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# On the family of $r$ -regular graphs with Grundy number $r + 1$

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## Abstract

The Grundy number of a graph  $G$ , denoted by  $\Gamma(G)$ , is the largest  $k$  such that there exists a partition of  $V(G)$ , into  $k$  independent sets  $V_1, \dots, V_k$  and every vertex of  $V_i$  is adjacent to at least one vertex in  $V_j$ , for every  $j < i$ . The objects which are studied in this article are families of  $r$ -regular graphs such that  $\Gamma(G) = r + 1$ . Using the notion of independent module, a characterization of this family is given for  $r = 3$ . Moreover, we determine classes of graphs in this family, in particular the class of  $r$ -regular graphs without induced  $C_4$ , for  $r \leq 4$ . Furthermore, our propositions imply results on partial Grundy number.

## 1 Introduction

We consider only undirected connected graphs in this paper. Given a graph  $G = (V, E)$ , a *proper  $k$ -coloring* of  $G$  is a surjective mapping  $c : V \rightarrow \{1, \dots, k\}$  such that  $c(u) \neq c(v)$  for any  $uv \in E$ ; the *color class*  $V_i$  is the set  $\{u \in V | c(u) = i\}$  and a vertex  $v$  has color  $i$  if  $v \in V_i$ . A vertex  $v$  of color  $i$  is a *Grundy vertex* if  $v$  is adjacent to at least one vertex colored  $j$ , for every  $j < i$ . A *Grundy  $k$ -coloring* is a proper  $k$ -coloring such that every vertex is a Grundy vertex. A *partial Grundy  $k$ -coloring* is a proper  $k$ -coloring such that every color class contains a Grundy vertex. The *Grundy number* (*partial Grundy number*, respectively) of  $G$  denoted by  $\Gamma(G)$  ( $\partial\Gamma(G)$ , respectively) is the largest  $k$  such that  $G$  admits a Grundy  $k$ -coloring (partial Grundy  $k$ -coloring, respectively).

Let  $N(v) = \{u \in V(G) | uv \in E(G)\}$  be the neighborhood of  $v$ . A set  $X$  of vertices is an *independent module* if  $X$  is an independent set and all vertices

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in  $X$  have the same neighborhood. The vertices in an independent module of size 2 are called *false twins*. Let  $P_n$ ,  $C_n$ ,  $K_n$  and  $I_n$  be respectively, the path, cycle complete and empty graph of order  $n$ . The concepts of Grundy  $k$ -coloring and domination are connected. In a Grundy coloring,  $V_1$  is a dominating set. Given a graph  $G$  and an ordering  $\phi$  on  $V(G)$  with  $\phi = v_1, \dots, v_n$ , the greedy algorithm assigns to  $v_i$  the minimum color that was not assigned in the set  $\{v_1, \dots, v_{i-1}\} \cap N(v_i)$ . Let  $\Gamma_\phi(G)$  be the number of colors used by the greedy algorithm with the ordering  $\phi$  on  $G$ . We obtain the following result [7]:  $\Gamma(G) = \max_{\phi \in S_n} (\Gamma_\phi(G))$ .

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed  $t$ , determining if a given graph has Grundy number at least  $t$  is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called  $t$ -atoms, such that any graph with Grundy number at least  $t$  contains a  $t$ -atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number [8, 15], that there exist upper bounds for  $d$ -degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number [6, 1, 4]. Recently, Havet and Sampaio [9] have proven that the problem of deciding if for a given graph  $G$  we have  $\Gamma(G) = \Delta(G) + 1$ , even if  $G$  is bipartite, is NP-complete. Moreover, they have proven that the dual of Grundy  $k$ -coloring problem is in FPT by finding an algorithm in  $O(2k^{2k} \cdot |E| + 2^{2k} k^{3k+5/2})$  time. Note that a Grundy  $k$ -coloring is a partial Grundy  $k$ -coloring, hence  $\Gamma(G) \leq \partial\Gamma(G)$ . Given a graph  $G$  and a positive integer  $k$ , the problem of determining if a partial Grundy  $k$ -coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the *b-chromatic number*, denoted by  $\varphi(G)$ , which is the largest  $k$  such that there exists a proper  $k$ -coloring and for every color class  $V_i$ , there exists a vertex adjacent to at least one vertex colored  $j$ , for every  $j$ , with  $j \neq i$ . Note that a  $b$ -coloring is a partial Grundy  $k$ -coloring, hence  $\varphi(G) \leq \partial\Gamma(G)$ . The  $b$ -chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For  $r \leq 4$ , every  $r$ -regular graphs without induced  $C_4$  has Grundy number  $r + 1$ . We conjecture that this assertion is also true for  $r > 4$ .

**Conjecture 1.** *For any integer  $r \geq 1$ , every  $r$ -regular graph without induced  $C_4$  has Grundy number  $r + 1$ .*

Section 2 gives characterizations of some classes of graphs with Grundy number at most  $k$ ,  $2 \leq k \leq \Delta(G)$ , using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except

$K_{3,3}$  has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most  $k$ ,  $3 \leq k \leq r$ . To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced  $C_4$  have Grundy number 5.

## 2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy  $k$ -coloring.

**Definition 2.1.** *Let  $G$  be a graph. A Grundy partial  $k$ -coloring is a Grundy  $k$ -coloring of a subset  $S$  of  $V(G)$ .*

**Observation 2.2** ([1],[6]). *If  $G$  admits a Grundy partial  $k$ -coloring, then  $\Gamma(G) \geq k$ .*

This property has an important consequence: For a graph  $G$ , with  $\Gamma(G) \geq t$  and any Grundy partial  $t$ -coloring, there exist smallest subgraphs  $H$  of  $G$  such that  $\Gamma(H) = t$ . The family of  $t$ -atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of  $t$ -atoms is finite and the presence of a  $t$ -atom can be determined in polynomial time for a fixed  $t$ . The following definition is slightly different from Zaker's one, insisting more on the construction of every  $t$ -atom.

**Definition 2.3** ([15]). *For any integer  $t$ , we define the family of  $t$ -atoms, denoted by  $\mathcal{A}_t$ ,  $t = 1, \dots$  by induction. Let the family  $\mathcal{A}_1$  contain only  $K_1$ . A graph  $G$  is in  $\mathcal{A}_{t+1}$  if there exists a graph  $G'$  in  $\mathcal{A}_t$  and an integer  $m$ ,  $m \leq |V(G')|$ , such that  $G$  is composed of  $G'$  and an independent set  $I_m$  of order  $m$ , adding edges between  $G'$  and  $I_m$  such that every vertex in  $G'$  is connected to at least one vertex in  $I_m$ . Moreover a  $t$ -atom  $A$  is minimal, if there is no  $t$ -atom included in  $A$  other than itself.*

**Theorem 1** ([15]). *For a given graph  $G$ ,  $\Gamma(G) \geq t$  if and only if  $G$  contains an induced minimal  $t$ -atom.*

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

**Proposition 2.4** ([1]). *Let  $G$  be a graph and  $X$  be an independent module. In every Grundy coloring of  $G$ , the vertices in  $X$  must have the same color.*

**Definition 2.5.** *Let  $G$  be an  $r$ -regular graph. A vertex  $v$  is a  $(0, \ell)$ -twin-vertex if there exists an independent module of cardinality  $r + 2 - \ell$  that contains  $v$ .*

**Proposition 2.6.** *Let  $G$  be an  $r$ -regular graph. The color of an  $(0, \ell)$ -twin-vertex is at most  $\ell$  in every Grundy coloring of  $G$ .*

*Proof.* Let  $v$  be a  $(0, \ell)$ -twin-vertex colored  $\ell + 1$  in  $G$ . By Definition,  $v$  is in an independent module  $X$  of cardinality  $r + 2 - \ell$  and by Proposition 2.4, every other vertex of  $X$  should be colored  $\ell + 1$ . Let  $u$  be a neighbor of  $v$ . There are at most  $\ell - 2$  neighbors of  $u$  in  $V(G - X)$ . Therefore,  $u$  cannot be colored  $\ell$ .  $\square$

**Definition 2.7.** A vertex  $v$  of a graph  $G$  is a  $(1, \ell)$ -twin-vertex if  $N(v)$  can be partitioned into at least  $\ell - 1$  independent modules.

**Proposition 2.8.** Let  $G$  be a graph. The color of an  $(1, \ell)$ -twin-vertex is at most  $\ell$  in every Grundy coloring of  $G$ .

*Proof.* By Proposition 2.4, vertices of the neighborhood of  $v$  can only have  $\ell - 1$  different colors. Therefore, the color of  $v$  is at most  $\ell$ .  $\square$

**Definition 2.9.** A vertex  $v$  of a graph  $G$  is a  $(2, \ell)$ -twin-vertex if  $N(v)$  is independent and every vertex in  $N(v)$  is a  $(1, \ell)$ -twin-vertex.

**Proposition 2.10.** Let  $G$  be a graph. The color of an  $(2, \ell)$ -twin-vertex is at most  $\ell$  in every Grundy coloring of  $G$ .

*Proof.* Let  $v$  be a  $(2, \ell)$ -twin-vertex in  $G$ . Every vertex in  $N(v)$  is a  $(1, \ell)$ -twin-vertex. If a vertex in  $N(v)$  is colored  $\ell$ , then  $v$  could only have a color at most  $\ell - 1$ . If the vertices in the neighborhood of  $v$  have colors at most  $\ell - 1$ , then in every Grundy coloring of  $G$ ,  $v$  has a color at most  $\ell$ .  $\square$

**Corollary 2.11.** Let  $G$  be a graph. If every vertex is a  $(1, \ell)$ -twin-vertex or a  $(2, \ell)$ -twin-vertex, then  $\Gamma(G) \leq \ell$ .

**Corollary 2.12.** Let  $G$  be a regular graph. If every vertex is an  $(i, \ell)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ , then  $\Gamma(G) \leq \ell$ .

**Proposition 2.13** ([1],[15]). Let  $G$  be a graph. We have  $\Gamma(G) \leq 2$  if and only if  $G = K_{n,m}$  for some integers  $n > 0$  and  $m > 0$ .

### 3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial  $k$ -colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

**Proposition 3.1** ([6]). Let  $G$  be a connected 2-regular graph.  $\partial\Gamma(G) = \Gamma(G) = 2$  if and only if  $G = C_4$ .

The following definition gives a construction of the cubic graphs in which every vertex is an  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ . Figure 2 gives the list of every graph of order at most 16 in this family.

**Definition 3.2.** Let  $K_{2,3}$  and  $K_{3,3}^*$  be the graphs from Figure 1. We define recursively the family of graphs  $\mathcal{F}_3^*$  as follows:

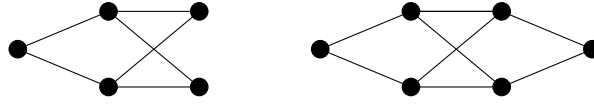


Figure 1: The graphs  $K_{2,3}$  (on the left) and  $K_{3,3}^*$  (on the right).

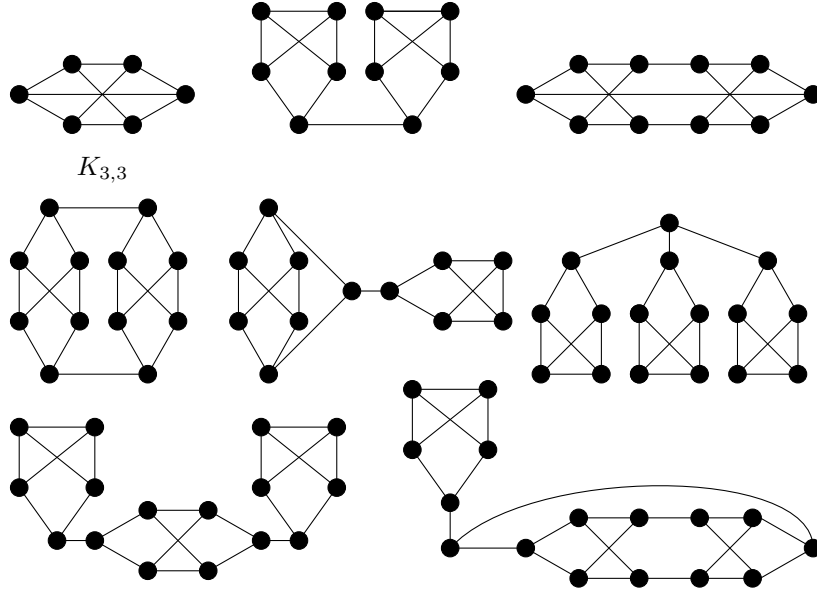


Figure 2: The cubic graphs  $G$  such that  $|V(G)| < 18$  and  $\Gamma(G) < 4$ .

1.  $K_{2,3} \in \mathcal{F}_3^*$  and  $K_{3,3}^* \in \mathcal{F}_3^*$ ;
2. the disjoint union of two elements of  $\mathcal{F}_3^*$  is in  $\mathcal{F}_3^*$ ;
3. if  $G$  is a graph in  $\mathcal{F}_3^*$ , then the graph  $H$  obtained from  $G$  by adding an edge between two vertices of degree at most 2 is also in  $\mathcal{F}_3^*$ ;
4. if  $G$  is a graph in  $\mathcal{F}_3^*$ , then the graph  $H$  obtained from  $G$  by adding a new vertex adjacent to three vertices of degree at most 2 is in  $\mathcal{F}_3^*$ .

The family  $\mathcal{F}_3$  is the subfamily of cubic graphs in  $\mathcal{F}_3^*$ .

**Proposition 3.3.** *Let  $G$  be a cubic graph. Every vertex of  $V(G)$  is an  $(i, 3)$ -twin vertex, for some  $i$ ,  $0 \leq i \leq 2$ , if and only if  $G \in \mathcal{F}_3$ .*

*Proof.* Every graph  $G$  in  $\mathcal{F}_3$  has three kind of vertices:  $(0, 3)$ -twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices

added by Point 4. Vertices where an edge is added by Point 3 are  $(1, 3)$ -twin-vertex and vice versa. Vertices added by Point 4 are  $(2, 3)$ -twin-vertices and vice versa.  $\square$

**Theorem 2.** *Let  $G$  be a cubic graph.  $\Gamma(G) \leq 3$  if and only if every vertex is an  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ .*

*Proof.* By Corollary 2.12, the "if" part is proven. Assume that  $G$  contains a vertex  $v$  which is not an  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$  and  $\Gamma(G) < 4$ . In every configuration we want to either find a Grundy partial 4-coloring, contradicting  $\Gamma(G) < 4$  or proving that  $v$  is an  $(i, 3)$ -twin-vertex, for some  $i$ , with  $0 \leq i \leq 2$ . We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases:  $v$  or a neighbor of  $v$  is in a  $C_3$ ,  $v$  is in an induced  $C_4$  and  $v$  or a neighbor of  $v$  are not in a  $C_3$  and  $v$  is not in an induced  $C_4$ . Let  $C$  be an induced cycle of order 3 or 4 which contains  $v$  or a neighbor of  $v$  and let  $D_1 = \{x \in V(G) | d(x, C) = 1\}$ , where  $d(x, C)$  is the distance from  $x$  to  $C$  in the graph  $G$ . To simplify notation,  $D_1$  will also denote the subgraph of  $G$  induced by  $D_1$ .

**Case 1:** Assume that  $v$  or a neighbor of  $v$  is in  $C$  and  $C = C_3$ . If  $|D_1| = 1$ , then  $G = K_4$  and  $\Gamma(K_4) = 4$ . If  $|D_1| = 2$  and  $D_1 = P_2$ , then  $v$  is a  $(0, 3)$ -twin-vertex or a  $(1, 3)$ -twin-vertex. If  $D_1 = I_2$  then Figure 3.1.a yields a Grundy partial 4-coloring of  $G$ . If  $|D_1| = 3$ , then we have four subcases:  $D_1$  is  $C_3$  or  $P_3$  (Figure 3.1.b),  $P_2 \cup I_1$  (Figure 3.1.c) or  $I_3$  (Figure 3.1.d). In every case  $G$  admits a Grundy partial 4-coloring.

**Case 2:** Assume that  $v$  is in  $C$  and  $C = C_4$ . Note that for two non adjacent vertices of  $C$  who have a common neighbor in  $D_1$ , the vertex  $v$  is a  $(0, 3)$ -twin-vertex or a  $(1, 3)$ -twin-vertex. Hence, we will not consider these cases. If  $|D_1| = 2$ , then  $D_1 = P_2$  or  $D_1 = I_2$  (Figure 3.2.a) and in both cases,  $G$  admits a Grundy partial 4-coloring. If  $|D_1| = 3$ , Figure 3.2.b yields a Grundy partial 4-coloring of  $G$ . In the case  $|D_1| = 4$ , we first assume that two adjacent vertices of  $C$  have their neighbors in  $D_1$  adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of  $C$  have their neighbors in  $D_1$  adjacent (Figure 3.2.d). In the case  $D_1 = I_4$ , we first suppose that two vertices of  $D_1$  which have two adjacent vertices of  $C$  as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

**Case 3:** Assume that  $v$  or a neighbor of  $v$  is not in a  $C_3$  and  $v$  is not in an induced  $C_4$ . Firstly, suppose that a neighbor  $u$  of  $v$  is in an induced  $C_4$ . Using the coloring from the previous case,  $G$  admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of  $v$  in the  $C_4$  have a common neighbor outside the  $C_4$ . However, this case cannot happen for every neighbor of  $v$ , otherwise  $v$  would be a  $(2, 3)$ -twin-vertex. Assume that  $u$  is the neighbor of  $v$  not in the previous configuration. If  $u$  is in an induced  $C_4$ , then using the coloring from the previous case,  $G$

admits a Grundy partial 4-coloring. If  $u$  is not in an induced  $C_4$ , then Figure 3.3.a yields a Grundy partial 4-coloring of  $G$ . In this figure, the color 2 is given to a neighbor of  $u$  not adjacent to both  $f_1$  and  $f_2$ . Secondly, suppose that  $v$  is in an induced  $C_5$ . Figure 3.3.b yields a Grundy partial 4-coloring of  $G$ . Thirdly, if  $v$  is not in an induced  $C_5$ , then Figure 3.3.c yields a Grundy partial 4-coloring of  $G$ .

Therefore, if  $\Gamma(G) \leq 3$ , then every vertex is an  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ .  $\square$

Observe that if an edge is added between the two vertices of degree 2 in  $K_{3,3}^*$ , then we obtain  $K_{3,3}$  which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

**Corollary 3.4.** *A cubic graph  $G$  does not contain any induced minimal subcubic 4-atom if and only if every vertex is an  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ .*

**Corollary 3.5.** *Let  $G$  be a cubic graph. If  $G$  is without induced  $C_4$ , then  $\Gamma(G) = 4$ .*

*Proof.* As every graph  $G$  with  $\Gamma(G) < 4$  is composed of copies of  $K_{2,3}$  or  $K_{3,3}^*$ , the graph  $G$  always contains a square if  $\Gamma(G) < 4$ .  $\square$

For a fixed integer  $t$ , the largest  $(t+1)$ -atom has order  $2^t$ . Thus, for a graph  $G$  of maximum degree  $t$ , there exists an  $O(n^{2^t})$ -time algorithm to determine if  $\Gamma(G) < t+1$  (which verifies if the graph contains an induced  $(t+1)$ -atom). For a cubic graph, we obtain an  $O(n^8)$ -time algorithm, whereas our characterization yields a linear-time algorithm.

**Observation 3.6.** *Let  $G$  be a cubic graph of order  $n$ . There exists an  $O(n)$ -time algorithm<sup>1</sup> to determine the Grundy number of  $G$ .*

*Proof.* Suppose we have a cubic graph  $G$  with its adjacency list. Verifying if  $G$  is  $K_{3,3}$  can be done in constant time. We suppose now that  $G$  is not  $K_{3,3}$ . For each vertex  $v$ , the algorithm verifies that  $v$  is an  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ . If the condition is true for all vertices, then  $\Gamma(G) = 3$ , else  $\Gamma(G) = 4$ . To determine if a vertex  $v$  is a  $(0, 3)$ -twin-vertex, it suffices to verify that there is a common vertex other than  $v$  in the adjacency lists of the neighbors of  $v$ . To determine if a vertex  $v$  is a  $(1, 3)$ -twin-vertex, it suffices to verify that there are two neighbors of  $v$  which have the same adjacency list. To determine if a vertex  $v$  is a  $(2, 3)$ -twin-vertex, it suffices to verify that the neighborhood of  $v$  is independent and that every neighbor is a  $(1, 3)$ -twin-vertex. Hence, checking if a vertex is an  $(i, 3)$ -twin vertex can be done in constant time, so the algorithm runs in linear time.  $\square$

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<sup>1</sup>Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.



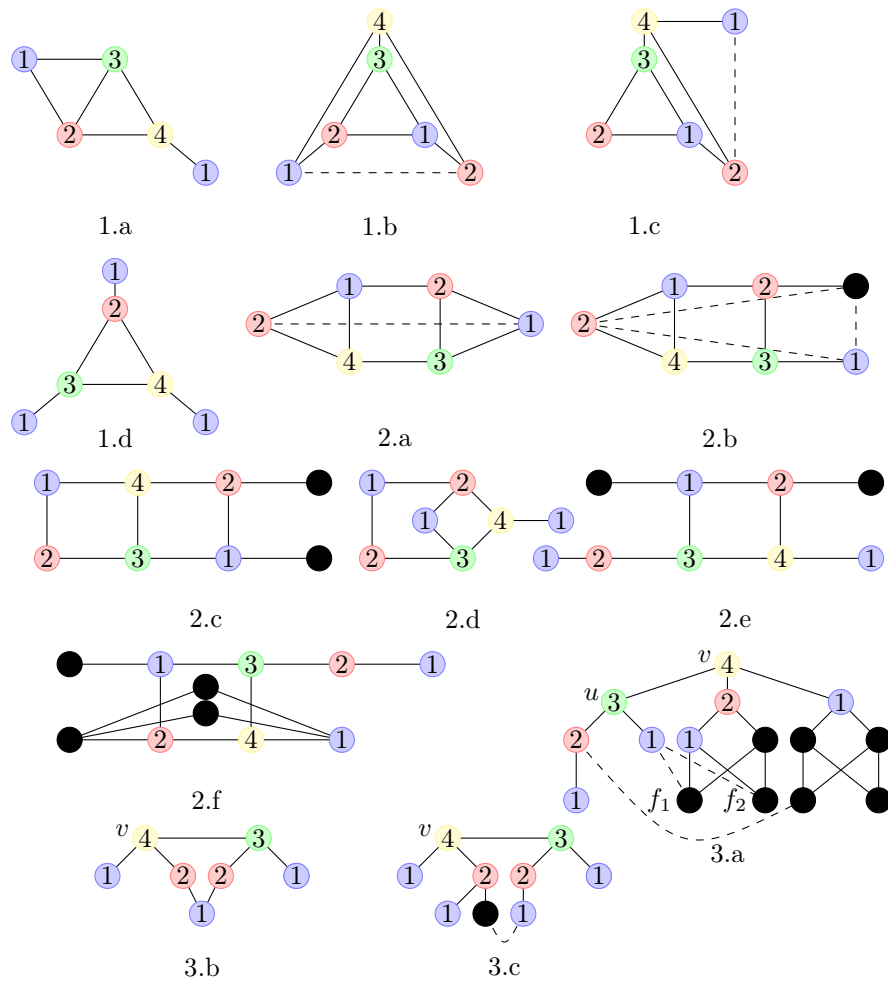


Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number  $i$ : Vertices of color  $i$ ).

**Proposition 3.7.** *If  $G$  is a connected cubic graph and  $G \neq K_{3,3}$ , then  $\partial\Gamma(G) = 4$ .*

*Proof.* Let  $G$  be a cubic connected graph. Note that if  $\Gamma(G) = 4$  then  $\partial\Gamma(G) = 4$ . Every graph  $G$  with  $\Gamma(G) < 4$  is composed of copies of  $K_{2,3}$  or  $K_{3,3}^*$ . If  $G$  contains more than two copies (so it is different from  $K_{3,3}$ ), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence,  $\partial\Gamma(G) = 4$ .  $\square$

Only  $K_{3,3}$  and three other cubic graphs have  $b$ -chromatic number at most 3 [10]. Thus, our result is coherent with the results on the  $b$ -chromatic number. Shi et al. [13] proved that there exists a smallest integer  $N_r$  such that every  $r$ -regular graph  $G$  with more than  $N_r$  vertices has  $\partial\Gamma(G) = r + 1$ . Observe that we have  $N_2 = 4$  and  $N_3 = 6$ . It is an open question to determine  $N_r$  for  $r \geq 4$ . However, using results on  $b$ -chromatic number [3], we have  $N_r \leq 2r^3 - r^2 + r$ .

## 4 Properties on the Grundy number of $r$ -regular graphs

**Definition 4.1.** *Let  $r \geq 2$  be an integer. We define recursively the family of graphs  $\mathcal{G}_r^*$  as follows:*

1.  $K_{r-k, k+2} \in \mathcal{G}_r^*$ , for any  $k$ ,  $0 \leq k \leq (r-2)/2$ ;
2. the disjoint union of two elements of  $\mathcal{G}_r^*$  is in  $\mathcal{G}_r^*$ ;
3. if  $G$  is a graph in  $\mathcal{G}_r^*$ , then the graph  $H$  obtained from  $G$  by adding an edge between two vertices of degree at most  $r-1$  is also in  $\mathcal{G}_r^*$ ;
4. if  $G$  is a graph in  $\mathcal{G}_r^*$ , then the graph  $H$  obtained from  $G$  by adding a new vertex adjacent to  $r$  vertices of degree at most  $r-1$  is in  $\mathcal{G}_r^*$ .

The family  $\mathcal{G}_r$  is the subfamily of  $r$ -regular graphs in  $\mathcal{G}_r^*$ .

**Proposition 4.2.** *Let  $G$  be an  $r$ -regular graph. If  $G \in \mathcal{G}_r$ , then  $\Gamma(G) < r + 1$ .*

*Proof.* By  $I_{r-k}$  and  $I_{k+2}$ , with  $|I_{r-k}| = r - k$  and  $|I_{k+2}| = k + 2$ , we denote the two sets of vertices in the bipartition of an induced subgraph  $K_{r-k, k+2}$  in  $G$ . Firstly, suppose there exists a vertex  $u$  in an induced subgraph  $K_{r-k, k+2}$  colored  $r + 1$ . Without loss of generality, suppose  $u$  is in  $I_{r-k}$ . The  $r$  neighbors of  $u$  should have colors from 1 to  $r$ . Among the neighbors of  $u$ ,  $k + 2$  neighbors are in  $I_{k+2}$ . Let  $v$  be the neighbor of  $u$  in  $I_{k+2}$  with the largest color in  $I_{k+2}$ . The vertex  $v$  has color at least  $k + 2$ . Hence, there exists an integer  $s \geq 0$  such that the color of  $v$  is  $k + 2 + s$ . Note that there are  $s$  vertices in  $N(u) \setminus I_{k+2}$  which have colors at most  $k + 2 + s$ . The colors of the  $s$  vertices are the only one possible remaining colors at most  $k + 2 + s$  in  $I_{r-k}$ . Hence, as there are  $k$  vertices in  $N(v) \setminus I_{r-k}$ , the neighbors of  $v$  can only have at most  $k + s$  different colors at most  $k + 2 + s$ . Therefore, we have a contradiction and  $u$  cannot have

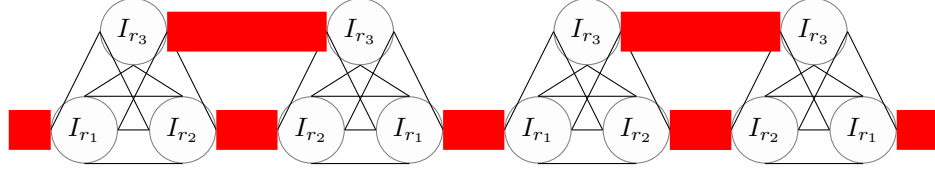


Figure 4: The Graph  $G_{r,4,i}$ ,  $i \geq 2$ ,  $r = r_1 + r_2 + r_3$ .

color  $r + 1$ . Secondly, suppose there exists a vertex  $u$  added by Point 4 which has color  $r + 1$ . As a neighbor of  $u$  in an induced  $K_{r-k,k+2}$  should be colored  $r$ , the argument is completely similar to the previous one.  $\square$

**Corollary 4.3.** *Let  $G$  be a 4-regular graph. If  $G \in \mathcal{G}_4$ , then  $\Gamma(G) < 5$ .*

The reader can believe that the family of 4-regular graphs with  $\Gamma(G) < 5$  contains only the family  $\mathcal{G}_4$ . However, there exist graphs with Grundy number  $r$  which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle  $C_7^2$  satisfies  $\Gamma(C_7^2) < 5$  and is not in  $\mathcal{G}_4$ .

The next proposition shows that unlike the  $b$ -chromatic number,  $r$ -regular graphs of order arbitrarily large with Grundy number  $k$  can be constructed for any  $r$  and any  $k$ ,  $3 \leq k \leq r + 1$ .

**Proposition 4.4.** *Let  $r \geq 4$  and  $3 \leq k \leq r + 1$  be integers. There exists an infinite family  $\mathcal{H}$  of connected  $r$ -regular graphs such that for all  $G$  in  $\mathcal{H}$ ,  $\Gamma(G) = k$ .*

*Proof.* Let  $i \geq 2$  be a positive integer and  $r_1, \dots, r_{k-1}$  be a sequence of positive integers such that  $r = r_1 + \dots + r_{k-1}$ . We construct a graph  $G_{r,k,i}$  as follows: Take  $2i$  copies of  $K_{r_1, \dots, r_{k-1}}$ . Let  $H_{j-1}$  be the copy number  $j$  of  $K_{r_1, \dots, r_{k-1}}$  and  $H_{j,r_l}$  be the independent  $r_l$ -set in  $H_j$ . If  $j \equiv 1 \pmod{2}$ , do the graph join of  $H_{j \pmod{2i}, r_1}$  and  $H_{j-1 \pmod{2i}, r_1}$  and for an integer  $l$ ,  $1 < l < k$ , do the graph join of  $H_{j \pmod{2i}, r_l}$  and  $H_{j+1 \pmod{2i}, r_l}$ . The  $r$ -regular graph obtained is the graph  $G_{r,k,i}$ . Figure 4 gives  $G_{r,k,i}$ , for  $k = 4$  and  $i \geq 2$ . Note that  $H_{j,r_i}$  is an independent module. Thus, every vertex is a  $(0, k)$ -twin-vertex. By Proposition 2.6,  $\Gamma(G_{r,k,i}) \leq k$ .

For an integer  $l$ ,  $1 < l < k$ , color one vertex  $l - 1$  in  $H_{1,r_l}$  and  $H_{2,r_l}$ . Afterwards, color one vertex  $k - 1$  in  $H_{1,r_1}$  and one vertex  $k$  in  $H_{2,r_1}$ . The given coloring is a Grundy partial  $k$ -coloring of  $G_{r,k,i}$  for  $i \geq 2$ . Therefore,  $\Gamma(G_{r,k,i}) = k$ , for  $i \geq 2$ .  $\square$

## 5 Grundy number of 4-regular graphs without induced $C_4$

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4-regular graphs without induced  $C_4$  contains only graphs with Grundy number 5.

**Lemma 5.1.** *Let  $G$  be a 4-regular graph without induced  $C_4$ . If  $G$  contains (an induced)  $K_4$  then  $\Gamma(G) = 5$ .*

*Proof.* Note that if  $G = K_5$ , we have  $\Gamma(G) = 5$ . If  $G$  is not  $K_5$  then every pair of neighbors of vertices of  $K_4$  cannot be adjacent ( $G$  would contain a  $C_4$ ). Giving the color 1 to each neighbor of the vertices of  $K_4$  and colors 2, 3, 4, 5 to the vertices of  $K_4$ , we obtain a Grundy partial 5-coloring of  $G$ .  $\square$

**Lemma 5.2.** *Let  $G$  be a 4-regular graph without induced  $C_4$  and let  $W$  be the graph from Figure 5. If  $G$  contains an induced  $W$  then  $\Gamma(G) = 5$ .*

*Proof.* The names of the vertices of  $W$  come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of  $G$  will be given using their references on Figure 5. Let  $D_1$  be the set of vertices at distance 1 from vertices of  $W$  in  $G - W$ . Suppose that two vertices of  $W$  have a common neighbor in  $D_1$ . This two vertices could only be  $u_4$  and  $u_5$  or  $u_3$  and  $u_5$  (or  $u_1$  and  $u_4$ , by symmetry). In the case that  $u_4$  and  $u_5$  have a common neighbor in  $D_1$ , colors will be given to neighbors of  $u_3$  in  $D_1$ , depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that  $u_3$  and  $u_5$  have a common neighbor  $w$  in  $D_1$ ,  $w$  can be adjacent with a neighbor of  $u_3$  in  $D_1$  (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in  $W$  have a common neighbor in  $D_1$ . Let  $w_1$  and  $w_2$  be the neighbors of  $u_3$  in  $D_1$ . We first consider that  $w_1$  and  $w_2$  are adjacent (Figure 5.3.a). Secondly, we consider that  $w_1$  and  $w_2$  are not adjacent and that  $u_5$ ,  $u_3$  and  $w_1$  are in an induced  $C_5$  (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c).  $\square$

**Proposition 5.3.** *Let  $G$  be a 4-regular graph without induced  $C_4$ . If  $G$  contains  $C_3$  then  $\Gamma(G) = 5$ .*

*Proof.* Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of  $G$  in Figure 6 will be given. Let  $M_i$ ,  $i = 2$  or  $3$ , be the graph of order  $2 + i$  containing two adjacent vertices  $u_1$  and  $u_2$  which have exactly  $i$  common neighbors,  $\{v_1, \dots, v_i\}$ , that form an independent set. Let  $D_1$  be the set of vertices at distance 1 from an induced  $M_i$  in  $G - M_i$ , for  $2 \leq i \leq 3$ .

**Case 1:** Firstly, assume that  $G$  contains an induced  $M_3$  and a vertex of  $M_3$  has its two neighbors in  $D_1$  adjacent (Figure 6.1.a). Secondly, assume that  $G$  contains an induced  $M_2$  and a vertex of  $M_2$  has its two neighbors in  $D_1$

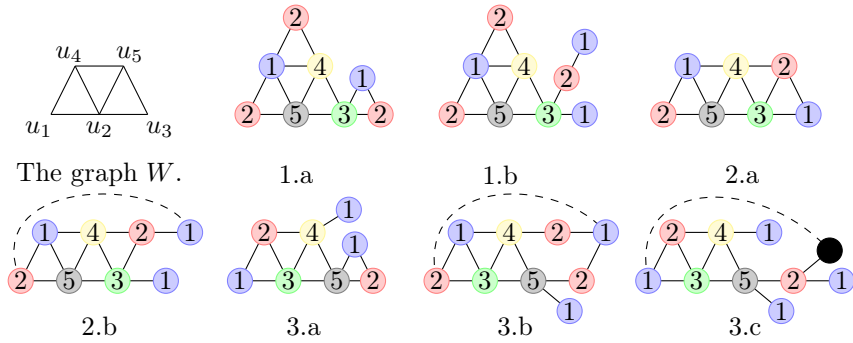


Figure 5: Possible configurations when  $G$  contains an induced  $W$ .

adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that  $G$  cannot contain a  $K_4$  by Lemma 5.1.

**Case 2:** Assume that  $G$  contains an induced  $M_3$  excluding the previous configuration. There are three cases:  $u_1, v_2$  and  $v_3$  are in an induced  $C_5$  (Figure 6.2.a),  $u_1, v_2$  and  $v_3$  are in an induced  $C_6$  and not in an induced  $C_5$  (Figure 6.2.b) and  $u_1, v_2$  and  $v_3$  are neither in an induced  $C_5$  nor  $C_6$  (Figure 6.2.c).

**Case 3:** Suppose that  $G$  contains an induced  $M_2$  excluding the previous configurations. Firstly, we suppose that  $u_1, v_1$  and  $v_2$  are in an induced  $C_5$  (Figure 6.3.a). Secondly, we suppose that  $u_1, v_1$  are in an induced  $C_5$  excluding the previous case (Figure 6.3.b). Thirdly, we suppose that  $u_1, v_1$  and  $v_2$  are in an induced  $C_6$  and not in an induced  $C_5$  (Figure 6.3.c) and finally neither in an induced  $C_5$  nor  $C_6$  (Figure 6.3.d).

Suppose that  $G$  contains a 3-cycle  $C$  and no induced  $M_2$ . Let  $u_1, u_2$  and  $u_3$  be the vertices of  $C$ . Let  $w_1$  and  $w_2$  be the neighbors of  $u_1$  outside  $C$ , let  $w'_1$  and  $w'_2$  be the neighbors of  $u_2$  outside  $C$  and let  $w''_1$  and  $w''_2$  be the neighbors of  $u_3$  outside  $C$ .

**Case 4:** Firstly, suppose that  $u_1, u_2, w_1$  and  $w'_1$  are in a 5-cycle and a neighbor of  $u_1$ , say  $w_1$ , has a common neighbor with  $w'_1$  (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that  $u_1, u_2, w_1$  and  $w'_1$  are in a 5-cycle;  $w''_1, v_1, u_1$  and  $w_1$  are in another 5-cycle and  $w_1$  is in a triangle (Figure 6.4.b). We suppose that  $w_1$  is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of  $C$  are in a 5-cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of  $C$  cannot be in a 5-cycle (Figure 6.4.e).

□

In the following two lemmas, we consider a graph  $G$  of girth  $g = 5$  and possibly containing an induced Petersen graph. Let  $u_1, u_2, u_3, u_4$  and  $u_5$  be

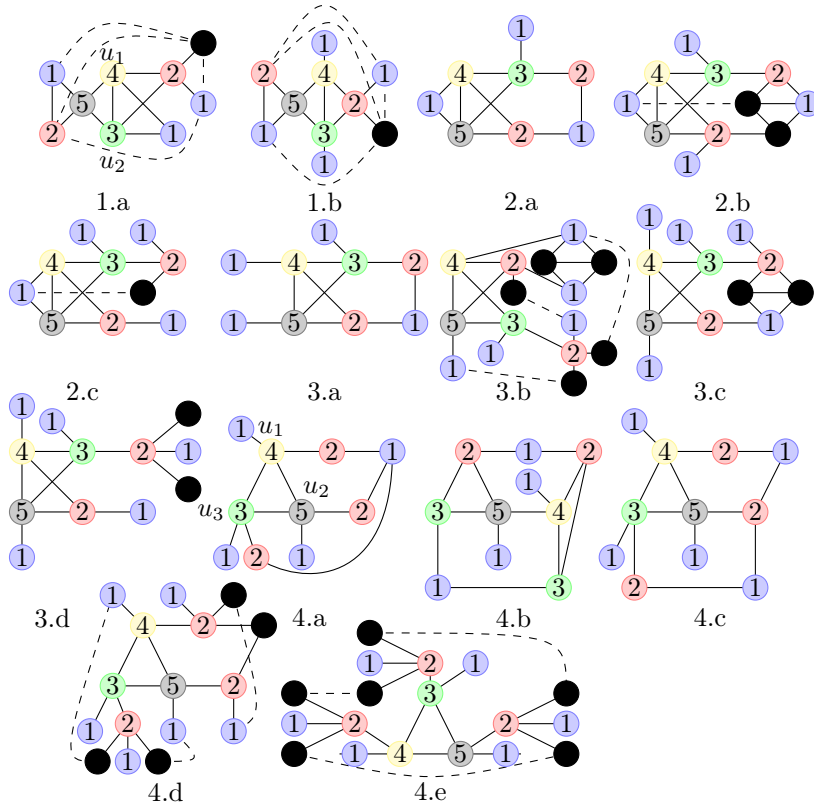


Figure 6: Possible configurations when  $G$  is an induced  $C_3$ .

the vertices in an induced  $C_5$  (or in the outer cycle of a Petersen graph, if any). Let  $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$  and  $v'_5$  be the remaining neighbors of respectively  $u_1, u_2, u_3, u_4$  and  $u_5$  (all different as  $g = 5$ ).

**Lemma 5.4.** *Let  $G$  be a 4-regular graph with girth  $g = 5$ . If  $G$  contains the Petersen graph as induced subgraph then  $\Gamma(G) = 5$ .*

*Proof.* Suppose that  $v_1, v_2, v_3, v_4$  and  $v_5$  form an induced  $C_5$  (the inner cycle of the Petersen graph). Let  $u'_2$  and  $u'_5$  be the remaining neighbors of respectively  $v_2$  and  $v_5$ . Observe that  $v'_1$  can be adjacent with no more than three vertices among  $v'_3, v'_4, u'_2$  and  $u'_5$ . Firstly, suppose that  $v'_1$  is not adjacent with  $v'_3$  (or  $v'_4$ , without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph  $G$ . Secondly, assume that  $v'_1$  is not adjacent with  $u'_5$  (or  $u'_2$ , without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph  $G$ .  $\square$

In a graph  $G$ , let a *neighbor-connected*  $C_n$  be an  $n$ -cycle  $C$  such that the set

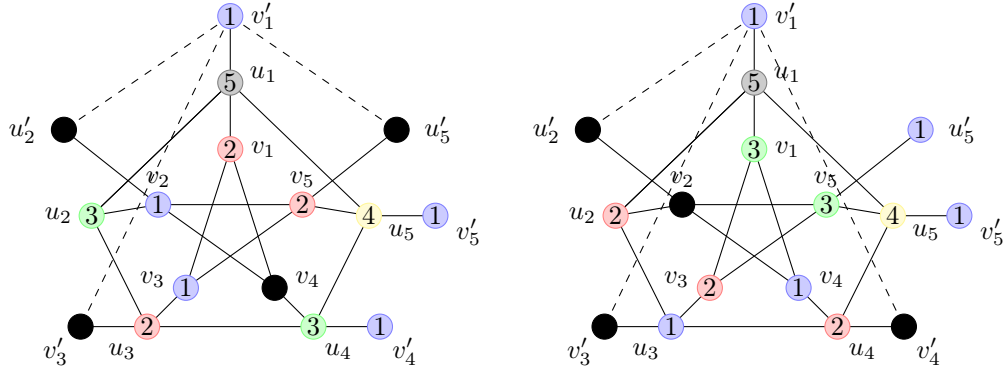


Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.

of vertices of  $G$  at distance 1 from  $C$  is not independent.

**Lemma 5.5.** *Let  $G$  be a 4-regular graph with girth  $g = 5$ . If  $G$  contains a neighbor-connected  $C_5$  as induced subgraph, then  $\Gamma(G) = 5$ .*

*Proof.* Let  $C$  be a neighbor-connected  $C_5$  in  $G$ . By Lemma 5.4 we can suppose that the neighbors of the vertices of  $C$  do not form an induced  $C_5$  (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of  $C$  form a subgraph of a  $C_{10}$ . If there are two edges between the neighbors of the vertices of  $C$ , then Figure 8 illustrates Grundy partial 5-colorings of the graph  $G$ . Suppose that two neighbors are adjacent, say  $v_1$  and  $v'_3$  and the graph  $G$  does not contain the previous configuration. Note that  $v'_3$  can be adjacent with  $v'_1$  and  $v'_5$ . Let  $w_1, w_2$  and  $w_3$  be the three neighbors of  $v_2$  different from  $u_2$ . We suppose that  $w_1$  can be possibly adjacent with  $v'_3$  and  $w_2$  can be possibly adjacent with  $v'_1$ . Figure 9 illustrates a Grundy partial 5-coloring of  $G$  in this case. In this figure, the vertex  $w_3$  can be possibly adjacent with  $v'_5$  or  $v_4$ , but in this case we can switch the color 1 from  $v'_5$  to  $v_5$  or from  $v'_4$  to  $v_4$ .  $\square$

**Proposition 5.6.** *If  $G$  is a 4-regular graph with girth  $g = 5$ , then  $\Gamma(G) = 5$ .*

*Proof.* Let  $C$  be a 5-cycle in  $G$ . Assume that two neighbors of consecutive vertices of  $C$ , for example  $v_1$  and  $v_5$ , have a common neighbor  $w_1$ . The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph  $G$ . In this figure the vertex  $w_1$  can be possibly adjacent with  $v'_2, v'_3$  or  $v_4$ , but in this case we can switch the color 1 from  $v'_2$  to  $v_2$ , from  $v'_3$  to  $v_3$  or from  $v_4$  to  $v'_4$ . Hence, we can suppose that no neighbors of consecutive vertices of  $C$  are adjacent. Among the neighbors of  $v_1$ , there exists one vertex  $w_1$  not adjacent with both  $v_4$  and  $v'_4$  (otherwise  $G$  would contain a  $C_4$ ). Among the neighbor of  $v'_5$ , there exists one

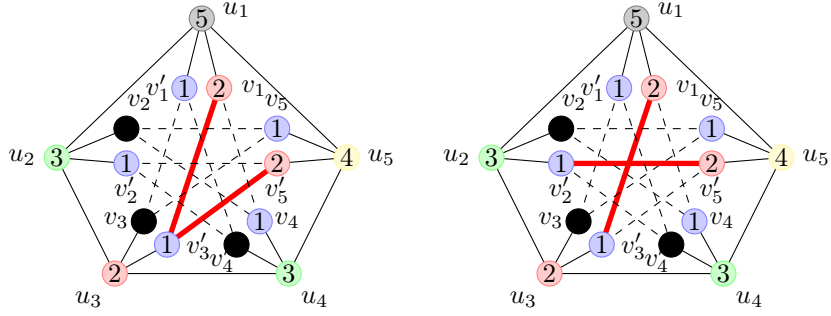


Figure 8: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected  $C_5$ .

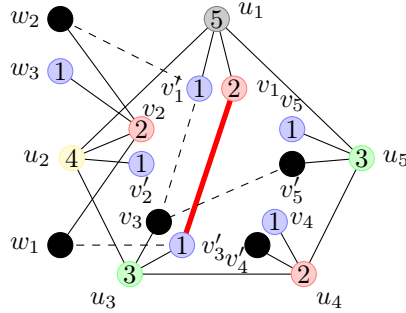


Figure 9: A Grundy partial 5-coloring of a subgraph containing an induced neighbor-connected  $C_5$ .

vertex, say  $w_2$ , not adjacent with  $w_1$ . The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph  $G$ . In this figure the vertex  $w_1$  can be possibly adjacent with  $v_4$  and the vertex  $w_2$  can be possibly adjacent with  $v'_2$  or  $v_4$ , but in these cases we can switch the color 1 from  $v'_2$  to  $v_2$  or from  $v_4$  to  $v'_4$ .  $\square$

In the following lemma and proposition, we consider a graph  $G$  of girth  $g = 6$ . Let  $u_1, u_2, u_3, u_4, u_5$  and  $u_6$  be the vertices in an induced  $C_6$ . Let  $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5, v'_5, v_6$  and  $v'_6$  be the remaining neighbors of respectively  $u_1, u_2, u_3, u_4, u_5$  and  $u_6$  (all different as  $g = 6$ ).

**Lemma 5.7.** *If  $G$  is a 4-regular graph with girth  $g = 6$  which contains a neighbor-connected  $C_6$  as induced subgraph, then  $\Gamma(G) = 5$ .*

*Proof.* Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let  $w_1$  be a neighbor of  $v'_1$  not adjacent with  $v_4$ . The graph  $G$  admits a Grundy partial 5-coloring as the



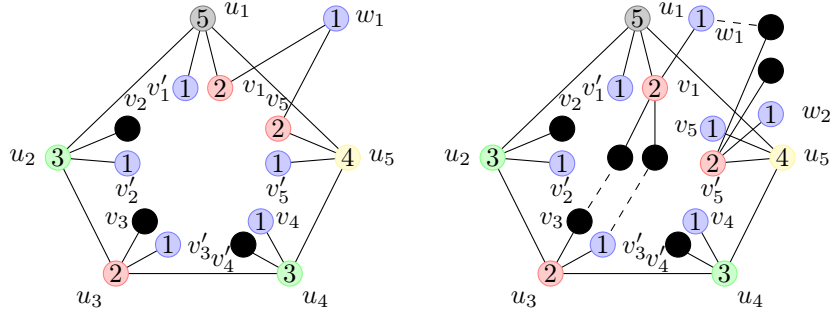


Figure 10: Two Grundy partial 5-colorings of a subgraph containing an induced  $C_5$ .

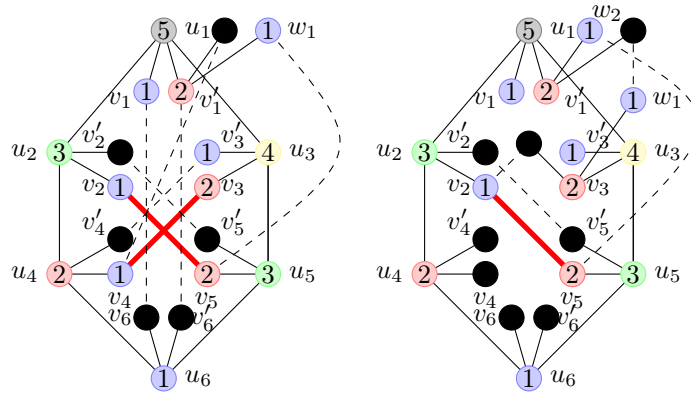


Figure 11: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected  $C_6$ .

left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let  $w_1$  be a neighbor of  $v_3$  not adjacent with  $v_2$  and let  $w_2$  be a neighbor of  $v'_1$  not adjacent with  $w_1$ . The graph  $G$  admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it.  $\square$

**Proposition 5.8.** *If  $G$  is a 4-regular graph with girth  $g = 6$ , then  $\Gamma(G) = 5$ .*

*Proof.* By Lemma 5.7, assume that no neighbors of the vertices of the induced  $C_6$  are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle  $C_6$ , for example  $v'_1$  and  $v_5$ , which have a common neighbor  $w_1$ . Let  $w_2$  be a neighbor of  $v_3$  not adjacent with  $w_1$ .  $G$  admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there

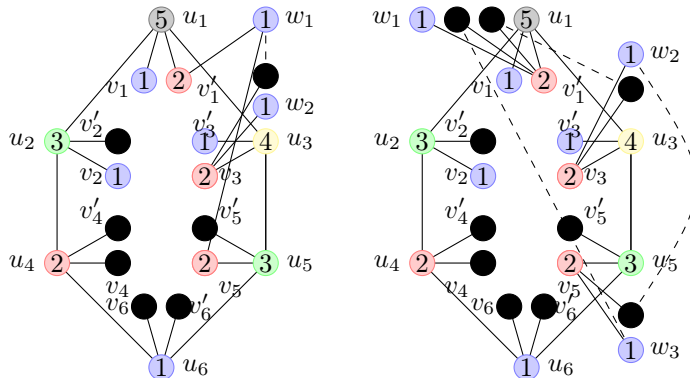


Figure 12: Two Grundy partial 5-colorings of a subgraph containing an induced  $C_6$ .

are no two neighbors at distance 4 along the cycle  $C_6$  which have a common neighbor. Let  $w_1$  be a neighbor of  $v'_1$  not adjacent with a neighbor of  $v_5$  or a neighbor of  $v_3$ , let  $w_2$  be a neighbor of  $v_3$  not adjacent with a neighbor of  $v_5$ , and let  $w_3$  be a neighbor of  $v_5$ . The graph  $G$  admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.  $\square$

**Proposition 5.9.** *If  $G$  is a 4-regular graph with girth  $g \geq 7$ , then  $\Gamma(G) = 5$ .*

*Proof.* Suppose that  $G$  contains a 7-cycle. We denote the 5-atom which is a tree by  $T_5$  (the binomial tree with maximum degree 4). It can be easily verified that  $G$  contains  $T_5$  where two leaves are merged (which is a 5-atom). Moreover, if  $G$  does not contain a 7-cycle, then it contains  $T_5$  as induced subgraph.  $\square$

**Theorem 3.** *Let  $G$  be a 4-regular graph. If  $G$  does not contain an induced  $C_4$ , then  $\Gamma(G) = 5$ .*

*Proof.* Suppose that  $G$  does not contain an induced  $C_4$ . Using Proposition 5.9 for the case  $g \geq 7$ , Propositions 5.6 and 5.8 for the case  $g = 5, 6$ , and Proposition 5.3 when  $G$  contains a  $C_3$  yields the desired result.  $\square$

By Proposition 3.1, Corollary 3.5 and Theorem 3, any  $r$ -regular graph with  $r \leq 4$  and without induced  $C_4$  has Grundy number  $r+1$ . Therefore, it is natural to propose Conjecture 1.

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