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Estimates $L^r - L^s$ for solutions of the $\bar\partial$ equation in strictly pseudo convex domains in $\mathbb{C}^n$.

Eric Amar

Abstract

We prove estimates for solutions of the $\bar\partial u = \omega$ equation in a strictly pseudo convex domain Ω in $\mathbb{C}^n$. For instance if the (p,q) current ω has its coefficients in $L^r(\Omega)$ with $1 \le r < 2(n+1)$ then there is a solution u in $L^s(\Omega)$ with $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{2(n+1)}$. We also have BMO and Lipschitz estimates for $r \ge 2(n+1)$. These results were already done by S. Krantz [12] in the case of $(0,1)$ forms and just for the $L^r - L^s$ part by L. Ma and S. Vassiliadou [14]for general (p,q) forms. To get the complete result we propose another approach, based on Carleson measures of order α introduced and studied in [4] and on the subordination lemma [5].

1 Introduction.

Let Ω be a bounded strictly pseudo convex domain with smooth $\mathcal{C}^\infty$ boundary. We shall denote these domains as s.p.c. domains in the sequel.

Ovrelid [15] proved that if we have a (p,q) current ω, $\bar\partial$ closed in Ω and such that its coefficients are in $L^r(\Omega)$ then there is a $(p,q-1)$ current u solution of the equation $\bar\partial u = \omega$ and with coefficients still in $L^r(\Omega)$. Let us define a norm on these currents :

$$\omega \in L^r_{(p,q)}(\Omega),\ \omega = \sum_{|I|=p,|J|=q} \omega_{I,J} dz^I \wedge d\bar z^J \Rightarrow \|\omega\|_r^r := \sum_{|I|=p,|J|=q} \|\omega_{I,J}\|_r^r.$$

Then Ovrelid proved that $\|u\|_r < C\|\omega\|_r$, where the constant C does not depend on ω.

In the case of $r=\infty$, this was done before by Lieb [13] and Romanov and Henkin [16] proved that still for $r=\infty$, there is a solution u in the space Lipschitz $1/2$. In the book of Henkin and Leiterer [11] we can find precise references for these topics.

The L^p results were strongly improved by Krantz [12] in the case of $(0,1)$ forms and the aim of this work is to generalise Krantz results to the case of (p,q) forms as a consequence of results on Carleson measures of order α.

A more general case was done by L. Ma and S. Vassiliadou [14] on q-convex intersections in $\mathbb{C}^n$, but only for the $L^r - L^s$ part, the Lipschitz one is not treated in their work. (Thanks to the referee who signals me this nice paper.)

Moreover, in the case of bounded convex domains of finite type, these results are already known, done by K. Diederich, B. Fischer and J-E. Fornaess [9], A. Cumenge [8] and B. Fischer [10].

So in the case of strictly convex domains, theorem 1.2 can also be seen as a corollary of their results, but for general strictly pseudo convex domains this is not the case and of course their proofs are much more involved than this one.

I shall reproof the $L^r - L^s$ part of this theorem and prove the BMO and Lipschitz one by another approach.

We already got this kind of results in [2] by use of Skoda's kernels [17] but we where dealing with boundary values instead of inside ones. Nevertheless using Skoda results we shall prove the following theorem, where $A \lesssim B$ means that there is a constant $C > 0$ independent of A and B such that $A \leq CB$.

Theorem 1.1 *Let Ω be a s.p.c. domain in $\mathbb{C}^n$ then for $1 < r < 2n+2$ we have*

$$\forall \omega \in L^r_{(p,q)}(\Omega),\ \bar\partial\omega = 0,\ \exists u \in L^s_{(p,q-1)}(\Omega) :: \bar\partial u = \omega,\ \|u\|_{L^s(\Omega)} \lesssim \|\omega\|_{L^r(\Omega)},$$

for any s such that $\dfrac{1}{s} > \dfrac{1}{r} - \dfrac{1}{2(n+1)}$.

We shall also generalise Krantz theorem [12] to (p,q) forms :

Theorem 1.2 *Let Ω be a s.p.c. domain in $\mathbb{C}^n$ then for $1 < r < 2n+2$ we have*

- $\forall \omega \in L^r_{(p,q)}(\Omega),\ \bar\partial\omega = 0,\ \exists u \in L^s_{(p,q-1)}(\Omega) :: \bar\partial u = \omega,\ \|u\|_{L^s(\Omega)} \lesssim \|\omega\|_{L^r(\Omega)},$

with $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{2(n+1)}$.

- *For $r = 2n+2$ we have*

$$\exists u \in BMO_{(p,q)}(\Omega) :: \bar\partial u = \omega,\ \|u\|_{BMO(\Omega)} \lesssim \|\omega\|_{L^{2n+2}(\Omega)}.$$

If ω is a $(p,1)$ form we have also :

- *for $r = 1$,*

$$\exists u \in L^{s,\infty}_{(p,0)}(\Omega) :: \bar\partial u = \omega,\ \|u\|_{L^{s,\infty}(\Omega)} \lesssim \|\omega\|_{L^1(\Omega)}$$

with $\dfrac{1}{s} = 1 - \dfrac{1}{2(n+1)}$.

- *for $r > 2n+2$,*

$$\exists u \in \Gamma^\beta_{(p,0)}(\Omega) :: \bar\partial u = \omega,\ \|u\|_{\Gamma^\beta(\Omega)} \lesssim \|\omega\|_{L^r(\Omega)},$$

where $\beta = 1 - \dfrac{2n+2}{r}$ *and Γ^β is an anisotropic Lipschitz class of functions.*

Moreover the solution u is linear on the data ω.

The classes $BMO(\Omega)$ and $\Gamma^\beta(\Omega)$ will be defined later. The space $L^{s,\infty}_{(p,0)}(\Omega)$ is the Lorentz space [7].

This theorem is stronger than theorem 1.1 because here, in the case $1 \leq r < 2(n+1)$ we get the result for the end point s such that $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{2(n+1)}$.

Of course if $u \in L^s_{(p,q-1)}(\Omega)$ for $s > r$ then $u \in L^r_{(p,q-1)}(\Omega)$ hence we also have an strong improvement to Ovrelid's theorem.

Because the class Lipschitz 1/2 is contained in $\Gamma^1(\Omega)$ we see that we recover the Romanov-Henkin result when $r = \infty$ in the case of $(p,1)$ forms.

Even if they do not appear in the statement, the Carleson measures of order α, A. Bonami and I introduced in [4], are at the heart of this proof.

2 Proof of the first theorem.

Let Ω be a s.p.c. in $\mathbb{C}^n$, defined by the function $\rho \in \mathcal{C}^\infty(\mathbb{C}^n)$, i.e. $\Omega := \{z \in \mathbb{C}^n :: \rho(z) < 0\}$ and $\forall z \in \partial\Omega,\ \partial\rho(z) \neq 0$.

Let $\Omega' := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} :: \rho'(z,w) := \rho(z) + |w|^2 < 0\}$ and lift a current ω to Ω' this way : $\omega'(z,w) := \omega(z)$.

Lemma 2.1 *Let Ω be a s.p.c. domain in $\mathbb{C}^n$, with the above notations we have*

$$\omega \in L^r_{(p,q)}(\Omega) \Rightarrow \omega'(z,w) \in L^r_{(p,q)}(\partial\Omega').$$

Proof.
This is an instance of the subordination principle [1], [5]. Let $f(z) \in L^r(\Omega)$ and set $f'(z,w) := f(z)$ in Ω', then, by the main lemma in [5], p. 6,

$$\|f'\|^r_{L^r(\partial\Omega')} := \int_{\partial\Omega'} |f'(z,w)|^r \, d\sigma(z,w) = \int_\Omega |f(z)|^r \sqrt{-\rho(z) + \frac{|\mathrm{grad}\rho(z)|^2}{4}} \{\int_{|w|^2=-\rho(z)} d\,|w|\} dm(z),$$

where $d\,|w|$ is the normalized Lebesgue measure [5] on the circle $|w|^2 = -\rho(z)$. Because $\bar{\Omega}$ is compact, we have $\forall z \in \bar{\Omega}, \sqrt{-\rho(z) + \frac{|\mathrm{grad}\rho(z)|^2}{4}} \leq C(\rho) < \infty$ hence we have

$$\|f'\|^r_{L^r(\partial\Omega')} \leq C(\rho) \int_\Omega |f(z)|^r \, dm(z) = C(\rho)\|f\|_{L^r(\Omega)}.$$

It remains to apply this taking for f any coefficient of ω. ■

Proof of theorem 1.1.
Since Ω is a s.p.c. domain so is Ω' by the subordination lemma [5]. By use of lemma 2.1 we have that $\omega' \in L^r_{(p,q)}(\partial\Omega')$ and still $\bar{\partial}\omega' = 0$, hence we can apply Skoda's theorem 2 in [17] to get that there is a solution u' of $\bar{\partial}_b u' = \omega'$ such that

$$u' \in L^s_{(p,q-1)}(\partial\Omega') \text{ with } \frac{1}{s} > \frac{1}{r} - \frac{1}{2(n+1)}.$$

We have

$$u'(z,w) = \sum_{I,J} a'_{I,J}(z,w) dz^I \wedge d\bar{z}^J.$$

Because ω' does not depend on w we have that the coefficients of u' are holomorphic in w, hence we can set (recall that u' is defined on $\partial\Omega'$)

$$\forall z \in \Omega,\ a_{I,J}(z) := \int_{|w|^2=-\rho(z)} a'_{I,J}(z,w) d\,|w|$$

and

$$u(z) := \sum_{I,J} a_{I,J}(z) dz^I \wedge d\bar{z}^J,$$

then exactly as in [3] we still have

$$\bar{\partial} u = \omega \text{ in } \Omega.$$

Moreover the subordination lemma [5] gives again $u \in L^s_{(p,q-1)}(\Omega)$, because $u' \in L^s_{(p,q-1)}(\partial\Omega')$. ■

3 Carleson measures of order α.

For Ω a s.p.c. domain in $\mathbb{C}^n$, let $V^0(\Omega)$ be the space of bounded measures in Ω, and $V^1(\Omega)$ the space of Carleson measures in Ω as defined for instance in [4]. We know that these spaces form a interpolating scale for the real method [4], and we set

$$V^\alpha(\Omega) := (V^0, V^1)_{(\alpha,\infty)} \ ; \ W^\alpha(\Omega) := (V^0, V^1)_{(\alpha,p)} \text{ with } p = \frac{1}{1-\alpha}.$$

Recall that a (p,q) form ω is in $W^\alpha_{(p,q)}(\Omega)$ (resp. $V^\alpha_{(p,q)}(\Omega)$) if its coefficients and the coefficients of $\dfrac{\omega \wedge \bar\partial\rho}{\sqrt{-\rho}}$ are measures in $W^\alpha(\Omega)$ (resp. $V^\alpha(\Omega)$) see [4] and [6].
A (p,q) form is in $L^r_{(p,q)}(\Omega)$ if just its coefficients are in $L^r(\Omega)$.

Let $\Omega' := \{(z,w) \in \mathbb{C}^n \times \mathbb{C} :: \rho'(z,w) := \rho(z) + |w|^2 < 0\}$ and lift a current ω to Ω' as before : $\omega'(z,w) := \omega(z)$.
Our first result links L^r estimates to Carleson α ones.

Theorem 3.1 *Let Ω be a s.p.c. domain in $\mathbb{C}^n$ then we have*

$$\omega \in L^r_{(p,q)}(\Omega) \Rightarrow \omega'(z,w) := \omega(z) \in W^\alpha_{(p,q)}(\Omega')$$

with $\alpha = \dfrac{1}{r'} + \dfrac{1}{2(n+1)}$.

Proof.
Let $U' := \bigcup_{j=1}^N Q'(\zeta'_j, h_j) \cap \partial\Omega'$ be an open set in $\partial\Omega'$ and $T(U') = \bigcup_{j=1}^N Q'(\zeta'_j, h_j)$ be its associated "tent" set inside [4] ; in order to see that a measure $d\mu = f dm$, with m the Lebesgue measure in $\mathbb{C}^n$, belongs to $V^\alpha(\Omega')$ we have to show, see [4],

$$\int_{T(U')} |f(z')|\, dm(z') \le C\, |U'|^\alpha$$

where $|U'| := \sigma(U')$ is the Lebesgue measure of U' on $\partial\Omega$, and with a constant C independent of U'.
Because we are dealing with (p,q) currents here, this means that we have to estimate

$$A := \int_{T(U')} \frac{|\omega(z)|}{\sqrt{-\rho'(z,w)}} dm(z,w)$$

with $\rho'(z,w) := \rho(z) + |w|^2$ is equivalent to the distance of $(z,w) \in \Omega'$ to the boundary $\partial\Omega'$.

Back to A,

$$A := \int_{T(U')} \frac{|\omega(z)|}{\sqrt{-\rho'(z,w)}} dm(z,w) \le \sum_{j=1}^N \int_{Q'_j} \frac{|\omega(z)|}{\sqrt{-\rho'(z,w)}} dm(z,w).$$

The Carleson window Q'_j is equivalent to the product $(Q'_j \cap \partial\Omega') \times [h_j]_{\nu_j}$ with $[h_j]_{\nu_j}$ the real segment of length h_j supported by the real normal ν_j to $\partial\Omega'$ at ζ'_j. Set $h := \max_{j=1,\dots,N} h_j$, we shall replace Q'_j by $Q''_j := (Q'_j \cap \partial\Omega') \times [h]_{\nu_j}$.

So we have

$$A \leq \sum_{j=1}^{N} \int_{Q'_j} \frac{|\omega(z)|}{\sqrt{-\rho'(z,w)}} dm(z,w) \leq \sum_{j=1}^{N} \int_{Q''_j} \frac{|\omega(z)|}{\sqrt{-\rho'(z,w)}} dm(z,w),$$

where now all the depths have the same value h. Hence by Fubini we have

$$A \leq \int_0^h \frac{1}{\sqrt{t}} \{ \int_{U'_t} |\omega(z)| \, d\sigma(z,w) \} dt$$

with $U'_t := \bigcup_{j=1}^{N} Q''_j \cap \partial\Omega'_t$ and $\partial\Omega'_t := \{(z,w) \in \Omega' :: \rho(z) + |w|^2 = -t\}$.

We can estimate the inner integral by Hölder

$$\int_{U'_t} |\omega(z)| \, d\sigma(z,w) \leq \left(\int_{U'_t} |\omega(z)|^r \, d\sigma(z,w) \right)^{1/r} \left(\int_{U'_t} d\sigma(z,w) \right)^{1/r'} \tag{3.1}$$

but

$$\int_{U'_t} |\omega(z)|^r \, d\sigma(z,w) \leq \int_{\partial\Omega_t} |\omega(z)|^r \, d\sigma(z,w) \leq C(\rho) \int_{\Omega_t} |\omega(z)|^r \, \{ \int_{|w|^2 = -\rho(z) - t} d\,|w| \} dm(z)$$

where $d\,|w|$ is the normalized Lebesgue measure on the circle $|w|^2 = -\rho(z) - t$. Hence, with $\Omega_t := \{z \in \Omega :: \rho(z) < -t\}$,

$$\int_{U'_t} |\omega(z)|^r \, d\sigma(z,w) \leq C(\rho) \int_{\Omega_t} |\omega(z)|^r \, dm(z) = C(\rho) \|\omega\|^r_{L^r(\Omega)}.$$

For the last factor of (3.1) we have

$$\textstyle\int_{U'_t} d\sigma(z,w) = \sigma(U'_t) \lesssim \sigma(U'),$$

so

$$A \leq \int_0^h \frac{1}{\sqrt{t}} \{ \int_{U'_t} |\omega(z)| \, d\sigma(z,w) \} dt \lesssim \|\omega\|_{L^r(\Omega)} (\sigma(U'))^{1/r'} \int_0^h \frac{dt}{\sqrt{t}} = \frac{1}{2} \|\omega\|_{L^r(\Omega)} \sqrt{h} \sigma(U')^{1/r'}.$$

Recall that $\sigma(Q'_j) \simeq h_j^{(n+1)}$ then we have

$$\sqrt{h} = \sqrt{\max_j h_j} \lesssim \max \sigma(Q'_j)^{1/2(n+1)} \leq \sigma(\textstyle\bigcup_{j=1}^{N} Q'_j \cap \partial\Omega')^{1/2(n+1)},$$

so finally we get

$$A := \textstyle\int_{T(U')} \frac{|\omega(z)|}{\sqrt{-\rho'(z,w)}} dm(z,w) \lesssim \|\omega\|_{L^r(\Omega)} \sigma(U')^{\frac{1}{r'} + \frac{1}{2(n+1)}}.$$

This means that $\dfrac{|\omega(z)|}{\sqrt{-\rho'(z,w)}}$ is a Carleson measure in Ω' of order α with

$$\alpha = \frac{1}{r'} + \frac{1}{2(n+1)}.$$

To get a usual Carleson measure, we need $\alpha = 1$ hence

$$\frac{1}{r'} + \frac{1}{2(n+1)} = 1 \iff r = 2(n+1).$$

We have by theorem 1 in [4], written in our situation, that if $\mu \in V^\alpha(\Omega')$ then $P^{0*}(\mu) \in L^{r,\infty}(\partial\Omega')$, where $P^{0*}(\mu)$ is the "balayage" of μ by the Hardy Littlewood kernel P_t^0. Hence we have that the linear operator P^{0*} sends $V^{\alpha_0}(\Omega')$ to $L^{r_0,\infty}(\partial\Omega')$, and $V^{\alpha_1}(\Omega')$ to $L^{r_1,\infty}(\partial\Omega')$ with, as usual, $\alpha_j = 1 - \dfrac{1}{r_j}$. This means that

$$f \in L^r(\Omega) \Rightarrow \mu := f/\sqrt{-\rho'} dm \in V^\alpha(\Omega') \Rightarrow P^{0*}(\mu) \in L^{s,\infty}(\partial\Omega')$$

with control of the norms.

So we have a linear operator T such that, with $r_0 < r_1$,

$$T \ : \ L^{r_0}(\Omega) \to L^{s_0,\infty}(\partial\Omega'), \text{ with } \frac{1}{s_0} = \frac{1}{r_0} - \frac{1}{2(n+1)} \ ;$$

$$T \ : \ L^{r_1}(\Omega) \to L^{s_1,\infty}(\partial\Omega'), \text{ with } \frac{1}{s_1} = \frac{1}{r_1} - \frac{1}{2(n+1)} \ ;$$

hence we can apply Marcinkiewich interpolation theorem between these two values of $r \in]1, 2(n+1)[$ i.e.

$$T \ : \ L^{r}(\Omega) \to L^{s}(\partial\Omega'), \text{ with } \frac{1}{s} = \frac{1}{r} - \frac{1}{2(n+1)} \text{ and } r \leq s$$

which is needed to apply Marcinkiewich theorem, with control of norms.
But this implies by theorem 2 in [4], that $\mu := f/\sqrt{-\rho'}dm \in W^{\alpha}(\Omega')$. ■

4 The main result.

Let Ω be a domain in $\mathbb{C}^n$ defined by the function ρ as above ; define $\Omega' \subset \mathbb{C}^{n+1}$ the lifted domain : we shall define the anisotropic class $\Gamma^{\beta}(\partial\Omega')$ as in [4] ; we say that a vector field X on $\partial\Omega'$ is *admissible* if X is of class $\mathcal{C}^k$ and at any point of $\zeta \in \partial\Omega'$, $X(\zeta)$ belongs to the complex tangent space of $\partial\Omega'$ at ζ.

We say that $u \in \Gamma^{\beta}(\partial\Omega')$ if u is bounded on $\partial\Omega'$ and u belongs to the usual Lipschitz $\Lambda^{\beta/2}(\partial\Omega')$, where $\partial\Omega'$ is viewed as a real manifold, and on any integral curve of an admissible vector field, $t \in [0,1] \to \gamma(t) \in \partial\Omega'$, the function $u \circ \gamma$ belongs to $\Lambda^{\beta}(0,1)$.

We can now define the class $\Gamma^{\beta}(\Omega)$: take a function u defined in Ω and lift it as $u'(z,w) := u(z)$ in Ω' ; then $u \in \Gamma^{\beta}(\Omega)$ if $u' \in \Gamma^{\beta}(\partial\Omega')$. We have that $u \in \Gamma^{\beta}(\Omega)$ implies that $u \in L^{\infty}(\Omega)$ and $u \in \Lambda^{\beta/2}(\Omega)$ with a Lipschitz constant uniform in Ω.

The same way we define function $u \in BMO(\Omega)$ if $u' \in BMO(\partial\Omega')$. We have that $u \in BMO(\Omega)$ implies that $u \in \bigcap_{r \geq 1} L^r(\Omega)$.

Now we are in position to prove our main result.

Theorem 4.1 *Let Ω be a s.p.c. domain in $\mathbb{C}^n$ then for $1 < r < 2n+2$ we have*

$$\forall \omega \in L^r_{(p,q)}(\Omega), \ \bar\partial\omega = 0, \ \exists u \in L^s_{(p,q-1)}(\Omega) :: \bar\partial u = \omega, \ \|u\|_{L^s(\Omega)} \lesssim \|\omega\|_{L^r(\Omega)},$$

with $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{2(n+1)}$.
For $r = 2n+2$ we have

$$\exists u \in BMO_{(p,q)}(\Omega) :: \bar\partial u = \omega, \ \|u\|_{BMO(\Omega)} \lesssim \|\omega\|_{L^{2n+2}(\Omega)}.$$

If ω is a $(p,1)$ form we have also :
for $r = 1$, we have

$$\exists u \in L^{s,\infty}_{(p,0)}(\Omega) :: \bar\partial u = \omega, \ \|u\|_{L^{s,\infty}(\Omega)} \lesssim \|\omega\|_{L^1(\Omega)}$$

with $\dfrac{1}{s} = 1 - \dfrac{1}{2(n+1)}$.
for $r > 2n+2$ we have

$$\exists u \in \Gamma^{\beta}_{(p,0)}(\Omega) :: \bar\partial u = \omega, \ \|u\|_{\Gamma^{\beta}(\Omega)} \lesssim \|\omega\|_{L^r(\Omega)},$$

where $\beta = 1 - \dfrac{2(n+1)}{r}$ *and* Γ^{β} *is an anisotropic Lipschitz class of functions.*
Moreover the solution u is linear on the data ω.

Proof.
By use of theorem 3.1 we have that $\omega' \in W^{\alpha}_{(p,q)}(\Omega')$ with $\alpha = \dfrac{1}{r'} + \dfrac{1}{2(n+1)}$ where Ω' is still s.p.c. [5], hence we can apply the theorem 7 in [4] if ω is a $(p,1)$ current or the generalisation to (p,q) current done in theorem 4.1 in [6] to get that there is a solution u' of $\bar{\partial}_b u' = \omega'$ such that

$$u' \in L^s_{(p,q-1)}(\partial\Omega') \text{ with } \frac{1}{s} = 1 - \alpha = \frac{1}{r} - \frac{1}{2(n+1)}.$$

Because ω' does not depend on w we have that the coefficients of u' are holomorphic in w hence with

$$u'(z,w) = \sum_{I,J} a'_{I,J}(z,w) dz^I \wedge d\bar{z}^J$$

we can set (recall that u' is defined on $\partial\Omega'$)

$$\forall z \in \Omega,\ a_{I,J}(z) := \int_{|w|^2 = -\rho(z)} a'_{I,J}(z,w) d\,|w|$$

and we set also

$$u(z) := \sum_{I,J} a_{I,J}(z,w) dz^I \wedge d\bar{z}^J,$$

then exactly as in [3] we still have

$$\bar{\partial} u = \omega \text{ in } \Omega.$$

Moreover the subordination lemma [5], gives us $u \in L^s_{(p,q-1)}(\Omega)$.
The last two results came directly from [4], theorem 7 and theorem 8 with the fact that we apply them in $\Omega' \subset \mathbb{C}^{n+1}$ so we have from theorem 8 that $\beta = 2(n+1)(\alpha - 1)$. ■

Remark 4.2 *In the range $1 < r < 2n+2$ theorem 1.2 is stronger than theorem 1.1 because we get the result with $\dfrac{1}{s} = \dfrac{1}{r} - \dfrac{1}{2(n+1)}$ and not only for $\dfrac{1}{s} > \dfrac{1}{r} - \dfrac{1}{2(n+1)}$.*

References

[1] E. Amar. Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbb{C}^n$. *Canadian J. Math.*, 30:711–737, 1978.

[2] E. Amar. Extension de formes $\bar{\partial}_b$ fermées et solutions de l'équation $\bar{\partial}_b u = f$.. *Annali Scuola Normale Superiore Pisa*, 7(1):155–179, 1980.

[3] E. Amar. Extension de fonctions holomorphes et courants. *Bull. Sc. Math.*, 107:24–48, 1983.

[4] E. Amar and A. Bonami. Mesure de Carleson d'ordre α et solution au bord de l'équation $\bar{\partial}$. *Bull. Soc. Math. France*, 107:23–48, 1979.

[5] Eric Amar. A subordination principle. *arXiv:1105.1932v3*, 2012.

[6] M. Andersson and H. Carlsson. Estimates of solutions of the H^p and BMOA corona problem. *Math. Ann.*, 316:83–102, 2000.

[7] J. Bergh and J. LöfStröm. *Interpolation Spaces*, volume 223. Grundlehren der mathematischen Wissenchaften, 1976.

[8] A. Cumenge. Sharp estimates for $\bar{\partial}$ on convex domains of finite type. *Ark. Mat.*, 39(1):1–25, 2001.

[9] K. Diederich, B. Fischer, and J.-E. Fornaess. Hölder estimates on convex domains of finite type. *Math. Z.*, 232(1):43–61, 1999.

[10] B. Fischer. L^p estimates on convex domains of finite type. *Mathematische Zeitschrift*, 236(2):401–418, 2001.

[11] G. Henkin and J. Leiterer. *Theory of functions on complex manifolds.* Mathematische Monographien. Akademie-Verlag Berlin, 1984.

[12] S. Krantz. Optimal Lipschitz and L^p regularity for the equation $\overline{\partial} u = f$ on stongly pseudoconvex domains. *Math. Ann.*, 219(3):233–260, 1976.

[13] I. Lieb. Die cauchy-riemannschen differentialgleichungen auf streng pseudokonvexen gebieten. beschrnkte lsungen. *Math. Ann.*, 190:6–44, 1971.

[14] L. Ma and S. Vassiliadou. L^p estimates for Cauchy-Riemann operator on q-convex intersections in $\mathbb{C}^n$. *Manuscripta math*, 103:413–433, 2000.

[15] N. Ovrelid. Integral representation formulas and L^p estimates for the $\bar{\partial}$ equation. *Math. Scand.*, 29:137–160, 1971.

[16] A. V. Romanov and G. M. Henkin. Exact hölder estimates of the solutions of the $\bar{\delta}$-equation. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:1171–1183, 1971.

[17] H. Skoda. Valeurs au bord pour les solutions de l'opérateur d¨ et caractérisation des zéros de la classe de Nevanlinna. *Bull. Soc. Math. France*, 104:225–299, 1976.