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# High-quality construction of analysis-suitable trivariate NURBS solids by reparameterization methods

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## Abstract

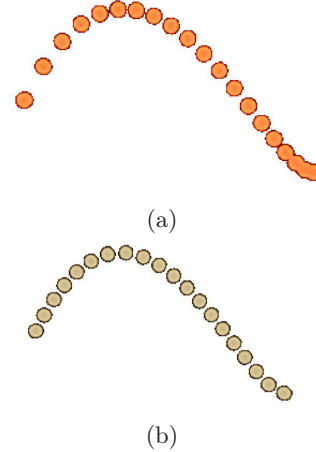
High-quality volumetric parameterization of computational domain plays an important role in three-dimensional isogeometric analysis. Reparameterization technique can improve the distribution of isoparametric curves/surfaces without changing the geometry. In this paper, using the reparameterization method, we investigate the high-quality construction of analysis-suitable NURBS volumetric parameterization. Firstly, we introduce the concept of volumetric reparameterization, and propose an optimal Möbius transformations to improve the quality of isoparametric structure based on a new uniformity metric. Secondly, from given boundary NURBS surfaces, we present a two-stage scheme to construct the analysis-suitable volumetric parameterization: in the first step, uniformity-improved reparameterization is performed on the boundary surfaces to achieve high-quality isoparametric structure without changing the shape; in the second step, from a new variational harmonic metric and the reparameterized boundary surfaces, we construct the optimal inner control points and weights to achieve an analysis-suitable NURBS solid. Several examples with complicated geometry are presented to illustrate the effectiveness of proposed methods.

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## 1 Introduction

The isogeometric analysis method proposed by Hughes et al. [14] employs the same type of spline representation both for the geometry and for the physical solutions. Based this unified data representation, it provides the possibility of seamless integration between geometric design and numerical analysis. Since high-order basis function is involved in this framework, compared with the traditional finite element analysis with the same degree of freedom, isogeometric approach can achieve a better accuracy especially for shell problems with the requirement of high-order continuity. Moreover, the number of parameters needed to describe the geometry is reduced, which is of particular interest for analysis-aware shape optimization.

In finite element analysis, mesh generation generates a discretized geometry as a computational domain from a given CAD object; it is a key and the most time-consuming step. In the isogeometric analysis framework, the parameterization of the computational domain, is also an important issue, which corresponds to the mesh generation in finite element analysis. In the first book on isogeometric analysis [7], it is pointed out that constructing analysis-suitable parameterization from given CAD boundary representation is one of the most significant challenges towards isogeometric analysis. In [6, 30], the authors study the parameterization of computational domain in IGA, and show that the quality of parameterization has great impact on analysis results and efficiency. Pilgerstorfer and Jüttler show that in isogeometric analysis the condition number of the stiffness matrix, which is a key factor for the stability of the linear system, depends strongly on the quality of domain parameterization [23].



**Figure 1:** Reparameterization example without changing the shape: (a) Sampling points on a NURBS curve with non-arc-length parameterization; (b) Sampling points on the same NURBS curve after arc-length reparameterization.

Based on above investigation, an analysis-suitable parameterization of computational domain in isogeometric analysis should satisfy three requirements: 1) it should have no self-intersections, i.e., the mapping from the parametric domain to physical domain should be injective; 2) the iso-parametric elements should be as uniform as possible; 3) the iso-parametric structure should be as orthogonal as possible. Several methods, such as constraint optimization method, variational harmonic method and gradient-based optimization method are proposed to construct analysis-suitable planar and volumetric parameterization from given CAD boundary in [35, 36, 32, 33, 27]. Previous work mainly focus on the construction of inner control points. As far as we know, the effect of the boundary parameterization on the interior volumetric parameterization has not been studied before. The quality of boundary parameterization has great effect on the subsequent volumetric parameterization results. Reparameterization technique can improve the quality of boundary parameterization without changing the geometry. As shown in Figure 1, a curve with arc-length parameterization can be obtained by reparameterization techniques. On the other hand, the weights in NURBS solid can be also considered as extra degree of freedom to obtain high-quality volumetric parameterization. In this paper, using the reparameterization method, we investigate the high-quality construction of analysis-suitable NURBS volumetric parameterization. Our main contributions are:

- Two kinds of new volumetric metrics are introduced. A uniformity metric is proposed from the geometric interpretation of the Jacobian and the concept of *variance* in statistics science; a new harmonic metric for

volumetric smoothing is introduced from the variational formulation of harmonic equation.

- Volumetric spline reparameterization is introduced, and an optimal Möbius transformation is proposed to improve the uniformity of isoparametric structure.
- By using uniformity-improved reparameterization of NURBS surfaces and the proposed variational harmonic metric, a two-stage scheme with multi-objective functions is proposed to construct the optimal inner control points and weights for analysis-suitable NURBS volumetric parameterization.

The rest of the paper is organized as follows. Section 2 reviews the related work on volumetric parameterization in isogeometric analysis. After a new uniformity metric is introduced, Section 3 describes the optimal Möbius volumetric reparameterization method for analysis-suitable NURBS solids. For volumetric parameterization problem from given boundaries, a two-stage framework with multi-objective function is proposed to construct the optimal analysis-suitable NURBS solid in Section 4. Some examples and comparisons are also presented in corresponding sections to illustrate the effectiveness of the proposed methods. Finally, we conclude this paper and outline future works in Section 5.

## 2 Related work

In this section, some related works on parameterization of computational domain in isogeometric analysis will be reviewed.

Since isogeometric analysis was firstly proposed by Hughes et al. [14] in 2005, many researchers working on computational mechanical and geometric modeling were involved in this field. We can classify the current work on isogeometric analysis into four categories: (1) isogeometric application in multi-physical problems [3, 9, 13]; (2) application of different spline models in isogeometric analysis [4, 5, 10, 21, 15]; (3) improving the accuracy and efficiency of IGA framework by refinement operations and parallel computing [2, 6, 8, 30, 31]; (4) constructing analysis-suitable parameterization of computational domain from given boundary [1, 19, 20, 26, 31, 32, 35, 36].

The work presented in this paper belongs to the forth topic. Volumetric parameterization of triangle mesh models has been investigated in [17, 29, 28] from the viewpoint of graphics applications. From the viewpoint of isogeometric applications, the parameterization of computational domain plays an important role as mesh generation in finite element analysis. Using volumetric harmonic functions, Martin et al. [19] proposed a fitting method for triangular mesh by B-spline parametric volumes. Aigner et al. [1] proposed a variational approach to construct NURBS parameterization of swept volumes. The concept of *analysis-aware modeling* is proposed by Cohen et al. [6], and the influence of parameterization of computational domains is demonstrated by several examples. In [12], a method is proposed to construct trivariate T-spline volumetric parameterization for genus-zero solid based on an adaptive tetrahedral meshing and mesh untangling technique. Zhang et al. proposed a robust and efficient approach to construct injective solid T-splines for genus-zero geometry from a boundary triangulation [35]. For mesh model with arbitrary topology, volumetric parameterization methods are proposed from the Morse theory [26] and Boolean operations [18]. The input data of above proposed methods is triangle mesh. For the product modeling by CAD software, its boundary is usually in spline form. For parameterization problem with spline boundary, Xu

et al. proposed a constraint optimization framework to construct analysis-suitable volume parameterization [32]. Spline volume faring is proposed by Pettersen and Skytt to obtain high-quality volume parameterization for isogeometric applications [24]. The construction of conformal solid T-spline from boundary T-spline representation is studied by using octree structure and boundary offset [36]. Variational harmonic method is proposed in [33] to construct analysis-suitable parameterization of computational domain from given CAD boundary information. Wang and Qian proposed an efficient method by combining divide-and-conquer, constraint aggregation and the hierarchical optimization technique to obtain valid trivariate B-spline solids from six boundary B-spline surfaces [27]. In this paper, by reparameterization technique, we introduce the optimal volumetric Möbius transformation, and propose a two-stage scheme to construct analysis-suitable NURBS volumetric parameterization based on uniform-improved boundary reparameterization method.

## 3 Volumetric reparameterization for analysis-suitable NURBS solids

### 3.1 Problem statement

The problem studied in this section can be stated as follows: given a trivariate NURBS solid, find an optimal Möbius parameter transformation such that the isoparametric structure of reparameterized NURBS solid is as uniform as possible.

In the following subsection, we will review the definition of NURBS solids and introduce the Möbius volumetric transformation.

### 3.2 NURBS solids and Möbius volumetric transformation

A NURBS solid can be defined as follows,

$$\mathbf{S}(u, v, w) = \frac{\sum_{i=0}^n \sum_{j=0}^m \sum_{k=0}^l \lambda_{i,j,k} \mathbf{C}_{i,j,k} N_i^p(u) N_j^q(v) N_k^\nu(w)}{\sum_{i=0}^n \sum_{j=0}^m \sum_{k=0}^l \lambda_{i,j,k} N_i^p(u) N_j^q(v) N_k^\nu(w)}$$

in which  $\mathbf{C}_{i,j,k}$  are control points,  $\lambda_{i,j,k}$  are the weights,  $N_i^p(u)$ ,  $N_j^q(v)$  and  $N_k^\nu(w)$  are B-spline basis function with degree  $p, q$  and  $\nu$  respectively defined on the knot vectors

$$\mathbf{U} = \{0, \dots, 0, u_{p+1}, \dots, u_l, 1, \dots, 1\}$$

$$\mathbf{V} = \{0, \dots, 0, v_{q+1}, \dots, v_m, 1, \dots, 1\}$$

and

$$\mathbf{W} = \{0, \dots, 0, w_{r+1}, \dots, w_n, 1, \dots, 1\}$$

**Definition 1.** (*Möbius volumetric transformation*) Suppose that  $\alpha, \beta, \gamma \in [0, 1]$ , the Möbius volumetric transformation can be defined as

$$u = \frac{(1-\alpha)\xi}{\alpha(1-\xi) + (1-\alpha)\xi} \quad (1)$$

$$v = \frac{(1-\beta)\eta}{\beta(1-\eta) + (1-\beta)\eta} \quad (2)$$

$$w = \frac{(1-\gamma)\zeta}{\gamma(1-\zeta) + (1-\gamma)\zeta} \quad (3)$$

After applying the Möbius transformation in (1)(2)(3) on the NURBS solid  $\mathbf{S}(u, v, w)$ , we can obtain a new parametric representation  $\tilde{\mathbf{S}}(\xi, \eta, \zeta)$  of the NURBS solid with

the same control points as follows [16],

$$\begin{aligned}\tilde{\mathbf{S}}(\xi, \eta, \zeta) &= (x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \\ &= \frac{\sum_{i=0}^n \sum_{j=0}^m \sum_{k=0}^l \tilde{\lambda}_{i,j,k} \mathbf{C}_{i,j,k} N_i^p(\xi) N_j^q(\eta) N_k^r(\zeta)}{\sum_{i=0}^n \sum_{j=0}^m \sum_{k=0}^l \tilde{\lambda}_{i,j,k} N_i^p(\xi) N_j^q(\eta) N_k^r(\zeta)}\end{aligned}$$

in which the new weights

$$\tilde{\lambda}_{i,j,k} = \frac{\lambda_{i,j,k}}{\prod_{r=1}^p K_{i,r} \prod_{s=1}^q L_{j,s} \prod_{t=1}^r M_{k,t}}$$

with

$$K_{i,r} = (1 - \alpha)(1 - u_{i+r}) + \alpha u_{i+r}, \quad (4)$$

$$L_{j,s} = (1 - \beta)(1 - v_{j+s}) + \beta v_{j+s}, \quad (5)$$

$$M_{k,t} = (1 - \gamma)(1 - w_{k+t}) + \gamma w_{k+t}. \quad (6)$$

And the corresponding knot vectors are changed into

$$\tilde{\mathbf{U}} = \left\{ \underbrace{0, \dots, 0}_{p+1}, \frac{\alpha u_{p+1}}{(1 - \alpha)(1 - u_{p+1}) + \alpha u_{p+1}}, \dots, \frac{\alpha u_l}{(1 - \alpha)(1 - u_l) + \alpha u_l}, \underbrace{1, \dots, 1}_{p+1} \right\}$$

$$\tilde{\mathbf{V}} = \left\{ \underbrace{0, \dots, 0}_{q+1}, \frac{\beta v_{q+1}}{(1 - \beta)(1 - v_{q+1}) + \beta v_{q+1}}, \dots, \frac{\beta v_m}{(1 - \beta)(1 - v_m) + \beta v_m}, \underbrace{1, \dots, 1}_{q+1} \right\}$$

and

$$\tilde{\mathbf{W}} = \left\{ \underbrace{0, \dots, 0}_{\nu+1}, \frac{\gamma w_{\nu+1}}{(1 - \gamma)(1 - w_{\nu+1}) + \gamma w_{\nu+1}}, \dots, \frac{\gamma w_n}{(1 - \gamma)(1 - w_n) + \gamma w_n}, \underbrace{1, \dots, 1}_{\nu+1} \right\}.$$

### 3.3 Improve the uniformity of isoparametric structure by volumetric reparameterization

In this subsection, we will propose a uniformity-improved volumetric reparameterization method based a new uniformity metric.

#### 3.3.1 Uniformity metric for NURBS solids

In order to achieve a volumetric parameterization with uniform isoparametric structure, a new uniformity metric is firstly proposed. The uniform isoparametric structure means that each isoparametric element has the same volume value. In probability theory and statistics, *variance* measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Hence, the uniformity of isoparametric structure means that the variance between the volume value of each isoparametric element should be as small as possible. Suppose that the  $V_i$  is the volume of  $i$ -th element, and  $V_{ave}$  is the average element volume in the NURBS solid  $\tilde{\mathbf{S}}(\xi, \eta, \zeta)$ , the discrete variance  $\sigma_{dis}$  can be defined as follows,

$$\sigma_{dis} = \frac{1}{N} \sum_{i=0}^N (V_i - V_{ave})^2 \quad (7)$$

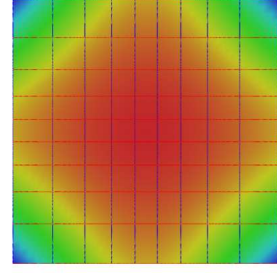


Figure 2: The uniformity metric illustrated by color-map.

in which  $N$  is the number of sampling elements.

From the geometric interpretation of Jacobian, the Jacobian determinant can be considered as a scaling factor that relates the volume change of the parametric element to the physical element. Hence, the variance of element volume in Eq. (7) can be replaced by the variance of Jacobian determinant of the NURBS solid  $\tilde{\mathbf{S}}(\xi, \eta, \zeta)$ , which can be defined in the form the continuous function as follows,

$$\sigma = \frac{\int_{\mathcal{P}} (\det \tilde{\mathbf{J}} - J_{mean})^2 d\mathcal{P}}{\int_{\mathcal{P}} d\xi d\eta d\zeta} \quad (8)$$

in which  $J_{mean}$  is the average value of Jacobian determinant at each sampling point on the NURBS solid. Based on the geometric meaning of Jacobian,  $J_{mean}$  can be computed as the ratio between the volume value of physical domain  $V_{physical}$  and parametric domain  $V_{parametric}$ ,

$$\begin{aligned}J_{mean} &= \frac{V_{physical}}{V_{parametric}} \\ &= \frac{\int_{\mathcal{P}} \tilde{\mathbf{S}}_{\zeta} \cdot (\tilde{\mathbf{S}}_{\xi} \times \tilde{\mathbf{S}}_{\eta}) d\xi d\eta d\zeta}{\int_{\mathcal{P}} d\xi d\eta d\zeta} \\ &= \frac{\int_{\mathcal{P}} \det \tilde{\mathbf{J}} d\xi d\eta d\zeta}{\int_{\mathcal{P}} d\xi d\eta d\zeta}\end{aligned}$$

in which  $\mathcal{P}$  is the parametric domain with knot vectors  $\tilde{\mathbf{U}}, \tilde{\mathbf{V}}$  and  $\tilde{\mathbf{W}}$ .  $\tilde{\mathbf{J}}$  is the Jacobian matrix of the NURBS solid  $\tilde{\mathbf{S}}(\xi, \eta, \zeta)$  as follows,

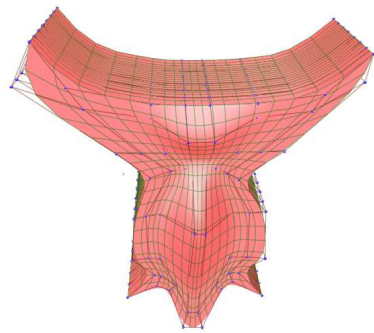
$$\tilde{\mathbf{J}} = \begin{pmatrix} x_{\xi} & x_{\eta} & x_{\zeta} \\ y_{\xi} & y_{\eta} & y_{\zeta} \\ z_{\xi} & z_{\eta} & z_{\zeta} \end{pmatrix} \quad (9)$$

As shown in [27, 32], the uniformity is also related to the second order derivative of the parameterization. By combining the variance of Jacobian in (8), a new uniformity metric at  $(\xi, \eta, \zeta)$  can be defined as follows,

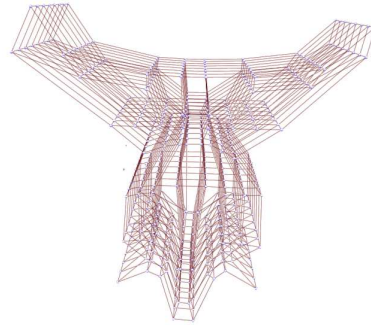
$$\mu(\tilde{\mathbf{S}}) = (\det \tilde{\mathbf{J}} - J_{mean})^2 + \omega (\|\tilde{\mathbf{S}}_{\xi\xi}\|^2 + \|\tilde{\mathbf{S}}_{\eta\eta}\|^2 + \|\tilde{\mathbf{S}}_{\zeta\zeta}\|^2), \quad (10)$$

in which  $\omega$  is a positive weight.

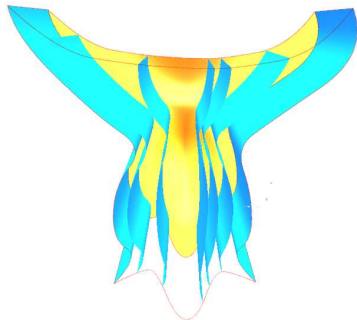
In order to show the effectiveness of the proposed metric, we present an example in Figure 2. The uniformity metric is illustrated with color-map, which is rendered according to the value of  $\mu(\tilde{\mathbf{S}})$ . The red part has smallest value and the best uniformity, the blue part has the biggest value and the worst uniformity. We can find that the uniformity color-map is consistent with the size change of the isoparametric element.



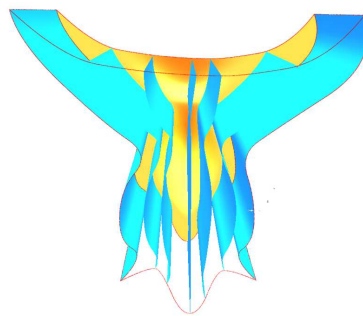
(a) NURBS solid



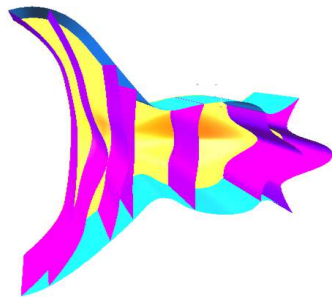
(b) Control lattice



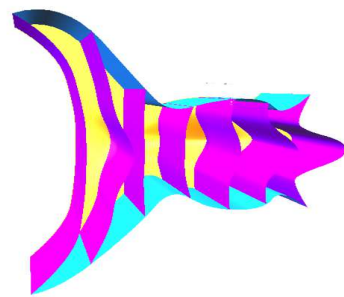
(c) Initial isoparametric surfaces in  $u$  direction



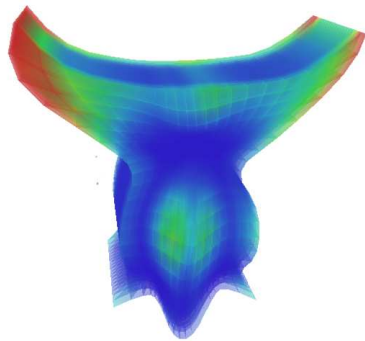
(d) Final isoparametric surfaces in  $\xi$  direction



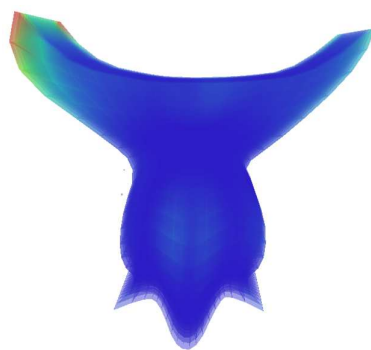
(e) Initial isoparametric surfaces in  $v$  direction



(f) Final isoparametric surfaces in  $\eta$  direction



(g) Initial color-map of uniform metric



(h) Final color-map of uniform metric

**Figure 3:** Volumetric Möbius reparameterization method for NURBS solid.

### 3.3.2 Optimal volumetric Möbius reparameterization

From the volumetric Möbius transformation and uniformity metric presented in previous sections, the optimal volumetric Möbius reparameterization problem can be stated as: given the initial parameterization  $\mathbf{S}(u, v, w)$  of NURBS solid, find the optimal  $\alpha, \beta, \gamma$  in the Möbius volumetric transformation (3)(4)(5), such that the resulted parameterization  $\tilde{\mathbf{S}}(\xi, \eta, \zeta)$  minimizes the following objective function

$$F_{\text{unif}}(\alpha, \beta, \gamma) = \int_{\mathcal{P}} \mu(\tilde{\mathbf{S}}) d\mathcal{P} \quad (11)$$

in which  $\mu(\tilde{\mathbf{S}})$  is defined in (10).

For solving the above non-linear optimization problem with three variables  $\alpha, \beta$  and  $\gamma$ , we use the Levenberg-Marquardt Method. The algorithm combines advantages of the steepest descent method, in which minimization is performed along the direction of the gradient, with the Newton method, in which a quadratic model is used to speed up the process of finding the minimum of a function. Hence, this algorithm obtained its operating stability from the steepest descent method, and adopted its accelerated convergence in the minimum vicinity from the Newton method.

### 3.4 Example and comparison

Figure 3 shows an example and corresponding comparison result for volumetric Möbius reparameterization. The given NURBS solid and the control lattice are shown in Figure 3 (a) and 3 (b). Figure 3 (c) presents the initial isoparametric surfaces in  $u$  direction of the given NURBS volumetric parametrization. Figure 3 (d) shows the final isoparametric surfaces in  $\xi$  direction of the NURBS volumetric parametrization constructed by the Möbius reparameterization method with  $\alpha = 0.323$ ,  $\beta = 0.494$  and  $\gamma = 0.217$ . The comparison of the isoparametric surfaces in  $v$  direction are also presented in Figure 3 (e) and 3 (f). We use the uniformity colormap to show the uniformity of isoparametric structure in the volume parameterizations. The uniformity colormap is computed according to the value of  $\mu(\tilde{\mathbf{S}})$  defined in (10). From Fig.3(g) and Fig.3 (h) with the same scale, we can find that the volumetric parameterization obtained by the optimal Möbius reparameterization method gives more uniform iso-parametric structure than the initial given volume parameterization.

From the chain rule, we can directly prove that if the initial volumetric parameterization has self-intersections, then Möbius reparameterization method can not remove the self-intersections. In the following section, we will propose a two-stage scheme to construct high-quality volumetric parameterization without self-intersections by boundary reparameterization.

## 4 Constructing analysis-suitable NURBS solids by boundary reparameterization

### 4.1 Main framework

Suppose that  $\mathcal{S}$  is a simply connected bounded domain in three dimensional space with Cartesian coordinates  $(x; y; z)^T$ , and is bounded by six NURBS surfaces. The volume parameterization problem of three-dimensional computational domain in isogeometric analysis can be stated as: given six boundary NURBS surfaces, find the optimal inner control points and weights such that the resulting trivariate NURBS parametric volume is a good computational domain for 3D isogeometric analysis.

The quality of boundary parameterization has great effect on the subsequent volumetric parameterization of

computational domain. In this section, we will present a two-stage scheme to construct analysis-suitable NURBS solids: in the first step, boundary surface reparameterization is performed to improve the quality of the boundary isoparametric structure; in the second step, from the reparameterized boundary surfaces, we construct the optimal inner control points and weights to achieve an analysis-suitable NURBS solid.

### 4.2 boundary reparameterization

The boundary reparameterization in this part can be viewed as the degenerated case of the volumetric reparameterization in Section 3. For each given boundary NURBS surface

$$\mathbf{R}(u, v) = \frac{\sum_{i=0}^n \sum_{j=0}^m \lambda_{i,j} \mathbf{C}_{i,j} N_i^p(u) N_j^q(v)}{\sum_{i=0}^n \sum_{j=0}^m \lambda_{i,j} N_i^p(u) N_j^q(v)},$$

The following Möbius transformation is performed on  $\mathbf{R}(u, v)$

$$u = \frac{(1 - \alpha)\xi}{\alpha(1 - \xi) + (1 - \alpha)\xi}$$

$$v = \frac{(1 - \beta)\eta}{\beta(1 - \eta) + (1 - \beta)\eta}$$

in which  $\alpha, \beta \in [0, 1]$ . Then we can obtain a new parametric NURBS surface  $\tilde{\mathbf{R}}(\xi, \eta)$  with the same geometry as  $\mathbf{R}(u, v)$ .  $\tilde{\mathbf{R}}(\xi, \eta)$  has the same control points but different weights with  $\mathbf{R}(u, v)$ . The new weights  $\tilde{\lambda}_{i,j,k}$  can be computed from the old weights  $\lambda_{i,j,k}$  as follows,

$$\tilde{\lambda}_{i,j,k} = \frac{\lambda_{i,j,k}}{\prod_{r=1}^p K_{i,r} \prod_{s=1}^q L_{j,s}}$$

with  $K_{i,r}$  and  $L_{j,s}$  defined in (4) and (5).

Then we seek for the optimal parameter  $\alpha$  and  $\beta$ , such that the isoparametric net of resulting NURBS surface  $\tilde{\mathbf{R}}(\xi, \eta)$  is as uniform as possible. That is, find the optimal  $\alpha$  and  $\beta$  to minimize the following objective function as shown in Section

$$\int_{\mathcal{P}} (\det \tilde{\mathbf{J}} - J_{\text{avg}})^2 + \omega_1 (\|\tilde{\mathbf{R}}_{\xi\xi}\|^2 + \|\tilde{\mathbf{R}}_{\eta\eta}\|^2) d\xi d\eta, \quad (12)$$

in which

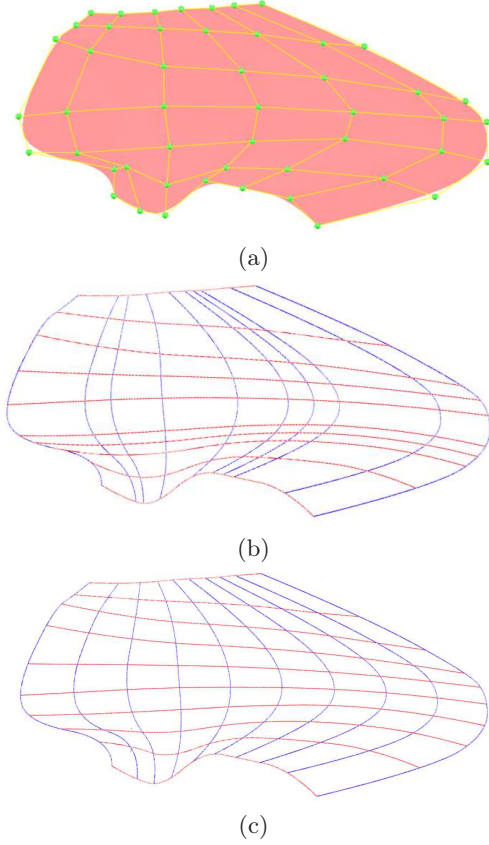
$$J_{\text{avg}} = \frac{\int_{\mathcal{P}} \det \mathbf{J} d\xi d\eta}{\int_{\mathcal{P}} d\xi d\eta},$$

and

$$\tilde{\mathbf{J}} = \begin{pmatrix} x_{\xi} & x_{\eta} \\ y_{\xi} & y_{\eta} \end{pmatrix}.$$

Similar with the trivariate case, the Levenberg-Marquardt method is used to solve this nonlinear optimization problem.

Figure 4 presents an example of boundary surface reparameterization. Figure 4 (a) presents the given planar NURBS surface and its control mesh. Figure 4(b) presents the initial isoparametric net on the surface; the corresponding iso-parametric structure of the reparameterized NURBS surface obtained by optimal Möbius transformation is shown in Figure 4(c). Obviously, more uniform iso-parametric structure can be achieved without changing the boundary shape by optimal reparameterization technique.



**Figure 4:** Reparameterization example of a planar NURBS surfaces: (a) NURBS surface and control mesh; (b) initial isoparametric net on the surface; (c) final isoparametric net after optimal M\"obius reparameterization.

### 4.3 Initial construction of NURBS solids

After boundary reparameterization, we need construct the initial control points and weights for the subsequent optimization process. In [32], the discrete Coons method is proposed to construct B-spline volume by linear combination of boundary control points. In this approach, the compatible boundary surfaces with the same degree, knot vectors and the number of control points are required. However, in practice, such requirements are usually not satisfied. Hence, some pre-processing operation must be performed for the given opposite NURBS surfaces according to the following operation procedure:

- make the given opposite NURBS surfaces have the consistent parametric direction
- perform degree elevation to have the same degree
- perform knot insertion to have the same number of control points

When all the opposite surfaces on the boundary are compatible, the discrete coons method can be employed to construct the NURBS volumes [30][11]. That is, the interior control points  $\mathbf{C}_{i,j,k}$  and weights  $\lambda_{i,j,k}$  can be constructed as linear combination of boundary control points and weights. If we introduce the four-dimensional notation  $\mathbf{P}_{i,j,k} = (\mathbf{C}_{i,j,k}, \lambda_{i,j,k})$ , the corresponding construc-

tion formula can be written as

$$\begin{aligned} \mathbf{P}_{i,j,k} = & (1 - i/l)\mathbf{P}_{0,j,k} + i/l\mathbf{P}_{l,j,k} + (1 - j/m)\mathbf{P}_{i,0,k} \\ & + j/m\mathbf{P}_{i,m,k} + (1 - k/n)\mathbf{P}_{i,j,0} + k/n\mathbf{P}_{i,j,n} \\ & - [1 - i/l, i/l] \begin{bmatrix} \mathbf{P}_{0,0,k} & \mathbf{P}_{0,m,k} \\ \mathbf{P}_{l,0,k} & \mathbf{P}_{l,m,k} \end{bmatrix} \begin{bmatrix} 1 - j/m \\ j/m \end{bmatrix} \\ & - [1 - j/m, j/m] \begin{bmatrix} \mathbf{P}_{i,0,0} & \mathbf{P}_{i,0,n} \\ \mathbf{P}_{i,m,0} & \mathbf{P}_{i,m,n} \end{bmatrix} \begin{bmatrix} 1 - k/n \\ k/n \end{bmatrix} \\ & - [1 - k/n, k/n] \begin{bmatrix} \mathbf{P}_{0,j,0} & \mathbf{P}_{l,j,0} \\ \mathbf{P}_{0,j,n} & \mathbf{P}_{l,j,n} \end{bmatrix} \begin{bmatrix} 1 - i/l \\ i/l \end{bmatrix} \\ & + (1 - k/n) \left[ [1 - i/l, i/l] \begin{bmatrix} \mathbf{P}_{0,0,0} & \mathbf{P}_{0,m,0} \\ \mathbf{P}_{l,0,0} & \mathbf{P}_{l,m,0} \end{bmatrix} \begin{bmatrix} 1 - j/m \\ j/m \end{bmatrix} \right] \\ & + k/n \left[ [1 - i/l, i/l] \begin{bmatrix} \mathbf{P}_{0,0,n} & \mathbf{P}_{0,m,n} \\ \mathbf{P}_{l,0,n} & \mathbf{P}_{l,m,n} \end{bmatrix} \begin{bmatrix} 1 - j/m \\ j/m \end{bmatrix} \right] \end{aligned}$$

As shown in [32], the initial NURBS solid constructed by discrete Coons method may have self-intersections and low quality. In the following we will propose a method to construct the optimal inner control points and weights to achieve an analysis-suitable NURBS solid based on a new variational harmonic metric.

## 4.4 Construction of analysis-suitable NURBS solids

### 4.4.1 Variational harmonic metric

The proposed volumetric parameterization method is based on the concept of *harmonic mapping*, which is a one-to-one transformation for three-dimensional domains. From the harmonic mapping theory, if  $f : \mathcal{S} \mapsto \mathcal{P}$  is a harmonic mapping from  $\mathcal{S}$  to  $\mathcal{P}$ , then the inverse mapping  $f^{-1} : \mathcal{P} \mapsto \mathcal{S}$  should be bijective.

The mapping  $f : \mathcal{S} \mapsto \mathcal{P}$  is called *harmonic mapping*, if  $f$  satisfies

$$\begin{aligned} \Delta\xi(x, y, z) &= \xi_{xx} + \xi_{yy} + \xi_{zz} = 0 \\ \Delta\eta(x, y, z) &= \eta_{xx} + \eta_{yy} + \eta_{zz} = 0 \\ \Delta\zeta(x, y, z) &= \zeta_{xx} + \zeta_{yy} + \zeta_{zz} = 0 \end{aligned}$$

Different from the method in [33], in this paper, a new harmonic metric is proposed based on the variational formulation of the PDEs (13)(13)(13), which is the classical Dirichlet integral as follows,

$$G = \int_{\Omega} (\nabla\xi)^2 + (\nabla\eta)^2 + (\nabla\zeta)^2 dx dy dz \quad (13)$$

The above function can be transformed from physical domain to the parametric domain using Jacobian transformation, which is denoted as *variational harmonic metric* as follows,

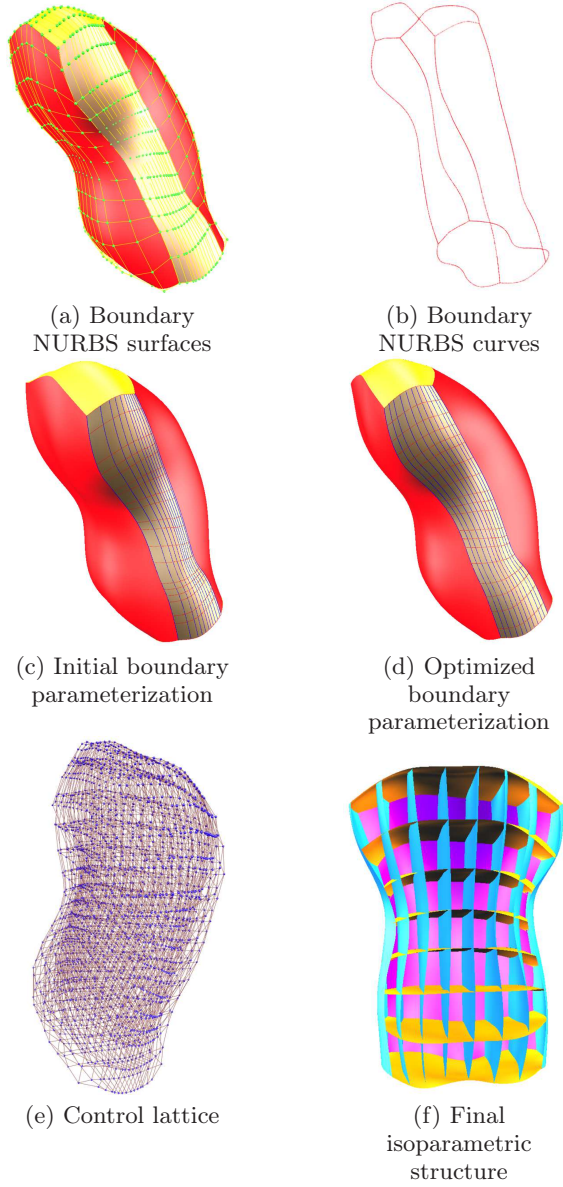
$$\begin{aligned} F_{\text{harmonic}} &= \int_{\mathcal{P}} \frac{[\frac{1}{3}(\|\mathbf{S}_{\xi}\|^2 + \|\mathbf{S}_{\eta}\|^2 + \|\mathbf{S}_{\zeta}\|^2)]^{\frac{3}{2}}}{\det \mathbf{J}} d\mathcal{P} \\ &= \int_{\mathcal{P}} \frac{[\frac{1}{3}\text{tr}(\mathbf{J}^T \mathbf{J})]^{\frac{3}{2}}}{\det \mathbf{J}} d\xi d\eta d\zeta \end{aligned} \quad (14)$$

in which  $\mathbf{J}$  is the Jacobian matrix of  $\mathbf{S}(\xi, \eta, \zeta)$  as defined in Eqn. (9).

### 4.4.2 Multi-objective optimization method for volumetric parameterization

The orthogonality of iso-parametric structure is also a key quality measure of analysis-suitable volumetric parameterization in numerical simulation [37]. The orthogonality measure can be defined according to the differential geometry property of parametric volumes as follows ,

$$F_{\text{orth}} = \int_{\mathcal{P}} \|\mathbf{S}_{\xi} \cdot \mathbf{S}_{\eta}\|^2 + \|\mathbf{S}_{\eta} \cdot \mathbf{S}_{\zeta}\|^2 + \|\mathbf{S}_{\xi} \cdot \mathbf{S}_{\zeta}\|^2 d\mathcal{P}. \quad (15)$$



**Figure 5:** Volumetric parameterization of human body model.

By combining the metrics defined in (11)(14)(15), a nonlinear optimization problem with multi-objective functions is achieved as follows

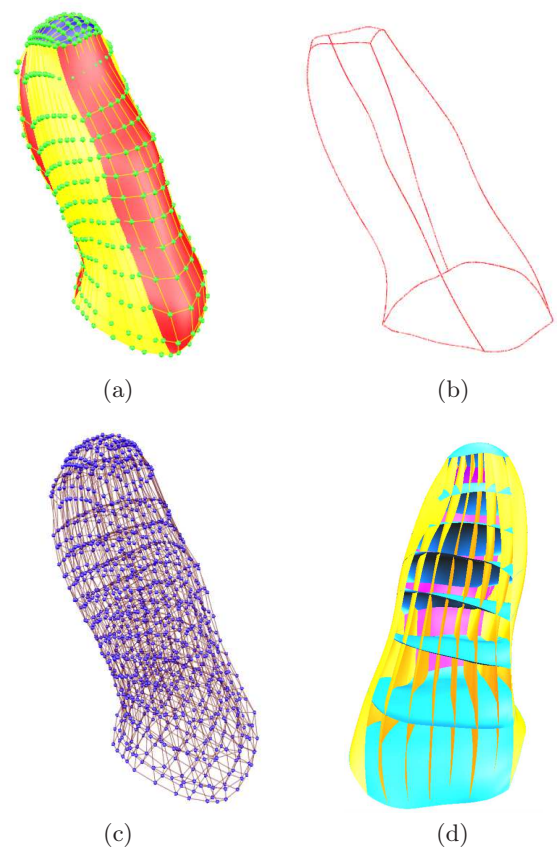
$$\min_{\mathbf{C}_{i,j,k}, \lambda_{i,j,k}} (1 - \theta_1 - \theta_2)F_{\text{harmonic}} + \theta_1 F_{\text{unif}} + \theta_2 F_{\text{orth}} \quad (16)$$

where  $\mathbf{C}_{i,j,k}$  and  $\lambda_{i,j,k}$  are control points and weights as design variables to be solved,  $\theta_1$  and  $\theta_2$  are weights for the balance between the harmonic metric, uniformity metric and orthogonality metric.

Since the problem in (16) is usually a large-scale optimization problem, we adapt L-BFGS method to obtain the optimal solution, which is a quasi-Newton method for solving unconstrained nonlinear minimization problems. In L-BFGS framework, the inverse Hessian matrix of the objective function is approximated by a sequence of gradient vectors from previous iterations. For more details, the reader can refer to [22].

#### 4.5 Experimental results

Figure 5 shows an example for volumetric parameterization of human body. The given boundary NURBS surfaces and



**Figure 6:** Volumetric parameterization of thumb model : (a) boundary NURBS surfaces and control mesh; (b) boundary NURBS curves; (c)resulting control lattice; (d) final isoparametric structure.

**Table 1:** Quantitative data in Figure 3, Figure 5 , Figure 6 and Figure 7. # deg.: degree of B-spline parameterization; # con.: number of control points; # iter.: number of optimization iterations.

Example	# Deg.	# Con.	#Iter.
Figure 3	4	8×8×8	4
Figure 5	3	13×14×13	13
Figure 6	3	8×10×13	9
Figure 7	3	8×14×11	11

curves are shown in Figure 5 (a) and Figure 5 (b). Figure 5 (c) presents initial isoparametric net on one of the given boundary NURBS surfaces. Figure 5 (d) shows the uniformity-improved isoparametric net on the reparameterized boundary surface by optimal Möbius transformation. The control lattice of the final NURBS volumetric parameterization is shown in Figure 5 (e). To illustrate the quality of the parameterization, the iso-parametric surfaces of the resulting NURBS volume are presented in Figure 5 (f). More volumetric parameterization examples with complex geometry are shown in Figure 6 and Figure 7.

Quantitative data of the examples presented in Figure 3, Figure 5 Figure 6 and Figure 7 are summarized in Table 1. Overall, the volumetric parameterization obtained by the proposed two-stage method has high-quality, and is suitable for isogeometric applications.



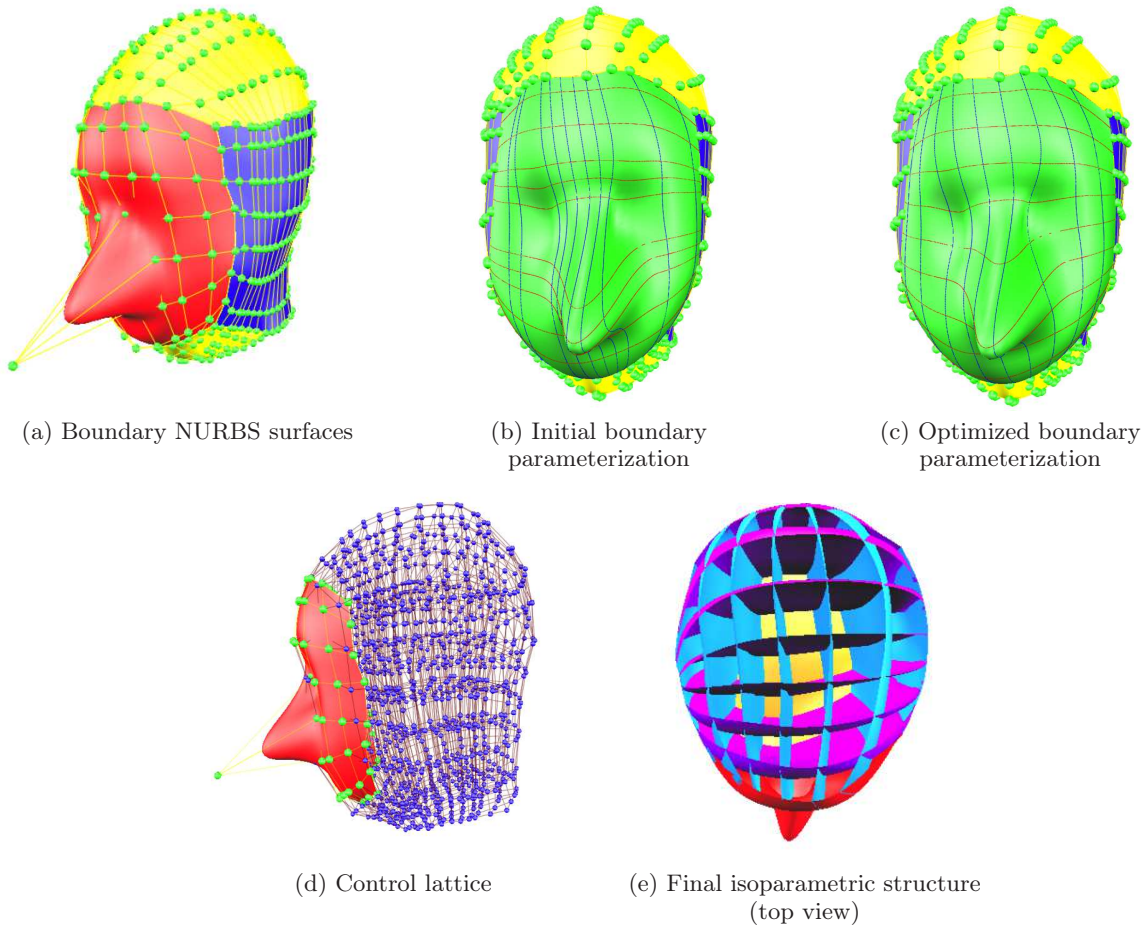


Figure 7: Volumetric parameterization of head model.

## 5 Conclusion

The quality of boundary parameterization has great effect on the subsequent volumetric parameterization results. Reparameterization methods can improve the quality of boundary parameterization without changing the geometry. In this paper, NURBS volumetric reparameterization is introduced into isogeometric analysis by using optimal Möbius transformation, and then the boundary surface reparameterization is performed as a pre-processing before constructing the inner control points and weights. Moreover, new uniformity metric and variational harmonic metric are also proposed for analysis-suitable volumetric parameterization. Experimental results illustrate that based on the reparameterization methods, we can obtain high-quality NURBS volumetric parameterization results, which are suitable for subsequent isogeometric analysis.

In the future, we will study the piece-wise reparameterization method for high-quality NURBS volumetric parameterization of computational domain [34]. The application of reparameterization technique in isogeometric solving on NURBS surfaces is also a part of our future work.

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## References

- [1] M. Aigner, C. Heinrich, B. Jüttler, E. Pilgerstorfer, B. Simeon and A.-V. Vuong. Swept volume parametrization for isogeometric analysis. In E. Hancock and R. Martin (eds.), *The Mathematics of Surfaces (MoS XIII 2009)*, LNCS vol. 5654(2009), Springer, 19-44.
- [2] Y. Bazilevs, L. Beirao de Veiga, J.A. Cottrell, T.J.R. Hughes, and G. Sangalli. Isogeometric analysis: approximation, stability and error estimates for refined meshes. *Mathematical Models and Methods in Applied Sciences*, 6(2006) 1031-1090.
- [3] Y. Bazilevs, V.M. Calo, T.J.R. Hughes, and Y. Zhang. Isogeometric fluid structure interaction: Theory, algorithms, and computations. *Computational Mechanics*, 43(2008) 3-37.
- [4] Y. Bazilevs, V.M. Calo, J.A. Cottrell, J. Evans, T.J.R. Hughes, S. Lipton, M.A. Scott, and T.W. Sederberg. Isogeometric analysis using T-Splines. *Computer Methods in Applied Mechanics and Engineering*, 199(2010) 229-263.
- [5] D. Burkhart, B. Hamann and G. Umlauf. Iso-geometric analysis based on Catmull-Clark subdivision solids. *Computer Graphics Forum*, 29(2010) 1575-1584.
- [6] E. Cohen, T. Martin, R.M. Kirby, T. Lyche and R.F. Riesenfeld, *Analysis-aware Modeling: Understanding Quality Considerations in Modeling for Isogeometric Analysis*. *Computer Methods in Applied Mechanics and Engineering*, 199(2010) 334-356.
- [7] J.A. Cottrell, T.J.R. Hughes, Y. Bazilevs. *Isogeometric Analysis: Toward Integration of CAD and FEA*. Wiley, 2009
- [8] J.A. Cottrell, T.J.R. Hughes, and A. Reali. Studies of refinement and continuity in isogeometric analysis. *Com-*

- puter Methods in Applied Mechanics and Engineering, 196(2007) 4160-4183.
- [9] J.A. Cottrell, A. Reali, Y. Bazilevs, and T.J.R. Hughes. Isogeometric analysis of structural vibrations. *Computer Methods in Applied Mechanics and Engineering*, 195(2006) 5257-5296.
- [10] M. Dörfler, B. Jüttler, and B. Simeon. Adaptive isogeometric analysis by local h-refinement with T-splines. *Computer Methods in Applied Mechanics and Engineering*, 199(2010) 264-275.
- [11] G. Elber, Y. Kim, M. Kim. Volumetric boolean sum. *Computer Aided Geometric Design*, 29(7): 532-540, 2012.
- [12] J.M. Escobara, J.M. Cascónb, E. Rodrígueza, R. Montenegro. A new approach to solid modeling with trivariate T-spline based on mesh optimization. *Computer Methods in Applied Mechanics and Engineering*, 200(2011) 3210-3222.
- [13] H. Gomez, V.M. Calo, Y. Bazilevs, and T.J.R. Hughes. Isogeometric analysis of the Cahn-Hilliard phase-field model. *Computer Methods in Applied Mechanics and Engineering*, 197(2008) 4333-4352.
- [14] T.J.R. Hughes, J.A. Cottrell, Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry, and mesh refinement. *Computer Methods in Applied Mechanics and Engineering*, 194(2005) 4135-4195.
- [15] Y. Jia, Y. Zhang, G. Xu, X. Zhuang, T. Rabczuk. Reproducing kernel triangular B-spline-based FEM for solving PDEs. *Computer Methods in Applied Mechanics and Engineering*, 267(2013) 342-358.
- [16] Lee, E. and Lucian, M. Möbius reparameterization of rational B-splines. *Computer Aided Geometric Design*, 8(1991): 213 -238.
- [17] X. Li, X. Guo, H. Wang, Y. He, X. Gu, H. Qin. Harmonic Volumetric Mapping for Solid Modeling Applications. *Proc. of ACM Solid and Physical Modeling Symposium 2007*, 109-120.
- [18] L. Liu, Y. Zhang, T. J.R. Hughes, M. A. Scott, T. W. Sederberg. Volumetric T-Spline construction Using Boolean operations. *Engineering with Computers*, 2014. DOI: 10.1007/s00366-013-0346-6
- [19] T. Martin, E. Cohen, and R.M. Kirby. Volumetric parameterization and trivariate B-spline fitting using harmonic functions. *Computer Aided Geometric Design*, 26(2009):648-664.
- [20] T. Nguyen, B. Jüttler. Using approximate implicitization for domain parameterization in isogeometric analysis. *International Conference on Curves and Surfaces*, Avignon, France, 2010.
- [21] N. Nguyen-Thanh, H. Nguyen-Xuan, S.P.A. Bordas and T. Rabczuk. Isogeometric analysis using polynomial splines over hierarchical T-meshes for two-dimensional elastic solids. *Computer Methods in Applied Mechanics and Engineering*, 200(2011) 1892-1908.
- [22] J. Nocedal and S. J. Wright. *Numerical optimization*. Springer verlag, 1999.
- [23] E. Pilgerstorfer, B. Jüttler . Bounding the influence of domain parameterization and knot spacing on numerical stability in Isogeometric Analysis. *Computer Methods in Applied Mechanics and Engineering*, 268(2014), 589-613
- [24] K.F. Pettersen, V. Skytt. Spline volume fairing. 7th International Conference on Curves and Surfaces. Avignon, France, June 24-30, 2010, *Lecture Notes in Computer Science*, 6920(2012) 553-561.
- [25] C.L. Wang, K. Tang. Non-self-overlapping Hermite interpolation mapping: a practical solution for structured quadrilateral meshing. *Computer-Aided Design*, 37(2005) 271-283.
- [26] W. Wang, Y. Zhang, L. Liu, T. J.R. Hughes. Trivariate solid T-spline construction from boundary triangulations with arbitrary genus topology. *Computer-Aided Design*, 45(2013) 351-360.
- [27] X. Wang, X. Qian. An optimization approach for constructing trivariate B-spline solids. *Computer-Aided Design*, 46(2014) 179-191.
- [28] J. Xia, Y. He, X. Yin, S. Han, and X. Gu. Direct product volume parameterization using harmonic fields. *Proceedings of IEEE International Conference on Shape Modeling and Applications 2010*, 3-12.
- [29] J. Xia, Y. He, S. Han, C.W. Fu, F. Luo, and X. Gu. Parameterization of star shaped volumes using Green's functions. *Proceedings of Geometric Modeling and Processing 2010*, 6130(2010) 219-235.
- [30] G. Xu, B. Mourrain, R. Duvigneau, A. Galligo. Optimal analysis-aware parameterization of computational domain in 3D isogeometric analysis. *Computer-Aided Design*, 45(2013), 812-821.
- [31] G. Xu, B. Mourrain, R. Duvigneau, A. Galligo. Parameterization of computational domain in isogeometric analysis: methods and comparison. *Computer Methods in Applied Mechanics and Engineering*, 200(2011), 2021-2031.
- [32] G. Xu, B. Mourrain, R. Duvigneau, A. Galligo. Analysis-suitable volume parameterization of multi-block computational domain in isogeometric applications. *Computer-Aided Design*, 45(2013), 395-404.
- [33] G. Xu, B. Mourrain, R. Duvigneau, A. Galligo. Constructing analysis-suitable parameterization of computational domain from CAD boundary by variational harmonic method. *Journal of Computational Physics*, 2013, 252, 275-289
- [34] J. Yang, D. Wang, H. Hong. Improving angular speed uniformity by  $C^1$  piecewise reparameterization. *Automated Deduction in Geometry 2012*: 33-47
- [35] Y. Zhang, W. Wang, T. J.R. Hughes. Solid T-spline construction from boundary representations for genus-zero geometry. *Computer Methods in Applied Mechanics and Engineering*, 201(2012) 185-197.
- [36] Y. Zhang, W. Wang, T. J.R. Hughes. Conformal solid T-spline construction from boundary T-spline representations. *Computational Mechanics*, 51(2013) 1051-1059.
- [37] Y. Zhang, Y. Jia, S.Y. Wang. An improved nearly-orthogonal structured mesh generation system with smoothness control functions. *Journal of Computational Physics*, 231(2012) 5289-5305.