



HAL
open science

The raising steps method. Application to the $\bar{\partial}$ equation in Stein manifolds.

Eric Amar

► **To cite this version:**

Eric Amar. The raising steps method. Application to the $\bar{\partial}$ equation in Stein manifolds.. 2013.
hal-00922689v1

HAL Id: hal-00922689

<https://hal.science/hal-00922689v1>

Preprint submitted on 29 Dec 2013 (v1), last revised 9 Jan 2014 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The raising steps method. Application to the $\bar{\partial}$ equation in Stein manifolds.

Eric Amar

Abstract

In order to get estimates on the solutions of the equation $\bar{\partial}u = \omega$ on Stein manifold, we introduce a new method the "raising steps method", to get global results from local ones. In particular it allows us to transfer results from open sets in \mathbb{C}^n to open sets in a Stein manifold.

Using it we get $L^r - L^s$ results for solutions of equation $\bar{\partial}u = \omega$ with a gain, $s > r$, in strictly pseudo convex domains in Stein manifolds.

1 Introduction.

We shall introduce a new method, the "raising steps method" to get from local results on solutions u of equation $Du = \omega$, global ones in a smooth manifold X . This method works because we get a quantified better regularity on the solution u of the D equation with respect to the regularity of the data.

On any complex manifold X we define first the "Lebesgue measure" as in Hörmander's book [7] section 5.2, with a hermitian metric locally equivalent to the usual one on any analytic coordinates patch. Associated to this metric there is a volume form dm and we take it for the Lebesgue measure on X .

There already exist results on solutions of the $\bar{\partial}$ equation in bounded strictly pseudo convex domains in complex manifolds. For instance Hörmander [7] solve the $\bar{\partial}$ equation with $L^2 - L^2$ estimates for any (p, q) currents. Kerzman [8] proved a $L^r - L^r$ estimate on solution of $\bar{\partial}$ for any $r \in [1, \infty[$ and a Hölder estimate in case $r = \infty$ for $(0, 1)$ currents. In [6], chap. 4, Henkin and Leiterer built global kernels on a Stein manifold for $(0, q)$ forms and get some uniform estimates. Demailly and Laurent [4], built global kernels on a Stein manifold for (p, q) forms and get $L^r - L^r$ estimate on solution of $\bar{\partial}$ for any $r \in [1, \infty[$ in the case the Stein manifold is equipped with a hermitian metric with null curvature.

As an application of this "raising steps" method we get the following theorems

Theorem 1.1 *Let Ω be a smoothly bounded relatively compact strictly pseudo convex domain in the Stein manifold X .*

There is a constant $C = C(p, q, r) > 0$ such that if $\omega \in L^r_{(p, q)}(\Omega)$, $\bar{\partial}\omega = 0$ if $1 \leq r \leq 2$, there is a $(p, q - 1)$ current u such that, with $\gamma := \min(\frac{1}{2(n+1)}, \frac{1}{r} - \frac{1}{2})$, and $\frac{1}{s} = \frac{1}{r} - \gamma$,

$$\bar{\partial}u = \omega \text{ and } u \in L^s_{(p,q-1)}(\Omega), \quad \|u\|_{L^s_{(p,q-1)}(\Omega)} \leq C\|\omega\|_{L^r_{(p,q)}(\Omega)}.$$

Let $\mathcal{H}_p^r(\Omega)$ be the set of all $(p, 0)$ $\bar{\partial}$ closed forms in Ω and in $L^r(\Omega)$.

To deal with $r > 2$ we have to make the assumption that ω has a compact support, i.e. $\omega \in L^{r,c}_{(p,q)}(\Omega)$ if $q < n$ and $\omega \perp \mathcal{H}_p^{r'}(\Omega)$ if $q = n$.

Theorem 1.2 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X . Let $\frac{2(n+1)}{n+2} \leq r < 2(n+1)$, there is a constant $C = C_r > 0$ such that if $\omega \in L^{r,c}_{(p,q)}(\Omega)$, $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and $\omega \in L^r_{(p,q)}(\Omega)$, $\omega \perp \mathcal{H}_p^{r'}(\Omega)$ if $q = n$, then there is a $(p, q-1)$ current u such that, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$,*

$$\bar{\partial}u = \omega \text{ and } u \in L^s_{(p,q-1)}(\Omega), \quad \|u\|_s \leq C\|\omega\|_r.$$

In the case of $(0, 1)$ currents, this assumption is not necessary, thanks to the results of N. Kerzman [8].

Theorem 1.3 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X .*

- *If $1 \leq r < 2(n+1)$ there is a constant $C = C_r > 0$ such that if $\omega \in L^r_{(0,1)}(\Omega)$, $\bar{\partial}\omega = 0$ then there is a function u such that, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$,*

$$\bar{\partial}u = \omega \text{ and } u \in L^s(\Omega), \quad \|u\|_{L^s(\Omega)} \leq C\|\omega\|_{L^r_{(0,1)}(\Omega)}.$$

- *If $r = 2(n+1)$ then for any $s < \infty$ there is a function u such that*

$$\bar{\partial}u = \omega \text{ and } u \in L^s(\Omega), \quad \|u\|_{L^s(\Omega)} \leq C\|\omega\|_{L^r_{(0,1)}(\Omega)}.$$

- *if $2(n+1) < r < \infty$ there is a function u such that*

$$u \in \Lambda^\beta(\Omega) :: \bar{\partial}u = \omega, \quad \|u\|_{\Lambda^\beta(\Omega)} \leq C\|\omega\|_{L^r_{(0,1)}(\Omega)},$$

where $\beta = \frac{1}{2} - \frac{(n+1)}{r}$ and Λ^β is the Hölder class of functions of order β .

Another way to release the compact support assumption is to have a Stein manifold equipped with a hermitian metric of null curvature in the sense of Demailly Laurent [4], because then we can use their $L^r - L^r$ estimates for any $r \in [1, \infty[$.

Theorem 1.4 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X equipped with a metric with null curvature.*

- *If $1 \leq r < 2(n+1)$ and $\omega \in L^r_{(p,q)}(\Omega)$, $\bar{\partial}\omega = 0$. Then there is a $(p, q-1)$ current u such that*

$$\bar{\partial}u = \omega \text{ and } u \in L^s_{(p,q-1)}(\Omega), \quad \|u\|_s \leq C\|\omega\|_{L^r_{(p,q)}(\Omega)} \text{ with } \frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}.$$

- *If $r = 2(n+1)$ and $\omega \in L^r_{(p,q)}(\Omega)$, $\bar{\partial}\omega = 0$. Then for any $s < \infty$ there is a $(p, q-1)$ current u such that*

$$\bar{\partial}u = \omega \text{ and } u \in L^s_{(p,q-1)}(\Omega), \quad \|u\|_{L^s_{(p,q-1)}(\Omega)} \leq C\|\omega\|_{L^r_{(p,q)}(\Omega)}.$$

We notice that N. Kerzman in [8], in order to solve $\bar{\partial}$ for $(0, 1)$ forms, also use local solutions and to find a global one he used also the Hörmander L^2 solution, but his method is based on "bump" around point at the boundary and is completely different from the raising steps method introduced here.

2 The "raising steps" method.

We shall deal with the following situation : we have a C^∞ smooth manifold X admitting partitions of unity and a decreasing scale $\{B_r\}_{r \geq 1}$, $s \geq r \Rightarrow B_s \subset B_r$ of Banach spaces of functions or forms defined on open sets of X . These Banach spaces must be modules over \mathcal{D} , the space of C^∞ functions with compact support, i.e.

$$\forall \Omega \text{ open in } X, \forall \chi \in \mathcal{D}(\Omega), \exists C(\chi) > 0 :: \forall f \in B_r, \chi f \in B_r(\Omega) \text{ and } \|\chi f\|_{B_r} \leq C(\chi)\|f\|_{B_r}.$$

For instance $B_r = L^r$ the Lebesgue spaces, or $B_r = H_r^2$ the Sobolev spaces, etc...

We are interested in solution of the linear equation $Du = \omega$, where D is a linear operator, with eventually the constraint $\Delta\omega = 0$, where Δ is also a linear operator such that $\Delta^2 = 0$. In case there is no constraint we take $\Delta \equiv 0$. One aim is that we want to apply this to the $\bar{\partial}$ equation.

We shall put the following hypotheses on D , for any domain $\Omega \subset X$:

$$(i) \forall \chi \in \mathcal{D}(\Omega), D\chi \in \mathcal{D}(\Omega) ;$$

$$(ii) \forall \chi \in \mathcal{D}(\Omega), \forall f \in B_r(\Omega), D(\chi f) = D\chi \cdot f + \chi Df ;$$

as can be easily seen, a linear differential operator D verifies these assumptions.

Let Ω be a relatively compact domain in X . Now we shall make the following assumptions on X and Ω . There is a $r_0 > 1$ and a $\delta > 0$ such that, setting $\frac{1}{s} = \frac{1}{r} - \delta$,

(iii) there is a covering $\{U_j\}_{j=1, \dots, N}$ of $\bar{\Omega}$ such that, $\forall r \leq r_0$, if $\omega \in B_r(\Omega)$, $\Delta\omega = 0$, we can solve $Du_j = \omega$ in $U_j \cap \Omega$ with $B_r(\Omega) - B_s(\Omega \cap U_j)$ estimates, i.e. $\exists C_0 > 0$ such that

$$Du_j = \omega \text{ in } \Omega_j := U_j \cap \Omega \text{ and } \|u_j\|_{B_s(\Omega_j)} \leq C_0 \|\omega\|_{B_r(\Omega)}.$$

(iv) We can solve $Dw = \omega$ globally in Ω with $B_{r_0} - B_{r_0}$ estimates, i.e.

$$\exists E > 0, \exists w :: Dw = \omega \text{ in } \Omega \text{ and } \|w\|_{B_{r_0}(\Omega)} \leq E \|\omega\|_{B_{r_0}(\Omega)} \text{ provided that } \Delta\omega = 0.$$

Then we have

Theorem 2.1 (*Raising steps theorem*) *Under the assumptions above, there is a constant $C > 0$, for $r \leq r_0$, if $\omega \in B_r(\Omega)$, $\Delta\omega = 0$ there is a $u \in B_s(\Omega)$ with $\gamma := \min(\delta, \frac{1}{r} - \frac{1}{r_0})$, and $\frac{1}{s} = \frac{1}{r} - \gamma$, such that*

$$Du = \omega \text{ and } u \in B_s(\Omega), \|u\|_{B_s(\Omega)} \leq C \|\omega\|_{B_r(\Omega)}.$$

Proof.

Let $r \leq r_0$ and $\omega \in B_r(\Omega)$, $\Delta\omega = 0$; we start with the covering $\{U_j\}_{j=1, \dots, N}$ and the local solution $Du_j = \omega$ with $u_j \in B_s(\Omega_j)$ given by hypothesis (iii).

Let χ_j be a C^∞ smooth partition of unity subordinate to $\{U_j\}_{j=1, \dots, N}$ and set

$$v_0 := \sum_{j=1}^N \chi_j u_j.$$

Then we have

- $v_0 \in B_{s_0}(\Omega)$ because B_s is a module over $\mathcal{D}(\Omega)$ and $\|v_0\|_{B_{s_0}(\Omega)} \leq C \|\omega\|_{B_r(\Omega)}$ with $\frac{1}{s_0} = \frac{1}{r} - \delta$ and $C = NC_0 \max_{j=1, \dots, N} C(\chi_j)$ by hypothesis (iii).

- $Dv_0 = \sum_{j=1}^N \chi_j Du_j + \sum_{j=1}^N D\chi_j \wedge u_j$ by hypothesis (ii) hence

$$Dv_0 = \sum_{j=1}^N \chi_j \omega + \sum_{j=1}^N D\chi_j \wedge u_j = \omega + \omega_1$$

with

$$\omega_1 := \sum_{j=1}^N D\chi_j \wedge u_j.$$

But, by hypothesis (i), $D\chi_j \in \mathcal{D}(\Omega)$ hence because B_s is a module over $\mathcal{D}(\Omega)$, we have $\omega_1 \in B_{s_0}(\Omega)$,

with $\frac{1}{s_0} = \frac{1}{r} - \delta$ and $\|\omega_1\|_{B_{s_0}(\Omega)} \leq G\|\omega\|_{B_r(\Omega)}$ with $G = C_0 N \max_{j=1, \dots, N} C(D\chi_j)$.

Hence the regularity of ω_1 is higher of one step δ than that of ω .

Moreover $\Delta\omega_1 = \Delta^2 v_0 - \Delta\omega = 0$ because $\Delta^2 = 0$ and $\Delta\omega = 0$.

Two cases

Case 1 : if $\frac{1}{s_0} = \frac{1}{r} - \delta \Rightarrow s_0 \geq r_0$ we can solve a global D in $B_{r_0}(\Omega)$, by assumption (iv) i.e.

$$\exists w \in B_{r_0}(\Omega) :: Dw = \omega_1, \|w\|_{B_{r_0}(\Omega)} \leq E\|\omega_1\|_{B_{r_0}(\Omega)} \Rightarrow \|w\|_{B_{r_0}(\Omega)} \leq EG\|\omega\|_{B_r(\Omega)}.$$

It remains to set

$$u := v_0 - w$$

to have, by the linearity of D ,

and, with $\frac{1}{s} = \frac{1}{r} - \gamma$, $\gamma := \min(\delta, \frac{1}{r} - \frac{1}{r_0})$, and $F := EG$,

$$u \in B_s(\Omega), \|u\|_{B_s(\Omega)} \leq F\|\omega\|_{B_r(\Omega)}.$$

Case 2 : if $\frac{1}{s_0} = \frac{1}{r} - \delta \Rightarrow s_0 < r_0$ then we continue :

$$\exists v_1 \in B_{s_1}(\Omega), Dv_1 = \omega_1 + \omega_2$$

with

$$\begin{aligned} \|v_1\|_{B_{s_1}(\Omega)} &\leq C\|\omega_1\|_{B_{s_0}(\Omega)} \leq CG\|\omega\|_{B_r(\Omega)}, \\ \omega_2 \in B_{s_1}(\Omega), \|\omega_2\|_{B_{s_1}(\Omega)} &\leq G\|\omega_1\|_{B_{s_0}(\Omega)}, \\ \frac{1}{s_1} &= \frac{1}{s_0} - \delta = \frac{1}{r} - 2\delta, \\ \Delta\omega_2 &= 0. \end{aligned}$$

And

$$D(v_0 - v_1) = \omega + \omega_1 - Dv_1 = \omega - \omega_2.$$

Hence by induction :

$$\exists v_0, \dots, v_N :: v_j \in B_{s_j}(\Omega), \frac{1}{s_j} = \frac{1}{r} - (j+1)\delta, \|v_j\|_{B_{s_j}(\Omega)} \leq CG^{j-1}\|\omega\|_{B_r(\Omega)}$$

and

$$D(\sum_{j=0}^N (-1)^j v_j) = \omega + (-1)^N \omega_N$$

with

$$\omega_N \in B_{s_{N-1}}(\Omega), \|\omega_N\|_{B_{s_{N-1}}(\Omega)} \leq G^N \|\omega\|_{B_r(\Omega)}, \frac{1}{s_{N-1}} = \frac{1}{r} - N\delta$$

and $\Delta\omega_N = 0$.

Hence the regularity of ω_N raises of N steps δ from that of ω .

Now let $s_0 :: \frac{1}{s_0} = \frac{1}{r} - \delta$ and suppose that $s_0 \leq r_0$ then take N such that $\frac{1}{s_{N-1}} = \frac{1}{r} - N\delta \leq \frac{1}{r_0}$

then $\omega_N \in B_{s_{N-1}}(\Omega) \subset B_{r_0}(\Omega)$ and now we can solve a global D in $B_{r_0}(\Omega)$, by assumption (iv) i.e.

$$\exists w \in B_{r_0}(\Omega) :: Dw = \omega_N, \|w\|_{B_{r_0}(\Omega)} \leq E\|\omega_N\|_{B_{r_0}(\Omega)} \Rightarrow \|w\|_{B_{r_0}(\Omega)} \leq ECG^N \|\omega\|_{B_r(\Omega)}.$$

It remains to set

$$u := \sum_{j=0}^N (-1)^j v_j + (-1)^N w$$

to have

$$\text{and, with } \frac{1}{s} = \frac{1}{r} - \delta \text{ and } F := ECG^N, \\ u \in B_s^r(\Omega), \|u\|_{B_s(\Omega)} \leq F \|\omega\|_{B_r(\Omega)}. \blacksquare$$

3 Applications.

We proved in [2] the following theorem for strictly pseudo convex (s.p.c.) domains in \mathbb{C}^n .

Theorem 3.1 *Let Ω be a s.p.c. domain in \mathbb{C}^n then*

- for $1 < r < 2n + 2$ we have

$$\forall \omega \in L_{(p,q)}^r(\Omega), \bar{\partial}\omega = 0, \exists u \in L_{(p,q-1)}^s(\Omega) :: \bar{\partial}u = \omega, \|u\|_{L^s(\Omega)} \lesssim \|\omega\|_{L_{(p,q)}^r(\Omega)},$$

$$\text{with } \frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}.$$

- For $r = 2n + 2$ we have

$$\exists u \in \bigcap_{r \geq 1} L_{(p,q-1)}^r(\Omega) :: \bar{\partial}u = \omega.$$

If ω is a $(p, 1)$ form we have also :

- for $r = 1$, we have

$$\exists u \in L_{(p,0)}^{s,\infty}(\Omega) :: \bar{\partial}u = \omega, \|u\|_{L^{s,\infty}(\Omega)} \lesssim \|\omega\|_{L^1(\Omega)}$$

$$\text{with } \frac{1}{s} = 1 - \frac{1}{2(n+1)}.$$

- for $r > 2n + 2$ we have

$$\exists u \in \Lambda^\beta(\Omega) :: \bar{\partial}u = \omega, \|u\|_{\Lambda^\beta(\Omega)} \lesssim \|\omega\|_{L^r(\Omega)},$$

where $\beta = \frac{1}{2} - \frac{(n+1)}{r}$ and Λ^β is the Hölder class of functions of order β .

Moreover the solution u is linear on the data ω .

In fact the theorem we have gave results on $BMO(\Omega)$ instead of $\bigcap_{r \geq 1} L_{(p,q-1)}^r(\Omega)$ with control of

the norm and on an anisotropic Hölder class $\Gamma^\beta(\Omega)$ instead of the usual Hölder class. But because we are mainly interested in classical results, it is not necessary to go further here. The aim is to have this kind of results with a Stein manifold instead of \mathbb{C}^n .

3.1 The transfer from \mathbb{C}^n to Stein manifold.

We shall apply the raising steps method to the case of $D = \bar{\partial}$, $\Delta = \bar{\partial}$, $B_r = L_{(p,q)}^r$ the space of (p, q) currents with coefficients in the Lebesgue space L^r and X a Stein manifold. Clearly (i) and (ii) are verified.

Lemma 3.2 *Let Ω be a relatively compact domain, C^∞ smoothly bounded and strictly pseudo convex in the complex manifold X . There is a covering $\{(U_j, \varphi_j)\}_{j=1, \dots, N}$ of $\bar{\Omega}$ by coordinates patches of X such that $\Omega_j := U_j \cap \Omega$ is still relatively compact, C^∞ smoothly bounded and strictly pseudo convex in X .*

Proof.

Let $z \in \bar{\Omega}$ and (G_z, φ_z) a coordinates patch such that $z \in G_z$. Set $V_z := \varphi_z(G_z) \subset \mathbb{C}^n$ and $\zeta := \varphi_z(z)$.

We have two cases

(i) if $z \in \Omega$ take a ball B_ζ centered at ζ and contained in V_z . B is spc in \mathbb{C}^n ; set $U_z := \varphi_z^{-1}(B)$ then $U_z = U_z \cap \Omega$ is still spc with smooth boundary in X , because φ_z is biholomorphic in G_z , and U_z is a neighbourhood of z .

(ii) if $z \in \partial\Omega$ we look at $V_z \subset \mathbb{C}^n$ and we make the completion of the part of the boundary of $\varphi_z(G_z \cap \Omega)$ in V_z to get a smoothly bounded strictly pseudo convex domain Γ_z in V_z as in [1].

Now we extend Γ_z on the other side of $\varphi_z(G_z \cap \partial\Omega)$ in V_z as an open set W_z such that $W_z \cap \varphi_z(U_z \cap \Omega) = \Gamma_z$. Now we set $U_z := \varphi_z^{-1}(W_z)$, then $\Omega_z := U_z \cap \Omega = \varphi_z^{-1}(\Gamma_z)$ is strictly pseudo convex with smooth boundary in X .

Hence we have that $\{U_z\}_{z \in \bar{\Omega}}$ is a covering of the compact set $\bar{\Omega}$ and we can extract of it a finite subset $\{U_j\}_{j=1, \dots, N}$ which is still a covering of $\bar{\Omega}$ with all the required properties. ■

Remark 3.3 *We have that the Lebesgue measure on X restricted to U_j is equivalent to the restriction of the Lebesgue measure of \mathbb{C}^n to $\varphi_j(U_j)$. This equivalence is uniform with respect to $j = 1, \dots, N$ of course. Moreover we have that the distance in $\Omega \cap U_j$ is also uniformly equivalent to the distance in Γ_j because φ_j is biholomorphic in U_j and there is only a finite number of U_j to deal with.*

3.2 Case with use of the L^2 estimates of Hörmander.

Let Ω be a relatively compact domain, \mathcal{C}^∞ smoothly bounded and strictly pseudo convex in the Stein manifold X . We plan to apply the raising steps method to extend estimates from domains in \mathbb{C}^n to domains in a Stein manifold.

By lemma 3.2 we have a covering $\{U_j\}_{j=1, \dots, N}$ of $\bar{\Omega}$ by coordinates patches (U_j, φ_j) such that $\Omega_j = U_j \cap \Omega$ is spc in X and $\Gamma_j = \varphi_j(\Omega_j)$ is spc in \mathbb{C}^n with \mathcal{C}^∞ smooth boundary.

Let $\omega \in L^r_{(p,q)}(\Omega)$, we have by remark 3.3, that Lebesgue measures on Ω_j and on Γ_j are equivalent, $\omega \in L^r_{(p,q)}(\Omega)$ implies $\varphi_j^* \omega \in L^r_{(p,q)}(\Gamma_j)$.

Hence we can apply theorem 3.1 to each Γ_j to get a $u'_j \in L^s_{(p,q-1)}(\Gamma_j)$, $\bar{\partial}u'_j = \varphi_j^* \omega$ with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$. So we have here $\delta = \frac{1}{2(n+1)}$.

Back to X , we have a $u_j \in L^s_{(p,q-1)}(\Omega_j)$, $\bar{\partial}u_j = \omega$ in $U_j \cap \Omega = \Omega_j$ with control of the norm. So assumption (iii) is fulfilled.

We get the assumption (iv) by the well known theorem [7], [5].

Theorem 3.4 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X . There is a constant $C > 0$ such that if $\omega \in L^2_{(p,q)}(\Omega)$, $\bar{\partial}\omega = 0$ there is a $(p, q-1)$ current $u \in L^2_{(p,q-1)}(\Omega)$, $\bar{\partial}u = \omega$ and $\|u\|_2 \leq C\|\omega\|_2$.*

So an application of the "raising steps" theorem 2.1 gives

Theorem 3.5 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X . There is a constant $C > 0$ such that if $\omega \in L^r_{(p,q)}(\Omega)$, $\bar{\partial}\omega = 0$, with $r \leq 2$,*

set $\gamma := \min(\frac{1}{2(n+1)}, \frac{1}{r} - \frac{1}{2})$ and $\frac{1}{s} = \frac{1}{r} - \gamma$, then there is a $(p, q-1)$ current u such that

$$\bar{\partial}u = \omega \text{ and } u \in L^s_{(p,q-1)}(\Omega), \quad \|u\|_{L^s_{(p,q-1)}(\Omega)} \leq C\|\omega\|_{L^r_{(p,q)}(\Omega)}.$$

To deal with the case $r > 2$ we shall proceed by duality and ask that ω has *compact support*, i.e. $\omega \in L^{r,c}_{(p,q)}(\Omega)$ when $q < n$.

Recall that $\mathcal{H}_p^r(\Omega)$ is the set of all $(p, 0)$ $\bar{\partial}$ closed forms in Ω and in $L^r(\Omega)$.

Theorem 3.6 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X .*

Let $\frac{2(n+1)}{n+2} \leq r \leq 2(n+1)$ and $\omega \in L^{r,c}_{(p,q)}(\Omega)$, $\bar{\partial}\omega = 0$ if $1 \leq q < n$ and $\omega \in L^r_{(p,q)}(\Omega)$, $\omega \perp \mathcal{H}_p^{r'}(\Omega)$

if $q = n$; then there is a $(p, q-1)$ current u such that, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$,

$$\bar{\partial}u = \omega \text{ and } u \in L^s_{(p,q-1)}(\Omega), \quad \|u\|_s \leq C\|\omega\|_r.$$

Proof.

We use the same technique of duality we already use in [3] inspired by the Serre duality theorem [9]. As usual let s' the conjugate exponent to s .

Lemma 3.7 *For Ω , ω as in the theorem, consider the function $\mathcal{L} = \mathcal{L}_\omega$, defined on $(n-p, n-q+1)$ form $\alpha \in L^{s'}(\Omega)$, $\bar{\partial}$ closed in Ω , as follows:*

$$\mathcal{L}(\alpha) := (-1)^{p+q-1} \langle \omega, \varphi \rangle, \text{ where } \varphi \in L^{r'}(\Omega) \text{ is such that } \bar{\partial}\varphi = \alpha \text{ in } \Omega.$$

Then \mathcal{L} is well defined and linear.

Proof.

First notice that if $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$ then $\frac{1}{r'} = \frac{1}{s'} - \frac{1}{2(n+1)}$. Such a current φ with $\bar{\partial}\varphi = \alpha$ exists since s' , the conjugate exponent of s verifies $s' \leq 2$, hence we can apply theorem 3.5, with the remark that this time $s' \leq r'$.

Suppose first that $q < n$.

In order for \mathcal{L} to be well defined we need

$$\forall \varphi, \psi \in L^{r'}_{(n-p, n-q)}(\Omega), \quad \bar{\partial}\varphi = \bar{\partial}\psi \Rightarrow \langle \omega, \varphi \rangle = \langle \omega, \psi \rangle.$$

This is meaningful because $\omega \in L^{r,c}(\Omega)$, $r > 1$, $\text{Supp } \omega \Subset \Omega$.

Then we have $\bar{\partial}(\varphi - \psi) = 0$ hence $\varphi - \psi \in L^{r'}(\Omega) \subset L^{s'}(\Omega)$ so we can solve $\bar{\partial}$ in $L^{r'}(\Omega)$ because $s' \leq 2$ by theorem 3.5 :

$$\exists \gamma \in L^{r'}_{(n-p, n-q-1)}(\Omega) :: \bar{\partial}\gamma = (\varphi - \psi) \in L^{r'}(\Omega).$$

So $\langle \omega, \varphi - \psi \rangle = \langle \omega, \bar{\partial}\gamma \rangle = (-1)^{p+q-1} \langle \bar{\partial}\omega, \gamma \rangle = 0$ because ω being compactly supported in Ω there is no boundary term.

Hence \mathcal{L} is well defined in that case.

Suppose now that $q = n$.

Of course $\bar{\partial}\omega = 0$ but we have that φ, ψ are $(p, 0)$ forms hence $\bar{\partial}(\varphi - \psi) = 0$ means that $h := \varphi - \psi$ is a $\bar{\partial}$ closed $(p, 0)$ form hence $h \in \mathcal{H}_p(\Omega)$. The hypothesis $\omega \perp \mathcal{H}_p^{r'}(\Omega)$ gives $\langle \omega, h \rangle = 0$, and \mathcal{L} is also well defined in that case. We notice that ω with compact support is not needed in that case, but in order that the scalar product be defined, we need $h \in \mathcal{H}_p^{r'}(\Omega)$.

It remains to see that \mathcal{L} is linear, so let $\alpha = \alpha_1 + \alpha_2$, with $\alpha_j \in L^{s'}(\Omega)$, $\bar{\partial}\alpha_j = 0$, $j = 1, 2$; we have $\alpha = \bar{\partial}\varphi$, $\alpha_1 = \bar{\partial}\varphi_1$ and $\alpha_2 = \bar{\partial}\varphi_2$, with $\varphi, \varphi_1, \varphi_2$ in $L^{r'}(\Omega)$ so, because $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$, we have

if $q < n$:

$$\varphi = \varphi_1 + \varphi_2 + \bar{\partial}\psi, \text{ with } \psi \text{ in } L^{r'}(\Omega), \text{ as we did above,}$$

so

$$\mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi_1 + \varphi_2 + \bar{\partial}\psi \rangle = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2) + (-1)^{p+q-1} \langle \omega, \bar{\partial}\psi \rangle,$$

but $\langle \omega, \bar{\partial}\psi \rangle = \langle \bar{\partial}\omega, \psi \rangle = 0$, because $\text{Supp } \omega \Subset \Omega$ implies there is no boundary term.

Hence $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$.

if $q = n$:

because $\bar{\partial}(\varphi - \varphi_1 - \varphi_2) = 0$, we have $h := \varphi - \varphi_1 - \varphi_2 \in \mathcal{H}_p^{r'}(\Omega)$ and the hypothesis $\omega \perp \mathcal{H}_p^{r'}(\Omega)$ gives $\langle \omega, h \rangle = 0$, so

$$\mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle = (-1)^{p+q-1} \langle \omega, \varphi_1 + \varphi_2 + h \rangle = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2) + (-1)^{p+q-1} \langle \omega, h \rangle,$$

hence $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_1) + \mathcal{L}(\alpha_2)$.

We notice again that ω with compact support is not needed in that case.

The same for $\alpha = \lambda\alpha_1$ and the linearity. ■

Lemma 3.8 *Still with the same hypotheses as above there is a $(p, q - 1)$ current u such that*

$$\forall \alpha \in L_{(n-p, n-q+1)}^{s'}(\Omega), \langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle,$$

and

$$\sup_{\alpha \in L^{s'}(\Omega), \|\alpha\|_{L^{s'}(\Omega)} \leq 1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}.$$

Proof.

By lemma 3.7 we have that \mathcal{L} is a linear form on $(n - p, n - q + 1)$ forms $\alpha \in L^{s'}(\Omega)$, $\bar{\partial}$ closed in Ω .

We have

$$\exists \varphi \in L_{(n-p, n-q)}^{r'}(\Omega) :: \bar{\partial}\varphi = \alpha, \|\varphi\|_{L^{r'}(\Omega)} \leq C_2 \|\alpha\|_{L^{s'}(\Omega)}$$

and by its very definition

$$\mathcal{L}(\alpha) = \langle \omega, \varphi \rangle.$$

By Hölder inequalities

$$|\mathcal{L}(\alpha)| \leq \|\omega\|_{L^r(\Omega)} \|\varphi\|_{L^{r'}(\Omega)} = \|\omega\|_{L^r(\Omega)} \|\alpha\|_{L^{s'}(\Omega)}.$$

But there is a constant C such that

$$\|\varphi\|_{L^{r'}(\Omega)} \leq C \|\alpha\|_{L^{s'}(\Omega)}.$$

Hence

$$|\mathcal{L}(\alpha)| \leq C \|\omega\|_{L^r(\Omega)} \|\alpha\|_{L^{s'}(\Omega)}.$$

So we have that the norm of \mathcal{L} is bounded on the subspace of $\bar{\partial}$ closed forms in $L^{s'}(\Omega)$ by $C \|\omega\|_{L^r(\Omega)}$.

We apply the Hahn-Banach theorem to extend \mathcal{L} with the same norm to all $(n - p, n - q + 1)$ forms in $L^{s'}(\Omega)$. As in Serre duality theorem ([9], p. 20) this is one of the main ingredient in the proof.

This means, by the definition of currents, that there is a $(p, q - 1)$ current u which represents the extended form \mathcal{L} , i.e.

$$\forall \alpha \in L_{(n-p, n-q+1)}^{s'}(\Omega), \langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle,$$

and such that

$$\sup_{\alpha \in L^{s'}(\Omega), \|\alpha\|_{L^{s'}(\Omega)} \leq 1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}.$$

■

The lemma 3.8 says that

$$\forall \alpha \in L^{s'}_{(n-p, n-q+1)}(\Omega), \langle u, \alpha \rangle = \mathcal{L}(\alpha) = (-1)^{p+q-1} \langle \omega, \varphi \rangle,$$

hence applied to $\varphi \in \mathcal{D}_{(n-p, n-q)}(\Omega)$ we get $\alpha = \bar{\partial}\varphi \in \mathcal{D}_{(n-p, n-q+1)} \subset L^{s'}(\Omega)$ and

$$\langle u, \bar{\partial}\varphi \rangle = \mathcal{L}(\bar{\partial}\varphi) = (-1)^{p+q-1} \langle \omega, \varphi \rangle \Rightarrow \langle \bar{\partial}u, \varphi \rangle = \langle \omega, \varphi \rangle$$

because φ has compact support in Ω , hence $\bar{\partial}u = \omega$ in the sense of distributions.

Moreover the fact that

$$\sup_{\alpha \in L^{s'}(\Omega), \|\alpha\|_{L^{s'}(\Omega)} \leq 1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}$$

gives, by an easy duality, that

$$\|u\|_{L^s_{(p, q-1)}(\Omega)} \leq C \|\omega\|_{L^r_{(p, q)}(\Omega)}$$

which completes the proof of theorem 3.6. ■

Remark 3.9 *The theorem is valid up to $r = 2(n+1)$ because there we have to take $s' = 1$ and we get*

$$\sup_{\alpha \in L^1(\Omega), \|\alpha\|_{L^1(\Omega)} \leq 1} |\langle u, \alpha \rangle| \leq C \|\omega\|_{L^r(\Omega)}$$

which implies that $u \in L^\infty_{(p, q-1)}(\Omega)$ with control of the norm.

3.3 Case of $(0, 1)$ forms by use of Kerzman's estimates.

In the special case of $(0, 1)$ forms ω , we apply the raising steps method with Kerzman's estimates [8] for the global solution.

Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X . As above we have the assumption (iii) fulfilled with $\delta = \frac{1}{2(n+1)}$. We shall prove

Theorem 3.10 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X .*

• *If $1 \leq r < 2(n+1)$ there is a constant $C = C_r > 0$ such that if $\omega \in L^r_{(0,1)}(\Omega)$, $\bar{\partial}\omega = 0$ then there is a function u such that, with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}$,*

$$\bar{\partial}u = \omega \text{ and } u \in L^s(\Omega), \|u\|_{L^s(\Omega)} \leq C \|\omega\|_{L^r_{(0,1)}(\Omega)}.$$

• *If $r = 2(n+1)$ then for any $s < \infty$ there is a function u such that*

$$\bar{\partial}u = \omega \text{ and } u \in L^s(\Omega), \|u\|_{L^s(\Omega)} \leq C \|\omega\|_{L^r_{(0,1)}(\Omega)}.$$

• *if $r > 2(n+1)$ there is a function u such that*

$$u \in \Lambda^\beta_{(p,0)}(\Omega) :: \bar{\partial}u = \omega, \|u\|_{\Lambda^\beta(\Omega)} \lesssim \|\omega\|_{L^r_{(0,1)}(\Omega)},$$

where $\beta = \frac{1}{2} - \frac{(n+1)}{r}$ and Λ^β is the Hölder class of functions of order β .

Proof.

The assumption (iv) is true for any $r_0 \in [1, \infty]$ by Kerzman's estimates [8] so starting with a $r \in [1, 2(n+1)[$, we choose $r_0 = s$ with $\frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)} > 0$, hence an application of the raising steps theorem 2.1 gives the first point.

For the second point the proof of the raising steps theorem gives that we have $v_0 := \sum_{j=1}^N \chi_j u_j$ with $u_j \in \bigcap_{1 \leq t} L^t(\Omega \cap U_j)$ hence $v_0 \in \bigcap_{1 \leq t} L^t(\Omega \cap U_j)$ because the χ_j are in $\mathcal{D}(\Omega \cap U_j)$. Choose a $s < \infty$ and $r_0 = s$, then apply Kerzman's result : there is a correction $w \in L^s(\Omega)$ such that $\bar{\partial}(v_0 - w) = \omega$ and $v_0 - w \in L^s(\Omega)$.

For the third point the proof of the raising steps theorem gives that we have again $v_0 := \sum_{j=1}^N \chi_j u_j$ with $u_j \in \Lambda^\beta(\Omega \cap U_j)$ because the distance in \mathbb{C}^n and in $\Omega \cap U_j$ are equivalent and the φ_j are biholomorphic, as in remark 3.3, hence the φ_j send Hölder classes to the same Hölder classes.

Then we apply the Hölder part of Kerzman's result [8] :

$\forall \varphi \in L_{(0,1)}^\infty(\Omega)$, $\bar{\partial}\varphi = 0$, $\exists w \in L^\infty(\Omega) :: \bar{\partial}w = \omega$ and $\forall \alpha < 1/2$, $w \in \Lambda^\alpha(\Omega)$, $\|w\|_{\Lambda^\alpha(\Omega)} \leq C_\alpha \|\varphi\|_{L^\infty(\Omega)}$;

i.e. we choose $r_0 = \infty$, and we have a correction $w \in \Lambda^\alpha$ for any $\alpha < 1/2$. Hence again $\bar{\partial}(v_0 - w) = \omega$ and $v_0 - w \in \Lambda^\beta$ because $\beta < 1/2$ and we choose of course $\alpha = \beta$. ■

3.4 Case of (p, q) forms by use of Demailly-Laurent's estimates.

We can also remove the fact that ω must have compact support when $r \geq 2$ by using a theorem of Demailly-Laurent([4], Remarque 4, page 596) but the price to paid here is that the manifold has to be equipped with a metric with null curvature, in order to avoid parasitic terms.

The proof is identical to the previous one and we get

Theorem 3.11 *Let Ω be a smoothly bounded relatively compact spc domain in the Stein manifold X equipped with a metric with null curvature.*

- If $1 \leq r < 2(n+1)$ and $\omega \in L_{(p,q)}^r(\Omega)$, $\bar{\partial}\omega = 0$. Then there is a $(p, q-1)$ current u such that

$$\bar{\partial}u = \omega \text{ and } u \in L_{(p,q-1)}^s(\Omega), \|u\|_s \leq C \|\omega\|_{L_{(p,q)}^r(\Omega)} \text{ with } \frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)}.$$

- If $r = 2(n+1)$ and $\omega \in L_{(p,q)}^r(\Omega)$, $\bar{\partial}\omega = 0$. Then for any $s < \infty$ there is a $(p, q-1)$ current u such that

$$\bar{\partial}u = \omega \text{ and } u \in L_{(p,q-1)}^s(\Omega), \|u\|_s \leq C \|\omega\|_{L_{(p,q)}^r(\Omega)}.$$

References

- [1] E. Amar. Cohomologie complexe et applications. *J. London Math. Soc.*, 2(29):127–140, 1984.
- [2] E. Amar. Estimates $L^r - L^s$ for solutions of the $\bar{\partial}$ equation in strictly pseudo convex domains in C^n . *HAL : hal-00922356*, 2013.

- [3] Eric Amar. An Andreotti-Grauert theorem with L^r estimates. *arXiv:1203.0759v5*, 2012.
- [4] J-P. Demailly and C. Laurent-Thiébaud. Formules intégrales pour les formes différentielles de type (p, q) dans les variétés de Stein. *Ann. Sci. Ecole Norm. Sup.*, 4(4):579–598, 1987.
- [5] G. B. Folland and J. J. Kohn. *The Neumann problem for the Cauchy-Riemann complex.*, volume 75 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1972.
- [6] G. Henkin and J. Leiterer. *Theory of functions on complex manifolds*. Mathematische Monographien. Akademie-Verlag Berlin, 1984.
- [7] L. Hörmander. *An introduction to complex analysis in several variables*. North-Holland/American Elsevier, 1994.
- [8] N. Kerzman. Hölder and L^p estimates for solutions of $\bar{\partial}u = f$ in strongly pseudoconvex domains. *Comm. Pure. Appl. Math.*, 24:301–379, 1971.
- [9] J-P. Serre. Un théorème de dualité. *Comment. Math. Helv.*, 29:9–26, 1955.