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On the family of r -regular graphs with Grundy number $r + 1$

Nicolas Gastineau^{*1,2}, Hamamache Kheddouci² and Olivier Togni¹

¹*LE2I, UMR CNRS 6303 , Université de Bourgogne, 21078 Dijon cedex, France*

²*LIRIS, UMR CNRS 5205 , Université Claude Bernard Lyon 1, Université de Lyon, F-69622 , France*

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Abstract

The Grundy Number of a graph G , denoted by $\Gamma(G)$, is the largest k such that there exists a partition of $V(G)$, into k independent sets V_1, \dots, V_k and every vertex in V_i is adjacent to at least one vertex in V_j , for every $j < i$. We consider the following question for which we give some partial answers: Does the equality $\Gamma(G) = r + 1$ hold for any r -regular graph without induced 4-cycle?

Keywords: Grundy number, Partial Grundy number, Regular graph, False twins.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G = (V, E)$, a *proper k -coloring* of G is a surjective mapping $c : V \rightarrow \{1, \dots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E$; the *color class* V_i is the set $\{u \in V | c(u) = i\}$ and a vertex v has color i if $v \in V_i$. A vertex v of color i is a *Grundy vertex* if v is adjacent to at least one vertex colored j , for every $j < i$. A *Grundy k -coloring* is a proper k -coloring such that every vertex is a Grundy vertex. A *partial Grundy k -coloring* is a proper k -coloring such that every color class contains a Grundy vertex. The *Grundy number* (*partial Grundy number*, respectively) of G denoted by $\Gamma(G)$ ($\partial\Gamma(G)$, respectively) is the largest k such that G admits a Grundy k -coloring (partial Grundy k -coloring, respectively).

Let $N(v) = \{u \in V(G) | uv \in E(G)\}$ be the neighborhood of v . A set X of vertices is an *independent module* if X is an independent set and all vertices

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in X have the same neighborhood. The vertices in an independent module of size 2 are called *false twins*. Let P_n , C_n , K_n and I_n be respectively, the path, cycle complete and empty graph of order n . The concepts of Grundy k -coloring and domination are connected. In a Grundy coloring, V_1 is a dominating set. Given a graph G and an ordering ϕ on $V(G)$ with $\phi = v_1, \dots, v_n$, the greedy algorithm assigns to v_i the minimum color that was not assigned in the set $\{v_1, \dots, v_{i-1}\} \cap N(v_i)$. Let $\Gamma_\phi(G)$ be the number of colors used by the greedy algorithm with the ordering ϕ on G . We have the following result [7]: $\Gamma(G) = \max_{\phi \in S_n} (\Gamma_\phi(G))$.

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a graph is a NP-complete problem, even for complement of bipartite graphs. However, for a fixed t , determining if a given graph has Grundy number at least t is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called t -atoms, such that any graph with Grundy number at least t contains a t -atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for Grundy number [15, 8], that there exist upper bounds for d -degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between products of graphs and Grundy number [6, 1, 4]. Recently, Havet and Sampaio [9] have proven that the problem of deciding if for a given graph G , $\Gamma(G) = \Delta + 1$ is NP-complete, even for bipartite graphs. Moreover, they have proven that the dual of Grundy k -coloring problem is in FPT, finding an algorithm in $O(2k^{2k} \cdot |E| + 2^{2k} k^{3k+5/2})$ time.

Notice that a Grundy k -coloring is a partial Grundy k -coloring, hence $\Gamma(G) \leq \partial\Gamma(G)$. Given a graph G and a positive integer k , the problem of determining if a partial Grundy k -coloring exists is NP-complete even for chordal graphs, but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the b -chromatic number, denoted by $\varphi(G)$, which is the largest k such that there exists a proper k -coloring and for every color class V_i , there exists a vertex adjacent to at least one vertex colored j , for every j , $j \neq i$. Notice that a b -coloring is a partial Grundy k -coloring, hence $\varphi(G) \leq \partial\Gamma(G)$. The b -chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: a characterization of the Grundy number of every cubic graph and a theorem: for $r \leq 4$, every r -regular graphs without induced C_4 has Grundy number $r + 1$. We conjecture that this assertion is also true for $r > 4$.

Section 2 gives general properties about the Grundy coloring. Section 3 contains the first main Theorem: a description of the cubic graphs with Grundy number at most 3. Section 4 gives properties about the Grundy number of r -regular graphs. The last section contains the second main theorem: 4-regular graphs without induced C_4 have Grundy number 5.

2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy k -coloring.

Definition 2.1. *Let G be a graph. A Grundy partial k -coloring is a Grundy k -coloring of a subset S of $V(G)$.*

Observation 2.2 ([6],[1]). *If G admits a Grundy partial k -coloring, then $\Gamma(G) \geq k$.*

This property has a consequence: for a graph G , with $\Gamma(G) = t$ and any Grundy partial t -coloring, there exist smallest subgraphs H of G such that $\Gamma(H) = t$. The family of t -atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of t -atoms is finite and the presence of a t -atom can be determined in polynomial time for a fixed t . The following definition is slightly different from Zaker's one, insisting more on the construction of every t -atom.

Definition 2.3 ([15]). *For any integer t , we define the family of t -atoms, denoted by \mathcal{A}_t , $t = 1, \dots$ by induction. Let the family \mathcal{A}_1 contain only K_1 . A graph G is in \mathcal{A}_{t+1} if there exists a graph G' in \mathcal{A}_t and an integer m , $m \leq |V(G')|$, such that G is composed of G' and an independent set I_m of order m , adding edges between G' and I_m such that every vertex in G' is connected to at least one vertex in I_m . Moreover a t -atom A is minimal, if there is no t -atom included in A other than itself.*

Theorem 1 ([15]). *For a given graph G , $\Gamma(G) \geq t$ if and only if G contains an induced minimal t -atom.*

We now present conditions related to the presence of modules that allow to upper-bound the Grundy number.

Proposition 2.4 ([1]). *Let G be a graph and X be an independent module. In every Grundy coloring of G , the vertices in X must have the same color.*

Definition 2.5. *Let G be an r -regular graph. A vertex v is a $(0, \ell)$ -twin-vertex if there exists an independent module of cardinality $r + 2 - \ell$ which contain v .*

Proposition 2.6. *Let G be an r -regular graph. A $(0, \ell)$ -twin-vertex can only have a color at most ℓ in every Grundy coloring of G .*

Proof. Let v be a $(0, \ell)$ -twin-vertex colored $\ell + 1$ in G . By Definition, v is in an independent module X of cardinality $r + 2 - \ell$ and by Proposition 2.4, every other vertex of X should be colored $\ell + 1$. There exists a neighbor u of v in $V(G - X)$ colored ℓ and at most $\ell - 2$ neighbors of u are in $V(G - X)$. Therefore, u cannot be colored ℓ . \square

Definition 2.7. *A vertex v of a graph G is a $(1, \ell)$ -twin-vertex if $N(v)$ can be partitioned into at least $\ell - 1$ independent modules.*

Proposition 2.8. *Let G be a graph. A $(1, \ell)$ -twin-vertex can only have a color at most ℓ in every Grundy coloring of G .*

Proof. The neighborhood of v can only have $\ell - 1$ different colors. Therefore, v can only have a color at most ℓ . \square

Definition 2.9. *A vertex v of a graph G is a $(2, \ell)$ -twin-vertex if $N(v)$ is independent and every vertex in $N(v)$ is a $(1, \ell)$ -twin-vertex.*

Proposition 2.10. *Let G be a graph. A $(2, \ell)$ -twin-vertex can only have a color at most ℓ in every Grundy coloring of G .*

Proof. Let v be a $(2, \ell)$ -twin-vertex in G . Every vertex in $N(v)$ is a $(1, \ell)$ -twin-vertex. Hence, in every Grundy coloring of G , the neighborhood of v can have a color at most $\ell - 1$, excluding the independent module which contains v . Therefore, in every Grundy coloring of G , v has a color at most ℓ . \square

Corollary 2.11. *Let G be a graph. If every vertex is a $(1, \ell)$ -twin-vertex or a $(2, \ell)$ -twin-vertex, then $\Gamma(G) \leq \ell$.*

Corollary 2.12. *Let G be a regular graph. If every vertex is a (i, ℓ) -twin-vertex, for some i , $0 \leq i \leq 2$, then $\Gamma(G) \leq \ell$.*

Proposition 2.13. *Let G be a graph. We have $\Gamma(G) \leq 2$ if and only if $G = K_{n,m}$ for some integers n and m .*

Proof. If $G = K_{n,m}$ for some integers n and m , then using Corollary 2.11, for $\ell = 2$, we have $\Gamma(G) \leq 2$ (also proven in [6]). If $G \neq K_{n,m}$ for some n and m , then either G contains a C_3 or there exists one vertex v , with $N(v)$ independent, which is not a $(1, 2)$ -twin-vertex. In the latter case, v has at least two non adjacent neighbors, u_1 and u_2 , with a non common neighbor w . Giving color 1 to w , color 2 to the vertex among u_1 and u_2 adjacent with w , 1 to the other one and 3 to v , we obtain a Grundy partial 3-coloring. \square

3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial k -colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures are not adjacent if it contradicts the hypothesis.

Proposition 3.1 ([6]). *For every connected 2-regular graph G , $\partial\Gamma(G) = \Gamma(G) = 2$ if and only if $G = C_4$.*

Definition 3.2. *Let r be an integer and \mathcal{F} be a disjoint union of connected graphs such that every vertex has degree at most r . The free ports are the vertices of degree at most $r - 1$. A graph G is obtained by matching and $K_{1,r}$ -joining from \mathcal{F} if two free ports can be connected by an edge (a matching) and r free ports can be connected to a new vertex ($K_{1,r}$ -joining) so that G be a r -regular graph.*

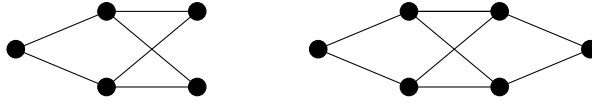


Figure 1: The graph $K_{2,3}$ (on the left) and $K_{3,3}^*$ (on the right).

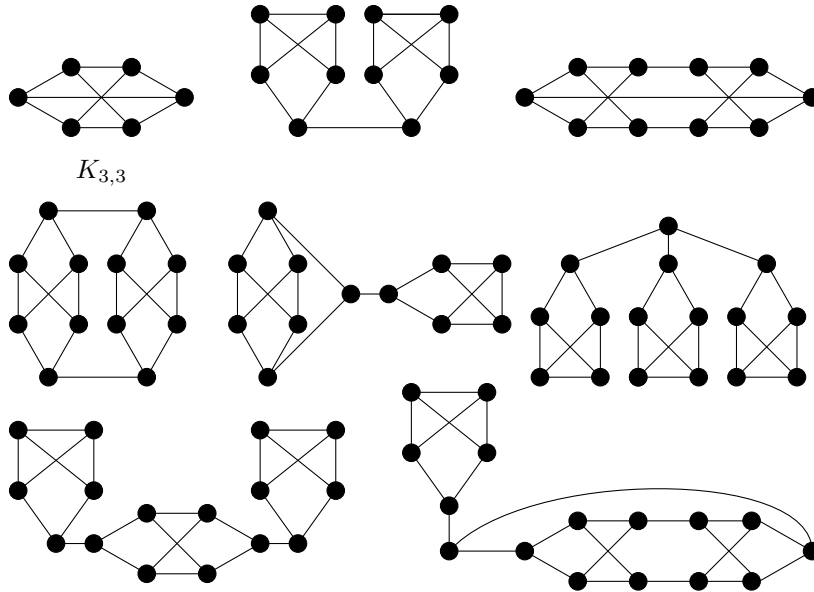


Figure 2: The cubic graphs such that $|V(G)| < 18$ and $\Gamma(G) < 4$.

The following proposition gives a construction of the cubic graphs in which every vertex is a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$. Figure 2 gives the list of such graphs of order at most 16.

Proposition 3.3. *Let G be a cubic graph and \mathcal{F} be a disjoint union of $K_{2,3}$ or $K_{3,3}^*$ from Figure 1. Every vertex of G is a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$, if and only if G can be obtained by matching and $K_{1,3}$ -joining from \mathcal{F} .*

Proof. Every graph G obtained by matching and $K_{1,3}$ -joining from \mathcal{F} has three kind of vertices: false twins, free ports and vertices added by $K_{1,3}$ -joining. False twins are $(0, 3)$ -twin-vertex and reciprocally. A free port is a $(1, 3)$ -twin-vertex and reciprocally. Vertices added by $K_{1,3}$ -joining are $(2, 3)$ -twin-vertices and reciprocally. \square

Theorem 2. *Let G be a cubic graph. $\Gamma(G) \leq 3$ if and only if every vertex is a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$.*

Proof. By Corollary 2.12, the "if" part is proven. Suppose that G contains a vertex v which is not a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$ and $\Gamma(G) < 4$. We want to have a contradiction in every configuration either with $\Gamma(G) < 4$, giving a Grundy partial 4-coloring, or proving that v is a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$. We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: v or a neighbor of v is in a C_3 , v is in an induced C_4 and v or a neighbor of v are not in a C_3 and v is not in an induced C_4 . Let C be a cycle of order 3 or 4 which contains v or a neighbor of v and let $D_1 = \{x \in V(G) | d(x, C) = 1\}$. In short, D_1 will also denote the subgraph of G induced by D_1 .

Case 1: suppose that v or a neighbor of v is in C and $C = C_3$. If $|D_1| = 1$, then $G = K_4$ and $\Gamma(K_4) = 4$. If $|D_1| = 2$ and $D_1 = P_2$, then v is a $(0, 3)$ -twin-vertex or a $(1, 3)$ -twin-vertex. If $D_1 = I_2$ then Figure 4.1.a gives a Grundy partial 4-coloring of G . If $|D_1| = 3$, then we have four subcases: D_1 is C_3 or P_3 (Figure 4.1.b), $P_2 \cup I_1$ (Figure 4.1.c) or I_3 (Figure 4.1.d) and in every case G admits a Grundy partial 4-coloring.

Case 2: suppose that v is in C and $C = C_4$. Notice that if two non adjacent vertices of C have a common neighbor in D_1 , then v is a $(0, 3)$ -twin-vertex or a $(1, 3)$ -twin-vertex. Hence, we will not consider these cases. If $|D_1| = 2$, then $D_1 = P_2$ or $D_1 = I_2$ (Figure 4.2.a) and in both cases G admits a Grundy partial 4-coloring. If $|D_1| = 3$, Figure 4.2.b gives a Grundy partial 4-coloring of G . In the case $|D_1| = 4$, we first suppose that two consecutive vertices of C have their neighbors in D_1 adjacent (Figure 4.2.c). Afterwards, we suppose that the previous case does not happen and two non consecutive vertices of C have their neighbors in D_1 adjacent (Figure 4.2.d). In the case $D_1 = I_4$, we first suppose that two vertices of D_1 which have two consecutive vertices of C as neighborhood, are not adjacent to two common vertices (Figure 4.2.e) and after consider they are (Figure 4.2.f).

Case 3: suppose that v or a neighbor of v is not in a C_3 and v is not in an induced C_4 . First, suppose that a neighbor u of v is in an induced C_4 . Using the coloring from the previous case, G admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of v in the C_4 have a common neighbor outside the C_4 . However, this case cannot happen for every neighbor of v , otherwise v would be a $(2, 3)$ -twin-vertex. Suppose that u is the neighbor of v not in the previous configuration. If u is in an induced C_4 , then using the coloring from the previous case, G admits a Grundy partial 4-coloring. If u is not in an induced C_4 , then Figure 4.3.a gives a Grundy partial 4-coloring of G . In this figure, the color 2 is given to a neighbor of u not adjacent to both f_1 and f_2 . Second, suppose that v is in an induced C_5 . Figure 4.3.b gives a Grundy partial 4-coloring of G . Third, if v is not in an induced C_5 , then Figure 4.3.c gives a Grundy partial 4-coloring of G .

Therefore, if $\Gamma(G) \leq 3$, then every vertex is a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$. \square

Observe that if all the free ports of a same $K_{2,3}$ are joined to a same vertex then we obtain $K_{3,3}$ which has Grundy number 2. By Proposition 2.13, in all

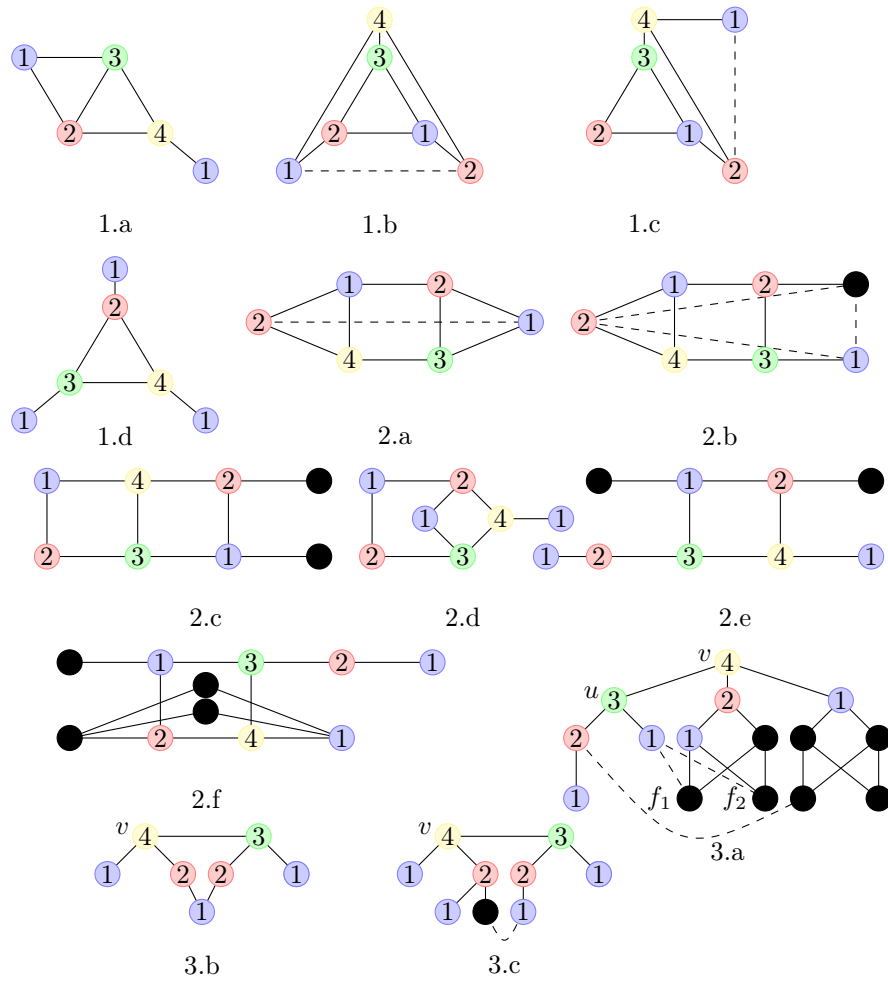


Figure 3: Possible configurations in a cubic graph (bold vertices: uncolored vertices, vertices with number i : vertices of color i).

the remaining cases, the cubic graphs which have Grundy number less than 4 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

Corollary 3.4. *A cubic graph G does not contain any induced minimal subcubic 4-atom if and only if every vertex is a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$.*

Corollary 3.5. *For every cubic graph G without induced C_4 , $\Gamma(G) = 4$.*

Proof. As every graph G with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K_{3,3}^*$, G always contains a square if $\Gamma(G) < 4$. \square

For a fixed t , the largest $(t + 1)$ -atom has order 2^t . Thus, for a graph G of maximum degree t , there exists an $O(n^{2^t})$ -time algorithm to determine if $\Gamma(G) < t+1$ (consisting to verify if the graph contains a $(t+1)$ -atom). For a cubic graph, that gives us an $O(n^8)$ -time algorithm, whereas our characterization yields a linear-time algorithm.

Observation 3.6. *Let G be a cubic graph of order n . There exists an $O(n)$ -time algorithm¹ to determine the Grundy number of G .*

Proof. Suppose we have a cubic graph G with its adjacency list. Testing if G is $K_{3,3}$ can be done in constant time. We suppose now that G is not $K_{3,3}$. For each vertex v , the algorithm consists to verify that v is a $(i, 3)$ -twin-vertex, for some i , $0 \leq i \leq 2$ (which can be checked in constant time). If the condition is true for all vertices, then $\Gamma(G) = 3$, else $\Gamma(G) = 4$. To determine if a vertex v is a $(0, 3)$ -twin-vertex, it suffices to verify that there is a common vertex other than v in the adjacency lists of the neighbors of v . To determine if a vertex v is a $(1, 3)$ -twin-vertex, it suffices to verify that there are two neighbors of v which have the same adjacency list. To determine if a vertex v is a $(2, 3)$ -twin-vertex, it suffices to verify that the neighborhood of v is independent and that every neighbor is a $(1, 3)$ -twin-vertex. \square

Proposition 3.7. *If G is a connected cubic graph and $G \neq K_{3,3}$, then $\partial\Gamma(G) = 4$.*

Proof. Notice that if $\Gamma(G) = 4$ then $\partial\Gamma(G) = 4$. Every graph G with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K_{3,3}^*$. If G contains more than two copies (is different from $K_{3,3}$), then a vertex can be colored 4 in a the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial\Gamma(G) = 4$. \square

Only $K_{3,3}$ and three other cubic graphs have b -chromatic number at most 3 [10]. Thus, our result is coherent with the results on the b -chromatic number. Shi et al. [13] proved that there exists a smallest integer N_r such that every r -regular graph G with more than N_r vertices has $\partial\Gamma(G) = \Delta + 1$. Notice that we have proven that $N_2 = 4$ and $N_3 = 6$. It is an open question to determine N_r for $r \geq 4$. However, using results on b -chromatic number [3], we have $N_r \leq 2r^3 - r^2 + r$.

4 Properties on the Grundy number of r -regular graphs

Proposition 4.1. *Let G be an r -regular graph and \mathcal{F} be a disjoint union of copies of $K_{r-k, k+2}$, for (possibly different) k , $0 \leq k \leq (r - 2)/2$. If G can be obtained by matching and $K_{1,r}$ -joining from \mathcal{F} , then $\Gamma(G) < r + 1$.*

¹Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.

Proof. By I_{r-k} and I_{k+2} , $|I_{r-k}| = r - k$ and $|I_{k+2}| = k + 2$, we denote the two sets of vertices in the bipartition of a $K_{r-k, k+2}$. First, suppose there exists a vertex u in a $K_{r-k, k+2}$ colored $r + 1$. Without loss of generality, suppose u is in I_{r-k} . The r neighbors of u should have colors from 1 to r . Among the neighbors of u , $k + 2$ are in I_{k+2} . Let v be the neighbor of u in I_{k+2} with the largest color in I_{k+2} . The vertex v has color at least $k + 2$. Hence, there exists an integer $s \geq 0$ such that the color of v is $k + 2 + s$. Notice that there are s vertices in $N(u) \setminus I_{k+2}$ which have colors at most $k + 2 + s$. The colors of the s vertices are the only one possible remaining colors at most $k + 2 + s$ in I_{r-k} . Hence, as there are k neighbors in $N(v) \setminus I_{r-k}$, the neighbors of v can only have at most $k + s$ different colors at most $k + 2 + s$. Therefore, we have a contradiction and u cannot have color $r + 1$. Second, suppose there exists a vertex u added by $K_{1, r}$ -joining which has color $r + 1$. As a neighbor of u in a $K_{r-k, k+2}$ should be colored r , the argument is totally similar to the previous one. \square

Corollary 4.2. *Let G be a 4-regular graph and \mathcal{F} be a disjoint union of $K_{3,3}$ and $K_{2,4}$. If G can be obtained by matching and $K_{1,4}$ -joining from \mathcal{F} , then $\Gamma(G) < 5$.*

The reader can believe that the family of 4-regular graphs with $\Gamma(G) < 5$ contains only the family of 4-regular graphs obtained by matching and $K_{1,4}$ -joining from a disjoint union of $K_{3,3}$ and $K_{2,4}$. However, there exist graphs with Grundy number r not inside this family. For example, the power graph C_7^2 (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle satisfies $\Gamma(C_7^2) < 5$ and is not inside this family.

The next result shows that unlike the b -chromatic number, r -regular graphs of order arbitrarily large with Grundy number k can be constructed for any r and any k , $3 \leq k \leq r + 1$.

Proposition 4.3. *Let $r \geq 4$ and $3 \leq k \leq r + 1$ be integers. There exists a infinite family \mathcal{F} of r -regular graphs such that $\forall G \in \mathcal{F}$, $\Gamma(G) = k$.*

Proof. Let i be a positive integer and r_1, \dots, r_{k-1} be a sequence of positive integers such that $r = r_1 + \dots + r_{k-1}$. We construct a graph $G_{r, k, i}$ as follows: Take $2i$ copies of $K_{r_1, \dots, r_{k-1}}$. Let H_{j-1} be the copy number j of $K_{r_1, \dots, r_{k-1}}$ and H_{j, r_l} be the independent r_l -set in H_j . If $j \equiv 1 \pmod{2}$, make the graph join of $H_{j \pmod{2i}, r_1}$ and $H_{j-1 \pmod{2i}, r_1}$ and for an integer l , $1 < l < k$, make the graph join of $H_{j \pmod{2i}, r_l}$ and $H_{j+1 \pmod{2i}, r_l}$. The r -regular graph obtained is the graph $G_{r, k, i}$. Figure 4 gives $G_{r, k, i}$, for $k = 4$ and $i \geq 2$. Notice that H_{j, r_i} is an independent module, hence every vertex is a $(0, k)$ -twin-vertex. Then, by Proposition 2.6, $\Gamma(G_{r, k, i}) \leq k$.

There exists a Grundy partial k -coloring of $G_{r, k, i}$ for $i \geq 2$. For an integer l , $1 < l < k$, color one vertex $l - 1$ in H_{1, r_l} and H_{2, r_l} . Afterwards, color one vertex $k - 1$ in H_{1, r_1} and one vertex k in H_{2, r_1} . The given coloring is a Grundy partial k -coloring of $G_{r, k, i}$ for $i \geq 2$. Therefore, $\Gamma(G_{r, k, i}) = k$, for $i \geq 2$. \square

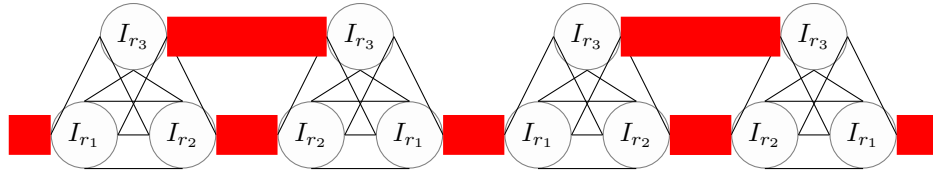


Figure 4: The Graph $G_{r,4,i}$, $i \geq 2$, $r = r_1 + r_2 + r_3$.

5 Grundy number of 4-regular graphs without induced C_4

The following Lemmas will be useful to prove our second main theorem: the family of 4-regular graphs without induced C_4 contains only graphs with Grundy number 5.

Lemma 5.1. *Let G be a 4-regular graph without induced C_4 . If G contains (an induced) K_4 then $\Gamma(G) = 5$.*

Proof. Notice that if $G = K_5$, we have $\Gamma(G) = 5$. If G is not K_5 then every two neighbors of vertices of K_4 cannot be adjacent (G would contain a C_4). Giving the color 1 to each neighbor of the vertices of K_4 and 2, 3, 4, 5 to the vertices of K_4 , we obtain a Grundy partial 5-coloring of G . \square

Lemma 5.2. *Let G be a 4-regular graph without induced C_4 and W be the graph from Figure 5. If G contains an induced W then $\Gamma(G) = 5$.*

Proof. The names of the vertices of W come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of G will be given using their references on Figure 5. Let D_1 be the set of vertices at distance 1 from vertices of W in $G - W$. Suppose that two vertices of W have a common neighbor in D_1 . It could only be u_4 and u_5 or u_3 and u_5 (or u_1 and u_4 , by symmetry). In the case u_4 and u_5 have a common neighbor, colors will be given to neighbors of u_3 in D_1 , either if they are adjacent (Figure 6.1.a) or not (Figure 6.1.b). In the case u_3 and u_5 have a common neighbor w in D_1 , w can be adjacent with a neighbor of u_3 in D_1 (Figure 6.2.a) or not (Figure 6.2.b). Suppose now, that no vertices in W have a common neighbor in D_1 . Let w_1 and w_2 be the neighbors of u_3 in D_1 . We first consider that w_1 and w_2 are adjacent (Figure 6.3.a). Second, we consider that w_1 and w_2 are not adjacent and that u_5 , u_3 and w_1 are in an induced C_5 (Figure 6.3.b). Last, we consider that the previous configurations are impossible (Figure 6.3.c). \square

Proposition 5.3. *Let G be a 4-regular graph without induced C_4 . If G contains C_3 then $\Gamma(G) = 5$.*

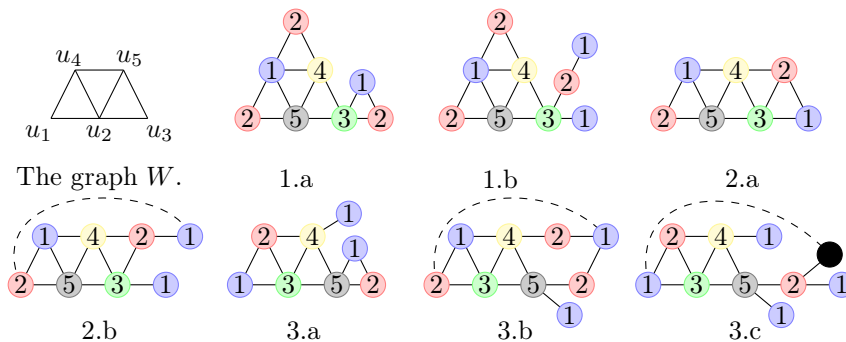


Figure 5: Possible configurations when G contains W .

Proof. Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of G in Figure 6 will be given. Let M_i , $i = 2$ or 3 , be the graph of order $2 + i$ containing two adjacent vertices u_1 and u_2 which have exactly i common neighbors, $\{v_1, \dots, v_i\}$, that form an independent set. Let D_1 be the set of vertices at distance 1 from an induced M_i in $G - M_i$, for $2 \leq i \leq 3$. First case, suppose that G contains an induced M_3 and a vertex of M_3 has its two neighbors in D_1 adjacent (Figure 7.1.a). Afterwards, suppose that G contains an induced M_2 and a vertex of M_2 has its two neighbors in D_1 adjacent (Figure 7.1.b). Notice that these Grundy partial 5-colorings use the fact that G cannot contain a K_4 by Lemma 5.1. Second case, suppose that G contains an induced M_3 excluding the previous configuration. At the beginning, we suppose that u_1, v_2 and v_3 are in an induced C_5 (Figure 7.2.a), after in an induced C_6 and not in an induced C_5 (Figure 7.2.b) and finally neither in an induced C_5 nor C_6 (Figure 7.2.c). Third case, suppose that G contains an induced M_2 excluding the previous configuration. At the beginning, we suppose that u_1, v_1 and v_2 are in an induced C_5 (Figure 7.3.a) and that u_1, v_1 are in an induced C_5 excluding the previous case (Figure 7.3.b). After we suppose that u_1, v_1 and v_2 are in an induced C_6 and not in an induced C_5 (Figure 7.3.c) and finally neither in an induced C_5 nor C_6 (Figure 6.3.d). Last case, suppose that G contains a 3-cycle C and no induced M_2 . Let u_1, u_2 and u_3 be the vertices of C . Let w_1 and w_2 be the neighbors of u_1 outside C , let w'_1 and w'_2 be the neighbors of u_2 outside C and w''_1 and w''_2 be the neighbors of u_3 outside C . Suppose that u_1, u_2, w_1 and w'_1 are in a 5-cycle and a neighbor of u_1 , say w_1 , has a common neighbor with w'_1 (Figure 7.4.a). Excluding the previous configuration, suppose that u_1, u_2, w_1 and w'_1 are in a 5-cycle; w''_1, v_1, u_1 and w_1 are in another 5-cycle and w_1 is in a triangle (Figure 7.4.b). Afterwards, we suppose that w_1 is not in a triangle (Figure 7.4.c). Finally, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of C are in a 5-cycle (Figure 7.4.d). Therefore, we suppose that two vertices of C cannot be in a 5-cycle (Figure 7.4.e). \square

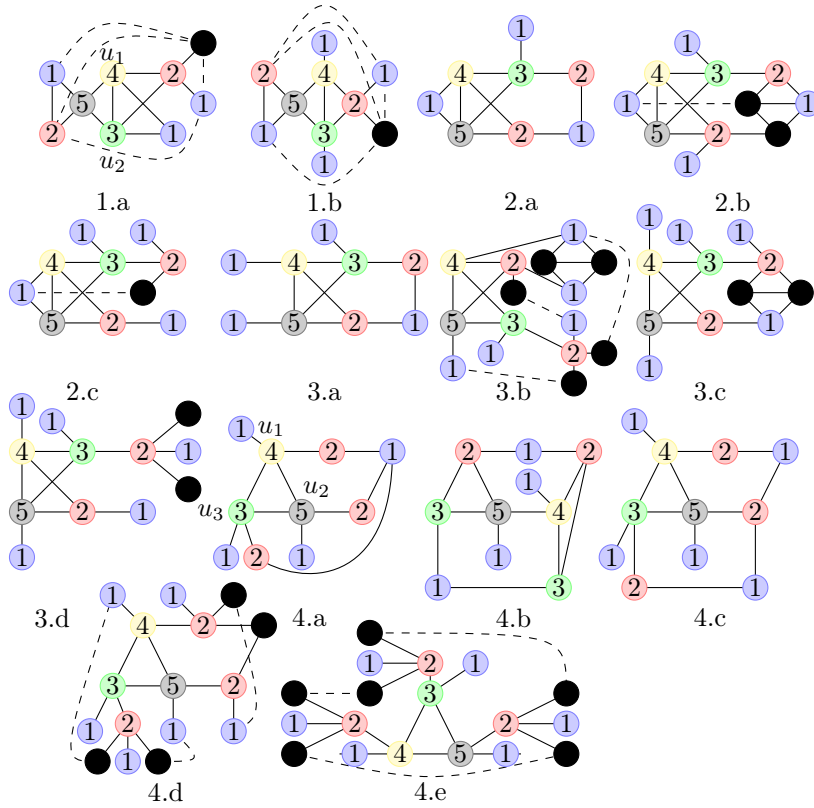


Figure 6: Possible configurations when G contains a C_3 .

In the following two Lemmas, we suppose that the graph G of girth $g = 5$ contains an induced C_5 and eventually an induced Petersen graph. Let u_1, u_2, u_3, u_4 and u_5 be the vertices in the C_5 (or in the outer cycle of the Petersen graph). Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$ and v'_5 be the remaining neighbors of respectively u_1, u_2, u_3, u_4 and u_5 (all different as $g = 5$).

Lemma 5.4. *Let G be a 4-regular graph with girth $g = 5$. If G contains the Petersen graph as induced subgraph then $\Gamma(G) = 5$.*

Proof. Suppose that v_1, v_2, v_3, v_4 and v_5 form an induced C_5 (the inner cycle of the Petersen graph). Let u'_2 and u'_5 be the remaining neighbors of respectively v_2 and v_5 . Observe that v'_1 can be adjacent with no more than three vertices among v'_3, v'_4, u'_2 and u'_5 . First, suppose that v'_1 is not adjacent with v'_3 (or v'_4 , without loss of generality since the graph is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph G . Second, suppose that v'_1 is not adjacent with u'_5 (or u'_2 , without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph G . \square

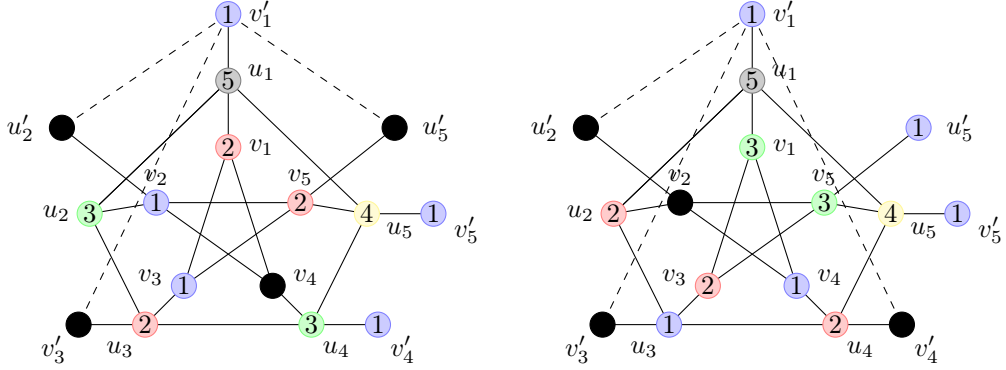


Figure 7: Two Grundy partial 5-colorings of a subgraph containing the Petersen graph as induced subgraph.

In a graph G , let a *neighbor-connected* C_n be an n -cycle C such that the set of vertices of G at distance 1 from C is not independent.

Lemma 5.5. *Let G be a 4-regular graph with girth $g = 5$. If G contains a neighbor-connected C_5 as induced subgraph, then $\Gamma(G) = 5$.*

Proof. Let C be a neighbor-connected C_5 in G . By Lemma 5.4 we can suppose that the neighbors of the vertices of C do not form an induced C_5 (otherwise a Petersen would be an induced subgraph). Hence, we can suppose that the neighbors of C form a subgraph of a C_{10} . If there are two edges between the neighbors of the vertices of C , then Figure 8 illustrates Grundy partial 5-colorings of the graph G . Suppose that two neighbors are adjacent, say v_1 and v'_3 and the graph G does not contain the previous configuration. Then v'_3 can be adjacent with v'_1 and v'_5 . Let w_1, w_2 and w_3 be the three neighbors of v_2 different from u_2 . We suppose that w_1 can be eventually adjacent with v'_3 and w_2 can be eventually adjacent with v'_1 . Figure 9 illustrates a Grundy partial 5-coloring of G in this case. In this figure, the vertex w_3 can be eventually adjacent with v'_5 or v_4 , but in this case we can switch the color 1 from v'_5 to v_5 or from v'_4 to v_4 . \square

Proposition 5.6. *If G is a 4-regular graph with girth $g = 5$, then $\Gamma(G) = 5$.*

Proof. Let C be a 5-cycle in G . Suppose that two consecutive neighbors, for example, v_1 and v_5 have a common neighbor w_1 . The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph G . In this figure the vertex w_1 can be eventually adjacent with v'_2, v'_3 or v_4 , but in this case we can switch the color 1 from v'_2 to v_2 , from v'_3 to v_3 or from v_4 to v'_4 . Hence, we can suppose that no neighbors of consecutive vertices of C are adjacent. Among the neighbors of v_1 , there exists one vertex w_1 not adjacent with both v_4 and v'_4

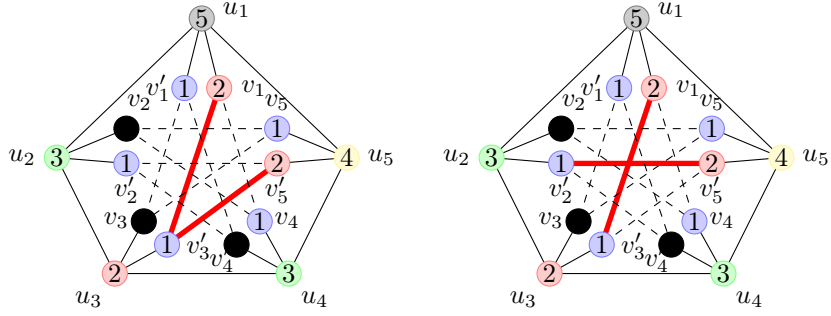


Figure 8: Two Grundy partial 5-colorings of a subgraph containing a neighbor-connected C_5 .

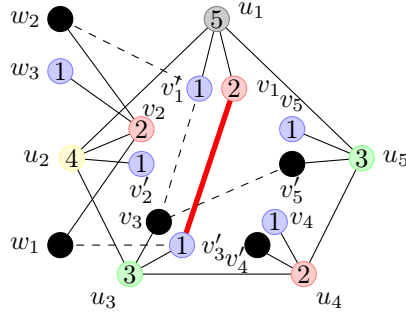


Figure 9: A Grundy partial 5-coloring of a subgraph containing a neighbor-connected C_5 .

(otherwise G would contain a C_4). Among the neighbor of v'_5 , there exists one vertex, say w_2 , not adjacent with w_1 . The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph G . In this figure the vertex w_1 can be eventually adjacent with v_4 and the vertex w_2 can be eventually adjacent with v'_2 or v_4 , but in these cases we can switch the color 1 from v'_2 to v_2 or from v_4 to v'_4 . \square

In the two following Lemmas, we suppose that a graph G contains an induced C_6 . Let u_1, u_2, u_3, u_4, u_5 and u_6 be the vertices in the C_6 . Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5, v'_5, v_6$ and v'_6 be the remaining neighbors of respectively u_1, u_2, u_3, u_4, u_5 and u_6 (all different as $g = 6$).

Lemma 5.7. *If G is a 4-regular graph with girth $g = 6$ which contains a neighbor-connected C_6 as induced subgraph, then $\Gamma(G) = 5$.*

Proof. First, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let w_1 be a neighbor of v'_1

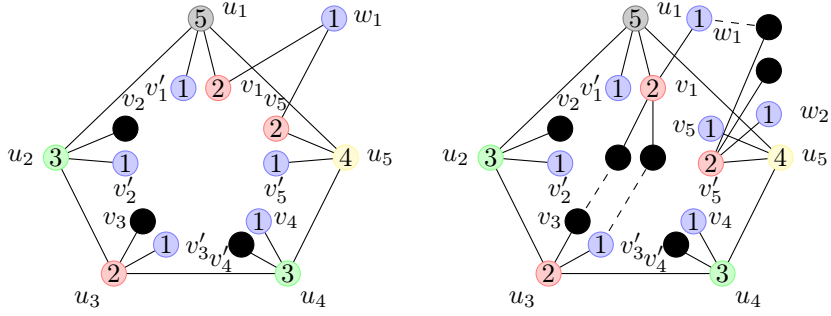


Figure 10: Two Grundy partial 5-colorings of a subgraph containing a C_5 .

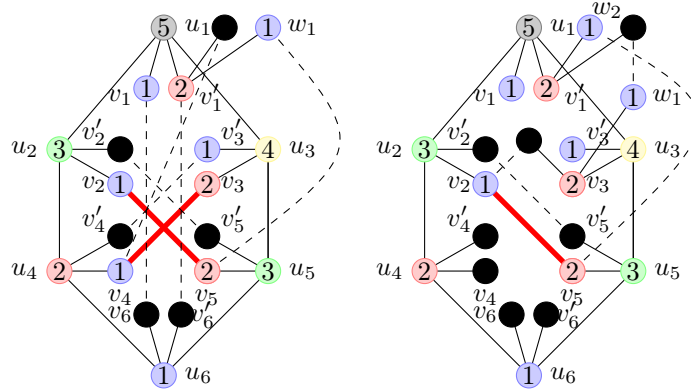


Figure 11: Two Grundy partial 5-colorings of a subgraph containing a neighbor-connected C_6 .

not adjacent with v_4 . The graph G admits a Grundy partial 5-coloring as the left part of Figure 11 illustrates it. Second, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let w_1 be a neighbor of v_3 not adjacent with v_2 and w_2 be a neighbor of v'_1 not adjacent with w_1 . The graph G admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it. \square

Proposition 5.8. *If G is a 4-regular graph with girth $g = 6$, then $\Gamma(G) = 5$.*

Proof. By Lemma 5.7, suppose that no neighbors of the C_6 are adjacent. First, suppose that there are two neighbors at distance 4 along the cycle C_6 , for example v'_1 and v_5 , which have a common neighbor w_1 . Let w_2 be a neighbor of v_3 not adjacent with w_1 . G admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Second, suppose that there are no two neighbors

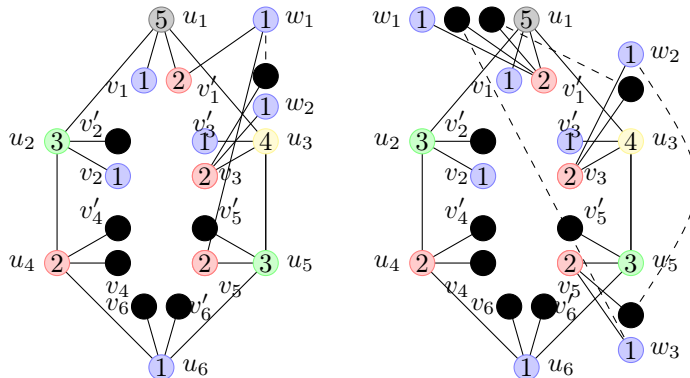


Figure 12: Two Grundy partial 5-colorings of a subgraph containing a C_6 .

at distance 4 along the cycle C_6 which have a common neighbor. Let w_1 be a neighbor of v'_1 not adjacent with a neighbor of v_5 or a neighbor of v_3 and w_2 be a neighbor of v_3 not adjacent with a neighbor of v_5 and w_3 be a neighbor of v_5 . The graph G admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it. \square

Proposition 5.9. *If G be a 4-regular graph with girth $g \geq 7$, then $\Gamma(G) = 5$.*

Proof. First, suppose that G contains a 7-cycle. By T_5 , we denote the 5-atom which is a tree (the binomial tree with maximum degree 4). It can be easily verified that G contains T_5 where two leaves are merged (which is a 5-atom). Moreover, if G does not contain a 7-cycle, then it contains T_5 as induced subgraph. \square

Theorem 3. *Let G be a 4-regular graph. If G does not contain an induced C_4 , then $\Gamma(G) = 5$.*

Proof. Suppose that G does not contain an induced C_4 . By Proposition 5.9 if $g \geq 7$, Propositions 5.6 and 5.8 if $g = 5, 6$, and Proposition 5.3 if G contains a C_3 , we have $\Gamma(G) = 5$. \square

By Proposition 3.1, Corollary 3.5 and Theorem 3, any r -regular graph with $r \leq 4$ and without induced C_4 has Grundy number $r + 1$. Therefore, it is natural to propose the following conjecture.

Conjecture 1. *For any $r \geq 1$, every r -regular graph without induced C_4 has Grundy number $r + 1$.*

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