sup_{[\pi] \in \widehat{A}} \dim \pi = n$.

We shall now define the similar notions for $C(X)$ -algebras. To do this, first recall that if D is a finite-dimensional C^* -algebra, then there is a finite number of central pairwise orthogonal projections $p_1, \dots, p_m \in Z(D)$ with $\sum_{i=1}^m p_i = 1_D$, such that

$$(2.1) \quad D = p_1 D \oplus \dots \oplus p_m D,$$

and each $p_i D$ is $*$ -isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$ (see e.g. [21, Theorem I.11.9]). We define the *rank* of D as

$$r(D) := \sum_{i=1}^m n_i = \sum_{[\pi] \in \widehat{D}} \dim \pi.$$

Definition 2.5. Let A be a $C(X)$ -algebra. We say that A is

- *homogeneous* all the fibres of A are $*$ -isomorphic to the same finite-dimensional C^* -algebra.
- *subhomogeneous* if there exists a positive integer N such that every fibre A_x of A is finite-dimensional with $\dim A_x \leq N$.

Remark 2.6. Let A be a $C(X)$ -algebra.

- (i) A is subhomogeneous if and only if

$$r(A) := \sup\{r(A_x) : x \in X\} < \infty$$

As in the finite-dimensional case, we call the number $r(A)$ the *rank* of A .

- (ii) If A is continuous and homogeneous, then by [10, Lemma 3.1] the underlying C^* -bundle \mathfrak{A} is locally trivial.

3. RESULTS

Remark 3.1. If A is a unital $C(X)$ -algebra, we always assume in this section that the map $\psi_A : C(X) \rightarrow Z(A)$ is injective, so that we can identify $C(X)$ with the unital C^* -subalgebra $\psi_A(C(X))$ of $Z(A)$.

In order to prove Theorem 1.4 we shall need the following two auxiliary results.

Lemma 3.2. *Let D be a unital C^* -algebra. Then $K(D, \mathbb{C}) := K(D, \mathbb{C}1_D) < \infty$ if and only if D is finite-dimensional. In this case we have:*

- (i) *The constant $K(\omega)$ is finite for every faithful state ω on D , which we identify with the corresponding faithful C.E.*

$$d \in D \mapsto \omega(d) \cdot 1_D \in \mathbb{C} \cdot 1_D \quad (d \in D).$$

(ii) $K(D, \mathbb{C}) = r(D)$. Moreover, there exists a unique state τ on D such that

$$(3.1) \quad r(D) \cdot \tau(d)1_D \geq d \quad \text{for all } d \in D_+.$$

Proof. The equivalence $K(D, \mathbb{C}) < \infty \Leftrightarrow \dim D < \infty$ follows from [13, Lemma 4.5]. Hence, suppose that D is finite-dimensional and let ω be a faithful state on D . The proof will now proceed in two steps.

Step 1. Assume that D is simple, i.e. $D = M_n(\mathbb{C})$ for some n . If $\text{tr}(\cdot)$ is the standard trace of $M_n(\mathbb{C})$, then there exists a strictly positive matrix $a \in M_n(\mathbb{C})$ with $\text{tr}(a) = 1$ such that

$$\omega(d) = \text{tr}(ad) \quad (d \in M_n(\mathbb{C})).$$

Let $a = u^* \cdot \text{diag}(\lambda_1, \dots, \lambda_n) \cdot u$ be a diagonalisation of a , where $u \in M_n(\mathbb{C})$ is a unitary and $\lambda_1, \dots, \lambda_n > 0$ are the eigenvalues of a . Then for all $d \in M_n(\mathbb{C})$ one has

$$(3.2) \quad \omega(u^*du) = \text{tr}(au^*du) = \text{tr}(uau^*d) = \text{tr}(\text{diag}(\lambda_1, \dots, \lambda_n)d).$$

The constant $K(\omega)$ is by definition the smallest $K \geq 1$ satisfying

$$(3.3) \quad K \cdot \omega(d)1_D \geq d \quad \text{for all } d \in D_+.$$

Thus, (3.2) and (3.3) for rank 1 projections in D imply that

$$K(\omega) = \max\{\lambda_i^{-1} : 1 \leq i \leq n\}.$$

As $1 = \omega(1) = \sum_{i=1}^n \lambda_i$, one has $K(\omega) \geq n$ for any faithful state ω on D . Also, $K(\omega) = n$ if and only if $\omega = \tau := \frac{1}{n}\text{tr}(\cdot)$. In particular, if $D = M_n(\mathbb{C})$, we have $K(D, \mathbb{C}) = r(D) = n$, and τ is the unique state on D satisfying (3.1).

Step 2. Suppose that D is an arbitrary finite-dimensional C^* -algebra. We decompose D as in (2.1). For each $1 \leq i \leq m$

$$\omega_i(p_i d) := \frac{1}{\omega(p_i)} \cdot \omega(p_i d) \quad (d \in D)$$

defines a faithful state on $p_i D$. By Step 1 we have $n_i \leq K(\omega_i) < \infty$ for all $1 \leq i \leq m$. Put

$$K_\omega := \max \left\{ \frac{K(\omega_i)}{\omega(p_i)} : 1 \leq i \leq m \right\}.$$

We claim that $K(\omega) = K_\omega$. Indeed, for all $d \in D_+$ we have

$$\begin{aligned} K_\omega \cdot \omega(d)1_D &= \sum_{i=1}^m K_\omega \cdot \omega(p_i)\omega_i(p_i d)1_D \geq \sum_{i=1}^m K(\omega_i) \cdot \omega_i(p_i d)p_i \\ &\geq \sum_{i=1}^m p_i d = d, \end{aligned}$$

which shows $K(\omega) \leq K_\omega$. On the other hand, for each $d \in D_+$ we have

$$[\omega(p_i)K(\omega)] \cdot \omega_i(p_i d)p_i \geq p_i d,$$

so that

$$(3.4) \quad \omega(p_i)K(\omega) \geq K(\omega_i) \quad (1 \leq i \leq m).$$

This shows $K(\omega) = K_\omega$, as wanted. Also,

$$K(\omega) = \sum_{i=1}^m \omega(p_i)K(\omega) \geq \sum_{i=1}^m K(\omega_i) \geq \sum_{i=1}^m n_i = r(D),$$

so that $K(D, \mathbb{C}) \geq r(D)$.

It remains to show that there exists a unique state τ on D satisfying (3.1). To do this, suppose that $r(D) = n$, and for each $1 \leq i \leq m$ let τ_i be the only faithful tracial state on $p_i D \cong M_{n_i}(\mathbb{C})$. Define the state τ on D by

$$(3.5) \quad \tau(d) := \frac{1}{n} \sum_{i=1}^m n_i \cdot \tau_i(p_i d) \quad (d \in D).$$

As $\tau(p_i) = \frac{n_i}{n}$ and $K(\tau_i) = n_i$ for all $1 \leq i \leq m$, we have $K(\tau) = K_\tau = n$. In particular, $K(D, \mathbb{C}) = n = r(D)$.

To show the uniqueness of this state τ , suppose that ω is another state on D with $K(\omega) = n$. Then using (3.4) we have

$$\sum_{i=1}^m K(\omega_i) \leq \sum_{i=1}^m \omega(p_i)K(\omega) = K(\omega) = n.$$

But since $K(\omega_i) \geq n_i$ and $\sum_{i=1}^m n_i = n$, we must have $K(\omega_i) = n_i$ for all $1 \leq i \leq m$. By the uniqueness part of Step 1 we conclude that

$$(3.6) \quad \omega_i = \tau_i \quad \text{for all } 1 \leq i \leq m.$$

Also, $K_\omega = K(\omega) = n$ and $K(\omega_i) = n_i$ imply $\omega(p_i) \geq \frac{n_i}{n}$ for all $1 \leq i \leq m$. Since ω is a state on D and $\sum_{i=1}^m p_i = 1_D$, we must have

$$(3.7) \quad \omega(p_i) = \frac{n_i}{n} \quad \text{for all } 1 \leq i \leq m.$$

Finally, (3.6) and (3.7) imply that

$$\omega(d) = \sum_{i=1}^m \omega(p_i)\omega_i(p_i d) = \frac{1}{n} \sum_{i=1}^m n_i \cdot \tau_i(p_i d) = \tau(d),$$

for all $d \in D$, which finishes the proof. \square

Proposition 3.3. *Let A be a unital $C(X)$ -algebra. If A admits a faithful $C(X)$ -valued C.E., then A is a continuous $C(X)$ -algebra.*

Proof. This can be deduced from [5, Section 2]. For completeness, we include a short proof of this fact. It suffices to show that all norm functions $x \mapsto \|a_x\|$ ($a \in A$) are lower semicontinuous on X . To prove this, let $E : A \rightarrow C(X)$ be a faithful C.E. and let $L^2(A, E)$ be the completion of the pre-Hilbert $C(X)$ -module A , with respect to the inner product (1.1). For $a \in A$ let $\Phi(a) : L^2(A, E) \rightarrow L^2(A, E)$ denote the continuous extension of the left multiplication map $a_1 \mapsto aa_1$ ($a \in A$). Since E is faithful and since

$$\begin{aligned} \langle \Phi(a)(a_1), a_2 \rangle_E &= \langle aa_1, a_2 \rangle_E = E(a_1^* a^* a_2) = \langle a_1, a^* a_2 \rangle_E \\ &= \langle a_1, \Phi(a^*)(a_2) \rangle_E, \end{aligned}$$

for all $a_1, a_2 \in A$, the map Φ defines an injective $C(X)$ -linear morphism from A to the $C(X)$ -algebra $\mathbb{B}_{C(X)}(L^2(A, E))$ of bounded adjointable $C(X)$ -linear operators on $L^2(A, E)$. Therefore, for $a \in A$ and $x \in X$ we have

$$\begin{aligned} \|a_x\| &= \|\Phi(a)_x\| \\ &= \sup\{|\langle \Phi(a)(a_1), a_2 \rangle_E(x)| : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\} \\ &= \sup\{|E(a_1^* a^* a_2)(x)| : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\}. \end{aligned}$$

In particular, the function $x \mapsto \|a_x\|$ is a supremum of continuous functions $x \mapsto |E(a_1^* a^* a_2)(x)|$ ($\|a_1\|_E = \|a_2\|_E = 1$), so it must be lower semicontinuous on X . \square

Proof of Theorem 1.4. Let $E : A \rightarrow C(X)$ be a C.E.F.I.. As the conditional expectation E is faithful, Proposition 3.3 implies that the $C(X)$ -algebra A is continuous (note that in this case $(A, \langle \cdot, \cdot \rangle_E)$ is already a complete Hilbert $C(X)$ -module by Theorem 1.1). It remains to show that each fibre A_x ($x \in X$) is finite-dimensional and satisfies $r(A_x) \leq K(E)$. Indeed, for a fixed point $x \in X$ and $\varepsilon > 0$,

$$\omega_x : a_x \mapsto E(a)(x)$$

defines a state on a fibre A_x satisfying

$$(K(E) + \varepsilon) \cdot \omega_x(a_x)1_x \geq a_x$$

for all $a_x \in (A_x)_+$. Lemma 3.2 now yields $r(A_x) \leq K(E)$, as wanted. \square

We shall now give some sufficient conditions on a continuous unital subhomogeneous $C(X)$ -algebra A to ensure the existence of a $C(X)$ -valued C.E.F.I..

Proposition 3.4. *Every continuous homogeneous unital $C(X)$ -algebra A admits a unique C.E. $E : A \rightarrow C(X)$ such that the map $r(A) \cdot E - \text{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.*

Proof. The construction of such a C.E. $E : A \rightarrow C(X)$ can be deduced from the proof of [13, Lemma 4.6]. But we include here the main steps of the proof for completeness. By assumption all fibres of A are $*$ -isomorphic to a fixed finite-dimensional C^* -algebra D . Suppose that $r(D) = n$, and let τ be a state on D defined by (3.5). It is easy to check that τ is invariant under the group $\text{Aut}(D)$ of $*$ -automorphisms of D . Since the $C(X)$ -algebra A is continuous and homogeneous, its underlying bundle \mathfrak{A} is locally trivial by Remark 2.6. Hence, there exists an open covering $\{U_\alpha\}$ of X such that $\Phi_\alpha : \mathfrak{A}|_{U_\alpha} \cong U_\alpha \times D$, where

- Φ_α is an isomorphism of C^* -bundles, and
- $\mathfrak{A}|_U$ is the restriction bundle over a subset $U \subseteq X$.

Fix an element $a \in A$. For $x \in X$ choose an index α such that $x \in U_\alpha$, and define

$$E(a)(x) := \tau(\Phi_\alpha(a_x)).$$

Since τ is invariant under the group $\text{Aut}(D)$, the value $E(a)(x)$ is well defined, and the local triviality of \mathfrak{A} implies that the function $E(a) : x \mapsto E(a)(x)$ is continuous on X . It is now easy to see that the map $E : a \rightarrow E(a)$ defines a $C(X)$ -valued C.E.F.I. on A . Moreover, by (3.1) we have

$$n \cdot E(a)(x)1_x \geq a_x, \quad \text{for all } a \in A_+ \text{ and } x \in X.$$

Thus, the map $n \cdot E - \text{id}_A$ is positive and E is the only C.E. with this property (Lemma 3.2). In particular, $K(A, C(X)) \leq r(A)$, so Theorem 1.4 yields that $K(A, C(X)) = n$. \square

Corollary 3.5. *If the unital $C(X)$ -algebra A admits a $C(X)$ -linear embedding into some homogeneous continuous unital $C(X)$ -algebra A' , then A admits a $C(X)$ -valued C.E.F.I..*

Proof. By Proposition 3.4 there exists a C.E. $E' : A' \rightarrow C(X)$ of finite index. Then the restriction $E'|_A : A \rightarrow C(X)$ defines a convenient C.E.F.I.. \square

Note that the embedding condition of Corollary 3.5 is not necessary for the existence of a $C(X)$ -valued C.E.F.I.. Indeed, in Example 3.6 we show that there exists a continuous unital $C(X)$ -algebra A of rank 2 which does not admit a $C(X)$ -linear embedding into any continuous homogeneous unital $C(X)$ -algebra. On the other hand, a direct consequence of Proposition 3.7 is that A admits a $C(X)$ -valued C.E.F.I..

To do this, first recall that a C^* -algebra A is said to be *central* if it satisfies the following two conditions:

- (i) A is *quasi-central* (i.e. no primitive ideal of A contains $Z(A)$);
- (ii) If $P, Q \in \text{Prim}(A)$ and $P \cap Z(A) = Q \cap Z(A)$, then $P = Q$

(see [1, 8, 12, 15]). By [8, Proposition 3] a quasi-central C^* -algebra A is central if and only if $\text{Prim}(A)$ is Hausdorff.

Example 3.6. By [18, Example 3.5] there exists a continuous $M_2(\mathbb{C})$ -bundle \mathfrak{A}_0 over the second countable locally compact space $X_0 := \bigsqcup_{n=1}^{\infty} \mathbb{C}P^n$, where $\mathbb{C}P^n$ is the complex projective space of dimension n , which is not of finite type (that is, X_0 does not admit a finite open cover $\{U_i\}$ such that each restriction bundle $\mathfrak{A}_0|_{U_i}$ is trivial, as a C^* -bundle). Let A_0 be the C^* -algebra $\Gamma_0(\mathfrak{A}_0)$ consisting of all continuous sections of \mathfrak{A}_0 which vanish at infinity. Then A_0 is a 2-homogeneous C^* -algebra with $\text{Prim}(A_0) = X_0$. In particular A_0 is a central C^* -algebra with centre $C_0(X_0)$. Let $X := X_0 \sqcup \{\infty\}$ be the one-point compactification of X_0 , and let A be the minimal unitisation of A_0 . By [8, Proposition 3] (or [12, Proposition 3.12]) A is also a central C^* -algebra with $\text{Prim}(A) = X$ and centre $C(X)$. In particular, by [4, II.6.5.8] all norm functions $x \mapsto \|a_x\|$ ($a \in A$) are continuous on X , so that A is a continuous unital $C(X)$ -algebra with fibres $A_x = M_2(\mathbb{C})$ ($x \in X_0$) and $A_\infty = \mathbb{C}$. Suppose that A is $C(X)$ -subalgebra of some continuous homogeneous $C(X)$ -algebra A' . Then the underlying C^* -bundle \mathfrak{A} of A over X is a C^* -subbundle of the underlying C^* -bundle \mathfrak{A}' of A' over X . Since A' is continuous and homogeneous, \mathfrak{A}' is locally trivial by Remark 2.6. Hence, since X is compact, \mathfrak{A}' is of finite type. Using [18, Lemma 2.6] we conclude that \mathfrak{A} is of finite type as a vector bundle. In particular, \mathfrak{A}_0 is of finite type as a vector bundle, since $\mathfrak{A}_0 = \mathfrak{A}|_{X_0}$. As \mathfrak{A}_0 is a $M_2(\mathbb{C})$ -bundle, this implies by [18, Proposition 2.9] that \mathfrak{A}_0 is also of finite type as a C^* -bundle; a contradiction.

On the other hand, the $C(X)$ -algebra A of Example 3.6 also admits a $C(X)$ -valued C.E.F.I.. This follows from the following more general fact:

Proposition 3.7. *Let A be a continuous unital $C(X)$ -algebra. If $r(A) = 2$, then there exists a conditional expectation $E : A \rightarrow C(X)$ such that the map $2 \cdot E - \text{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.*

In order to prove Proposition 3.7, let us first make the following observation:

Lemma 3.8. *Let A be a unital $C(X)$ -algebra and let $a \in A_{sa}$. For each point $x \in X$ let $\lambda_{\max}(a)$ and $\lambda_{\min}(a)$ respectively denote the largest and the smallest numbers in*

the spectrum of a_x . Then the functions $x \mapsto \lambda_{\max}(a_x)$ and $x \mapsto \lambda_{\min}(a_x)$ are upper semicontinuous on X . Furthermore, these functions are continuous on X , whenever A is a continuous $C(X)$ -algebra.

Proof. This follows directly from the equations

$$\lambda_{\max}(a_x) = \| \|a\|1_x + a_x \| - \|a\| \quad \text{and} \quad \lambda_{\min}(a_x) = \|a\| - \| \|a\|1_x - a_x \|. \quad \square$$

Proof of Proposition 3.7. As $r(A) = 2$, any fibre A_x is isomorphic to \mathbb{C} , $\mathbb{C} \oplus \mathbb{C}$ or $M_2(\mathbb{C})$. Therefore, for each point $x \in X$ we can choose a unital embedding $\varphi_x : A_x \hookrightarrow M_2(\mathbb{C})$. For $a \in A$ and $x \in X$ we define

$$E(a)(x) := \frac{1}{2} \operatorname{tr}(\varphi_x(a_x)).$$

Obviously $E(a)$ is a $C(X)$ -linear map. If $a \in A_{sa}$, note that

$$(3.8) \quad E(a)(x) = \frac{1}{2} (\lambda_{\min}(a_x) + \lambda_{\max}(a_x))$$

for all $x \in X$. By Remark 3.8, $E(a)$ is a continuous function on X for all $a \in A_{sa}$. As A is the linear span of A_{sa} , we conclude that $E(a) \in C(X)$ for all $a \in A$. Therefore, E defines a C. E. from A onto $C(X)$. Further, by (3.8) for all $a \in A_+$ and $x \in X$ we have

$$2 \cdot E(a)(x)1_x = (\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \cdot 1_x \geq a_x.$$

This shows that the map $2 \cdot E - \operatorname{id}_A$ is positive, so that $K(A, C(X)) = 2$ by Theorem 1.4. \square

Let A be a unital C^* -algebra and let \check{Z} be the maximal ideal space of $Z(A)$. We may consider A as a $C(\check{Z})$ -algebra, with respect to the action

$$f \cdot a := \mathcal{G}^{-1}(f)a \quad (f \in C(X), a \in A),$$

where $\mathcal{G} : Z(A) \rightarrow C(\check{Z})$ is the Gelfand transform. We say that A is *quasi-standard* if A is a continuous $C(\check{Z})$ -algebra and each (Glimm) ideal $J_x = C_x(\check{Z})A$ is primal (see [2]).

Corollary 3.9. *For a unital C^* -algebra A the following conditions are equivalent:*

- (i) *There exist a C.E. $E : A \rightarrow Z(A)$ such that the map $2 \cdot E - \operatorname{id}_A$ is positive.*
- (ii) *A is either commutative or quasi-standard and 2-subhomogeneous.*

Proof. (i) \Rightarrow (ii). Suppose that there exists a C.E. $E : A \rightarrow Z(A)$ such that the map $2 \cdot E - \operatorname{id}_A$ is positive. Then by Theorem 1.4 A is a continuous $C(\check{Z})$ -algebra and $r(A_x) \leq 2$ for all $x \in \check{Z}$. In particular, A as a C^* -algebra is n -subhomogeneous, where $n \in \{1, 2\}$. Hence, by [13, Proposition 4.1] every Glimm ideal of A is primal. Also, $n = 1$ if and only if A is commutative.

(ii) \Rightarrow (i). If A is commutative we have nothing to prove, so suppose that A is quasi-standard and 2-subhomogeneous. Then by [9, Corollary 1, p. 388] for each point $x \in X$ we have

$$r(A_x) = \sum_{[\pi_x] \in \widehat{A_x}} \dim \pi_x \leq 2.$$

It remains to apply Proposition 3.7. \square

Remark 3.10. At the end of this paper we note that every separable continuous unital $C(X)$ -algebra A admits a faithful C.E. $E : A \rightarrow C(X)$ (see e.g. [5]). In particular, this result applies to continuous subhomogeneous unital $C(X)$ -algebras, when X is second-countable. In this case for each point $x \in X$, the map $E_x : a_x \mapsto E(a)(x)$ defines a faithful state on A_x , so Lemma 3.2 implies $K(E_x) < \infty$. However, this does not imply that E is of finite index. That is, it may happen that $\sup_{x \in X} K(E_x) = \infty$. Consider for instance the following example:

- Let X be the closed compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $[0, 1]$.
- Let A be the continuous $C(X)$ -subalgebra of $C(X) \oplus C(X)$ consisting of all pairs $(f, g) \in C(X) \oplus C(X)$ such that $f(0) = g(0)$.
- Let $E : A \rightarrow C(X)$ be a C.E. fixed by the relations

$$E(f \oplus g) \left(\frac{1}{n} \right) = \begin{cases} \frac{n}{n+1} f\left(\frac{1}{n}\right) + \frac{1}{n+1} g\left(\frac{1}{n}\right) & \text{if } n \text{ is odd} \\ \frac{1}{n+1} f\left(\frac{1}{n}\right) + \frac{n}{n+1} g\left(\frac{1}{n}\right) & \text{otherwise} \end{cases}$$

where $(f, g) \in A$.

Then E is a faithful C.E. which is not of finite index. Indeed, one has

$$E(f \oplus 0) \left(\frac{1}{2n} \right) = \frac{1}{2n+1} f \left(\frac{1}{2n} \right)$$

for all $f \in C_0(X \setminus \{0\})$ and all integers $n \in \mathbb{N}$. Consequently, a convenient constant K would satisfy $K \geq 2n + 1$ for all $n \in \mathbb{N}$, which is impossible.

We end this paper with some unresolved problems:

Problem 3.11. Is the converse of Theorem 1.4 also true? Moreover, does every continuous subhomogeneous unital $C(X)$ -algebra A admit a C.E. $E : A \rightarrow C(X)$ such that the map $r(A) \cdot E - \text{id}_A$ is positive? In particular, do we always have $K(A, C(X)) = r(A)$?

REFERENCES

1. R. J. Archbold, *Density theorems for the centre of a C^* -algebra*, J. London Math. Soc. (2), **10** (1975), 189–197.
2. R. J. Archbold and D. W. B. Somerset, *Quasi-standard C^* -algebras*, Math. Proc. Cambridge Philos. Soc., **107** (1990), 349–360.
3. M. Baillel, Y. Denizeau, J-F Havet, *Indice d'une espérance conditionnelle*, Compositio Math. **66** (1988), 199–236.
4. B. Blackadar, *Operator Algebras. Theory of C^* -Algebras and von Neumann Algebras*, Encycl. Math. Sciences 122, Springer-Verlag, Berlin-Heidelberg, 2006.
5. E. Blanchard, *Déformations de C^* -algèbres de Hopf*, Bull. Soc. Math. France **124** (1996), 141–215.
6. D. P. Blecher and C. Le Merdy, *Operator algebras and Their modules*, Clarendon Press, Oxford, 2004.
7. P.J. Cohen, *Factorization in group algebras*, Duke Math. J. **26** (1959), 199–205.
8. C. Delaroche, *Sur les centres des C^* -algèbres*, Bull. Sc. Math., **91** (1967), 105–112.
9. J. M. G. Fell, *The dual spaces of C^* -algebras*, Trans. Amer. Math. Soc. **94** (1960), 365–403.
10. J. M. G. Fell, *The structure of algebras of operator fields*, Acta Math., **106** (1961), 233–280.
11. M. Frank and E. Kirchberg, *On Conditional Expectations of Finite Index*, J. Oper. Theory **40** (1998), 87–111.
12. I. Gogić, *Derivations which are inner as completely bounded maps*, Oper. Matrices, **4** (2010), 193–211.
13. I. Gogić, *On derivations and elementary operators on C^* -algebras*, Proc. Edinb. Math. Soc. (2), **56** (2013), 515–534.

14. V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), 1–25.
15. I. Kaplansky, *Normed algebras*, Duke Math. J., **16** (1949), 399–418.
16. H. Kosaki, *Extension of Jones theory on index to arbitrary factors*, J. Funct. Anal. **66** (1986), 123–140.
17. A. Pavlov and E. V. Troitsky, *Quantization of branched coverings*, Russ. J. Math. Phys. **18** (2011), 338–352.
18. N. C. Phillips, *Recursive subhomogeneous algebras*, Trans. Amer. Math. Soc. **359** (2007), 4595–4623.
19. M. Pimsner, S. Popa, *Entropy and index for subfactors*, Ann. Scient. Ec. Norm. Sup. **19** (1986), 57–106.
20. S. Popa, *Classification of Subfactors and Their Endomorphisms*, Conf. Board Math. Sci. (Reg. Conf. Ser. Math.) **86**, Amer. Math. Soc., Providence, R.I., 1995.
21. M. Takesaki, *Theory of Operator Algebras I*, Springer, 1979.
22. Y. Tomiyama, *On the projection of norm one in W^* -algebras*, Proc. Japan Acad. **33**, (1957), 608–612.
23. Y. Watatani, *Index for C^* -subalgebras*, Mem. Amer. Math. Soc. **83** (1990), no 424.
24. D. P. Williams, *Crossed Products C^* -Algebras*, Mathematical Surveys and Monographs 134, Amer. Math. Soc., Providence, RI, 2007.

INSTITUT DE MATHÉMATIQUES DE JUSSIEU, BÂTIMENT SOPHIE GERMAIN, CASE 7012, F-75205
PARIS CEDEX 13

E-mail address: Etienne.Blanchard@math.jussieu.fr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA 30, 10000 ZAGREB, CROA-
TIA, AND DEPARTMENT OF MATHEMATICS AND INFORMATICS, UNIVERSITY OF NOVI SAD, TRG
DOSITEJA OBRADOVIĆA 4, 21000 NOVI SAD, SERBIA

E-mail address: ilja@math.hr