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► **To cite this version:**

Nicolas Gastineau, Hamamache Kheddouci, Olivier Togni. On the family of  $r$ -regular graphs with Grundy number  $r + 1$ . 2013. hal-00922022v1

**HAL Id: hal-00922022**

**<https://hal.science/hal-00922022v1>**

Preprint submitted on 23 Dec 2013 (v1), last revised 19 May 2014 (v2)

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# On the family of $r$ -regular graphs with Grundy number $r + 1$

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December 23, 2013

## Abstract

The Grundy Number of a graph  $G$ , denoted by  $\Gamma(G)$ , is the largest  $k$  such that there exists a partition of  $V(G)$ , into  $k$  independent sets  $V_1, \dots, V_k$  and every vertex in  $V_i$  is adjacent to at least one vertex in  $V_j$ , for every  $j < i$ . We consider the following question for which we give some partial answers: Does the equality  $\Gamma(G) = r + 1$  hold for any  $r$ -regular graph without induced 4-cycle?

**Keywords:** Grundy number, Partial Grundy number, Regular graph, False twins.

## 1 Introduction

We consider only undirected connected graphs in this paper. Given a graph  $G = (V, E)$ , a *proper  $k$ -coloring* of  $G$  is a surjective mapping  $c : V \rightarrow \{1, \dots, k\}$  such that  $c(u) \neq c(v)$  for any  $uv \in E$ ; the *color class*  $V_i$  is the set  $\{u \in V | c(u) = i\}$  and a vertex  $v$  has color  $i$  if  $v \in V_i$ . A vertex  $v$  of color  $i$  is a *Grundy vertex* if  $v$  is adjacent to at least one vertex colored  $j$ , for every  $j < i$ . A *Grundy  $k$ -coloring* is a proper  $k$ -coloring such that every vertex is a Grundy vertex. A *partial Grundy  $k$ -coloring* is a proper  $k$ -coloring such that every color class contains a Grundy vertex. The *Grundy number* (*partial Grundy number*, respectively) of  $G$  denoted by  $\Gamma(G)$  ( $\partial\Gamma(G)$ , respectively) is the largest  $k$  such that  $G$  admits a Grundy  $k$ -coloring (*partial Grundy  $k$ -coloring*, respectively).

Let  $N(v) = \{u \in V(G) | uv \in E(G)\}$  be the neighborhood of  $v$ . A set  $X$  of vertices is an *independent module* if  $X$  is an independent set and all vertices

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\*Author partially supported by the Burgundy Council

in  $X$  have the same neighborhood. The vertices in an independent module of size 2 are called *false twins*. Let  $P_n$ ,  $C_n$ ,  $K_n$  and  $I_n$  be respectively, the path, cycle complete and empty graph of order  $n$ . The concepts of Grundy  $k$ -coloring and domination are connected. In a Grundy coloring,  $V_1$  is a dominating set. Given a graph  $G$  and an ordering  $\phi$  on  $V(G)$  with  $\phi = v_1, \dots, v_n$ , the greedy algorithm assigns to  $v_i$  the minimum color that was not assigned in the set  $\{v_1, \dots, v_{i-1}\} \cap N(v_i)$ . Let  $\Gamma_\phi(G)$  be the number of colors used by the greedy algorithm with the ordering  $\phi$  on  $G$ . We have the following result [7]:  $\Gamma(G) = \max_{\phi \in S_n} (\Gamma_\phi(G))$ .

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a graph is a NP-complete problem, even for complement of bipartite graphs. However, for a fixed  $t$ , determining if a given graph has Grundy number at least  $t$  is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called  $t$ -atoms, such that any graph with Grundy number at least  $t$  contains a  $t$ -atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for Grundy number [15, 8], that there exist upper bounds for  $d$ -degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between products of graphs and Grundy number [6, 1, 4]. Recently, Havet and Sampaio [9] have proven that the problem of deciding if for a given graph  $G$ ,  $\Gamma(G) = \Delta + 1$  is NP-complete, even for bipartite graphs. Moreover, they have proven that the dual of Grundy  $k$ -coloring problem is in FPT, finding an algorithm in  $O(2k^{2k} \cdot |E| + 2^{2k} k^{3k+5/2})$  time.

Notice that a Grundy  $k$ -coloring is a partial Grundy  $k$ -coloring, hence  $\Gamma(G) \leq \partial\Gamma(G)$ . Given a graph  $G$  and a positive integer  $k$ , the problem of determining if a partial Grundy  $k$ -coloring exists is NP-complete even for chordal graphs, but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the  $b$ -chromatic number, denoted by  $\varphi(G)$ , which is the largest  $k$  such that there exists a proper  $k$ -coloring and for every color class  $V_i$ , there exists a vertex adjacent to at least one vertex colored  $j$ , for every  $j$ ,  $j \neq i$ . Notice that a  $b$ -coloring is a partial Grundy  $k$ -coloring, hence  $\varphi(G) \leq \partial\Gamma(G)$ . The  $b$ -chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: a characterization of the Grundy number of every cubic graph and a theorem: for  $r \leq 4$ , every  $r$ -regular graphs without induced  $C_4$  has Grundy number  $r + 1$ . We conjecture that this assertion is also true for  $r > 4$ .

Section 2 gives general properties about the Grundy coloring. Section 3 contains the first main Theorem: a description of the cubic graphs with Grundy number at most 3. Section 4 gives properties about the Grundy number of  $r$ -regular graphs. The last section contains the second main theorem: 4-regular graphs without induced  $C_4$  have Grundy number 5.

## 2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy  $k$ -coloring.

**Definition 2.1.** *Let  $G$  be a graph. A Grundy partial  $k$ -coloring is a Grundy  $k$ -coloring of a subset  $S$  of  $V(G)$ .*

**Observation 2.2** ([6],[1]). *If  $G$  admits a Grundy partial  $k$ -coloring, then  $\Gamma(G) \geq k$ .*

This property has a consequence: for a graph  $G$ , with  $\Gamma(G) = t$  and any Grundy partial  $t$ -coloring, there exist smallest subgraphs  $H$  of  $G$  such that  $\Gamma(H) = t$ . The family of  $t$ -atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of  $t$ -atoms is finite and the presence of a  $t$ -atom can be determined in polynomial time for a fixed  $t$ . The following definition is slightly different from Zaker's one, insisting more on the construction of every  $t$ -atom.

**Definition 2.3** ([15]). *For any integer  $t$ , we define the family of  $t$ -atoms, denoted by  $\mathcal{A}_t$ ,  $t = 1, \dots$  by induction. Let the family  $\mathcal{A}_1$  contain only  $K_1$ . A graph  $G$  is in  $\mathcal{A}_{t+1}$  if there exists a graph  $G'$  in  $\mathcal{A}_t$  and an integer  $m$ ,  $m \leq |V(G')|$ , such that  $G$  is composed of  $G'$  and an independent set  $I_m$  of order  $m$ , adding edges between  $G'$  and  $I_m$  such that every vertex in  $G'$  is connected to at least one vertex in  $I_m$ . Moreover a  $t$ -atom  $A$  is minimal, if there is no  $t$ -atom included in  $A$  other than itself.*

**Theorem 1** ([15]). *For a given graph  $G$ ,  $\Gamma(G) \geq t$  if and only if  $G$  contains an induced minimal  $t$ -atom.*

We now present conditions related to the presence of modules that allow to upper-bound the Grundy number.

**Proposition 2.4** ([1]). *Let  $G$  be a graph and  $X$  be an independent module. In every Grundy coloring of  $G$ , the vertices in  $X$  must have the same color.*

**Definition 2.5.** *Let  $G$  be an  $r$ -regular graph. A vertex  $v$  is a  $(0, \ell)$ -twin-vertex if there exists an independent module of cardinality  $r + 2 - \ell$  which contain  $v$ .*

**Proposition 2.6.** *Let  $G$  be an  $r$ -regular graph. A  $(0, \ell)$ -twin-vertex can only have a color at most  $\ell$  in every Grundy coloring of  $G$ .*

*Proof.* Let  $v$  be a  $(0, \ell)$ -twin-vertex colored  $\ell + 1$  in  $G$ . By Definition,  $v$  is in an independent module  $X$  of cardinality  $r + 2 - \ell$  and by Proposition 2.4, every other vertex of  $X$  should be colored  $\ell + 1$ . There exists a neighbor  $u$  of  $v$  in  $V(G - X)$  colored  $\ell$  and at most  $\ell - 2$  neighbors of  $u$  are in  $V(G - X)$ . Therefore,  $u$  cannot be colored  $\ell$ .  $\square$

**Definition 2.7.** *A vertex  $v$  of a graph  $G$  is a  $(1, \ell)$ -twin-vertex if  $N(v)$  can be partitioned into at least  $\ell - 1$  independent modules.*

**Proposition 2.8.** *Let  $G$  be a graph. A  $(1, \ell)$ -twin-vertex can only have a color at most  $\ell$  in every Grundy coloring of  $G$ .*

*Proof.* The neighborhood of  $v$  can only have  $\ell - 1$  different colors. Therefore,  $v$  can only have a color at most  $\ell$ .  $\square$

**Definition 2.9.** *A vertex  $v$  of a graph  $G$  is a  $(2, \ell)$ -twin-vertex if  $N(v)$  is independent and every vertex in  $N(v)$  is a  $(1, \ell)$ -twin-vertex.*

**Proposition 2.10.** *Let  $G$  be a graph. A  $(2, \ell)$ -twin-vertex can only have a color at most  $\ell$  in every Grundy coloring of  $G$ .*

*Proof.* Let  $v$  be a  $(2, \ell)$ -twin-vertex in  $G$ . Every vertex in  $N(v)$  is a  $(1, \ell)$ -twin-vertex. Hence, in every Grundy coloring of  $G$ , the neighborhood of  $v$  can have a color at most  $\ell - 1$ , excluding the independent module which contains  $v$ . Therefore, in every Grundy coloring of  $G$ ,  $v$  has a color at most  $\ell$ .  $\square$

**Corollary 2.11.** *Let  $G$  be a graph. If every vertex is a  $(1, \ell)$ -twin-vertex or a  $(2, \ell)$ -twin-vertex, then  $\Gamma(G) \leq \ell$ .*

**Corollary 2.12.** *Let  $G$  be a regular graph. If every vertex is a  $(i, \ell)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ , then  $\Gamma(G) \leq \ell$ .*

**Proposition 2.13.** *Let  $G$  be a graph. We have  $\Gamma(G) \leq 2$  if and only if  $G = K_{n,m}$  for some integers  $n$  and  $m$ .*

*Proof.* If  $G = K_{n,m}$  for some integers  $n$  and  $m$ , then using Corollary 2.11, for  $\ell = 2$ , we have  $\Gamma(G) \leq 2$  (also proven in [6]). If  $G \neq K_{n,m}$  for some  $n$  and  $m$ , then either  $G$  contains a  $C_3$  or there exists one vertex  $v$ , with  $N(v)$  independent, which is not a  $(1, 2)$ -twin-vertex. In the latter case,  $v$  has at least two non adjacent neighbors,  $u_1$  and  $u_2$ , with a non common neighbor  $w$ . Giving color 1 to  $w$ , color 2 to the vertex among  $u_1$  and  $u_2$  adjacent with  $w$ , 1 to the other one and 3 to  $v$ , we obtain a Grundy partial 3-coloring.  $\square$

### 3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial  $k$ -colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures are not adjacent if it contradicts the hypothesis.

**Proposition 3.1** ([6]). *For every connected 2-regular graph  $G$ ,  $\partial\Gamma(G) = \Gamma(G) = 2$  if and only if  $G = C_4$ .*

**Definition 3.2.** *Let  $r$  be an integer and  $\mathcal{F}$  be a disjoint union of connected graphs such that every vertex has degree at most  $r$ . The free ports are the vertices of degree at most  $r - 1$ . A graph  $G$  is obtained by matching and  $K_{1,r}$ -joining from  $\mathcal{F}$  if two free ports can be connected by an edge (a matching) and  $r$  free ports can be connected to a new vertex ( $K_{1,r}$ -joining) so that  $G$  be a  $r$ -regular graph.*

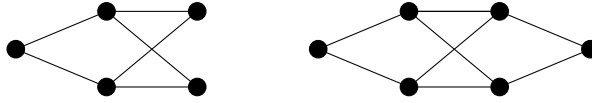


Figure 1: The graph  $K_{2,3}$  (on the left) and  $K_{3,3}^*$  (on the right).

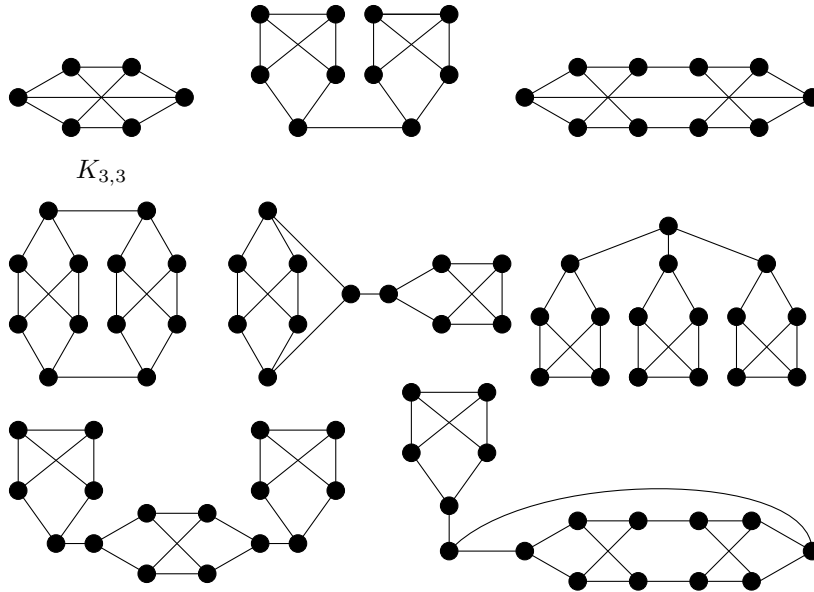


Figure 2: The cubic graphs such that  $|V(G)| < 18$  and  $\Gamma(G) < 4$ .

The following proposition gives a construction of the cubic graphs in which every vertex is a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ . Figure 2 gives the list of such graphs of order at most 16.

**Proposition 3.3.** *Let  $G$  be a cubic graph and  $\mathcal{F}$  be a disjoint union of  $K_{2,3}$  or  $K_{3,3}^*$  from Figure 1. Every vertex of  $G$  is a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ , if and only if  $G$  can be obtained by matching and  $K_{1,3}$ -joining from  $\mathcal{F}$ .*

*Proof.* Every graph  $G$  obtained by matching and  $K_{1,3}$ -joining from  $\mathcal{F}$  has three kind of vertices: false twins, free ports and vertices added by  $K_{1,3}$ -joining. False twins are  $(0, 3)$ -twin-vertex and reciprocally. A free port is a  $(1, 3)$ -twin-vertex and reciprocally. Vertices added by  $K_{1,3}$ -joining are  $(2, 3)$ -twin-vertices and reciprocally.  $\square$

**Theorem 2.** *Let  $G$  be a cubic graph.  $\Gamma(G) \leq 3$  if and only if every vertex is a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ .*

*Proof.* By Corollary 2.12, the "if" part is proven. Suppose that  $G$  contains a vertex  $v$  which is not a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$  and  $\Gamma(G) < 4$ . We want to have a contradiction in every configuration either with  $\Gamma(G) < 4$ , giving a Grundy partial 4-coloring, or proving that  $v$  is a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ . We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases:  $v$  or a neighbor of  $v$  is in a  $C_3$ ,  $v$  is in an induced  $C_4$  and  $v$  or a neighbor of  $v$  are not in a  $C_3$  and  $v$  is not in an induced  $C_4$ . Let  $C$  be a cycle of order 3 or 4 which contains  $v$  or a neighbor of  $v$  and let  $D_1 = \{x \in V(G) | d(x, C) = 1\}$ . In short,  $D_1$  will also denote the subgraph of  $G$  induced by  $D_1$ .

Case 1: suppose that  $v$  or a neighbor of  $v$  is in  $C$  and  $C = C_3$ . If  $|D_1| = 1$ , then  $G = K_4$  and  $\Gamma(K_4) = 4$ . If  $|D_1| = 2$  and  $D_1 = P_2$ , then  $v$  is a  $(0, 3)$ -twin-vertex or a  $(1, 3)$ -twin-vertex. If  $D_1 = I_2$  then Figure 4.1.a gives a Grundy partial 4-coloring of  $G$ . If  $|D_1| = 3$ , then we have four subcases:  $D_1$  is  $C_3$  or  $P_3$  (Figure 4.1.b),  $P_2 \cup I_1$  (Figure 4.1.c) or  $I_3$  (Figure 4.1.d) and in every case  $G$  admits a Grundy partial 4-coloring.

Case 2: suppose that  $v$  is in  $C$  and  $C = C_4$ . Notice that if two non adjacent vertices of  $C$  have a common neighbor in  $D_1$ , then  $v$  is a  $(0, 3)$ -twin-vertex or a  $(1, 3)$ -twin-vertex. Hence, we will not consider these cases. If  $|D_1| = 2$ , then  $D_1 = P_2$  or  $D_1 = I_2$  (Figure 4.2.a) and in both cases  $G$  admits a Grundy partial 4-coloring. If  $|D_1| = 3$ , Figure 4.2.b gives a Grundy partial 4-coloring of  $G$ . In the case  $|D_1| = 4$ , we first suppose that two consecutive vertices of  $C$  have their neighbors in  $D_1$  adjacent (Figure 4.2.c). Afterwards, we suppose that the previous case does not happen and two non consecutive vertices of  $C$  have their neighbors in  $D_1$  adjacent (Figure 4.2.d). In the case  $D_1 = I_4$ , we first suppose that two vertices of  $D_1$  which have two consecutive vertices of  $C$  as neighborhood, are not adjacent to two common vertices (Figure 4.2.e) and after consider they are (Figure 4.2.f).

Case 3: suppose that  $v$  or a neighbor of  $v$  is not in a  $C_3$  and  $v$  is not in an induced  $C_4$ . First, suppose that a neighbor  $u$  of  $v$  is in an induced  $C_4$ . Using the coloring from the previous case,  $G$  admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of  $v$  in the  $C_4$  have a common neighbor outside the  $C_4$ . However, this case cannot happen for every neighbor of  $v$ , otherwise  $v$  would be a  $(2, 3)$ -twin-vertex. Suppose that  $u$  is the neighbor of  $v$  not in the previous configuration. If  $u$  is in an induced  $C_4$ , then using the coloring from the previous case,  $G$  admits a Grundy partial 4-coloring. If  $u$  is not in an induced  $C_4$ , then Figure 4.3.a gives a Grundy partial 4-coloring of  $G$ . In this figure, the color 2 is given to a neighbor of  $u$  not adjacent to both  $f_1$  and  $f_2$ . Second, suppose that  $v$  is in an induced  $C_5$ . Figure 4.3.b gives a Grundy partial 4-coloring of  $G$ . Third, if  $v$  is not in an induced  $C_5$ , then Figure 4.3.c gives a Grundy partial 4-coloring of  $G$ .

Therefore, if  $\Gamma(G) \leq 3$ , then every vertex is a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ .  $\square$

Observe that if all the free ports of a same  $K_{2,3}$  are joined to a same vertex then we obtain  $K_{3,3}$  which has Grundy number 2. By Proposition 2.13, in all

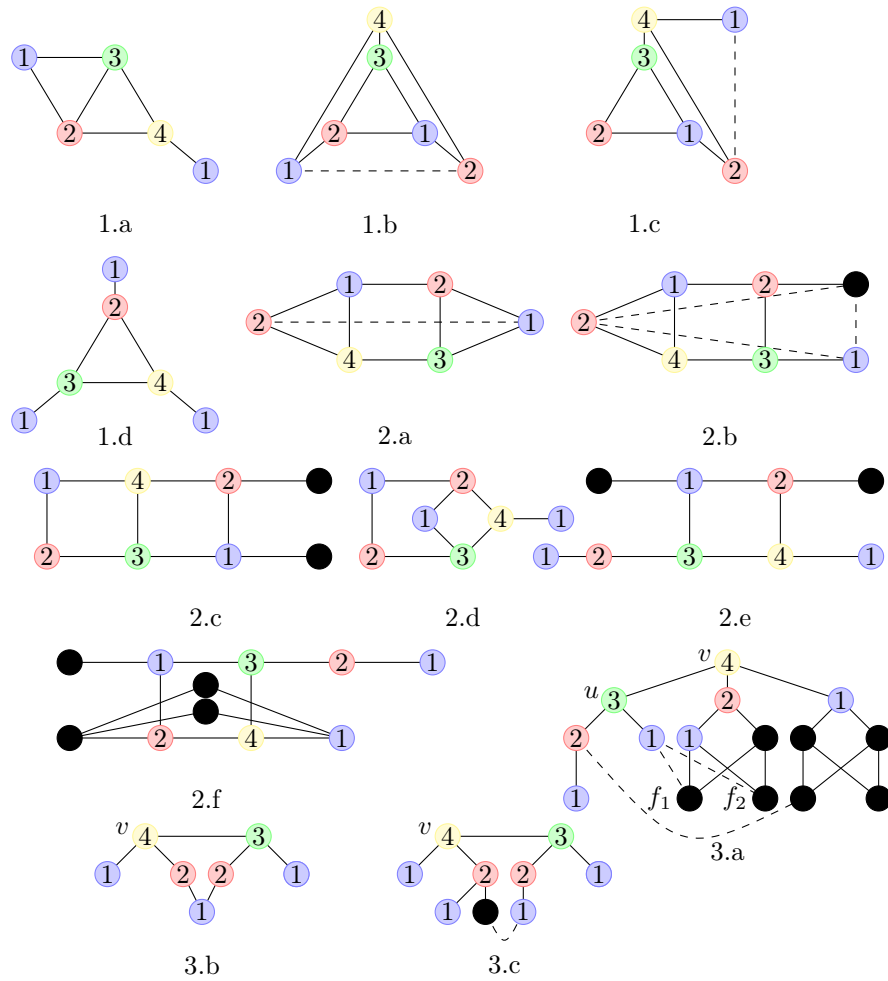


Figure 3: Possible configurations in a cubic graph (bold vertices: uncolored vertices, vertices with number  $i$ : vertices of color  $i$ ).

the remaining cases, the cubic graphs which have Grundy number less than 4 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

**Corollary 3.4.** *A cubic graph  $G$  does not contain any induced minimal subcubic 4-atom if and only if every vertex is a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$ .*

**Corollary 3.5.** *For every cubic graph  $G$  without induced  $C_4$ ,  $\Gamma(G) = 4$ .*

*Proof.* As every graph  $G$  with  $\Gamma(G) < 4$  is composed of copies of  $K_{2,3}$  or  $K_{3,3}^*$ ,  $G$  always contains a square if  $\Gamma(G) < 4$ .  $\square$



For a fixed  $t$ , the largest  $(t + 1)$ -atom has order  $2^t$ . Thus, for a graph  $G$  of maximum degree  $t$ , there exists an  $O(n^{2^t})$ -time algorithm to determine if  $\Gamma(G) < t+1$  (consisting to verify if the graph contains a  $(t+1)$ -atom). For a cubic graph, that gives us an  $O(n^8)$ -time algorithm, whereas our characterization yields a linear-time algorithm.

**Observation 3.6.** *Let  $G$  be a cubic graph of order  $n$ . There exists an  $O(n)$ -time algorithm<sup>1</sup> to determine the Grundy number of  $G$ .*

*Proof.* Suppose we have a cubic graph  $G$  with its adjacency list. Testing if  $G$  is  $K_{3,3}$  can be done in constant time. We suppose now that  $G$  is not  $K_{3,3}$ . For each vertex  $v$ , the algorithm consists to verify that  $v$  is a  $(i, 3)$ -twin-vertex, for some  $i$ ,  $0 \leq i \leq 2$  (which can be checked in constant time). If the condition is true for all vertices, then  $\Gamma(G) = 3$ , else  $\Gamma(G) = 4$ . To determine if a vertex  $v$  is a  $(0, 3)$ -twin-vertex, it suffices to verify that there is a common vertex other than  $v$  in the adjacency lists of the neighbors of  $v$ . To determine if a vertex  $v$  is a  $(1, 3)$ -twin-vertex, it suffices to verify that there are two neighbors of  $v$  which have the same adjacency list. To determine if a vertex  $v$  is a  $(2, 3)$ -twin-vertex, it suffices to verify that the neighborhood of  $v$  is independent and that every neighbor is a  $(1, 3)$ -twin-vertex.  $\square$

**Proposition 3.7.** *If  $G$  is a connected cubic graph and  $G \neq K_{3,3}$ , then  $\partial\Gamma(G) = 4$ .*

*Proof.* Notice that if  $\Gamma(G) = 4$  then  $\partial\Gamma(G) = 4$ . Every graph  $G$  with  $\Gamma(G) < 4$  is composed of copies of  $K_{2,3}$  or  $K_{3,3}^*$ . If  $G$  contains more than two copies (is different from  $K_{3,3}$ ), then a vertex can be colored 4 in a the first copy and a vertex can be colored 3 in the second copy. Hence,  $\partial\Gamma(G) = 4$ .  $\square$

Only  $K_{3,3}$  and three other cubic graphs have  $b$ -chromatic number at most 3 [10]. Thus, our result is coherent with the results on the  $b$ -chromatic number. Shi et al. [13] proved that there exists a smallest integer  $N_r$  such that every  $r$ -regular graph  $G$  with more than  $N_r$  vertices has  $\partial\Gamma(G) = \Delta + 1$ . Notice that we have proven that  $N_2 = 4$  and  $N_3 = 6$ . It is an open question to determine  $N_r$  for  $r \geq 4$ . However, using results on  $b$ -chromatic number [3], we have  $N_r \leq 2r^3 - r^2 + r$ .

## 4 Properties on the Grundy number of $r$ -regular graphs

**Proposition 4.1.** *Let  $G$  be an  $r$ -regular graph and  $\mathcal{F}$  be a disjoint union of copies of  $K_{r-k, k+2}$ , for (possibly different)  $k$ ,  $0 \leq k \leq (r - 2)/2$ . If  $G$  can be obtained by matching and  $K_{1,r}$ -joining from  $\mathcal{F}$ , then  $\Gamma(G) < r + 1$ .*

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<sup>1</sup>Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.

*Proof.* By  $I_{r-k}$  and  $I_{k+2}$ ,  $|I_{r-k}| = r - k$  and  $|I_{k+2}| = k + 2$ , we denote the two sets of vertices in the bipartition of a  $K_{r-k, k+2}$ . First, suppose there exists a vertex  $u$  in a  $K_{r-k, k+2}$  colored  $r + 1$ . Without loss of generality, suppose  $u$  is in  $I_{r-k}$ . The  $r$  neighbors of  $u$  should have colors from 1 to  $r$ . Among the neighbors of  $u$ ,  $k + 2$  are in  $I_{k+2}$ . Let  $v$  be the neighbor of  $u$  in  $I_{k+2}$  with the largest color in  $I_{k+2}$ . The vertex  $v$  has color at least  $k + 2$ . Hence, there exists an integer  $s \geq 0$  such that the color of  $v$  is  $k + 2 + s$ . Notice that there are  $s$  vertices in  $N(u) \setminus I_{k+2}$  which have colors at most  $k + 2 + s$ . The colors of the  $s$  vertices are the only one possible remaining colors at most  $k + 2 + s$  in  $I_{r-k}$ . Hence, as there are  $k$  neighbors in  $N(v) \setminus I_{r-k}$ , the neighbors of  $v$  can only have at most  $k + s$  different colors at most  $k + 2 + s$ . Therefore, we have a contradiction and  $u$  cannot have color  $r + 1$ . Second, suppose there exists a vertex  $u$  added by  $K_{1, r}$ -joining which has color  $r + 1$ . As a neighbor of  $u$  in a  $K_{r-k, k+2}$  should be colored  $r$ , the argument is totally similar to the previous one.  $\square$

**Corollary 4.2.** *Let  $G$  be a 4-regular graph and  $\mathcal{F}$  be a disjoint union of  $K_{3,3}$  and  $K_{2,4}$ . If  $G$  can be obtained by matching and  $K_{1,4}$ -joining from  $\mathcal{F}$ , then  $\Gamma(G) < 5$ .*

The reader can believe that the family of 4-regular graphs with  $\Gamma(G) < 5$  contains only the family of 4-regular graphs obtained by matching and  $K_{1,4}$ -joining from a disjoint union of  $K_{3,3}$  and  $K_{2,4}$ . However, there exist graphs with Grundy number  $r$  not inside this family. For example, the power graph  $C_7^2$  (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle satisfies  $\Gamma(C_7^2) < 5$  and is not inside this family.

The next result shows that unlike the  $b$ -chromatic number,  $r$ -regular graphs of order arbitrarily large with Grundy number  $k$  can be constructed for any  $r$  and any  $k$ ,  $3 \leq k \leq r + 1$ .

**Proposition 4.3.** *Let  $r \geq 4$  and  $3 \leq k \leq r + 1$  be integers. There exists a infinite family  $\mathcal{F}$  of  $r$ -regular graphs such that  $\forall G \in \mathcal{F}$ ,  $\Gamma(G) = k$ .*

*Proof.* Let  $i$  be a positive integer and  $r_1, \dots, r_{k-1}$  be a sequence of positive integers such that  $r = r_1 + \dots + r_{k-1}$ . We construct a graph  $G_{r, k, i}$  as follows: Take  $2i$  copies of  $K_{r_1, \dots, r_{k-1}}$ . Let  $H_{j-1}$  be the copy number  $j$  of  $K_{r_1, \dots, r_{k-1}}$  and  $H_{j, r_l}$  be the independent  $r_l$ -set in  $H_j$ . If  $j \equiv 1 \pmod{2}$ , make the graph join of  $H_{j \pmod{2i}, r_1}$  and  $H_{j-1 \pmod{2i}, r_1}$  and for an integer  $l$ ,  $1 < l < k$ , make the graph join of  $H_{j \pmod{2i}, r_l}$  and  $H_{j+1 \pmod{2i}, r_l}$ . The  $r$ -regular graph obtained is the graph  $G_{r, k, i}$ . Figure 4 gives  $G_{r, k, i}$ , for  $k = 4$  and  $i \geq 2$ . Notice that  $H_{j, r_i}$  is an independent module, hence every vertex is a  $(0, k)$ -twin-vertex. Then, by Proposition 2.6,  $\Gamma(G_{r, k, i}) \leq k$ .

There exists a Grundy partial  $k$ -coloring of  $G_{r, k, i}$  for  $i \geq 2$ . For an integer  $l$ ,  $1 < l < k$ , color one vertex  $l - 1$  in  $H_{1, r_l}$  and  $H_{2, r_l}$ . Afterwards, color one vertex  $k - 1$  in  $H_{1, r_1}$  and one vertex  $k$  in  $H_{2, r_1}$ . The given coloring is a Grundy partial  $k$ -coloring of  $G_{r, k, i}$  for  $i \geq 2$ . Therefore,  $\Gamma(G_{r, k, i}) = k$ , for  $i \geq 2$ .  $\square$

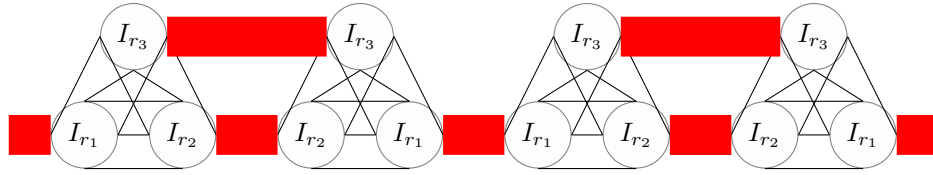


Figure 4: The Graph  $G_{r,4,i}$ ,  $i \geq 2$ ,  $r = r_1 + r_2 + r_3$ .

## 5 Grundy number of 4-regular graphs without induced $C_4$

The following Lemmas will be useful to prove our second main theorem: the family of 4-regular graphs without induced  $C_4$  contains only graphs with Grundy number 5.

**Lemma 5.1.** *Let  $G$  be a 4-regular graph without induced  $C_4$ . If  $G$  contains (an induced)  $K_4$  then  $\Gamma(G) = 5$ .*

*Proof.* Notice that if  $G = K_5$ , we have  $\Gamma(G) = 5$ . If  $G$  is not  $K_5$  then every two neighbors of vertices of  $K_4$  cannot be adjacent ( $G$  would contain a  $C_4$ ). Giving the color 1 to each neighbor of the vertices of  $K_4$  and 2, 3, 4, 5 to the vertices of  $K_4$ , we obtain a Grundy partial 5-coloring of  $G$ .  $\square$

**Lemma 5.2.** *Let  $G$  be a 4-regular graph without induced  $C_4$  and  $W$  be the graph from Figure 5. If  $G$  contains an induced  $W$  then  $\Gamma(G) = 5$ .*

*Proof.* The names of the vertices of  $W$  come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of  $G$  will be given using their references on Figure 5. Let  $D_1$  be the set of vertices at distance 1 from vertices of  $W$  in  $G - W$ . Suppose that two vertices of  $W$  have a common neighbor in  $D_1$ . It could only be  $u_4$  and  $u_5$  or  $u_3$  and  $u_5$  (or  $u_1$  and  $u_4$ , by symmetry). In the case  $u_4$  and  $u_5$  have a common neighbor, colors will be given to neighbors of  $u_3$  in  $D_1$ , either if they are adjacent (Figure 6.1.a) or not (Figure 6.1.b). In the case  $u_3$  and  $u_5$  have a common neighbor  $w$  in  $D_1$ ,  $w$  can be adjacent with a neighbor of  $u_3$  in  $D_1$  (Figure 6.2.a) or not (Figure 6.2.b). Suppose now, that no vertices in  $W$  have a common neighbor in  $D_1$ . Let  $w_1$  and  $w_2$  be the neighbors of  $u_3$  in  $D_1$ . We first consider that  $w_1$  and  $w_2$  are adjacent (Figure 6.3.a). Second, we consider that  $w_1$  and  $w_2$  are not adjacent and that  $u_5$ ,  $u_3$  and  $w_1$  are in an induced  $C_5$  (Figure 6.3.b). Last, we consider that the previous configurations are impossible (Figure 6.3.c).  $\square$

**Proposition 5.3.** *Let  $G$  be a 4-regular graph without induced  $C_4$ . If  $G$  contains  $C_3$  then  $\Gamma(G) = 5$ .*

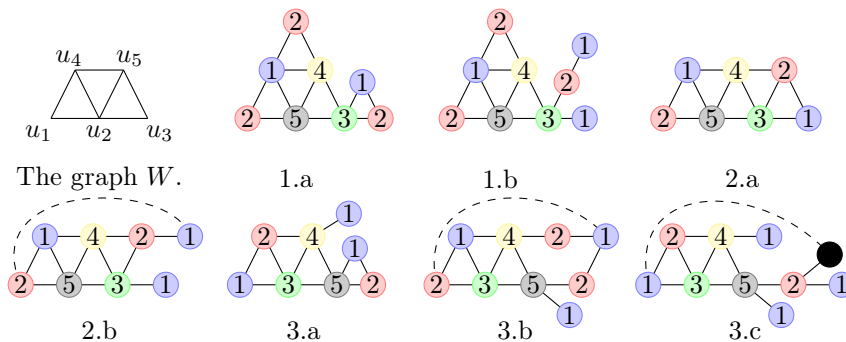


Figure 5: Possible configurations when  $G$  contains  $W$ .

*Proof.* Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of  $G$  in Figure 6 will be given. Let  $M_i$ ,  $i = 2$  or  $3$ , be the graph of order  $2 + i$  containing two adjacent vertices  $u_1$  and  $u_2$  which have exactly  $i$  common neighbors,  $\{v_1, \dots, v_i\}$ , that form an independent set. Let  $D_1$  be the set of vertices at distance 1 from an induced  $M_i$  in  $G - M_i$ , for  $2 \leq i \leq 3$ . First case, suppose that  $G$  contains an induced  $M_3$  and a vertex of  $M_3$  has its two neighbors in  $D_1$  adjacent (Figure 7.1.a). Afterwards, suppose that  $G$  contains an induced  $M_2$  and a vertex of  $M_2$  has its two neighbors in  $D_1$  adjacent (Figure 7.1.b). Notice that these Grundy partial 5-colorings use the fact that  $G$  cannot contain a  $K_4$  by Lemma 5.1. Second case, suppose that  $G$  contains an induced  $M_3$  excluding the previous configuration. At the beginning, we suppose that  $u_1, v_2$  and  $v_3$  are in an induced  $C_5$  (Figure 7.2.a), after in an induced  $C_6$  and not in an induced  $C_5$  (Figure 7.2.b) and finally neither in an induced  $C_5$  nor  $C_6$  (Figure 7.2.c). Third case, suppose that  $G$  contains an induced  $M_2$  excluding the previous configuration. At the beginning, we suppose that  $u_1, v_1$  and  $v_2$  are in an induced  $C_5$  (Figure 7.3.a) and that  $u_1, v_1$  are in an induced  $C_5$  excluding the previous case (Figure 7.3.b). After we suppose that  $u_1, v_1$  and  $v_2$  are in an induced  $C_6$  and not in an induced  $C_5$  (Figure 7.3.c) and finally neither in an induced  $C_5$  nor  $C_6$  (Figure 6.3.d). Last case, suppose that  $G$  contains a 3-cycle  $C$  and no induced  $M_2$ . Let  $u_1, u_2$  and  $u_3$  be the vertices of  $C$ . Let  $w_1$  and  $w_2$  be the neighbors of  $u_1$  outside  $C$ , let  $w'_1$  and  $w'_2$  be the neighbors of  $u_2$  outside  $C$  and  $w''_1$  and  $w''_2$  be the neighbors of  $u_3$  outside  $C$ . Suppose that  $u_1, u_2, w_1$  and  $w'_1$  are in a 5-cycle and a neighbor of  $u_1$ , say  $w_1$ , has a common neighbor with  $w'_1$  (Figure 7.4.a). Excluding the previous configuration, suppose that  $u_1, u_2, w_1$  and  $w'_1$  are in a 5-cycle;  $w''_1, v_1, u_1$  and  $w_1$  are in another 5-cycle and  $w_1$  is in a triangle (Figure 7.4.b). Afterwards, we suppose that  $w_1$  is not in a triangle (Figure 7.4.c). Finally, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of  $C$  are in a 5-cycle (Figure 7.4.d). Therefore, we suppose that two vertices of  $C$  cannot be in a 5-cycle (Figure 7.4.e).  $\square$

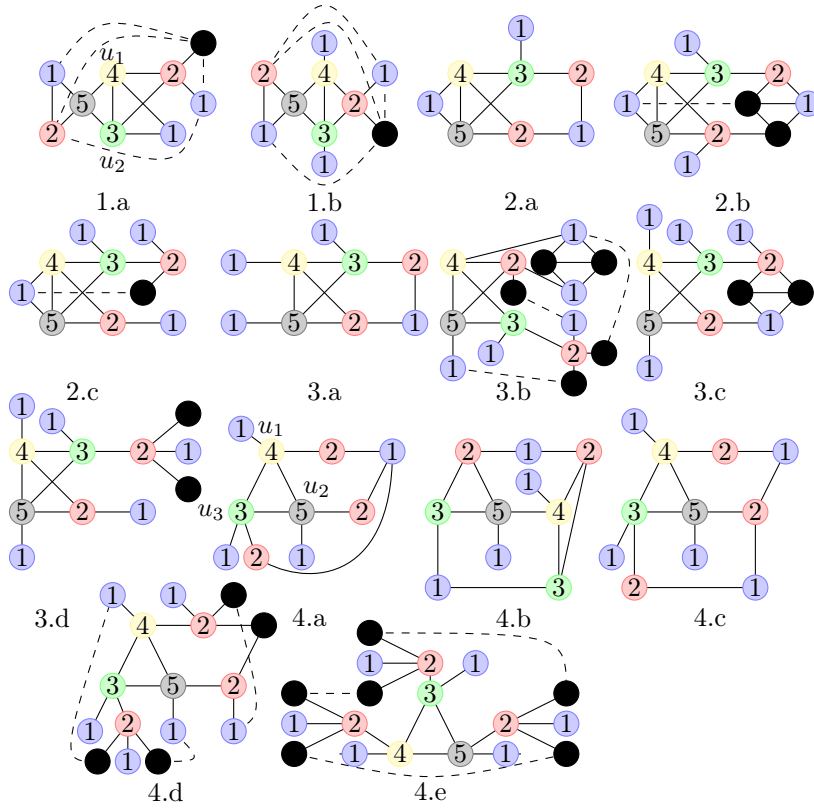


Figure 6: Possible configurations when  $G$  contains a  $C_3$ .

In the following two Lemmas, we suppose that the graph  $G$  of girth  $g = 5$  contains an induced  $C_5$  and eventually an induced Petersen graph. Let  $u_1, u_2, u_3, u_4$  and  $u_5$  be the vertices in the  $C_5$  (or in the outer cycle of the Petersen graph). Let  $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$  and  $v'_5$  be the remaining neighbors of respectively  $u_1, u_2, u_3, u_4$  and  $u_5$  (all different as  $g = 5$ ).

**Lemma 5.4.** *Let  $G$  be a 4-regular graph with girth  $g = 5$ . If  $G$  contains the Petersen graph as induced subgraph then  $\Gamma(G) = 5$ .*

*Proof.* Suppose that  $v_1, v_2, v_3, v_4$  and  $v_5$  form an induced  $C_5$  (the inner cycle of the Petersen graph). Let  $u'_2$  and  $u'_5$  be the remaining neighbors of respectively  $v_2$  and  $v_5$ . Observe that  $v'_1$  can be adjacent with no more than three vertices among  $v'_3, v'_4, u'_2$  and  $u'_5$ . First, suppose that  $v'_1$  is not adjacent with  $v'_3$  (or  $v'_4$ , without loss of generality since the graph is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph  $G$ . Second, suppose that  $v'_1$  is not adjacent with  $u'_5$  (or  $u'_2$ , without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph  $G$ .  $\square$

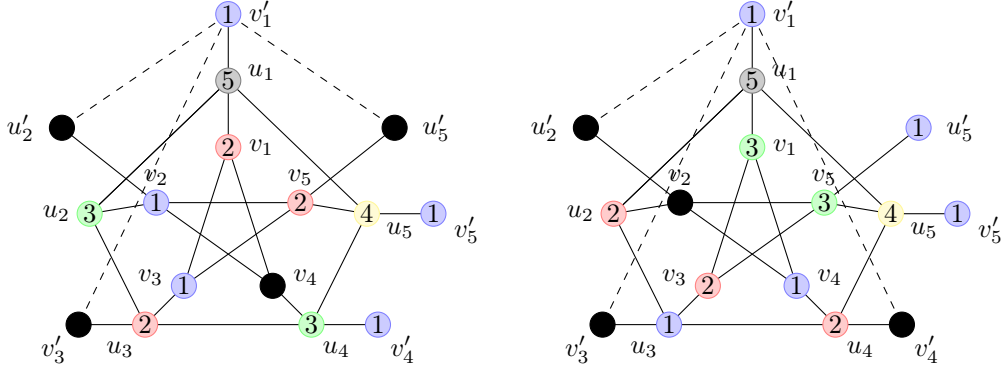


Figure 7: Two Grundy partial 5-colorings of a subgraph containing the Petersen graph as induced subgraph.

In a graph  $G$ , let a *neighbor-connected*  $C_n$  be an  $n$ -cycle  $C$  such that the set of vertices of  $G$  at distance 1 from  $C$  is not independent.

**Lemma 5.5.** *Let  $G$  be a 4-regular graph with girth  $g = 5$ . If  $G$  contains a neighbor-connected  $C_5$  as induced subgraph, then  $\Gamma(G) = 5$ .*

*Proof.* Let  $C$  be a neighbor-connected  $C_5$  in  $G$ . By Lemma 5.4 we can suppose that the neighbors of the vertices of  $C$  do not form an induced  $C_5$  (otherwise a Petersen would be an induced subgraph). Hence, we can suppose that the neighbors of  $C$  form a subgraph of a  $C_{10}$ . If there are two edges between the neighbors of the vertices of  $C$ , then Figure 8 illustrates Grundy partial 5-colorings of the graph  $G$ . Suppose that two neighbors are adjacent, say  $v_1$  and  $v'_3$  and the graph  $G$  does not contain the previous configuration. Then  $v'_3$  can be adjacent with  $v'_1$  and  $v'_5$ . Let  $w_1, w_2$  and  $w_3$  be the three neighbors of  $v_2$  different from  $u_2$ . We suppose that  $w_1$  can be eventually adjacent with  $v'_3$  and  $w_2$  can be eventually adjacent with  $v'_1$ . Figure 9 illustrates a Grundy partial 5-coloring of  $G$  in this case. In this figure, the vertex  $w_3$  can be eventually adjacent with  $v'_5$  or  $v_4$ , but in this case we can switch the color 1 from  $v'_5$  to  $v_5$  or from  $v'_4$  to  $v_4$ .  $\square$

**Proposition 5.6.** *If  $G$  is a 4-regular graph with girth  $g = 5$ , then  $\Gamma(G) = 5$ .*

*Proof.* Let  $C$  be a 5-cycle in  $G$ . Suppose that two consecutive neighbors, for example,  $v_1$  and  $v_5$  have a common neighbor  $w_1$ . The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph  $G$ . In this figure the vertex  $w_1$  can be eventually adjacent with  $v'_2, v'_3$  or  $v_4$ , but in this case we can switch the color 1 from  $v'_2$  to  $v_2$ , from  $v'_3$  to  $v_3$  or from  $v_4$  to  $v'_4$ . Hence, we can suppose that no neighbors of consecutive vertices of  $C$  are adjacent. Among the neighbors of  $v_1$ , there exists one vertex  $w_1$  not adjacent with both  $v_4$  and  $v'_4$

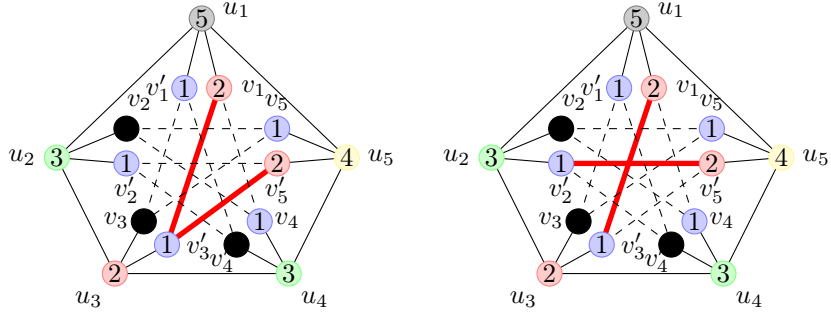


Figure 8: Two Grundy partial 5-colorings of a subgraph containing a neighbor-connected  $C_5$ .

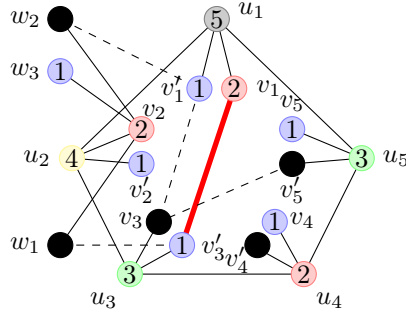


Figure 9: A Grundy partial 5-coloring of a subgraph containing a neighbor-connected  $C_5$ .

(otherwise  $G$  would contain a  $C_4$ ). Among the neighbor of  $v'_5$ , there exists one vertex, say  $w_2$ , not adjacent with  $w_1$ . The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph  $G$ . In this figure the vertex  $w_1$  can be eventually adjacent with  $v_4$  and the vertex  $w_2$  can be eventually adjacent with  $v'_2$  or  $v_4$ , but in these cases we can switch the color 1 from  $v'_2$  to  $v_2$  or from  $v_4$  to  $v'_4$ .  $\square$

In the two following Lemmas, we suppose that a graph  $G$  contains an induced  $C_6$ . Let  $u_1, u_2, u_3, u_4, u_5$  and  $u_6$  be the vertices in the  $C_6$ . Let  $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5, v'_5, v_6$  and  $v'_6$  be the remaining neighbors of respectively  $u_1, u_2, u_3, u_4, u_5$  and  $u_6$  (all different as  $g = 6$ ).

**Lemma 5.7.** *If  $G$  is a 4-regular graph with girth  $g = 6$  which contains a neighbor-connected  $C_6$  as induced subgraph, then  $\Gamma(G) = 5$ .*

*Proof.* First, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let  $w_1$  be a neighbor of  $v'_1$

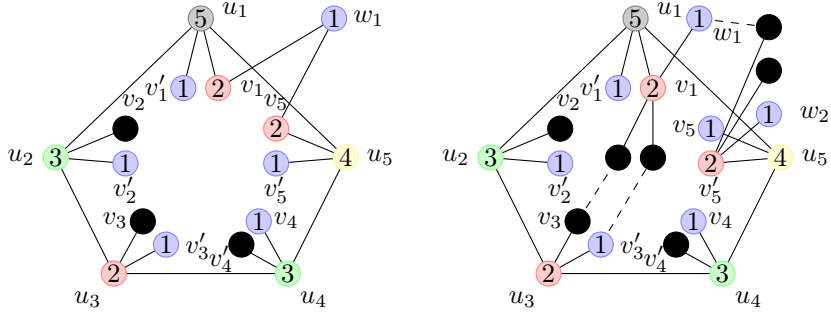


Figure 10: Two Grundy partial 5-colorings of a subgraph containing a  $C_5$ .

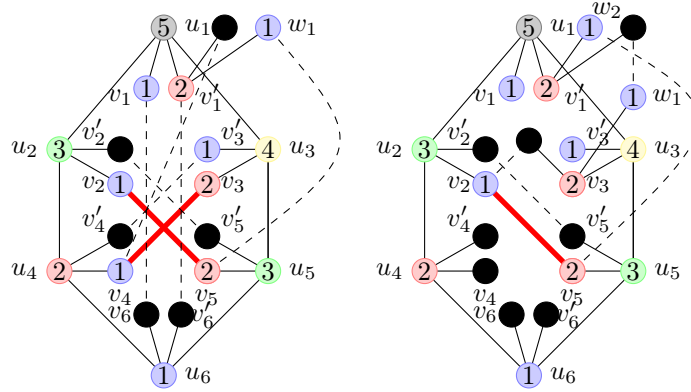


Figure 11: Two Grundy partial 5-colorings of a subgraph containing a neighbor-connected  $C_6$ .

not adjacent with  $v_4$ . The graph  $G$  admits a Grundy partial 5-coloring as the left part of Figure 11 illustrates it. Second, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let  $w_1$  be a neighbor of  $v_3$  not adjacent with  $v_2$  and  $w_2$  be a neighbor of  $v'_1$  not adjacent with  $w_1$ . The graph  $G$  admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it.  $\square$

**Proposition 5.8.** *If  $G$  is a 4-regular graph with girth  $g = 6$ , then  $\Gamma(G) = 5$ .*

*Proof.* By Lemma 5.7, suppose that no neighbors of the  $C_6$  are adjacent. First, suppose that there are two neighbors at distance 4 along the cycle  $C_6$ , for example  $v'_1$  and  $v_5$ , which have a common neighbor  $w_1$ . Let  $w_2$  be a neighbor of  $v_3$  not adjacent with  $w_1$ .  $G$  admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Second, suppose that there are no two neighbors



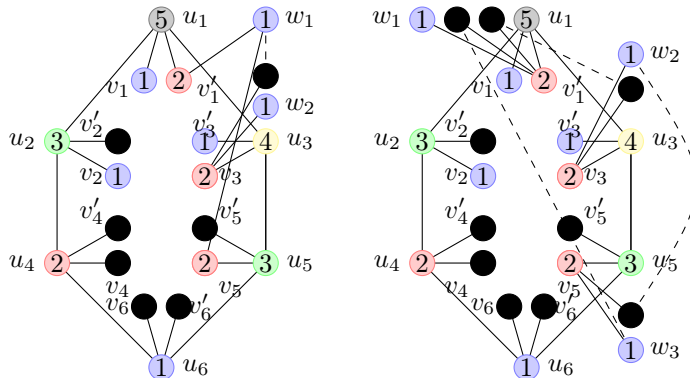


Figure 12: Two Grundy partial 5-colorings of a subgraph containing a  $C_6$ .

at distance 4 along the cycle  $C_6$  which have a common neighbor. Let  $w_1$  be a neighbor of  $v'_1$  not adjacent with a neighbor of  $v_5$  or a neighbor of  $v_3$  and  $w_2$  be a neighbor of  $v_3$  not adjacent with a neighbor of  $v_5$  and  $w_3$  be a neighbor of  $v_5$ . The graph  $G$  admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.  $\square$

**Proposition 5.9.** *If  $G$  be a 4-regular graph with girth  $g \geq 7$ , then  $\Gamma(G) = 5$ .*

*Proof.* First, suppose that  $G$  contains a 7-cycle. By  $T_5$ , we denote the 5-atom which is a tree (the binomial tree with maximum degree 4). It can be easily verified that  $G$  contains  $T_5$  where two leaves are merged (which is a 5-atom). Moreover, if  $G$  does not contain a 7-cycle, then it contains  $T_5$  as induced subgraph.  $\square$

**Theorem 3.** *Let  $G$  be a 4-regular graph. If  $G$  does not contain an induced  $C_4$ , then  $\Gamma(G) = 5$ .*

*Proof.* Suppose that  $G$  does not contain an induced  $C_4$ . By Proposition 5.9 if  $g \geq 7$ , Propositions 5.6 and 5.8 if  $g = 5, 6$ , and Proposition 5.3 if  $G$  contains a  $C_3$ , we have  $\Gamma(G) = 5$ .  $\square$

By Proposition 3.1, Corollary 3.5 and Theorem 3, any  $r$ -regular graph with  $r \leq 4$  and without induced  $C_4$  has Grundy number  $r+1$ . Therefore, it is natural to propose the following conjecture.

**Conjecture 1.** *For any  $r \geq 1$ , every  $r$ -regular graph without induced  $C_4$  has Grundy number  $r+1$ .*

## References

- [1] Marie Asté, Frédéric Havet, and Cláudia Linhares Sales. Grundy number and products of graphs. *Discrete Mathematics*, 310:1482–1490, 2010.
- [2] József Balogh, Stephen G. Hartke, Qi Liu, and Gexin Yu. On the first-fit chromatic number of graphs. *SIAM Journal on Discrete Math*, 22:887–900, 2008.
- [3] Sergio Cabello and Marko Jakovac. On the b-chromatic number of regular graphs. *Discrete Applied Mathematics*, 159:1303–1310, 2011.
- [4] Victor Campos, András Gyárfás, Frédéric Havet, Cláudia Linhares Sales, and Frédéric Maffray. New bounds on the Grundy number of graphs. *Journal of Graph Theory*, 71:78–88, 2012.
- [5] Gerard Jennhwa Chang and Hsiang-Chun Hsu. First-fit chromatic numbers of  $d$ -degenerate graphs. *Discrete Mathematics*, 312:2088–2090, 2012.
- [6] Brice Effantin and Hamamache Kheddouci. Grundy number of graphs. *Discussiones Mathematicae Graph theory*, 27:5–18, 2007.
- [7] Paul Erdős, Stephen T. Hedetniemi, Renu C. Laskar, and Geert C.E Prins. On the equality of the partial Grundy and upper chromatic number of graphs. *Discrete Mathematics*, 272:53–64, 2003.
- [8] Zoltán Füredi, András Gyárfás, Gábor N. Sárközy, and Stanley Selkow. Inequalities for the first-fit chromatic number. *Journal of Graph Theory*, 59:75–88, 2008.
- [9] Frédéric Havet and Leonardo Sampaio. On the Grundy and b-chromatic numbers of a graph. *Algorithmica*, 65:885–899, 2013.
- [10] Marko Jakovac and Sandi Klavžar. The b-Chromatic Number of Cubic Graphs. *Graphs and Combinatorics*, 26:107–118, 2010.
- [11] Amine El Sahili and Mekkia Kouidder. b-chromatic number of regular graphs. *Utilitas Mathematica*, 80:211–216, 2009.
- [12] Saeed Shaebani. On the b-chromatic number of regular graphs without 4-cycle. *Discrete Applied Mathematics*, 160:1610–1614, 2012.
- [13] Zhengnan Shi, Wayne Goddard, Stephen T. Hedetniemi, Ken Kennedy, Renu Laskar, and Alice McRae. An algorithm for partial Grundy number on trees. *Discrete Mathematics*, 304:108–116, 2005.
- [14] Said Yahiaoui and Brice Effantin. On the Grundy number of cubic graphs. *preprint*, 2012.
- [15] Manouchehr Zaker. Results on the Grundy chromatic number of graphs. *Discrete Mathematics*, 306:3166–3173, 2006.