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# LOCAL AND GLOBAL PROPER INFINITENESS FOR CONTINUOUS $C(X)$ -ALGEBRAS

ETIENNE BLANCHARD

ABSTRACT. We show that all unital continuous  $C^*$ -bundles with properly infinite fibres are properly infinite  $C^*$ -algebras if and only if the full unital free product of the Cuntz extensions  $\mathcal{T}_2$  and  $\mathcal{T}_3$  is a  $K_1$ -injective  $C^*$ -algebra.

## 1. INTRODUCTION

The classification programme of nuclear  $C^*$ -algebras through  $K$ -theoretical invariants launched by G. Elliott ([Ell94]) led A. Toms and W. Winter to introduce the strong self-absorption property for simple unital  $C^*$ -algebras ([TW07]). This notion is a pretty rigid one: Any separable unital continuous  $C(X)$ -algebra  $A$  the fibres of which are isomorphic to the same strongly self-absorbing  $C^*$ -algebra  $D$  is a trivial  $C(X)$ -algebra provided the compact Hausdorff base space  $X$  has finite topological dimension. (The strongly self-absorbing  $C^*$ -algebra  $D$  tensorially absorbs the Jiang-Su algebra  $\mathcal{Z}$  ([Win09]). Hence, the  $C^*$ -algebra  $D$  is  $K_1$ -injective ([Ror04]) and the  $C(X)$ -algebra  $A$  satisfies  $A \cong D \otimes C(X)$  ([DW08]).) But I. Hirshberg, M. Rørdam and W. Winter have built a non-trivial unital continuous  $C^*$ -bundle over the infinite dimensional compact product  $\prod_{n=0}^{\infty} S^2$  such that all its fibres are isomorphic to the strongly self-absorbing UHF algebra of type  $2^{\infty}$  ([HRW07, Example 4.7]). More recently, M. Dădărlat has constructed in [Dad09, §3] for all pair  $(\Gamma_0, \Gamma_1)$  of discrete countable torsion groups a unital separable continuous  $C(X)$ -algebra  $A$  such that:

- the base space  $X$  is the compact Hilbert cube  $X = \mathfrak{X}$  of infinite dimension,
- all the fibres  $A_x$  ( $x \in \mathfrak{X}$ ) are isomorphic to the strongly self-absorbing Cuntz  $C^*$ -algebra  $\mathcal{O}_2$  generated by two isometries  $s_1, s_2$  satisfying  $1_{\mathcal{O}_2} = s_1 s_1^* + s_2 s_2^*$ ,
- $K_i(A) \cong C(Y_0, \Gamma_i)$  for  $i = 0, 1$ , where  $Y_0 \subset [0, 1]$  is the canonical Cantor set.

These  $K$ -theoretical conditions imply that the  $C(\mathfrak{X})$ -algebra  $A$  is not a trivial one. But this argument does not anymore work when the stringy self-absorbing algebra  $D$  is the Cuntz algebra  $\mathcal{O}_{\infty}$  ([Cun77]), in so far as  $K_0(\mathcal{O}_{\infty}) = \mathbb{Z}$  is a torsion free group.

In this note, we attack this trivialization problem from another angle: All unital continuous  $C(X)$ -algebras with properly infinite fibres are properly infinite  $C^*$ -algebras if and only if the full unital free product  $\mathbf{A} = \mathcal{T}_2 *_C \mathcal{T}_2$  of two distinct copies of the unital extension  $\mathcal{T}_2$  of the Cuntz algebras  $\mathcal{O}_2$  by the compact operators ([Cun77]) is a  $K_1$ -injective  $C^*$ -algebra, *i.e.* the canonical map from  $\mathcal{U}(\mathbf{A})/\mathcal{U}^0(\mathbf{A})$  to  $K_1(\mathbf{A})$  is injective (Corollary 4.3). Works by M. Rieffel ([Rief83], [Rief87]) and K Dykema, U.

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Haagerup, M. Rørdam ([DHR97], [Roh09]) then enable us to prove that at least some reduced quotient  $\mathfrak{B}$  of the free product  $\mathcal{T}_2 *_C \mathcal{T}_3$  is a unital  $K_1$ -injective  $C^*$ -algebra which contains the algebraic unital free product  $\mathcal{T}_2 \otimes_C \mathcal{T}_3$  of the two Cuntz extensions  $\mathcal{T}_2$  and  $\mathcal{T}_3$  (Proposition 5.2).

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## 2. NOTATIONS

We present in this section the main notations which will be used in this article.

**Definition 2.1.** ([Dix69], [Kas88], [Blan97]) *Let  $X$  be a compact Hausdorff space and let  $C(X)$  be the  $C^*$ -algebra of continuous function on  $X$ .*

- *A unital  $C(X)$ -algebra is a unital  $C^*$ -algebra  $A$  endowed with a unital morphism of  $C^*$ -algebra from  $C(X)$  to the centre of  $A$ .*
- *For all closed subset  $F \subset X$  and all  $a \in A$ , one denotes by  $a|_F$  the image of  $a$  in the quotient  $A|_F := A/C_0(X \setminus F) \cdot A$ . If  $x \in X$  is a point in  $X$ , one calls fibre at  $x$  the quotient  $A_x := A|_{\{x\}}$  and one write  $a_x$  for  $a|_{\{x\}}$ .*
- *The  $C(X)$ -algebra  $A$  is said to be continuous if the upper semicontinuous map  $x \in X \mapsto \|a_x\| \in \mathbb{R}_+$  is continuous for all  $a \in A$ .*

**Definition 2.2.** ([Pim95]) *Let  $X$  be a compact Hausdorff space.*

a) *The full Fock Hilbert  $C(X)$ -module  $\mathcal{F}(E)$  of a Hilbert  $C(X)$ -module  $E$  is the direct sum of Hilbert  $C(X)$ -module*

$$\mathcal{F}(E) := \bigoplus_{m \in \mathbb{N}} E^{(\otimes_{C(X)} m)}, \quad (2.1)$$

where  $E^{(\otimes_{C(X)} m)} := \begin{cases} E \otimes_{C(X)} \dots \otimes_{C(X)} E & (m \text{ terms}) \text{ if } m \geq 1, \\ C(X) & \text{if } m = 0. \end{cases}$

b) *The Pimsner-Toeplitz  $C(X)$ -algebra  $\mathcal{T}(E)$  of a full Hilbert  $C(X)$ -module  $E$ , i.e. with non-zero fibres, is the unital subalgebra of the  $C(X)$ -algebra  $\mathcal{L}_{C(X)}(\mathcal{F}(E))$  of adjointable  $C(X)$ -linear operator acting on  $\mathcal{F}(E)$  generated by the creation operators  $\ell(\zeta)$  ( $\zeta \in E$ ), where:*

$$\begin{aligned} - \ell(\zeta)(f \cdot \hat{1}_{C(X)}) &:= f \cdot \zeta = \zeta \cdot f & \text{for } f \in C(X) & \text{and} \\ - \ell(\zeta)(\zeta_1 \otimes \dots \otimes \zeta_k) &:= \zeta \otimes \zeta_1 \otimes \dots \otimes \zeta_k & \text{for } \zeta_1, \dots, \zeta_k \in E & \text{if } k \geq 1. \end{aligned} \quad (2.2)$$

*Remarks 2.3.* a) ([Cun81], [BRR08]) For all integer  $n \geq 2$ , the  $C^*$ -algebra  $\mathcal{T}_n := \mathcal{T}(\mathbb{C}^n)$  is the universal unital  $C^*$ -algebra generated by  $n$  isometries  $s_1, \dots, s_n$  satisfying the relation

$$s_1 s_1^* + \dots + s_n s_n^* \leq 1. \quad (2.3)$$

b) A unital  $C^*$ -algebra  $A$  is *properly infinite* if and only if one the following equivalent conditions holds ([Cun77], [Ror03, Proposition 2.1]):

- $A$  contains two isometries with mutually orthogonal range projections,
- $A$  contains a unital copy of the simple  $C^*$ -algebra  $\mathcal{O}_\infty$ .

### 3. LOCAL PROPER INFINITENESS

Let  $(C^*(\mathbb{Z}), \Delta)$  be the unital Hopf  $C^*$ -algebra ([Wor95]) generated by a unitary  $\mathbf{u}$  with spectrum  $\mathbb{T} := \{z \in \mathbb{C}; z^*z = 1\}$  (often written  $S^1$ ) and with coproduct  $\Delta(\mathbf{u}) = \mathbf{u} \otimes \mathbf{u}$ .

If  $X$  is a compact Hausdorff space and  $E$  is a separable Hilbert  $C(X)$ -module with non-zero fibres, there is only one coaction  $\alpha_E$  of  $(C^*(\mathbb{Z}), \Delta)$  on the Pimsner-Toeplitz  $C(X)$ -algebra  $\mathcal{T}(E)$  such that  $\alpha_E(\ell(\zeta)) = \ell(\zeta) \otimes \mathbf{u}$  for all  $\zeta \in E$ , *i.e.*

$$\begin{aligned} \mathcal{T}(E) &\rightarrow \mathcal{T}(E) \otimes C^*(\mathbb{Z}) = C(\mathbb{T}, \mathcal{T}(E)) \\ \ell(\zeta) &\mapsto \ell(\zeta) \otimes \mathbf{u} = (z \mapsto \ell(z\zeta)) \end{aligned} \tag{3.1}$$

Equation (3.1) implies that the fixed point  $C(X)$ -subalgebra  $\mathcal{T}_{C(X)}(E)^{\alpha_E}$  under this coaction is the closed linear span  $\mathcal{T}_{C(X)}(E)^{\alpha_E} = [C(X).1 + \sum_{k \geq 1} \ell(E)^k \cdot (\ell(E)^k)^*]$ .

Besides, the following local proper infiniteness property holds.

**Proposition 3.1.** ([Blac04]) *Let  $X$  be a compact Hausdorff space and let  $E$  be a separable Hilbert  $C(X)$ -module with infinite dimensional fibres.*

*If  $x$  is a point in  $X$ , then there is a closed neighbourhood  $F \subset X$  of  $x$  such that the restriction  $\mathcal{T}(E)|_F$  of the Pimsner-Toeplitz  $C(X)$ -algebra  $\mathcal{T}(E)$  is properly infinite, *i.e.* there is a unital  $*$ -homomorphism  $\mathcal{O}_\infty \rightarrow \mathcal{T}(E)|_F$ .*

*Proof.* Let  $\zeta \in E$  be a norm 1 section satisfying  $\|\zeta_y\| = 1$  for all  $y$  in a closed neighbourhood  $F \subset X$  of the point  $x$ . Denote by  $\pi_F$  the quotient  $*$ -homomorphism  $C(X) \rightarrow C(X)/C_0(X \setminus F) \cong C(F)$ .

- (1) There is an isomorphism of Hilbert  $C(F)$ -module  $E|_F \cong C(F) \cdot \zeta \otimes_{\pi_F} 1 \oplus E'|_F$ , where  $E'$  is the orthocomplement of  $\zeta$  in the Hilbert  $C(X)$ -module  $E$ .
- (2) There is an isomorphism of  $C(F)$ -algebra

$$\mathcal{T}(E)|_F = C^* \langle \mathcal{T}(E)^{\alpha_E}|_F, \ell(\zeta)|_F \rangle \cong (\mathcal{T}(E)^{\alpha_E})|_F \rtimes \mathbb{N},$$

where the action of the semigroup  $\mathbb{N}$  on the  $C(F)$ -algebra  $\mathcal{T}(E)^{\alpha_E}|_F$  is given by  $a \mapsto \ell(\zeta)|_F \cdot a \cdot \ell(\zeta)|_F^*$ .

- (3) The restriction  $\mathcal{T}(E)|_F$  is properly infinite by [Blan13, Lemma 6.1].

□

*Remarks 3.2.* a) We can also prove Proposition 3.1 thanks to the semiprojectivity of the Cuntz  $C^*$ -algebra  $\mathcal{O}_\infty$  ([Blac04, Theorem 3.2]).

b) The restriction  $\mathcal{T}(E)|_F$  is properly infinite if the closed subset  $F \subset X$  is perfect (*i.e.* without any isolated point) and there is a section  $\zeta \in E$  with  $\|\zeta_x\| \geq 1$  for all points  $x \in F$ .

### 4. GLOBAL PROPER INFINITENESS

Proposition 3.1 of the present paper, Proposition 2.5 of [BRR08] and section 6 of [Blan13] entail the following:

**Proposition 4.1.** *Let  $X$  be a second countable perfect compact Hausdorff space and let  $A$  be a separable unital continuous  $C(X)$ -algebra with properly infinite fibres.*

1) *There exist:*

- (a) *a finite integer  $n \geq 1$ ,*
- (b) *a covering  $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$  by the interiors of closed balls  $F_1, \dots, F_n$ ,*
- (c) *unital embeddings of  $C^*$ -algebra  $\sigma_k : \mathcal{O}_\infty \rightarrow A|_{F_k}$  ( $1 \leq k \leq n$ ).*

2) *The tensor product  $M_p(\mathbb{C}) \otimes A$  is properly infinite for all large enough  $p$ .*

3) *For all  $i, j$  in  $\{1, \dots, n\}$ , there is a unitary  $u_{i,j} \in \mathcal{U}(A|_{F_i \cap F_j})$  such that*

$$\sigma_i(s_m)|_{F_i \cap F_j} = u_{i,j} \cdot \sigma_j(s_m)|_{F_i \cap F_j} \quad \text{for all } m \in \mathbb{N}.$$

*Proof.* 1) For all point  $x \in X$ , the semiprojectivity of the  $C^*$ -subalgebra  $\mathcal{O}_\infty \hookrightarrow A_x$  ([Blac04, Theorem 3.2]) implies that there are a closed neighbourhood  $F \subset X$  of the point  $x$  and a  $C(F)$ -linear unital embedding  $\mathcal{O}_\infty \otimes C(F) \hookrightarrow A|_F$ . The compactness of the topological space  $X$  enables to conclude.

2) derives from Proposition 3.1, [BRR08, Proposition 2.7] and [Ror97, Proposition 2.1].

3) One sets  $u_{i,j} := \sum_{m \in \mathbb{N}} \sigma_i(s_m)|_{F_i \cap F_j} \cdot \sigma_j(s_m)^*|_{F_i \cap F_j}$  ([Blan13, Proposition 6.3]).  $\square$

*Remark 4.2.* Assertion (2) alone does not imply that the  $C^*$ -algebra  $A$  is properly infinite: There exists a unital  $C^*$ -algebra  $A$  which is not properly infinite, but such that  $M_2(\mathbb{C}) \otimes A$  is properly infinite ([Ror03, Proposition 4.5]).

Note the following link with the  $K_1$ -injectivity for properly infinite  $C^*$ -algebras.

**Corollary 4.3.** *Let  $j_0, j_1$  denote the two canonical unital embeddings of the  $C^*$ -algebra  $\mathcal{T}_2$  in the unital free product  $\mathbf{A} := \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  and let  $u \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  be a  $K_1$ -trivial unitary such that  $j_1(s_1) = u \cdot j_0(s_1)$  ([BRR08, Lemma 2.4]).*

1) *The following conditions are equivalent:*

- (a) *The full unital free product  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  is  $K_1$ -injective.*
- (b) *The unitary  $u \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$  belongs to the connected component  $\mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ .*
- (c) *Every separable unital continuous  $C(X)$ -algebra with properly infinite fibres is a properly infinite  $C^*$ -algebra.*

2) *If the  $C^*$ -algebra  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$  is  $K_1$ -injective, then the Pimsner-Toeplitz algebra  $\mathcal{T}(E)$  of any Hilbert  $C(X)$ -module  $E$  with fibres of dimension at least 2 is properly infinite and there is an isomorphism of  $C(X)$ -algebra  $\mathcal{T}(E) \cong \mathcal{T}(E)^{\alpha_E} \rtimes \mathbb{N}$ .*

*Proof.* 1) (a) $\Rightarrow$ (b) A unital  $C^*$ -algebra  $A$  is  $K_1$ -injective if and only if every unitary  $v \in \mathcal{U}(A)$  is homotopic to  $1_A$  in the group  $\mathcal{U}(A)$ . Hence, (b) is a special case of (a).

(b) $\Rightarrow$ (c) Take a finite covering  $X = \overset{\circ}{F}_1 \cup \dots \cup \overset{\circ}{F}_n$  of the compact space  $X$  and  $n$  unital embeddings  $\sigma_k : \mathcal{T}_2 \rightarrow A|_{F_k}$ . Set  $G_k := F_1 \cup \dots \cup F_k \subset X$  for all  $1 \leq k \leq n$  and let us construct by induction isometries  $w_k \in A|_{G_k}$  such that the two projections  $w_k w_k^*$  and  $1_{G_k} - w_k w_k^*$  are properly infinite and full in  $A|_{G_k}$  :

- If  $k = 1$ , the isometry  $w_1 := \sigma_1(s_1)$  has the requested properties.
- If  $1 \leq k \leq n-1$  and the isometry  $w_k \in A_{|G_k}$  is already constructed, then Lemma 2.4 of [BRR08] implies that there exist a unital homomorphism of C\*-algebra  $\pi_k : \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2 \rightarrow A_{|G_k \cap F_{k+1}}$  and a  $K_1$ -trivial unitary  $u_{k+1} \in \mathcal{U}(A_{|G_k \cap F_{k+1}})$  satisfying:

- $\pi_k(j_0(s_1)) = w_k|_{G_k \cap F_{k+1}}$  and
- $\pi_k(j_1(s_1)) = \sigma_{k+1}(s_1)|_{G_k \cap F_{k+1}} = u_{k+1} \cdot w_k|_{G_k \cap F_{k+1}}$ .

If condition (b) holds, the unitary  $u_{k+1}$  can even be taken in  $\mathcal{U}^0(A_{|G_k \cap F_{k+1}})$ , so that it admits a unitary lifting  $z_{k+1} \in \mathcal{U}^0(A_{|F_{k+1}})$  ([LLR00, Lemma 2.1.7]). The only isometry  $w_{k+1} \in \mathcal{T}(E)|_{G_{k+1}}$  satisfying the two constraints:

$$w_{k+1}|_{G_k} = w_k \quad \text{and} \quad w_{k+1}|_{F_{k+1}} = (z_{k+1})^* \cdot \sigma_{k+1}(s_1)$$

verifies that the projections  $w_{k+1}w_{k+1}^*$  and  $1_{|G_{k+1}} - w_{k+1}w_{k+1}^*$  are properly infinite and full in  $A_{|G_{k+1}}$ .

The proper infiniteness of the projection  $w_n w_n^* \in A_{|G_n} = A$  implies that the unit  $1_A = w_n^* \cdot w_n w_n^* \cdot w_n$  is also properly infinite in  $A$ , *i.e.*  $A$  is a properly infinite C\*-algebra.

(c) $\Rightarrow$ (a) The C\*-algebra  $\mathcal{D} := \{f \in C([0, 1], \mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2); f(0) \in j_0(\mathcal{T}_2) \text{ and } f(1) \in j_1(\mathcal{T}_2)\}$  is a unital continuous  $C([0, 1])$ -algebra the fibres of which are all properly infinite. Thus, condition (c) implies that  $\mathcal{D}$  is a properly infinite C\*-algebra, a statement equivalent to the  $K_1$ -injectivity of the full free product  $\mathcal{T}_2 *_{\mathbb{C}} \mathcal{T}_2$  ([Blan10, Proposition 4.2]).

2) All the fibres of the  $C(X)$ -algebra  $\mathcal{T}(E)$  are simple and properly infinite. Thus, the C\*-algebra  $\mathcal{T}(E)$  is properly infinite as soon as condition (c) of assertion 1) holds. Besides, one can construct in that case by induction isometries  $w_k \in \mathcal{T}(E)|_{G_k}$  ( $1 \leq k \leq n$ ) such that:

- the projections  $w_k w_k^*$  and  $1_{|G_k} - w_k w_k^*$  are properly infinite and full in  $A_{|G_k}$ ,
- there is an isomorphism of  $C(G_k)$ -algebra  $\mathcal{T}(E)|_{G_k} \cong \mathcal{T}(E)^{\alpha_E}|_{G_k} \rtimes_{ad w_k} \mathbb{N}$ .

□

*Remarks 4.4.* a) The isometry  $w_n \in \mathcal{T}(E)$  built in assertion 2) does not belong to the subspace  $\ell(E) \subset \mathcal{T}(E)$  in general. For instance, if  $\mathfrak{X}$  is the compact Hilbert cube of infinite topological dimension and  $\mathcal{E}$  is the non-trivial Hilbert  $C(\mathfrak{X})$ -module constructed by J. Dixmier and A. Douady ([DD63], [BK04a, Proposition 3.6]), then there exist sections  $\zeta_1, \dots, \zeta_n$  in  $\mathcal{E}$  such that  $\mathcal{T}(\mathcal{E}) = C^* \langle \mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}}, \ell(\zeta_1), \dots, \ell(\zeta_n) \rangle$ .

But there is no  $\zeta \in \sum_{1 \leq k \leq n} C(X) \cdot \zeta_k \subset \mathcal{E}$  such that  $\mathcal{T}(\mathcal{E}) = C^* \langle \mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}}, \ell(\zeta) \rangle$  since any section  $\zeta \in \mathcal{E}$  satisfies  $\zeta_x = 0$  for at least one point  $x \in \mathfrak{X}$  ([DD63, 14], [BK04a, Proposition 3.6]) and  $C^* \langle (\mathcal{T}(\mathcal{E})^{\alpha_{\mathcal{E}}})_x, \ell(\zeta)_x \rangle \cong \mathcal{T}(\mathcal{E}_x)^{\alpha_{\mathcal{E}_x}} \subsetneq \mathcal{T}(\mathcal{E}_x) \cong \mathcal{O}_{\infty}$ .

b) If  $\mathfrak{A}$  denotes the universal UHF-algebra with  $K_1(\mathfrak{A}) = \mathbb{Q}$ , there is a non-trivial unital continuous  $C(\mathfrak{X})$ -algebra  $D$  with  $D_x \cong \mathcal{O}_{\infty} \otimes \mathfrak{A}$  for all  $x \in \mathfrak{X}$  ([Dad09]). But is there a non-trivial unital continuous  $C(\mathfrak{X})$ -algebra with fibres isomorphic to  $\mathcal{O}_{\infty}$ ? (Note that  $K_0(\mathcal{O}_{\infty} \otimes \mathfrak{A}) = \mathbb{Q} \neq \mathbb{Z} = K_0(\mathcal{O}_{\infty})$ .)

## 5. A $K_1$ -INJECTIVE QUOTIENT

We reduced in the previous section the proper infiniteness problem for all unital continuous fields with properly infinite fibres to a  $K_1$ -injectivity question for the full unital free product  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_3$ . We show in this section that some quotient of this unital free product  $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_3$  is a simple  $K_1$ -injective  $C^*$ -algebra which contains the algebraic unital free product  $\mathcal{T}_2 \otimes_{\mathbb{C}} \mathcal{T}_3$ .

Let  $\ell^2(\mathbb{N})$  be the Hilbert space of all sequences  $\xi = (\xi_k) \in \mathbb{C}^{\mathbb{N}}$  which satisfy  $\|\xi\|^2 = \sum_k |\xi_k|^2 < \infty$  and take an orthonormal basis  $\{e_k; k \in \mathbb{N}\}$  on  $\ell^2(\mathbb{N})$ . Define for any integer  $n \geq 2$ :

- the  $n$  isometries  $s_{n,1}, \dots, s_{n,n}$  in  $\mathbb{B}(\ell^2(\mathbb{N}))$  by  $s_{n,i} \cdot e_k = e_{nk+i}$  ( $k \in \mathbb{N}$ ),
- the Cuntz extension  $\mathcal{T}_n := C^*(s_{n,1}, \dots, s_{n,n})$  ([Cun77]),
- the state  $\psi_n(a) = \frac{1}{n} \sum_{i=1}^n \langle e_i, a \cdot e_i \rangle$  on the  $C^*$ -algebra  $\mathcal{T}_n \subset \mathbb{B}(\ell^2(\mathbb{N}))$ ,
- the unitary  $u_n = s_{n,1}s_{n,2}^* + \dots + s_{n,n}s_{n,1}^* + p \in \mathcal{T}_n \subset \mathbb{B}(\ell^2(\mathbb{N}))$ , where  $p$  is the rank 1 projection  $p = 1 - \sum_i s_{n,i}s_{n,i}^* = \theta_{e_0, e_0}$  in  $\mathbb{B}(\ell^2(\mathbb{N}))$ .

**Lemma 5.1.** 1)  $\psi_n(a) = \psi_n(u_n^* a u_n)$  for any operator  $a \in \mathcal{T}_n$  and  $\psi_n((u_n)^j) = 0$  for any integer  $j \in \{1, \dots, n-1\}$ ,

2) The state  $\psi_n$  on  $\mathcal{T}_n$  has a faithful GNS representation.

3) The restriction of the state  $\psi_n$  to  $C^*(u_n) \subset \mathcal{T}_n$  is faithful.

*Proof.* 1) The unitary  $u_n$  satisfies  $(u_n)^n = 1$  and  $(u_n)^j \cdot e_1 = e_{n-j+1}$  for all  $1 \leq j \leq n$ . Thus, there exists an isomorphism  $\alpha_n : C^*(u_n) \rightarrow C^*(\mathbb{Z}/n\mathbb{Z})$  and

$$\begin{aligned} \psi_n(a) &= \frac{1}{n} \sum_{i=1}^n \langle (u_n)^i \cdot e_1, a (u_n)^i \cdot e_1 \rangle \\ &= \frac{1}{n} \sum_{i=1}^n \langle (u_n)^{i+1} \cdot e_1, a (u_n)^{i+1} \cdot e_1 \rangle = \psi_n(u_n^* a u_n). \end{aligned}$$

We also have  $\psi_n((u_n)^j) = \langle e_1, (u_n)^j \cdot e_1 \rangle = \langle e_1, e_{n-j+1} \rangle = 0$  for all  $j \in \{1, \dots, n-1\}$ .

2) The  $C^*$ -algebra  $\mathcal{T}_n$  is contained in  $\mathbb{B}(\ell^2(\mathbb{N}))$  and the Hilbert space  $\ell^2(\mathbb{N})$  is the closure of  $\mathcal{T}_n \cdot e_0$ . Thus, if  $a \in \mathcal{T}_n \setminus \{0\}$  is non-zero, there exists an element  $b \in \mathcal{T}_n$  such that

$$\begin{aligned} 0 < \|ab \cdot e_0\|^2 &= \|ab s_{n,1}^* \cdot e_1\|^2 = \langle e_1, s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^* e_1 \rangle \\ &= \sum_{i=1}^n \langle e_i, s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^* e_i \rangle \\ &= n \cdot \psi_n(s_{n,1} b^* \cdot a^* a \cdot b s_{n,1}^*). \end{aligned}$$

3) The composition  $\psi_n \circ \alpha_n^{-1}$  equals the only Haar state  $h_n$  on the finite dimensional Hopf  $C^*$ -algebra  $(C^*(\mathbb{Z}/n\mathbb{Z}), \Delta)$ , which is faithful on  $C^*(\mathbb{Z}/n\mathbb{Z})$ .  $\square$

Accordingly, the following holds:

**Proposition 5.2.** Let  $\mathcal{B}$  be the reduced unital free product  $(\mathfrak{B}, \psi) := (\mathcal{T}_2, \psi_2) *_\mathbb{C} (\mathcal{T}_3, \psi_3)$  defined by  $D$ . Voiculescu ([Voi85]).

- 1) The  $C^*$ -algebra  $\mathfrak{B}$  contains the algebraic unital free product generated by  $\mathcal{T}_2$  and  $\mathcal{T}_3$
- 2) The  $C^*$ -algebra  $\mathfrak{B}$  is simple and has stable rank 1.
- 3) The  $C^*$ -algebra  $\mathfrak{B}$  is  $K_1$ -injective.

*Proof.* 1) This embedding result is essentially contained in Proposition 4.2 of [DS01]. But we include here the main steps of that proof for completeness. Define:

- the tensor product  $C^*$ -algebra  $B := \mathcal{T}_2 \otimes \mathcal{T}_3$ ,
- the state  $\phi := \psi_2 \otimes \psi_3$  on  $B$ ,
- the full countably generated Hilbert  $B$ -bimodule  $E := L^2(B, \phi) \otimes B$ ,
- the full Fock Hilbert  $B$ -bimodule  $\mathcal{F}(E) = B \oplus E \oplus (E \otimes E) \oplus \dots$ ,
- the Pimsner-Toeplitz  $C^*$ -algebra  $\mathcal{T}(E)$  generated in  $\mathcal{L}(\mathcal{F}(E))$  by the *creation* operators  $\ell(\xi)$ ,  $\xi \in E$  ([Pim95]).
- the conditional expectation  $\mathfrak{E} : \mathcal{T}(E) \rightarrow B$  given by compression with the orthogonal projection from  $\mathcal{F}_B(E)$  onto the first summand  $B \subset \mathcal{F}(E)$  ([DS01]).

Take a unitary  $v$  in the  $C^*$ -subalgebra  $C^*(\ell(\Lambda_\phi 1 \otimes 1)) \subset \mathcal{T}(E)$  such that  $\mathfrak{E}(v^j) = 0$  for any non-zero integer  $j$  and let  $\pi_2 : \mathcal{T}_2 \rightarrow \mathcal{T}(E)$ ,  $\pi_3 : \mathcal{T}_3 \rightarrow \mathcal{T}(E)$  be the two unital  $*$ -morphisms  $\pi_2(a) = v^2(1 \otimes a \otimes 1_{\mathcal{T}_3})v^{-2}$  and  $\pi_3(a) = v^3(1 \otimes 1_{\mathcal{T}_2} \otimes a)v^{-3}$ . Then, the algebraic amalgamated free product  $\mathcal{T}_2 \otimes_{\mathbb{C}} \mathcal{T}_3$  studied by B. Blackadar ([Blac78]) and G. Pedersen ([Ped94]) is (isomorphic to) the unital  $*$ -algebra generated by  $\pi_2(\mathcal{T}_2)$  and  $\pi_3(\mathcal{T}_3)$  in  $\mathcal{T}(E)$ , whereas Voiculescu's reduced unital free product  $(\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_3, \psi_3)$  ([Voi85]) is (isomorphic to) its closure (see *e.g.* [Blan09, Theorem 4.1]).

2) The previous Lemma and [BD01, Theorem 1.3] imply that there is a unital embedding  $(C^*(\mathbb{Z}/2\mathbb{Z}), h_2) *_C (C^*(\mathbb{Z}/3\mathbb{Z}), h_3) \hookrightarrow (\mathfrak{B}, \psi) = (\mathcal{T}_2, \psi_2) *_C (\mathcal{T}_3, \psi_3)$ , if  $h_2$  and  $h_3$  denote the Haar states on the compact groups  $(C^*(\mathbb{Z}/2\mathbb{Z}), \Delta_2)$  and  $(C^*(\mathbb{Z}/3\mathbb{Z}), \Delta_3)$ . Hence, the reduced free product  $\mathfrak{B}$  is simple ([Av82, Proposition 3.1+its corollary]) and this unital  $C^*$ -algebra has stable rank 1 ([DHR97, Theorem 3.8]), *i.e.* the open subset of invertible elements is dense in  $\mathfrak{B}$ .

3) With the notations of [Rief83], the set  $Lg_1(\mathfrak{B})$  of invertible elements is dense in the stable rank 1  $C^*$ -algebra  $\mathfrak{B}$ , so that this  $C^*$ -algebra  $\mathfrak{B}$  is  $K_1$ -injective by [Rief83, Proposition 1.6] and [Rief87, Theorem 2.10]. (A self-contained proof is also available in Theorem 3.2.11 of [Roh09].)  $\square$

*Remark 5.3.* The same proof shows that the reduced free product  $(\mathcal{T}_{n_1}, \psi_{n_1}) *_C (\mathcal{T}_{n_2}, \psi_{n_2})$  is simple and  $K_1$ -injective for all integers  $n_1 \geq 2$  and  $n_2 \geq 3$ .

**Question 5.4.** *Is the full free product  $\mathcal{T}_2 *_C \mathcal{T}_3$  also  $K_1$ -injective?*

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