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# On the family of $r$-regular graphs with Grundy number $r+1$ 

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#### Abstract

The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the largest $k$ such that there exists a partition of $V(G)$, into $k$ independent sets $V_{1}, \ldots, V_{k}$ and every vertex of $V_{i}$ is adjacent to at least one vertex in $V_{j}$, for every $j<i$. The objects which are studied in this article are families of $r$-regular graphs such that $\Gamma(G)=r+1$. Using the notion of independent module, a characterization of this family is given for $r=3$. Moreover, we determine classes of graphs in this family, in particular the class of $r$-regular graphs without induced $C_{4}$, for $r \leq 4$. Furthermore, our propositions imply results on partial Grundy number.


## 1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G=$ $(V, E)$, a proper $k$-coloring of $G$ is a surjective mapping $c: V \rightarrow\{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $u v \in E$; the color class $V_{i}$ is the set $\{u \in V \mid c(u)=i\}$ and a vertex $v$ has color $i$ if $v \in V_{i}$. A vertex $v$ of color $i$ is a Grundy vertex if $v$ is adjacent to at least one vertex colored $j$, for every $j<i$. A Grundy $k$-coloring is a proper $k$-coloring such that every vertex is a Grundy vertex. A partial Grundy $k$-coloring is a proper $k$-coloring such that every color class contains a Grundy vertex. The Grundy number (partial Grundy number, respectively) of $G$ denoted by $\Gamma(G)(\partial \Gamma(G)$, respectively) is the largest $k$ such that $G$ admits a Grundy $k$-coloring (partial Grundy $k$-coloring, respectively).

Let $N(v)=\{u \in V(G) \mid u v \in E(G)\}$ be the neighborhood of $v$. A set $X$ of vertices is an independent module if $X$ is an independent set and all vertices

[^0]in $X$ have the same neighborhood. The vertices in an independent module of size 2 are called false twins. Let $P_{n}, C_{n}, K_{n}$ and $I_{n}$ be respectively, the path, cycle complete and empty graph of order $n$. The concepts of Grundy $k$ coloring and domination are connected. In a Grundy coloring, $V_{1}$ is a dominating set. Given a graph $G$ and an ordering $\phi$ on $V(G)$ with $\phi=v_{1}, \ldots, v_{n}$, the greedy algorithm assigns to $v_{i}$ the minimum color that was not assigned in the set $\left\{v_{1}, \ldots, v_{i-1}\right\} \cap N\left(v_{i}\right)$. Let $\Gamma_{\phi}(G)$ be the number of colors used by the greedy algorithm with the ordering $\phi$ on $G$. We obtain the following result [7]: $\Gamma(G)=\max _{\phi \in S_{n}}\left(\Gamma_{\phi}(G)\right)$.

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed $t$, determining if a given graph has Grundy number at least $t$ is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called $t$-atoms, such that any graph with Grundy number at least $t$ contains a $t$-atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number $[8,15]$, that there exist upper bounds for $d$-degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number $[6,1,4]$. Recently, Havet and Sampaio [9] have proven that the problem of deciding if for a given graph $G$ we have $\Gamma(G)=\Delta(G)+1$, even if $G$ is bipartite, is NP-complete. Moreover, they have proven that the dual of Grundy $k$-coloring problem is in FPT by finding an algorithm in $O\left(2 k^{2 k} .|E|+2^{2 k} k^{3 k+5 / 2}\right)$ time.
Note that a Grundy $k$-coloring is a partial Grundy $k$-coloring, hence $\Gamma(G) \leq$ $\partial \Gamma(G)$. Given a graph $G$ and a positive integer $k$, the problem of determining if a partial Grundy $k$-coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number, denoted by $\varphi(G)$, which is the largest $k$ such that there exists a proper $k$-coloring and for every color class $V_{i}$, there exists a vertex adjacent to at least one vertex colored $j$, for every $j$, with $j \neq i$. Note that a $b$ coloring is a partial Grundy $k$-coloring, hence $\varphi(G) \leq \partial \Gamma(G)$. The $b$-chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For $r \leq 4$, every $r$-regular graphs without induced $C_{4}$ has Grundy number $r+1$. We conjecture that this assertion is also true for $r>4$.

Conjecture 1. For any integer $r \geq 1$, every $r$-regular graph without induced $C_{4}$ has Grundy number $r+1$.

Section 2 gives characterizations of some classes of graphs with Grundy number at most $k, 2 \leq k \leq \Delta(G)$, using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except
$K_{3,3}$ has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most $k, 3 \leq k \leq r$. To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced $C_{4}$ have Grundy number 5 .

## 2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy $k$-coloring.

Definition 2.1. Let $G$ be a graph. A Grundy partial $k$-coloring is a Grundy $k$-coloring of a subset $S$ of $V(G)$.

Observation 2.2 ([1],[6]). If $G$ admits a Grundy partial $k$-coloring, then $\Gamma(G) \geq$ $k$.

This property has an important consequence: For a graph $G$, with $\Gamma(G) \geq t$ and any Grundy partial $t$-coloring, there exist smallest subgraphs $H$ of $G$ such that $\Gamma(H)=t$. The family of $t$-atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from Zaker's one, insisting more on the construction of every $t$-atom.

Definition 2.3 ([15]). For any integer $t$, we define the family of $t$-atoms, denoted by $\mathscr{A}_{t}, t=1, \ldots$ by induction. Let the family $\mathscr{A}_{1}$ contain only $K_{1}$. A graph $G$ is in $\mathscr{A}_{t+1}$ if there exists a graph $G^{\prime}$ in $\mathscr{A}_{t}$ and an integer $m, m \leq\left|V\left(G^{\prime}\right)\right|$, such that $G$ is composed of $G^{\prime}$ and an independent set $I_{m}$ of order $m$, adding edges between $G^{\prime}$ and $I_{m}$ such that every vertex in $G^{\prime}$ is connected to at least one vertex in $I_{m}$. Moreover a t-atom $A$ is minimal, if there is no $t$-atom included in $A$ other than itself.

Theorem 1 ([15]). For a given graph $G, \Gamma(G) \geq t$ if and only if $G$ contains an induced minimal t-atom.

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

Proposition 2.4 ([1]). Let $G$ be a graph and $X$ be an independent module. In every Grundy coloring of $G$, the vertices in $X$ must have the same color.

Definition 2.5. Let $G$ be an r-regular graph. $A$ vertex $v$ is a $(0, \ell)$-twin-vertex if there exists an independent module of cardinality $r+2-\ell$ that contains $v$.

Proposition 2.6. Let $G$ be an r-regular graph. The color of an $(0, \ell)$-twinvertex is at most $\ell$ in every Grundy coloring of $G$.

Proof. Let $v$ be a $(0, \ell)$-twin-vertex colored $\ell+1$ in $G$. By Definition, $v$ is in an independent module $X$ of cardinality $r+2-\ell$ and by Proposition 2.4, every other vertex of $X$ should be colored $\ell+1$. Let $u$ be a neighbor of $v$. There are at most $\ell-2$ neighbors of $u$ in $V(G-X)$. Therefore, $u$ cannot be colored $\ell$.

Definition 2.7. A vertex $v$ of a graph $G$ is a $(1, \ell)$-twin-vertex if $N(v)$ can be partitioned into at least $\ell-1$ independent modules.

Proposition 2.8. Let $G$ be a graph. The color of an $(1, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$.

Proof. By Proposition 2.4, vertices of the neighborhood of $v$ can only have $\ell-1$ different colors. Therefore, the color of $v$ is at most $\ell$.

Definition 2.9. A vertex $v$ of a graph $G$ is a (2, $\ell)$-twin-vertex if $N(v)$ is independent and every vertex in $N(v)$ is a $(1, \ell)$-twin-vertex.

Proposition 2.10. Let $G$ be a graph. The color of an $(2, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$.

Proof. Let $v$ be a $(2, \ell)$-twin-vertex in $G$. Every vertex in $N(v)$ is a $(1, \ell)$-twinvertex. If a vertex in $N(v)$ is colored $\ell$, then $v$ could only have a color at most $\ell-1$. If the vertices in the neighborhood of $v$ have colors at most $\ell-1$, then in every Grundy coloring of $G, v$ has a color at most $\ell$.

Corollary 2.11. Let $G$ be a graph. If every vertex is a $(1, \ell)$-twin-vertex or a $(2, \ell)$-twin-vertex, then $\Gamma(G) \leq \ell$.

Corollary 2.12. Let $G$ be a regular graph. If every vertex is an (i, $\ell$ )-twinvertex, for some $i, 0 \leq i \leq 2$, then $\Gamma(G) \leq \ell$.

Proposition 2.13 ([1],[15]). Let $G$ be a graph. We have $\Gamma(G) \leq 2$ if and only if $G=K_{n, m}$ for some integers $n>0$ and $m>0$.

## 3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial $k$-colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

Proposition $3.1([6])$. Let $G$ be a connected 2-regular graph. $\partial \Gamma(G)=\Gamma(G)=$ 2 if and only if $G=C_{4}$.

The following definition gives a construction of the cubic graphs in which every vertex is an ( $i, 3$ )-twin-vertex, for some $i, 0 \leq i \leq 2$. Figure 2 gives the list of every graph of order at most 16 in this family.

Definition 3.2. Let $K_{2,3}$ and $K_{3,3}^{*}$ be the graphs from Figure 1. We define recursively the family of graphs $\mathscr{F}_{3}^{*}$ as follows:


Figure 1: The graphs $K_{2,3}$ (on the left) and $K_{3,3}^{*}$ (on the right).


Figure 2: The cubic graphs $G$ such that $|V(G)|<18$ and $\Gamma(G)<4$.

1. $K_{2,3} \in \mathscr{F}_{3}^{*}$ and $K_{3,3}^{*} \in \mathscr{F}_{3}^{*}$;
2. the disjoint union of two elements of $\mathscr{F}_{3}^{*}$ is in $\mathscr{F}_{3}^{*}$;
3. if $G$ is a graph in $\mathscr{F}_{3}^{*}$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most 2 is also in $\mathscr{F}_{3}^{*}$;
4. if $G$ is a graph in $\mathscr{F}_{3}^{*}$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to three vertices of degree at most 2 is in $\mathscr{F}_{3}^{*}$.

The family $\mathscr{F}_{3}$ is the subfamily of cubic graphs in $\mathscr{F}_{3}^{*}$.
Proposition 3.3. Let $G$ be a cubic graph. Every vertex of $V(G)$ is an (i,3)-twin vertex, for some $i, 0 \leq i \leq 2$, if and only if $G \in \mathscr{F}_{3}$.

Proof. Every graph $G$ in $\mathscr{F}_{3}$ has three kind of vertices: ( 0,3 )-twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices
added by Point 4. Vertices where an edge is added by Point 3 are ( 1,3 )-twinvertex and vice versa. Vertices added by Point 4 are (2,3)-twin-vertices and vice versa.

Theorem 2. Let $G$ be a cubic graph. $\Gamma(G) \leq 3$ if and only if every vertex is an ( $i, 3$ )-twin-vertex, for some $i, 0 \leq i \leq 2$.
Proof. By Corollary 2.12, the "if" part is proven. Assume that $G$ contains a vertex $v$ which is not an (i,3)-twin-vertex, for some $i, 0 \leq i \leq 2$ and $\Gamma(G)<$ 4. In every configuration we want to either find a Grundy partial 4 -coloring, contradicting $\Gamma(G)<4$ or proving that $v$ is an $(i, 3)$-twin-vertex, for some $i$, with $0 \leq i \leq 2$. We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: $v$ or a neighbor of $v$ is in a $C_{3}, v$ is in an induced $C_{4}$ and $v$ or a neighbor of $v$ are not in a $C_{3}$ and $v$ is not in an induced $C_{4}$. Let $C$ be an induced cycle of order 3 or 4 which contains $v$ or a neighbor of $v$ and let $D_{1}=\{x \in V(G) \mid d(x, C)=1\}$, where $d(x, C)$ is the distance from $x$ to $C$ in the graph $G$. To simplify notation, $D_{1}$ will also denote the subgraph of $G$ induced by $D_{1}$.

Case 1: Assume that $v$ or a neighbor of $v$ is in $C$ and $C=C_{3}$. If $\left|D_{1}\right|=1$, then $G=K_{4}$ and $\Gamma\left(K_{4}\right)=4$. If $\left|D_{1}\right|=2$ and $D_{1}=P_{2}$, then $v$ is a $(0,3)$ -twin-vertex or a (1,3)-twin-vertex. If $D_{1}=I_{2}$ then Figure 3.1.a yields a Grundy partial 4-coloring of $G$. If $\left|D_{1}\right|=3$, then we have four subcases: $D_{1}$ is $C_{3}$ or $P_{3}$ (Figure 3.1.b), $P_{2} \cup I_{1}$ (Figure 3.1.c) or $I_{3}$ (Figure 3.1.d). In every case $G$ admits a Grundy partial 4 -coloring.

Case 2: Assume that $v$ is in $C$ and $C=C_{4}$. Note that for two non adjacent vertices of $C$ who have a common neighbor in $D_{1}$, the vertex $v$ is a $(0,3)$ -twin-vertex or a $(1,3)$-twin-vertex. Hence, we will not consider these cases. If $\left|D_{1}\right|=2$, then $D_{1}=P_{2}$ or $D_{1}=I_{2}$ (Figure 3.2.a) and in both cases, $G$ admits a Grundy partial 4-coloring. If $\left|D_{1}\right|=3$, Figure 3.2.b yields a Grundy partial 4-coloring of $G$. In the case $\left|D_{1}\right|=4$, we first assume that two adjacent vertices of $C$ have their neighbors in $D_{1}$ adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of $C$ have their neighbors in $D_{1}$ adjacent (Figure 3.2.d). In the case $D_{1}=I_{4}$, we first suppose that two vertices of $D_{1}$ which have two adjacent vertices of $C$ as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

Case 3: Assume that $v$ or a neighbor of $v$ is not in a $C_{3}$ and $v$ is not in an induced $C_{4}$. Firstly, suppose that a neighbor $u$ of $v$ is in an induced $C_{4}$. Using the coloring from the previous case, $G$ admits a Grundy partial 4coloring in every cases except in the case where two neighbors of $v$ in the $C_{4}$ have a common neighbor outside the $C_{4}$. However, this case cannot happen for every neighbor of $v$, otherwise $v$ would be a (2,3)-twin-vertex. Assume that $u$ is the neighbor of $v$ not in the previous configuration. If $u$ is in an induced $C_{4}$, then using the coloring from the previous case, $G$
admits a Grundy partial 4-coloring. If $u$ is not in an induced $C_{4}$, then Figure 3.3.a yields a Grundy partial 4-coloring of $G$. In this figure, the color 2 is given to a neighbor of $u$ not adjacent to both $f_{1}$ and $f_{2}$. Secondly, suppose that $v$ is in an induced $C_{5}$. Figure 3.3.b yields a Grundy partial 4-coloring of $G$. Thirdly, if $v$ is not in an induced $C_{5}$, then Figure 3.3.c yields a Grundy partial 4-coloring of $G$.

Therefore, if $\Gamma(G) \leq 3$, then every vertex is an (i,3)-twin-vertex, for some $i$, $0 \leq i \leq 2$.

Observe that if an edge is added between the two vertices of degree 2 in $K_{3,3}^{*}$, then we obtain $K_{3,3}$ which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

Corollary 3.4. A cubic graph $G$ does not contain any induced minimal subcubic 4-atom if and only if every vertex is an (i,3)-twin-vertex, for some $i, 0 \leq i \leq 2$.

Corollary 3.5. Let $G$ be a cubic graph. If $G$ is without induced $C_{4}$, then $\Gamma(G)=4$.

Proof. As every graph $G$ with $\Gamma(G)<4$ is composed of copies of $K_{2,3}$ or $K_{3,3}^{*}$, the graph $G$ always contains a square if $\Gamma(G)<4$.

For a fixed integer $t$, the largest $(t+1)$-atom has order $2^{t}$. Thus, for a graph $G$ of maximum degree $t$, there exists an $O\left(n^{2^{t}}\right)$-time algorithm to determine if $\Gamma(G)<t+1$ (which verifies if the graph contains an induced $(t+1)$-atom). For a cubic graph, we obtain an $O\left(n^{8}\right)$-time algorithm, whereas our characterization yields a linear-time algorithm.

Observation 3.6. Let $G$ be a cubic graph of order $n$. There exists an $O(n)$-time algorithm ${ }^{1}$ to determine the Grundy number of $G$.

Proof. Suppose we have a cubic graph $G$ with its adjacency list. Verifying if $G$ is $K_{3,3}$ can be done in constant time. We suppose now that $G$ is not $K_{3,3}$. For each vertex $v$, the algorithm verifies that $v$ is an $(i, 3)$-twin-vertex, for some $i$, $0 \leq i \leq 2$. If the condition is true for all vertices, then $\Gamma(G)=3$, else $\Gamma(G)=4$. To determine if a vertex $v$ is a ( 0,3 )-twin-vertex, it suffices to verify that there is a common vertex other than $v$ in the adjacency lists of the neighbors of $v$. To determine if a vertex $v$ is a $(1,3)$-twin-vertex, it suffices to verify that there are two neighbors of $v$ which have the same adjacency list. To determine if a vertex $v$ is a (2,3)-twin-vertex, it suffices to verify that the neighborhood of $v$ is independent and that every neighbor is a (1,3)-twin-vertex. Hence, checking if a vertex is an $(i, 3)$-twin vertex can be done in constant time, so the algorithms runs in linear time.

[^1]
1.a

1.b

1.c



2.b



2.d
2.e


3.b

3.c

Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number $i$ : Vertices of color $i$ ).

Proposition 3.7. If $G$ is a connected cubic graph and $G \neq K_{3,3}$, then $\partial \Gamma(G)=$ 4.

Proof. Let $G$ be a cubic connected graph. Note that if $\Gamma(G)=4$ then $\partial \Gamma(G)=4$. Every graph $G$ with $\Gamma(G)<4$ is composed of copies of $K_{2,3}$ or $K_{3,3}^{*}$. If $G$ contains more than two copies (so it is different from $K_{3,3}$ ), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial \Gamma(G)=4$.

Only $K_{3,3}$ and three other cubic graphs have $b$-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the $b$-chromatic number. Shi et al. [13] proved that there exists a smallest integer $N_{r}$ such that every $r$-regular graph $G$ with more than $N_{r}$ vertices has $\partial \Gamma(G)=r+1$. Observe that we have $N_{2}=4$ and $N_{3}=6$. It is an open question to determine $N_{r}$ for $r \geq 4$. However, using results on $b$-chromatic number [3], we have $N_{r} \leq 2 r^{3}-r^{2}+r$.

## 4 Properties on the Grundy number of $r$-regular graphs

Definition 4.1. Let $r \geq 2$ be an integer. We define recursively the family of graphs $\mathscr{G}_{r}^{*}$ as follows:

1. $K_{r-k, k+2} \in \mathscr{G}_{r}^{*}$, for any $k, 0 \leq k \leq(r-2) / 2$;
2. the disjoint union of two elements of $\mathscr{G}_{r}^{*}$ is in $\mathscr{G}_{r}^{*}$;
3. if $G$ is a graph in $\mathscr{G}_{r}^{*}$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most $r-1$ is also in $\mathscr{G}_{r}^{*}$;
4. if $G$ is a graph in $\mathscr{G}_{r}^{*}$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to $r$ vertices of degree at most $r-1$ is in $\mathscr{G}_{r}^{*}$.

The family $\mathscr{G}_{r}$ is the subfamily of r-regular graphs in $\mathscr{G}_{r}^{*}$.
Proposition 4.2. Let $G$ be an $r$-regular graph. If $G \in \mathscr{G}_{r}$, then $\Gamma(G)<r+1$.
Proof. By $I_{r-k}$ and $I_{k+2}$, with $\left|I_{r-k}\right|=r-k$ and $\left|I_{k+2}\right|=k+2$, we denote the two sets of vertices in the bipartition of an induced subgraph $K_{r-k, k+2}$ in $G$. Firstly, suppose there exists a vertex $u$ in an induced subgraph $K_{r-k, k+2}$ colored $r+1$. Without loss of generality, suppose $u$ is in $I_{r-k}$. The $r$ neighbors of $u$ should have colors from 1 to $r$. Among the neighbors of $u, k+2$ neighbors are in $I_{k+2}$. Let $v$ be the neighbor of $u$ in $I_{k+2}$ with the largest color in $I_{k+2}$. The vertex $v$ has color at least $k+2$. Hence, there exists an integer $s \geq 0$ such that the color of $v$ is $k+2+s$. Note that there are $s$ vertices in $N(\bar{u}) \backslash I_{k+2}$ which have colors at most $k+2+s$. The colors of the $s$ vertices are the only one possible remaining colors at most $k+2+s$ in $I_{r-k}$. Hence, as there are $k$ vertices in $N(v) \backslash I_{r-k}$, the neighbors of $v$ can only have at most $k+s$ different colors at most $k+2+s$. Therefore, we have a contradiction and $u$ cannot have


Figure 4: The Graph $G_{r, 4, i}, i \geq 2, r=r_{1}+r_{2}+r_{3}$.
color $r+1$. Secondly, suppose there exists a vertex $u$ added by Point 4 which has color $r+1$. As a neighbor of $u$ in an induced $K_{r-k, k+2}$ should be colored $r$, the argument is completely similar to the previous one.

Corollary 4.3. Let $G$ be a 4-regular graph. If $G \in \mathscr{G}_{4}$, then $\Gamma(G)<5$.
The reader can believe that the family of 4-regular graphs with $\Gamma(G)<5$ contains only the family $\mathscr{G}_{4}$. However, there exist graphs with Grundy number $r$ which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7 -cycle $C_{7}^{2}$ satisfies $\Gamma\left(C_{7}^{2}\right)<5$ and is not in $\mathscr{G}_{4}$.

The next proposition shows that unlike the $b$-chromatic number, $r$-regular graphs of order arbitrarily large with Grundy number $k$ can be constructed for any $r$ and any $k, 3 \leq k \leq r+1$.

Proposition 4.4. Let $r \geq 4$ and $3 \leq k \leq r+1$ be integers. There exists an infinite family $\mathscr{H}$ of connected r-regular graphs such that for all $G$ in $\mathscr{H}$, $\Gamma(G)=k$.

Proof. Let $i \geq 2$ be a positive integer and $r_{1}, \ldots, r_{k-1}$ be a sequence of positive integers such that $r=r_{1}+\ldots+r_{k-1}$. We construct a graph $G_{r, k, i}$ as follows: Take $2 i$ copies of $K_{r_{1}, \ldots, r_{k-1}}$. Let $H_{j-1}$ be the copy number $j$ of $K_{r_{1}, \ldots, r_{k-1}}$ and $H_{j, r_{l}}$ be the independent $r_{l}$-set in $H_{j}$. If $j \equiv 1(\bmod 2)$, do the graph join of $H_{j(\bmod 2 i), r_{1}}$ and $H_{j-1}(\bmod 2 i), r_{1}$ and for an integer $l, 1<l<k$, do the graph join of $H_{j}(\bmod 2 i), r_{l}$ and $H_{j+1}(\bmod 2 i), r_{l}$. The $r$-regular graph obtained is the graph $G_{r, k, i}$. Figure 4 gives $G_{r, k, i}$, for $k=4$ and $i \geq 2$. Note that $H_{j, r_{i}}$ is an independent module. Thus, every vertex is a ( $0, k$ )-twin-vertex. By Proposition 2.6, $\Gamma\left(G_{r, k, i}\right) \leq k$.

For an integer $l, 1<l<k$, color one vertex $l-1$ in $H_{1, r_{l}}$ and $H_{2, r_{l}}$. Afterwards, color one vertex $k-1$ in $H_{1, r_{1}}$ and one vertex $k$ in $H_{2, r_{1}}$. The given coloring is a Grundy partial $k$-coloring of $G_{r, k, i}$ for $i \geq 2$. Therefore, $\Gamma\left(G_{r, k, i}\right)=k$, for $i \geq 2$.

## 5 Grundy number of 4-regular graphs without induced $C_{4}$

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4 -regular graphs without induced $C_{4}$ contains only graphs with Grundy number 5.

Lemma 5.1. Let $G$ be a 4-regular graph without induced $C_{4}$. If $G$ contains (an induced) $K_{4}$ then $\Gamma(G)=5$.

Proof. Note that if $G=K_{5}$, we have $\Gamma(G)=5$. If $G$ is not $K_{5}$ then every pair of neighbors of vertices of $K_{4}$ cannot be adjacent ( $G$ would contain a $C_{4}$ ). Giving the color 1 to each neighbor of the vertices of $K_{4}$ and colors $2,3,4,5$ to the vertices of $K_{4}$, we obtain a Grundy partial 5-coloring of $G$.

Lemma 5.2. Let $G$ be a 4-regular graph without induced $C_{4}$ and let $W$ be the graph from Figure 5. If $G$ contains an induced $W$ then $\Gamma(G)=5$.

Proof. The names of the vertices of $W$ come from Figure 5. Depending on the different cases that could happen, Grundy partial 5 -colorings of $G$ will be given using their references on Figure 5. Let $D_{1}$ be the set of vertices at distance 1 from vertices of $W$ in $G-W$. Suppose that two vertices of $W$ have a common neighbor in $D_{1}$. This two vertices could only be $u_{4}$ and $u_{5}$ or $u_{3}$ and $u_{5}$ (or $u_{1}$ and $u_{4}$, by symmetry). In the case that $u_{4}$ and $u_{5}$ have a common neighbor in $D_{1}$, colors will be given to neighbors of $u_{3}$ in $D_{1}$, depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that $u_{3}$ and $u_{5}$ have a common neighbor $w$ in $D_{1}, w$ can be adjacent with a neighbor of $u_{3}$ in $D_{1}$ (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in $W$ have a common neighbor in $D_{1}$. Let $w_{1}$ and $w_{2}$ be the neighbors of $u_{3}$ in $D_{1}$. We first consider that $w_{1}$ and $w_{2}$ are adjacent (Figure 5.3.a). Secondly, we consider that $w_{1}$ and $w_{2}$ are not adjacent and that $u_{5}, u_{3}$ and $w_{1}$ are in an induced $C_{5}$ (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c).

Proposition 5.3. Let $G$ be a 4-regular graph without induced $C_{4}$. If $G$ contains $C_{3}$ then $\Gamma(G)=5$.

Proof. Depending on the different cases that could happen, a reference to the Grundy partial 5 -coloring of $G$ in Figure 6 will be given. Let $M_{i}, i=2$ or 3, be the graph of order $2+i$ containing two adjacent vertices $u_{1}$ and $u_{2}$ which have exactly $i$ common neighbors, $\left\{v_{1}, \ldots, v_{i}\right\}$, that form an independent set. Let $D_{1}$ be the set of vertices at distance 1 from an induced $M_{i}$ in $G-M_{i}$, for $2 \leq i \leq 3$.

Case 1: Firstly, assume that $G$ contains an induced $M_{3}$ and a vertex of $M_{3}$ has its two neighbors in $D_{1}$ adjacent (Figure 6.1.a). Secondly, assume that $G$ contains an induced $M_{2}$ and a vertex of $M_{2}$ has its two neighbors in $D_{1}$


The graph $W$.

2.b

1.a

3.a





Figure 5: Possible configurations when $G$ contains an induced $W$.
adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that $G$ cannot contain a $K_{4}$ by Lemma 5.1.

Case 2: Assume that $G$ contains an induced $M_{3}$ excluding the previous configuration. There are three cases: $u_{1}, v_{2}$ and $v_{3}$ are in an induced $C_{5}$ (Figure 6.2.a), $u_{1}, v_{2}$ and $v_{3}$ are in an induced $C_{6}$ and not in an induced $C_{5}$ (Figure 6.2.b) and $u_{1}, v_{2}$ and $v_{3}$ are neither in an induced $C_{5}$ nor $C_{6}$ (Figure 6.2.c).

Case 3: Suppose that $G$ contains an induced $M_{2}$ excluding the previous configurations. Firstly, we suppose that $u_{1}, v_{1}$ and $v_{2}$ are in an induced $C_{5}$ (Figure 6.3.a). Secondly, we suppose that $u_{1}, v_{1}$ are in an induced $C_{5}$ excluding the previous case (Figure 6.3.b). Thirdly, we suppose that $u_{1}$, $v_{1}$ and $v_{2}$ are in an induced $C_{6}$ and not in an induced $C_{5}$ (Figure 6.3.c) and finally neither in an induced $C_{5}$ nor $C_{6}$ (Figure 6.3.d).
Suppose that $G$ contains a 3 -cycle $C$ and no induced $M_{2}$. Let $u_{1}, u_{2}$ and $u_{3}$ be the vertices of $C$. Let $w_{1}$ and $w_{2}$ be the neighbors of $u_{1}$ outside $C$, let $w_{1}^{\prime}$ and $w_{2}^{\prime}$ be the neighbors of $u_{2}$ outside $C$ and let $w_{1}^{\prime \prime}$ and $w_{2}^{\prime \prime}$ be the neighbors of $u_{3}$ outside $C$.

Case 4: Firstly, suppose that $u_{1}, u_{2}, w_{1}$ and $w_{1}^{\prime}$ are in a 5 -cycle and a neighbor of $u_{1}$, say $w_{1}$, has a common neighbor with $w_{1}^{\prime}$ (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that $u_{1}, u_{2}, w_{1}$ and $w_{1}^{\prime}$ are in a 5-cycle; $w_{1}^{\prime \prime}, v_{1}, u_{1}$ and $w_{1}$ are in another 5 -cycle and $w_{1}$ is in a triangle (Figure 6.4.b). We suppose that $w_{1}$ is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5 -coloring if two vertices of $C$ are in a 5 -cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of $C$ cannot be in a 5 -cycle (Figure 6.4.e).

In the following two lemmas, we consider a graph $G$ of girth $g=5$ and possibly containing an induced Petersen graph. Let $u_{1}, u_{2}, u_{3}, u_{4}$ and $u_{5}$ be


Figure 6: Possible configurations when $G$ an induced $C_{3}$.
the vertices in an induced $C_{5}$ (or in the the outer cycle of a Petersen graph, if any). Let $v_{1}, v_{1}^{\prime}, v_{2}, v_{2}^{\prime}, v_{3}, v_{3}^{\prime}, v_{4}, v_{4}^{\prime}, v_{5}$ and $v_{5}^{\prime}$ be the remaining neighbors of respectively $u_{1}, u_{2}, u_{3}, u_{4}$ and $u_{5}$ (all different as $g=5$ ).

Lemma 5.4. Let $G$ be a 4-regular graph with girth $g=5$. If $G$ contains the Petersen graph as induced subgraph then $\Gamma(G)=5$.

Proof. Suppose that $v_{1}, v_{2}, v_{3}, v_{4}$ and $v_{5}$ form an induced $C_{5}$ (the inner cycle of the Petersen graph). Let $u_{2}^{\prime}$ and $u_{5}^{\prime}$ be the remaining neighbors of respectively $v_{2}$ and $v_{5}$. Observe that $v_{1}^{\prime}$ can be adjacent with no more than three vertices among $v_{3}^{\prime}, v_{4}^{\prime}, u_{2}^{\prime}$ and $u_{5}^{\prime}$. Firstly, suppose that $v_{1}^{\prime}$ is not adjacent with $v_{3}^{\prime}$ (or $v_{4}^{\prime}$, without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5 -coloring of the graph $G$. Secondly, assume that $v_{1}^{\prime}$ is not adjacent with $u_{5}^{\prime}$ (or $u_{2}^{\prime}$, without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$.

In a graph $G$, let a neighbor-connected $C_{n}$ be an $n$-cycle $C$ such that the set


Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.
of vertices of $G$ at distance 1 from $C$ is not independent.
Lemma 5.5. Let $G$ be a 4-regular graph with girth $g=5$. If $G$ contains a neighbor-connected $C_{5}$ as induced subgraph, then $\Gamma(G)=5$.

Proof. Let $C$ be a neighbor-connected $C_{5}$ in $G$. By Lemma 5.4 we can suppose that the neighbors of the vertices of $C$ do not form an induced $C_{5}$ (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of $C$ form a subgraph of a $C_{10}$. If there are two edges between the neighbors of the vertices of $C$, then Figure 8 illustrates Grundy partial 5 -colorings of the graph $G$. Suppose that two neighbors are adjacent, say $v_{1}$ and $v_{3}^{\prime}$ and the graph $G$ does not contain the previous configuration. Note that $v_{3}^{\prime}$ can be adjacent with $v_{1}^{\prime}$ and $v_{5}^{\prime}$. Let $w_{1}, w_{2}$ and $w_{3}$ be the three neighbors of $v_{2}$ different from $u_{2}$. We suppose that $w_{1}$ can be possibly adjacent with $v_{3}^{\prime}$ and $w_{2}$ can be possibly adjacent with $v_{1}^{\prime}$. Figure 9 illustrates a Grundy partial 5 -coloring of $G$ in this case. In this figure, the vertex $w_{3}$ can be possibly adjacent with $v_{5}^{\prime}$ or $v_{4}$, but in this case we can switch the color 1 from $v_{5}^{\prime}$ to $v_{5}$ or from $v_{4}^{\prime}$ to $v_{4}$.

Proposition 5.6. If $G$ is a 4-regular graph with girth $g=5$, then $\Gamma(G)=5$.
Proof. Let $C$ be a 5 -cycle in $G$. Assume that two neighbors of consecutive vertices of $C$, for example $v_{1}$ and $v_{5}$, have a common neighbor $w_{1}$. The left part of Figure 10 illustrates a Grundy partial 5 -coloring of the graph $G$. In this figure the vertex $w_{1}$ can be possibly adjacent with $v_{2}^{\prime}, v_{3}^{\prime}$ or $v_{4}$, but in this case we can switch the color 1 from $v_{2}^{\prime}$ to $v_{2}$, from $v_{3}^{\prime}$ to $v_{3}$ or from $v_{4}$ to $v_{4}^{\prime}$. Hence, we can suppose that no neighbors of consecutive vertices of $C$ are adjacent. Among the neighbors of $v_{1}$, there exists one vertex $w_{1}$ not adjacent with both $v_{4}$ and $v_{4}^{\prime}$ (otherwise $G$ would contain a $C_{4}$ ). Among the neighbor of $v_{5}^{\prime}$, there exists one


Figure 8: Two Grundy partial 5 -colorings of a subgraph containing an induced neighbor-connected $C_{5}$.


Figure 9: A Grundy partial 5-coloring of a subgraph containing an induced neighbor-connected $C_{5}$.
vertex, say $w_{2}$, not adjacent with $w_{1}$. The right part of Figure 10 illustrates a Grundy partial 5 -coloring of the graph $G$. In this figure the vertex $w_{1}$ can be possibly adjacent with $v_{4}$ and the vertex $w_{2}$ can be possibly adjacent with $v_{2}^{\prime}$ or $v_{4}$, but in these cases we can switch the color 1 from $v_{2}^{\prime}$ to $v_{2}$ or from $v_{4}$ to $v_{4}^{\prime}$.

In the following lemma and proposition, we consider a graph $G$ of girth $g=6$. Let $u_{1}, u_{2}, u_{3}, u_{4}, u_{5}$ and $u_{6}$ be the vertices in an induced $C_{6}$. Let $v_{1}, v_{1}^{\prime}, v_{2}, v_{2}^{\prime}, v_{3}, v_{3}^{\prime}, v_{4}, v_{4}^{\prime}, v_{5}, v_{5}^{\prime}, v_{6}$ and $v_{6}^{\prime}$ be the remaining neighbors of respectively $u_{1}, u_{2}, u_{3}, u_{4}, u_{5}$ and $u_{6}$ (all different as $g=6$ ).

Lemma 5.7. If $G$ is a 4-regular graph with girth $g=6$ which contains a neighbor-connected $C_{6}$ as induced subgraph, then $\Gamma(G)=5$.

Proof. Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let $w_{1}$ be a neighbor of $v_{1}^{\prime}$ not adjacent with $v_{4}$. The graph $G$ admits a Grundy partial 5 -coloring as the


Figure 10: Two Grundy partial 5-colorings of a subgraph containing an induced $C_{5}$.


Figure 11: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected $C_{6}$.
left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let $w_{1}$ be a neighbor of $v_{3}$ not adjacent with $v_{2}$ and let $w_{2}$ be a neighbor of $v_{1}^{\prime}$ not adjacent with $w_{1}$. The graph $G$ admits a Grundy partial 5 -coloring as the right part of Figure 11 illustrates it.

Proposition 5.8. If $G$ is a 4-regular graph with girth $g=6$, then $\Gamma(G)=5$.
Proof. By Lemma 5.7, assume that no neighbors of the vertices of the induced $C_{6}$ are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle $C_{6}$, for example $v_{1}^{\prime}$ and $v_{5}$, which have a common neighbor $w_{1}$. Let $w_{2}$ be a neighbor of $v_{3}$ not adjacent with $w_{1}$. $G$ admits a Grundy partial 5coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there


Figure 12: Two Grundy partial 5-colorings of a subgraph containing an induced $C_{6}$.
are no two neighbors at distance 4 along the cycle $C_{6}$ which have a common neighbor. Let $w_{1}$ be a neighbor of $v_{1}^{\prime}$ not adjacent with a neighbor of $v_{5}$ or a neighbor of $v_{3}$, let $w_{2}$ be a neighbor of $v_{3}$ not adjacent with a neighbor of $v_{5}$, and let $w_{3}$ be a neighbor of $v_{5}$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.

Proposition 5.9. If $G$ is a 4-regular graph with girth $g \geq 7$, then $\Gamma(G)=5$.
Proof. Suppose that $G$ contains a 7 -cycle. We denote the 5 -atom which is a tree by $T_{5}$ (the binomial tree with maximum degree 4 ). It can be easily verified that $G$ contains $T_{5}$ where two leaves are merged (which is a 5 -atom). Moreover, if $G$ does not contain a 7 -cycle, then it contains $T_{5}$ as induced subgraph.

Theorem 3. Let $G$ be a 4-regular graph. If $G$ does not contain an induced $C_{4}$, then $\Gamma(G)=5$.

Proof. Suppose that $G$ does not contain an induced $C_{4}$. Using Proposition 5.9 for the case $g \geq 7$, Propositions 5.6 and 5.8 for the case $g=5,6$, and Proposition 5.3 when $G$ contains a $C_{3}$ yields the desired result.

By Proposition 3.1, Corollary 3.5 and Theorem 3, any $r$-regular graph with $r \leq 4$ and without induced $C_{4}$ has Grundy number $r+1$. Therefore, it is natural to propose Conjecture 1.

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[^1]:    ${ }^{1}$ Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4 .

