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On the complexity of determining the irregular chromatic index of a graph

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Abstract

An undirected simple graph G is locally irregular if adjacent vertices of G have different degrees. An edge-colouring ϕ of G is locally irregular if each colour class of ϕ induces a locally irregular subgraph of G . The irregular chromatic index $\chi'_{irr}(G)$ of G is the least number of colours used by a locally irregular edge-colouring of G (if any). We show that the problem of determining the irregular chromatic index of a graph can be handled in linear time when restricted to trees, but remains NP-complete in general.

1 Introduction

How to distinguish the vertices of some undirected simple graph G ? One natural way to proceed consists in considering the degrees of the vertices of G , namely to consider that any two vertices are distinguished whenever they have distinct degrees. But distinguishing via the degrees is not relevant in general as it can be easily proved that every simple graph with order at least 2 necessarily has two vertices with the same degree.

To overcome this issue, Chartrand *et al.* proposed the following approach [5]: transform the graph G into some totally irregular *multigraph* G' by replacing each edge e of G by a set of n_e parallel edges, with $n_e \geq 1$. Since two vertices are adjacent in G' if and only if they are adjacent in G , the structures of G and G' are similar. In that case, we are interested in finding such a multigraph G' which minimizes the quantity $\max\{n_e / e \in E(G)\}$.

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This problem can be expressed as an edge-weighting problem as follows. Let $w : E(G) \rightarrow \{1, \dots, k\}$ be a k -edge-weighting of G . For each vertex $v \in V(G)$, define

$$c_w(v) := \sum_{u \in N(v)} w(vu)$$

as the sum of the weights “incident” to v . If c_w is injective, that is $c_w(u) \neq c_w(v)$ for every two vertices u and v of G , we say that w is *vertex-sum-distinguishing*. Note that if w is vertex-sum-distinguishing, the multigraph G' obtained from G by replacing each edge e of G by $w(e)$ parallel edges is totally irregular. Regarding the problem introduced above, we are interested in finding a vertex-sum-distinguishing edge-weighting of G that minimizes the number k of weights. The smallest k for which a graph G admits a vertex-sum-distinguishing k -edge-weighting is called the *irregularity strength* of G in the literature.

The notion of vertex-sum-distinguishing edge-weighting of graphs gave birth to dozens of variants (see e.g. [1, 9, 11], or [7] for a complete survey dedicated to this topic). One such variant, considered by Karoński, Łuczak and Thomason in [9], is defined as follows. A k -edge-weighting w is *neighbour-sum-distinguishing* if $c_w(u) \neq c_w(v)$ for every two *adjacent* vertices of G . The multigraph G' obtained from G by replacing each edge e of G by $w(e)$ parallel edges is now *locally irregular*, in the sense that only adjacent vertices are distinguished by their degrees. The notion of local irregularity is sometimes referred to as *high irregularity* in the literature, see e.g. [3].

In [10], Nierhoff proved that graphs with large order have irregularity strength at most $n - 1$, this upper bound being tight. In contrast, Karoński, Łuczak and Thomason conjectured in [9] that for every graph with no isolated edge one can produce a neighbour-sum-distinguishing 3-edge-weighting:

1-2-3 Conjecture. *Every graph with no isolated edge admits a neighbour-sum-distinguishing 3-edge-weighting.*

We refer the interested reader to [12] for an up-to-date survey dedicated to the 1-2-3 Conjecture. It is still not known whether the 1-2-3 Conjecture is true for regular graphs in general. One way for dealing with this question is to consider the following edge-colouring notion. An (improper) edge-colouring ϕ of G is *locally irregular* if every colour class of ϕ induces a locally irregular subgraph of G . As pointed out in [4], a locally irregular 2-edge-colouring of a regular graph G is also a neighbour-sum-distinguishing 2-edge-weighting of G . Hence, studying locally irregular 2-edge-colourings of graphs can be a way to tackle the 1-2-3 Conjecture in the context of regular graphs.

For a given graph G , we are interested in finding the least number of colours needed by a locally irregular edge-colouring of G (if any), called the

irregular chromatic index of G and denoted by $\chi'_{irr}(G)$. If G does not admit any locally irregular edge-colouring, we say that G is *non-colourable* and let $\chi'_{irr}(G) = \infty$. It was shown in [4] that a graph G is non-colourable if and only if G is either an odd-length path, an odd-length cycle, or a “tree-like” graph obtained by connecting an arbitrary number of triangles in a specific way. Due to their simple structure, such graphs can be recognized in polynomial time.

All known colourable graphs have irregular chromatic index at most 3, and graphs with irregular chromatic index exactly k are known for every $k \in \{1, 2, 3\}$. For instance, we have $\chi'_{irr}(P_3) = 1$, $\chi'_{irr}(P_{2q+1}) = 2$ for every $q \geq 2$, and $\chi'_{irr}(C_{2q'}) = 3$ for every odd $q' \geq 3$, where, for every $n \geq 1$, P_n and C_n denote the path and the cycle on n vertices, respectively. The following conjecture was proposed in [4]:

Local-Irregularity Conjecture. *Every colourable graph has irregular chromatic index at most 3.*

The Local-Irregularity Conjecture was verified for several classes of graphs in [4]. In particular, we have the following result (recall that a tree is non-colourable if and only if it is an odd-length path):

Theorem 1.1 (Baudon *et al.* [4]). *If T is a colourable tree, then $\chi'_{irr}(T) \leq 3$.*

If the Local-Irregularity Conjecture turned out to be true, then all colourable graphs would have irregular chromatic index 1, 2 or 3. Therefore, a natural question is to find out whether it is easy to determine the irregular chromatic index of a given graph. This leads to the following decision problem:

LOCALLY-IRREGULAR k -EDGE-COLOURING

Instance: A graph G .

Question: Do we have $\chi'_{irr}(G) \leq k$?

Clearly, $\chi'_{irr}(G) = 1$ if and only if G is itself locally irregular. Since checking whether a graph is locally irregular can be done in polynomial time, LOCALLY-IRREGULAR 1-EDGE-COLOURING is in P. If the Local-Irregularity Conjecture were true, then any colourable graph would have irregular chromatic index less than k for every $k \geq 3$ and, for such a value of k , the problem LOCALLY-IRREGULAR k -EDGE-COLOURING would thus be equivalent to the problem of determining whether G is colourable (which is easy, as noticed above). Hence, if the Local-Irregularity Conjecture were true, then we would get that LOCALLY-IRREGULAR k -EDGE-COLOURING is in P for every $k \geq 3$.

In this paper, we investigate the status of the remaining problem LOCALLY-IRREGULAR 2-EDGE-COLOURING. We will prove in Section 3 that

LOCALLY-IRREGULAR 2-EDGE-COLOURING is easy when restricted to trees, and in Section 4 that LOCALLY-IRREGULAR 2-EDGE-COLOURING is NP-complete in general. More precisely, we prove the following:

Theorem 1.2. *There is a linear-time algorithm for solving LOCALLY-IRREGULAR 2-EDGE-COLOURING when restricted to trees.*

Theorem 1.3. *LOCALLY-IRREGULAR 2-EDGE-COLOURING is NP-complete in general, even when restricted to planar graphs with maximum degree at most 6.*

Theorem 1.3 notably meets a result proved independently in [2] and [6] stating that it is NP-complete to decide whether a graph admits a neighbour-sum-distinguishing 2-edge-weighting. Before proving Theorems 1.2 and 1.3, we introduce, in Section 2, the definitions, notation and terminology we will use throughout. We end up this paper in Section 5 by raising additional remarks and open questions related to our results.

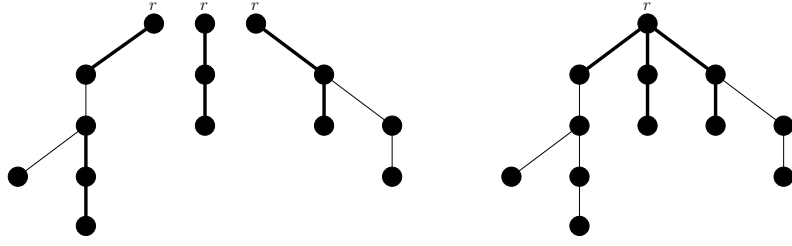
2 Definitions, notation, and terminology

For every graph G , we denote by $V(G)$, $E(G)$ and $\Delta(G)$ its vertex set, edge set and maximum degree, respectively. If $E_0 \subseteq E(G)$, we denote by $G[E_0]$ the subgraph of G induced by E_0 , that is the graph with vertex set $V(G)$ and edge set E_0 . If $u \in V(G)$, we denote by $\deg_G(u)$ the degree of the vertex u in G , or simply by $\deg(u)$ whenever the graph G is clear from the context. Let ϕ be a k -edge-colouring of G and $a \in \{1, \dots, k\}$. An edge e of G with $\phi(e) = a$ is said to be a -coloured. The a -subgraph of G is the subgraph induced by the a -coloured edges of G . If v is some vertex of G , the a -degree of v , denoted by $\deg_{\phi,a}(v)$, is the degree of v in the a -subgraph of G .

Let T be a tree. By choosing a particular node r of T as the root of T , one naturally define an orientation of T from its root to its leaves. The resulting rooted tree is denoted T_r . As usual, according to the orientation of T_r , we can speak of the father or the children of a node u . The father of a node $u \neq r$ will be denoted by u^- . If a non-leaf node u has only one child, this child will be denoted by u^+ . A rooted tree T_r is called a shrub if its root has only one child.

Let u be a node in T_r with $p \geq 1$ children denoted v_1, \dots, v_p . For every $i \in \{1, \dots, p\}$, we denote by $T_r[u, i]$ the subtree of T_r induced by u and the nodes in the subtree of T_r rooted at v_i . Every such $T_r[u, i]$ is a shrub, with $v_i = u^+$ and $u = v_i^-$. Clearly, T_r is isomorphic to the tree obtained by identifying the roots of the shrubs $T_r[r, 1], \dots, T_r[r, \deg(r)]$.

Let T_r be a shrub and $\phi : E(T_r) \rightarrow \{a, b\}$ a 2-edge-colouring of T_r . To make the colour of the edge rr^+ by ϕ explicit, we will denote ϕ by $\phi_{a,b}$ if



(a) Three almost locally irregular 2-edge-colourings $\phi_{a,b}^1$, $\phi_{a,b}^2$ and $\phi_{a,b}^3$ of $T_r[r, 1]$, $T_r[r, 2]$ and $T_r[r, 3]$, respectively.

(b) A locally irregular 2-edge-colouring $\phi_{a,b}$ of T_r .

Figure 1: Examples of almost locally irregular 2-edge-colourings of shrubs and of a locally irregular 2-edge-colouring of a rooted tree. Thick edges stand for a -coloured edges and thin edges for b -coloured edges.

$\phi(rr^+) = a$, or by $\phi_{b,a}$ if $\phi(rr^+) = b$. If $\phi_{a,b}$ is a 2-edge-colouring of T_r that uses colours a and b , and $\{c, d\}$ is a pair of distinct colours, then we can obtain a 2-edge-colouring $\phi_{c,d}$ of T_r using colours c and d by *swapping* $\{a, b\}$ and $\{c, d\}$: $\phi_{c,d}(uv) = c$ if $\phi_{a,b}(uv) = a$, or $\phi_{c,d}(uv) = d$ otherwise. A swapping of $\phi_{a,b}$ to $\phi_{b,a}$ is called an *inversion*. Clearly, a node with a -degree p in T_r by $\phi_{a,b}$ has b -degree p by $\phi_{b,a}$. We further say that $\phi_{a,b}$ is an *almost locally irregular 2-edge-colouring* of T_r if either $\phi_{a,b}$ is a locally irregular 2-edge-colouring of T_r , or rr^+ is isolated in the a -subgraph of T_r and $\phi_{a,b}$ is a locally irregular 2-edge-colouring of $T_r[r, 1]$.

Let T be a tree whose edge set $E(T)$ is partitioned into $p \geq 2$ disjoint subsets $E_1 \cup \dots \cup E_p$, and let $\phi_{a,b}^1, \dots, \phi_{a,b}^p$ be 2-edge-colourings of $T[E_1], \dots, T[E_p]$, respectively. The *union* $\phi_{a,b} = \phi_{a,b}^1 + \dots + \phi_{a,b}^p$ of $\phi_{a,b}^1, \dots, \phi_{a,b}^p$ is the 2-edge-colouring of T_r defined by $\phi_{a,b}(uv) = \phi_{a,b}^i(uv)$ if and only if $uv \in E_i$.

Figure 1 depicts how a locally irregular 2-edge-colouring $\phi_{a,b}$ of a tree T_r with $\deg(r) = 3$ can be obtained by decomposing T_r into three shrubs $T_r[r, 1]$, $T_r[r, 2]$ and $T_r[r, 3]$, computing three almost locally irregular 2-edge-colourings $\phi_{a,b}^1$, $\phi_{a,b}^2$ and $\phi_{a,b}^3$ of these shrubs, and considering the union $\phi_{a,b} = \phi_{a,b}^1 + \phi_{a,b}^2 + \phi_{a,b}^3$.

3 Proof of Theorem 1.2

We herein prove that the irregular chromatic index of every colourable tree can be determined in linear time. For this purpose, we first show, in Section 3.1, how to obtain an almost locally irregular 2-edge-colouring of every shrub. In Section 3.2, we then explain how to “decompose” every colourable

```

1 if  $p = 0$  then
2    $\phi_{a,b}(rr^+) = a$ ;
3 else
4   foreach  $i \in \{1, \dots, p\}$  do
5     compute an almost locally irregular 2-edge-colouring  $\phi_{a,b}^i$  of
6      $T_r[r^+, i]$  inductively;
7    $\phi_{a,b}^0(rr^+) = a$ ;
   choose  $\phi_{c_i, c'_i}^i = \phi_{a,b}^i$  or  $\phi_{b,a}^i$  for every  $i \in \{1, \dots, p\}$  in such a way
   that  $\phi_{a,b} = \phi_{a,b}^0 + \phi_{c_1, c'_1}^1 + \dots + \phi_{c_p, c'_p}^p$  is an almost locally irregular
   2-edge-colouring of  $T_r$ ;

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Algorithm 1: Algorithm for constructing an almost locally irregular 2-edge-colouring $\phi_{a,b}$ of a shrub T_r .

tree T into shrubs, and how to construct a locally irregular edge-colouring of T by composing the almost locally irregular 2-edge-colourings of these shrubs. Then, by carefully studying the situations in which this colouring strategy does not work with only three colours, we characterize the colourable trees with irregular chromatic index exactly 3 in Section 3.3. Using all the previous results and observations, we eventually propose a linear time algorithm for computing the irregular chromatic index of every colourable tree in Section 3.4.

3.1 Every shrub admits an almost locally irregular 2-edge-colouring

Algorithm 1 constructs an almost locally irregular 2-edge-colouring $\phi_{a,b}$ of any shrub T_r . In this algorithm, we denote by p , $p \geq 0$, the number of children of r^+ . Roughly speaking, the algorithm first inductively constructs almost locally irregular 2-edge-colourings $\phi_{a,b}^1, \dots, \phi_{a,b}^p$ of $T_r[r^+, 1], \dots, T_r[r^+, p]$, respectively. It then inverts some of the $\phi_{a,b}^i$'s so that their union is an almost locally irregular 2-edge-colouring of T_r when the edge rr^+ is a -coloured.

The keystone of Algorithm 1 is Line 7. Let us prove that the almost locally irregular 2-edge-colouring $\phi_{a,b}$ of T_r , obtained by inverting some of the $\phi_{a,b}^i$'s, necessarily exists.

Lemma 3.1. *The almost locally irregular 2-edge-colouring $\phi_{a,b}$ of T_r claimed at Line 7 of Algorithm 1 necessarily exists.*

Proof. If $p = 0$, then there is nothing to prove. Thus assume that r^+ has

$p \geq 1$ children v_1, v_2, \dots, v_p in T_r . We first consider small values of p , namely $p \in \{1, 2, 3\}$, before generalizing our arguments.

- Suppose $p = 1$. If $\phi_{a,b} = \phi_{a,b}^0 + \phi_{a,b}^1$ is not an almost locally irregular 2-edge-colouring of T_r , then v_1 has a -degree 2 in $T_r[r^+, 1]$ by $\phi_{a,b}^1$. Besides, $\phi_{a,b}^1$ is a locally irregular 2-edge-colouring of $T_r[r^+, 1]$. The colouring $\phi_{a,b} = \phi_{a,b}^0 + \phi_{b,a}^1$, obtained by inverting $\phi_{a,b}^1$, is thus clearly an almost locally irregular 2-edge-colouring of T_r .
- Suppose $p = 2$. If $\phi_{a,b} = \phi_{a,b}^0 + \phi_{a,b}^1 + \phi_{a,b}^2$ is not an almost locally irregular 2-edge-colouring of T_r , then a child of r^+ , say v_1 , has a -degree 3 in $T_r[r^+, 1]$ by $\phi_{a,b}^1$, and $\phi_{a,b}^1$ is a locally irregular 2-edge-colouring of $T_r[r^+, 1]$. Now consider $\phi_{a,b} = \phi_{a,b}^0 + \phi_{b,a}^1 + \phi_{a,b}^2$. If $\phi_{a,b}$ is not an almost locally irregular 2-edge-colouring of T_r , then the other child v_2 of r^+ has a -degree 2 in $T_r[r^+, 2]$ by $\phi_{a,b}^2$. Moreover, $\phi_{a,b}^2$ is a locally irregular 2-edge-colouring of $T_r[r^+, 2]$. Thus, $\phi_{a,b} = \phi_{a,b}^0 + \phi_{a,b}^1 + \phi_{b,a}^2$ is an almost locally irregular 2-edge-colouring of T_r .
- Suppose $p = 3$. If $\phi_{a,b} = \phi_{a,b}^0 + \phi_{a,b}^1 + \phi_{a,b}^2 + \phi_{a,b}^3$ is not an almost locally irregular 2-edge-colouring of T_r , then a child of r^+ , say v_1 , has a -degree 4 in $T_r[r^+, 1]$ by $\phi_{a,b}^1$, and $\phi_{a,b}^1$ is a locally irregular 2-edge-colouring of $T_r[r^+, 1]$. Now, if $\phi_{a,b}^0 + \phi_{b,a}^1 + \phi_{a,b}^2 + \phi_{a,b}^3$ is not an almost locally irregular 2-edge-colouring of T_r , then another child of r^+ , say v_2 , has a -degree 3 in $T_r[r^+, 2]$ by $\phi_{a,b}^2$, and $\phi_{a,b}^2$ is a locally irregular 2-edge-colouring of $T_r[r^+, 2]$. Again, the a -degree of the last child v_3 of r^+ in $T_r[r^+, 3]$ by $\phi_{a,b}^3$ is 3 if $\phi_{a,b}^0 + \phi_{a,b}^1 + \phi_{b,a}^2 + \phi_{a,b}^3$ is not an almost locally irregular 2-edge-colouring of T_r . Under all these assumptions, we clearly get that $\phi_{a,b} = \phi_{a,b}^0 + \phi_{a,b}^1 + \phi_{b,a}^2 + \phi_{b,a}^3$ is an almost locally irregular 2-edge-colouring of T_r .

By following the same scheme for any $p \geq 4$, i.e. by inverting none of the $\phi_{a,b}^i$'s, then one, two, three, ..., of them, we either find an almost locally irregular 2-edge-colouring $\phi_{a,b}$ of T_r or find out all the values of the a -degrees of v_1, v_2, \dots, v_p in $T_r[r^+, 1], T_r[r^+, 2], \dots, T_r[r^+, p]$ by $\phi_{a,b}^1, \phi_{a,b}^2, \dots, \phi_{a,b}^p$, respectively. More precisely, in this last situation, we get that one of these a -degrees is equal to $p + 1$, two of them are equal to p , three of them are equal to $p - 1$ (unless p is not big enough), and so on. Under the assumption that $p \geq 4$, note that the biggest $\lfloor \frac{p+1}{2} \rfloor$ values of the resulting a -degree sequence are strictly greater than $\lfloor \frac{p+1}{2} \rfloor + 1$, while the other values are strictly greater than $\lfloor \frac{p+1}{2} \rfloor - 1$. Considering that the a -degrees of v_1, v_2, \dots, v_p are ordered decreasingly, i.e. v_1 has a -degree $p + 1$, v_2 has a -degree p , ..., the 2-edge-colouring $\phi_{a,b} = \phi_{a,b}^0 + \phi_{a,b}^1 + \dots + \phi_{a,b}^{\lfloor \frac{p+1}{2} \rfloor} + \phi_{b,a}^{\lfloor \frac{p+1}{2} \rfloor + 1} + \dots + \phi_{b,a}^p$,

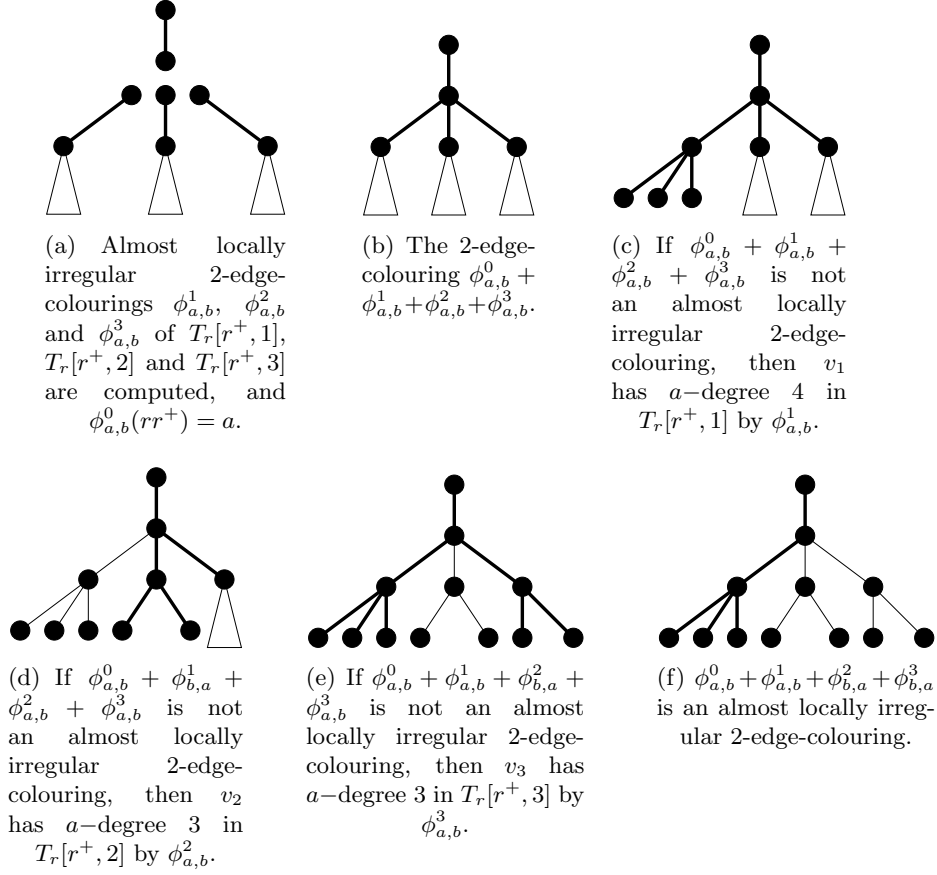


Figure 2: Application of Algorithm 1 on a shrub T_r with $\deg(r^+) = 4$.

obtained by inverting the last $(\lceil \frac{p+1}{2} \rceil - 1)$ almost locally irregular 2-edge-colourings, is an almost locally irregular 2-edge-colouring of T_r since r^+ has then a -degree $\lfloor \frac{p+1}{2} \rfloor + 1$ and b -degree $\lceil \frac{p+1}{2} \rceil - 1$. \square

Figure 2 shows an application of Algorithm 1 on a shrub. Using Algorithm 1, and thanks to Lemma 3.1, we get:

Theorem 3.2. *Every shrub admits an almost locally irregular 2-edge-colouring.*

3.2 From shrubs to trees

Consider the procedure based on Algorithm 1, for possibly computing a locally irregular 2-edge-colouring of any colourable tree T , described as follows. Let r be a node of T with $\deg(r) = p \geq 1$. Start by decomposing T_r into the p shrubs $T_r[r, 1], \dots, T_r[r, p]$, and then compute almost locally

irregular 2-edge-colourings $\phi_{a,b}^1, \dots, \phi_{a,b}^p$ of $T_r[r, 1], \dots, T_r[r, p]$, respectively. These colourings necessarily exist according to Theorem 3.2. Finally, invert some of the $\phi_{a,b}^i$'s so that their union is a locally irregular 2-edge-colouring of T_r .

The success of this colouring procedure is not guaranteed since, in some special cases, inverting the $\phi_{a,b}^i$'s in every possible way does not lead to a locally irregular 2-edge-colouring of T_r . However, the more children the vertex r has, the more possible ways for inverting the $\phi_{a,b}^i$'s there are. Hence, the choice of r for rooting T before applying the above colouring procedure is crucial. Because the number of possibilities for inverting the $\phi_{a,b}^i$'s grows exponentially compared to $\deg(r)$, this strategy actually leads to a locally irregular 2-edge-colouring of T_r whenever $\deg(r) \geq 5$. More precisely, we can prove the following:

Theorem 3.3. *If T is a tree with $\Delta(T) \geq 5$, then $\chi'_{irr}(T) \leq 2$.*

Proof. Let r be a node of T with $p \geq 5$ neighbours v_1, v_2, \dots, v_p . Let $\phi_{a,b}^1, \phi_{a,b}^2, \dots, \phi_{a,b}^p$ be almost locally irregular 2-edge-colourings of $T_r[r, 1], T_r[r, 2], \dots, T_r[r, p]$, respectively, which necessarily exist according to Theorem 3.2. Consider successively the 2-edge-colourings $\phi_{a,b}$ of T_r obtained by inverting none, one, two, ..., of the $\phi_{a,b}^i$'s. If, at some step, $\phi_{a,b}$ is a locally irregular 2-edge-colouring, then the claim is true for T . Otherwise, at each step, a conflict arises because, for at least one of the children v_i of r , the a -degree of v_i in $T_r[r, i]$ by $\phi_{a,b}^i$ is equal to the a -degree of r by $\phi_{a,b}$. In particular, if the 2-edge-colouring obtained by inverting none of the $\phi_{a,b}^i$'s is not a locally irregular 2-edge-colouring of T_r , then we reveal that one of the v_i 's has a -degree p . Similarly, if none of the 2-edge-colourings obtained by inverting one of the $\phi_{a,b}^i$'s is a locally irregular 2-edge-colouring of T_r , then we reveal that two of the v_i 's have a -degree $p-1$. If none of the 2-edge-colourings obtained by inverting two of the $\phi_{a,b}^i$'s is a locally irregular 2-edge-colouring of T_r , then we reveal that three of the v_i 's have a -degree $p-2$, and so on. We stop the procedure once all the a -degrees of the v_i 's have been revealed.

Once the procedure has stopped, we get that the a -degree sequence is $(p, p-1, p-1, p-2, p-2, p-2, \dots)$, where the element $p-k$ appears exactly $k+1$ times, except maybe in the case where $p-k$ is the last value of the sequence. When $p \geq 5$, each of the a -degrees is strictly greater than $\lfloor \frac{p}{2} \rfloor$. Hence, if the a -degrees of v_1, v_2, \dots, v_p are ordered decreasingly, then

$$\phi_{a,b} = \phi_{a,b}^1 + \phi_{a,b}^2 + \dots + \phi_{a,b}^{\lfloor \frac{p}{2} \rfloor} + \phi_{b,a}^{\lfloor \frac{p}{2} \rfloor + 1} + \dots + \phi_{b,a}^p,$$

obtained by inverting the last $\lfloor \frac{p}{2} \rfloor$ $\phi_{a,b}^i$'s, is a locally irregular 2-edge-colouring of T_r since the a - and b -degrees of r are then $\lceil \frac{p}{2} \rceil$ and $\lfloor \frac{p}{2} \rfloor$, respectively,

which are strictly less than the a - and b -degree of its neighbours in the a - and b -subgraphs, respectively. \square

Thanks to Theorems 3.2 and 3.3, we can also give an alternate proof of Theorem 1.1 [4].

Alternate proof of Theorem 1.1. Let T be a colourable tree. If $\Delta(T) \leq 2$, then T is a path with even length and $\chi'_{irr}(T) \leq 2$. If $\Delta(T) \geq 5$, then $\chi'_{irr}(T) \leq 2$ according to Theorem 3.3. Let us thus suppose that $\Delta(T) \in \{3, 4\}$, and let r be a node of T with degree $\deg(r) = p = \Delta(T)$ whose neighbours are denoted by v_1, v_2, \dots, v_p . As in the proof of Theorem 3.3, let $\phi_{a,b}^1, \phi_{a,b}^2, \dots, \phi_{a,b}^p$ be almost locally irregular 2-edge-colourings of $T_r[r, 1], T_r[r, 2], \dots, T_r[r, p]$, respectively (these exist by Theorem 3.2), and try out the inversion procedure. If no locally irregular 2-edge-colouring $\phi_{a,b}$ of T_r can be found, then the revealed a -degree sequence is necessarily $(3, 2, 2)$ if $p = 3$, or $(4, 3, 3, 2)$ if $p = 4$. Assuming that the a -degrees of v_1, v_2, \dots, v_p are ordered decreasingly, the colouring $\phi_{a,b} = \phi_{a,b}^1 + \phi_{b,a}^2 + \phi_{c,a}^3$, where c is a third colour, is a locally irregular 3-edge-colouring of T_r for $p = 3$ since r has then a -, b - and c -degree 1 while its neighbours have degree 3, 2, and 2 in the a -, b - and c -subgraph, respectively. When $p = 4$, a locally irregular 3-edge-colouring of T_r is given, for instance, by $\phi_{a,b} = \phi_{a,b}^1 + \phi_{b,a}^2 + \phi_{b,a}^3 + \phi_{c,a}^4$ since r has then a -, b - and c -degree 1, 2, and 1, respectively, while its neighbours have a -, b - and c -degree 4, 3 and 2, respectively. \square

3.3 Trees with irregular chromatic index 3

We now consider trees with maximum degree at most 4. In the proof of Theorem 1.1, we have pointed out that the colouring procedure presented in Section 3.2 does not always provide a locally irregular 2-edge-colouring of a tree T_r . This typically occurs when the inversion procedure of the $\phi_{a,b}^i$'s fails, that is when none of the possible inversions of some of the $\phi_{a,b}^i$'s is a locally irregular 2-edge-colouring. A simple computation shows that the inversion procedure fails if and only if the a -degree sequence of the v_i 's in the $T_r[r, i]$'s by the $\phi_{a,b}^i$'s is *bad*, namely is (1) if $\deg(r) = 1$, (2, 1) if $\deg(r) = 2$, (3, 2, 2) if $\deg(r) = 3$, or (4, 3, 3, 2) if $\deg(r) = 4$.

Consequently, if there exist almost locally irregular 2-edge-colourings $\psi_{a,b}^1, \psi_{a,b}^2, \dots, \psi_{a,b}^p$ of $T_r[r, 1], T_r[r, 2], \dots, T_r[r, p]$, respectively, leading to a a -degree sequence which is not bad, then inverting some of the $\psi_{a,b}^i$'s necessarily leads to a locally irregular 2-edge-colouring of T_r . We thus now focus on the structure of shrubs T_r with maximum degree at most 4 such that r^+ has the same a -degree by all of the possible almost locally irregular 2-edge-colourings of T_r . If r^+ always has a -degree k in this way, with $k \in \{1, 2, 3, 4\}$, we say that T_r is a k -*bad* shrub.

$\mathbf{D}_0 = \{\mathbf{k}\}$	\mathbf{p}	Signature
$\{1\}$	0	-
	1	$D_1 = \{2\}$
$\{2\}$	1	$D_1 = \{1\}$
	2	$D_1 = \{2\}, D_2 = \{3\}$
	3	$D_1 = \{3\}, D_2 = \{3\}, D_3 = \{4\}$
$\{3\}$	2	$D_1 = \{2\}, D_2 = \{2\}$
	3	$D_1 = \{2\}, D_2 = \{3\}, D_3 = \{4\}$
$\{4\}$	3	$D_1 = \{2\}, D_2 = \{3\}, D_3 = \{3\}$

Table 1: List of all k -bad signatures.

Signature $\mathbf{D}_1, \mathbf{D}_2$	\mathbf{D}_0	Signature $\mathbf{D}_1, \mathbf{D}_2$	\mathbf{D}_0
$\{1\}, \{1\}$	$\{1, 3\}$	$\{2\}, \{3\}$	$\{2\}$
$\{1\}, \{2\}$	$\{2, 3\}$	$\{2\}, \{4\}$	$\{2, 3\}$
$\{1\}, \{3\}$	$\{1, 2\}$	$\{3\}, \{3\}$	$\{1, 2\}$
$\{1\}, \{4\}$	$\{1, 2, 3\}$	$\{3\}, \{4\}$	$\{1, 2\}$
$\{2\}, \{2\}$	$\{3\}$	$\{4\}, \{4\}$	$\{1, 2, 3\}$

Table 2: All possible canonical signatures of T_r and the resulting set D_0 when $p = 2$.

Suppose that r^+ has children v_1, \dots, v_p with $p \geq 0$. For each of the nodes v_i , we denote by D_i the set of all possible a -degrees of v_i in $T_r[r^+, i]$ by all of the possible almost locally irregular 2-edge-colourings of $T_r[r^+, i]$. The set of D_i 's is the *signature* of T_r . Analogously, we denote by D_0 the set of all possible a -degrees of r^+ by all of the almost locally irregular 2-edge-colourings of T_r . According to these definitions, T_r is a k -bad shrub if and only if $D_0 = \{k\}$, that is D_0 is a singleton.

The set D_0 of any shrub T_r can easily be computed thanks to an inductive scheme inspired by Algorithm 1. Roughly speaking, we first compute inductively the sets D_0 of each of the p shrubs $T_r[r^+, 1], \dots, T_r[r^+, p]$. By definition, the set D_0 of $T_r[r^+, i]$ corresponds to the set D_i of T_r . Thanks to the signature D_1, \dots, D_p of T_r which, in some sense, is a compact way for representing the almost locally irregular 2-edge-colourings of the $T_r[r^+, i]$'s, the set D_0 of T_r can finally be deduced. Using this procedure, we are able to identify all k -bad signatures of T_r , that is all signatures making the set D_0 of T_r being $\{k\}$ for every $k \in \{1, 2, 3, 4\}$, as shown in the following:

Theorem 3.4. *All k -bad signatures are those given in Table 1.*

Proof. We consider each possible signature of T_r with regards to $p \leq 3$, the number of children of r^+ . For the sake of simplicity, we here only detail the proof for the easy cases, i.e. $p = 0$ and $p = 1$, so that the reader gets an

Signature D_1, D_2, D_3	D_0	Signature D_1, D_2, D_3	D_0
$\{1\}, \{1\}, \{1\}$	$\{1, 2, 4\}$	$\{2\}, \{2\}, \{2\}$	$\{1, 3, 4\}$
$\{1\}, \{1\}, \{2\}$	$\{1, 3, 4\}$	$\{2\}, \{2\}, \{3\}$	$\{3, 4\}$
$\{1\}, \{1\}, \{3\}$	$\{2, 3, 4\}$	$\{2\}, \{2\}, \{4\}$	$\{1, 3\}$
$\{1\}, \{1\}, \{4\}$	$\{1, 2, 3\}$	$\{2\}, \{3\}, \{3\}$	$\{4\}$
$\{1\}, \{2\}, \{2\}$	$\{1, 3, 4\}$	$\{2\}, \{3\}, \{4\}$	$\{3\}$
$\{1\}, \{2\}, \{3\}$	$\{3, 4\}$	$\{2\}, \{4\}, \{4\}$	$\{1, 3\}$
$\{1\}, \{2\}, \{4\}$	$\{1, 3\}$	$\{3\}, \{3\}, \{3\}$	$\{2, 4\}$
$\{1\}, \{3\}, \{3\}$	$\{2, 4\}$	$\{3\}, \{3\}, \{4\}$	$\{2\}$
$\{1\}, \{3\}, \{4\}$	$\{2, 3\}$	$\{3\}, \{4\}, \{4\}$	$\{2, 3\}$
$\{1\}, \{4\}, \{4\}$	$\{1, 2, 3\}$	$\{4\}, \{4\}, \{4\}$	$\{1, 2, 3\}$

Table 3: All possible canonical signatures of T_r and the resulting set D_0 when $p = 3$.

idea of the technique we use. The remaining cases, i.e. $p = 2$ and $p = 3$, are given in Tables 2 and 3, respectively. Signatures in bold are those which are k -bad for some k . All remaining cases that do not appear in these tables do not concern bad signatures and can be deduced from canonical cases thanks to the following two rules:

- if D_1, \dots, D_p is not a bad signature of T_r , then D'_1, \dots, D'_p is not a bad signature when $D_i \subseteq D'_i$ for every $i \in \{1, \dots, p\}$ (inclusion rule),
- if $D_1, \dots, D_i, D_{i+1}, \dots, D_p$ is a k -bad signature and $D_1, \dots, D'_i, D_{i+1}, \dots, D_p$ is a k' -bad signature with $k' \neq k$ for some $D'_i \neq D_i$, then $D_1, \dots, D_i \cup D'_i, D_{i+1}, \dots, D_p$ is not a bad signature (union rule).

If $p = 0$, then the edge rr^+ has to be coloured a and r^+ thus necessarily has a -degree 1. Therefore, the empty signature is a 1-bad signature.

Now suppose that $p = 1$. If $D_1 = \{1\}$ then, in every almost locally irregular 2-edge-colouring of $T_r[r^+, 1]$, v_1 has a -degree 1 and we have to colour rr^+ with colour a . Thus $D_0 = \{2\}$, and $D_1 = \{1\}$ is a 2-bad signature. Similarly, if $D_1 = \{2\}$, then every almost locally irregular 2-edge-colouring of $T_r[r^+, 1]$ is a locally irregular 2-edge-colouring and we have to invert it before colouring rr^+ with colour a . Therefore, $D_0 = \{1\}$, and $D_1 = \{2\}$ is a 1-bad signature.

If there exists an almost locally irregular 2-edge-colouring $\phi_{a,b}^1$ of $T_r[r^+, 1]$ such that v_1 has a -degree 3 or 4, then we may either colour rr^+ with colour a directly, so that r^+ has a -degree 2, or invert $\phi_{a,b}^1$ before, so that r^+ has a -degree 1. Therefore, $D_0 = \{1, 2\}$ if 3 or 4 belongs to D_1 . Thus, D_1 is not a bad signature whenever it contains 3 or 4.

Finally, $D_1 = \{1, 2\}$ is not a bad signature since we get $D_0 = \{1, 2\}$ by the union rule. Every other possibility for D_1 leads to a set D_0 which is

not a singleton by the inclusion and union rules. Therefore, $D_1 = \{1\}$ and $D_1 = \{2\}$ are the only bad signatures when $p = 1$. \square

Arbitrarily many k -bad shrubs can be constructed, thanks to Theorem 3.4, by connecting together “bad pieces” as follows. First choose a k -bad signature, that is some values p and $D_1 = \{d_1\}, \dots, D_p = \{d_p\}$, corresponding to one row of Table 1. Let then T_r be a rooted tree made of one single edge rr^+ , and T_1, \dots, T_p be d_1- , \dots , d_p -bad shrubs, respectively. Finally, identify the roots of T_1, \dots, T_p with r^+ . The resulting shrub T_r is clearly k -bad.

Suppose that the vertex r has $p \geq 1$ neighbours in a colourable tree T . As explained above, if the shrubs $T_r[r, 1], \dots, T_r[r, p]$ are k_1- , \dots , k_p -bad, respectively, and the sequence (k_1, \dots, k_p) is one of the bad a -degree sequences (1) , $(2, 1)$, $(3, 2, 2)$ or $(4, 3, 3, 2)$, then we cannot produce a locally irregular 2-edge-colouring of T_r by means of the colouring procedure introduced in Section 3.2. In this situation, we say that the vertex r is *bad*.

We end up this section by showing that if r is bad then every node $r' \neq r$ of T is also bad. This implies that $\chi'_{irr}(T) = 3$ if and only if any node of T is bad. By comparing the bad a -degree sequences and the bad signatures from Table 1, we get the following:

Observation 3.5. *If $\{d_1\}, \dots, \{d_p\}$ is a d_0 -bad signature, then (d_0, d_1, \dots, d_p) is a bad sequence. Conversely, if σ is any permutation of $\{d_0, d_1, \dots, d_p\}$ and (d_0, d_1, \dots, d_p) is a bad sequence, then $\{\sigma(d_1)\}, \dots, \{\sigma(d_p)\}$ is a $\sigma(d_0)$ -bad signature.*

Then we have:

Theorem 3.6. *If r is a bad node of T , then every other node $r' \neq r$ of T is also bad.*

Proof. Note that it suffices to prove the claim when r and r' are neighbours in T . Suppose that $\deg(r) = p \geq 1$, $\deg(r') = p' \geq 0$, r' is the first child of r in T_r , and r is the first child of r' in $T_{r'}$ (that is $r' = r^+$ in $T_r[r, 1]$ and $r = (r')^+$ in $T_{r'}[r', 1]$).

Because r is bad, the shrubs $T_r[r, 1], \dots, T_r[r, p]$ are k_1- , \dots , k_p -bad, respectively, and (k_1, \dots, k_p) is a bad a -degree sequence. According to Theorem 3.4, if $T_r[r, 1]$ is k_1 -bad, then $T_r[r', 1], \dots, T_r[r', p' - 1]$ are ℓ_1- , \dots , $\ell_{p'-1}$ -bad, respectively, and $\{\ell_1\}, \dots, \{\ell_{p'-1}\}$ is a k_1 -bad signature. Besides, according to Observation 3.5, the sequence $(k_1, \ell_1, \dots, \ell_{p'-1})$ is bad. Now, because r is bad, $\{k_2\}, \dots, \{k_p\}$ is a k_1 -bad signature, again by Observation 3.5, and $T_{r'}[r', 1]$ is thus a k_1 -bad shrub. Thus, $T_{r'}[r', 1], T_{r'}[r', 2], \dots, T_{r'}[r', p']$ are k_1- , ℓ_1- , \dots , $\ell_{p'-1}$ -bad shrubs, respectively, and $(k_1, \ell_1, \dots, \ell_{p'-1})$ is a bad sequence. Therefore, r' is bad. \square

```

1 if  $T$  is an odd-length path then
2    $\chi'_{irr}(T)$  is undefined;
3 else if  $T$  is locally irregular then
4    $\chi'_{irr}(T) = 1$ ;
5 else if  $\Delta(T) \leq 2$  or  $\Delta(T) \geq 5$  then
6    $\chi'_{irr}(T) = 2$ ;
7 else
8   choose an arbitrary node  $r$  of  $T$  with  $\deg(r) = p \geq 1$ ;
9   foreach  $i \in \{1, 2, \dots, p\}$  do
10    let  $D_i$  be the set  $D_0$  of  $T_r[r, i]$  computed inductively;
11    if  $D_i$  is not a singleton then
12       $\chi'_{irr}(T) = 2$ ;
13      exit algorithm;
14    let  $D_i = \{d_i\}$  for every  $i \in \{1, 2, \dots, p\}$ ;
15    if  $(d_1, d_2, \dots, d_p)$  is not a bad  $a$ -degree sequence then
16       $\chi'_{irr}(T) = 2$ ;
17    else
18       $\chi'_{irr}(T) = 3$ ;

```

Algorithm 2: Algorithm for determining the irregular chromatic index of any tree T .

Corollary 3.7. *If T is a colourable tree, then $\chi'_{irr}(T) = 3$ if and only if any node of T is bad.*

All trees with irregular chromatic index 3 can be constructed as follows. First choose one of the bad sequences (d_1, \dots, d_p) , and construct p shrubs T_1, \dots, T_p which are d_1 - , . . . , d_p -bad, respectively. Recall that there are infinitely many such shrubs as observed above. Finally, identify the roots of T_1, \dots, T_p . By construction, the node used for the identification is bad, and the obtained tree thus has irregular chromatic index 3 according to Corollary 3.7.

3.4 A linear-time algorithm for computing the irregular chromatic index of trees

We now propose an algorithm that determines, thanks to our previous results, the irregular chromatic index of an input tree T . Recall that the bad a -degree sequences are (1) , $(2, 1)$, $(3, 2, 2)$ and $(4, 3, 3, 2)$.

Theorem 3.8. *Algorithm 2 determines the irregular chromatic index of any tree T of order n in time $\mathcal{O}(n)$.*

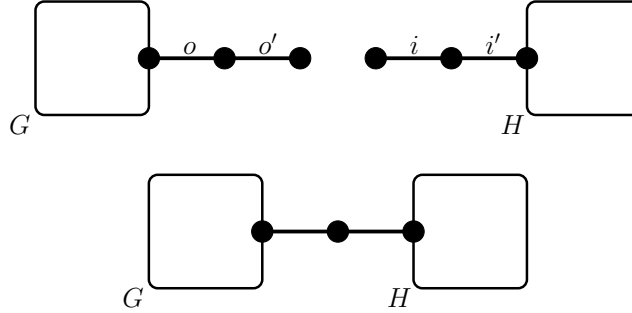


Figure 3: Two graphs G and H with output (o, o') and input (i, i') , respectively, and the connection of G and H along (o, o') and (i, i') .

Proof. The correctness of Algorithm 2 follows from the previous results and observations. In particular, the correctness of Lines 5-6 follows from Theorem 3.3, while the correctness of Lines 11-12 and Lines 15-16 follows from observations raised in Section 3.3. The correctness of Lines-17-18 follows from Corollary 3.7. The most costly instruction of Algorithm 2 is Line 10, which is achieved in time $\mathcal{O}(n)$ by computing the values of D_0 from the leaves to the root for each shrub, as in the proof of Theorem 3.4. Every other line of the algorithm runs either in time $\mathcal{O}(1)$ or $\mathcal{O}(n)$. Therefore, Algorithm 2 runs in time $\mathcal{O}(n)$. \square

4 Proof of Theorem 1.3

In this section, we prove Theorem 1.3 by exhibiting a polynomial-time reduction from 1-IN-3 SATISFIABILITY to LOCALLY-IRREGULAR 2-EDGE-COLOURING. For this purpose, we will need the upcoming graph constructions and additional definitions.

An *input* (resp. *output*) of some graph G is a pair of edges (i, i') (resp. (o, o')) such that $i = uv$ and $i' = vw$ (resp. $o = uv$ and $o' = vu$) and u and v have degree 1 and 2 in G , respectively. Consider now two graphs G and H such that (o, o') and (i, i') are an output of G and an input of H , respectively. The *connection of G and H along (o, o') and (i, i')* is the graph obtained by taking the disjoint union of G and H , and then identifying the edges o and i , and o' and i' . The inputs and outputs of the resulting graph are those of G and H that have not been used for the connection. This construction is depicted in Figure 3.

Given two outputs (o_1, o'_1) and (o_2, o'_2) of G with $o'_1 = v_1w_1$ and $o'_2 = v_2w_2$, the *identification of the two outputs (o_1, o'_1) and (o_2, o'_2)* is obtained by identifying the vertices w_1 and w_2 in G . The identification of more than two outputs of G is defined analogously. According to the original definition,

note that identified outputs of G are not outputs of the resulting graph.

We are now ready to prove the NP-completeness of LOCALLY-IRREGULAR 2-EDGE-COLOURING. Given a graph G and a 2-edge-colouring ϕ of G , one can check in polynomial time whether the two subgraphs of G induced by ϕ are locally irregular or not. Therefore, LOCALLY-IRREGULAR 2-EDGE-COLOURING is in NP. We now show the NP-completeness of LOCALLY-IRREGULAR 2-EDGE-COLOURING by reduction from the following NP-complete problem (refer e.g. to [8] for more details on computational complexity theory).

1-IN-3 SATISFIABILITY

Instance: A 3CNF formula F with clauses C_1, C_2, \dots, C_m and variables x_1, x_2, \dots, x_n .

Question: Is there a 1-in-3 truth assignment of the variables of F , that is a truth assignment such that each clause of F has exactly one true literal?

Let us make some observations on the structure of F . First, we can assume that every possible literal appears in F . Indeed, if ℓ_i does not appear in any clause of F , then the 3CNF formula

$$F' = F \wedge (\ell_i \vee \bar{\ell}_i \vee \bar{x}_{n+1}) \wedge (x_{n+1} \vee \bar{x}_{n+1} \vee \bar{x}_{n+1}),$$

where x_{n+1} is a new variable, clearly admits a 1-in-3 truth assignment of its variables if and only if F admits one too. Since there are $2n$ literals related to the variables of F , a formula equivalent to F that contains every possible literal over its variables can be obtained from F in polynomial time.

Next, note that no clause of the form $(x_i \vee x_i \vee x_i)$ can be satisfied by exactly one of its literals by any truth assignment of the variables of F . Therefore, we may assume that F does not contain any such clause. Finally, observe that if F contains a clause of the form $(x_i \vee x_i \vee x_j)$ then, in every 1-in-3 truth assignment of the variables of F , x_i and x_j have to be set to false and true, respectively. In that case, we say that x_i and x_j are *forced* to false and true, respectively.

For each formula F , we will build a graph G_F such that F is satisfiable in a 1-in-3 way if and only if $\chi'_{irr}(G_F) = 2$. It is convenient to view the graph G_F as an electrical circuit through which two opposite signals, namely *true* and *false*, are propagated through several components whose inputs and outputs are connected in a specific way. The graph G_F is constructed in such a way that the propagation of these signals has the same properties as the “propagation” of a locally irregular 2-edge-colouring in a graph.

The graph G_F is built from the following components. First, the *generator gadget* $G_F(S)$ of G_F spreads the *true* signal through m *clause gadgets* $G_F(C_1), G_F(C_2), \dots, G_F(C_m)$, where each clause C_i in F is associated with the clause gadget $G_F(C_i)$ in G_F . The clause gadgets then modulate their

input *true* signal, that is they can switch their input signal or not, and propagate it to the *literal gadgets* $G_F(\ell_1), G_F(\ell_2), \dots, G_F(\ell_{2n})$ of G_F . Similarly as for the clauses of F and the clause gadgets of G_F , there is a direct analogy between the literal ℓ_i and the literal gadget $G_F(\ell_i)$.

The clause and literal gadgets are linked in the following way:

- if $C_i = (\ell_{i_1} \vee \ell_{i_2} \vee \ell_{i_3})$ is a clause of F , then $G_F(C_i)$ is connected to $G_F(\ell_{i_1})$ along an output of $G_F(C_i)$. If $\ell_{i_2} \neq \ell_{i_1}$, then $G_F(C_i)$ is connected to $G_F(\ell_{i_2})$ along a second output of $G_F(C_i)$. Finally, if $\ell_{i_3} \neq \ell_{i_1}$ and $\ell_{i_3} \neq \ell_{i_2}$, then $G_F(C_i)$ is also connected to $G_F(\ell_{i_3})$ along a third output.
- Exactly one arbitrary output of each clause gadget spreads the *true* signal, while its other outputs (there are at most two of them) propagate the *false* signal.

For every $i \in \{1, 2, \dots, 2n\}$, the literal gadget $G_F(\ell_i)$ of G_F has exactly one output and, according to the connection with the clause gadgets, exactly n_i inputs, where n_i is the number of distinct clauses in F that contain the literal ℓ_i . The main property of a literal gadget $G_F(\ell_i)$ is that it outputs a signal if and only if the same signal comes in from its n_i inputs. Finally, the outputs of the two literal gadgets $G_F(x_i)$ and $G_F(\bar{x}_i)$ are linked in such a way that the propagation of the signal is correct if and only if the two output signals are different.

Hence, we have the following analogy between satisfying F in a 1-in-3 way and spreading the *true* signal through G_F :

- each clause C_i in F must have exactly one true literal and exactly one output of $G_F(C_i)$ must spread the *true* signal out,
- every literal ℓ_i must have the same truth value in all clauses it appears in and all the inputs of $G_F(\ell_i)$ must spread the same signal in,
- a variable x_i and its negation \bar{x}_i must have distinct truth values and the outputs of $G_F(x_i)$ and $G_F(\bar{x}_i)$ must spread different signals out.

We now go into the details of the proof by introducing the generator, clause and literal gadgets of G_F . The graph G_F is constructed step by step as it is augmented by connecting these gadgets. All along this proof, the function $\phi : E(G_F) \rightarrow \{0, 1\}$ is a locally irregular 2-edge-colouring of G_F , propagated along the edges of G_F . When augmenting G_F with a new gadget, it should be understood that ϕ is extended to this new gadget according to the lemmas we point out. The *true* (resp. *false*) signal is depicted as thick (resp. thin) edges in our schemas and corresponds to colour 1 (resp. 0) of ϕ .

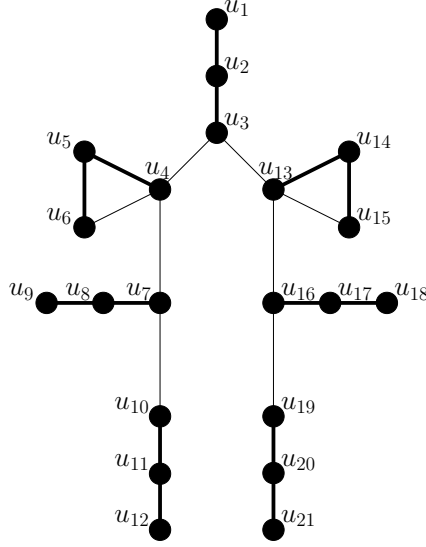


Figure 4: The gadget G^* and a locally irregular 2-edge-colouring of G^* .

The generator gadget $G_F(S)$ of G_F is obtained by connecting several copies of the graph G^* , depicted in Figure 4, with input (u_1u_2, u_2u_3) and outputs $(u_{10}u_{11}, u_{11}u_{12})$ and $(u_{19}u_{20}, u_{20}u_{21})$. The gadget G^* has the following property:

Lemma 4.1. *In every locally irregular 2-edge-colouring of G^* , the input and output edges of G^* have the same colour.*

Proof. We initiate the locally irregular 2-edge-colouring ϕ of G^* with the input of G^* . Let us suppose, without loss of generality, that $\phi(u_1u_2) = 1$. Then we have $\phi(u_2u_3) = 1$, since otherwise we would have $d_{\phi,1}(u_1) = d_{\phi,1}(u_2) = 1$, and $\phi(u_3u_4) = \phi(u_3u_{13})$, since otherwise we would have $d_{\phi,1}(u_2) = d_{\phi,1}(u_3) = 2$.

Let us first suppose that $\phi(u_3u_4) = \phi(u_3u_{13}) = 1$. We cannot have $\phi(u_4u_5) = \phi(u_4u_6)$ since, when colouring u_5u_6 , we would have either $d_{\phi,1}(u_5) = d_{\phi,1}(u_6)$ or $d_{\phi,0}(u_5) = d_{\phi,0}(u_6)$. Assume thus that $\phi(u_4u_5) = 1$ and $\phi(u_4u_6) = 0$. If $\phi(u_5u_6) = 1$, then we must set $\phi(u_4u_7) = 0$ to get $d_{\phi,0}(u_6) \neq d_{\phi,0}(u_4)$. But then $d_{\phi,1}(u_4) = d_{\phi,1}(u_5) = 2$. Similarly, if $\phi(u_5u_6) = 0$, then we must have $\phi(u_4u_7) = 1$ since otherwise we would have $d_{\phi,0}(u_6) = d_{\phi,0}(u_4) = 2$. But now we have $d_{\phi,1}(u_3) = d_{\phi,1}(u_4) = 3$. Hence, we cannot extend ϕ to G^* if $\phi(u_3u_4) = \phi(u_3u_{13}) = 1$.

Thus, we must have $\phi(u_3u_4) = \phi(u_3u_{13}) = 0$. For the same reasons as before, we can assume $\phi(u_4u_5) = 1$ and $\phi(u_4u_6) = 0$ without loss of generality. Now, if $\phi(u_5u_6) = 0$, then we must have $\phi(u_4u_7) = 0$, so that $d_{\phi,0}(u_4) \neq d_{\phi,0}(u_6)$. But then we get $d_{\phi,1}(u_4) = d_{\phi,1}(u_5) = 1$. Therefore,

we must set $\phi(u_5u_6) = 1$. To ensure that u_4 and u_3 have distinct degrees in the 0-subgraph, we need to have $\phi(u_4u_7) = 0$. Similarly as before, we must have $\phi(u_9u_8) = \phi(u_8u_7)$. If $\phi(u_9u_8) = \phi(u_8u_7) = 0$, then we must have $\phi(u_7u_{10}) = 0$ so that $d_{\phi,0}(u_7) \neq d_{\phi,0}(u_8)$. But then we get $d_{\phi,0}(u_4) = d_{\phi,0}(u_7) = 3$. Therefore, $\phi(u_9u_8) = \phi(u_8u_7) = 1$ and $\phi(u_7u_{10}) = 0$, since otherwise we would have $d_{\phi,1}(u_8) = d_{\phi,1}(u_7) = 2$. Because $d_{\phi,0}(u_7) = 2$, we must have $\phi(u_{10}u_{11}) = 1$. Besides, to get $d_{\phi,1}(u_{10}) \neq d_{\phi,1}(u_{11})$, we have to set $\phi(u_{11}u_{12}) = 1$.

The locally irregular 2-edge-colouring ϕ is propagated to the remaining edges of G^* in a symmetric way. We finally get that the input and outputs of G^* have the same colour via ϕ , as claimed. \square

According to Lemma 4.1, the gadget G^* propagates its input colour, say the thick one, into two directions. Moreover, by connecting two copies G_1 and G_2 of G^* along one output of G_1 and the input of G_2 , we can propagate the thick colour into three directions. By successively repeating this construction, one can spread the thick colour towards an arbitrary number of directions.

Now consider the path $P_5 = u_1u_2u_3u_4u_5$ on 5 vertices with input (u_1u_2, u_2u_3) and output (u_3u_4, u_4u_5) , and denote by G' the graph obtained by connecting G^* and P_5 along $(u_{10}u_{11}, u_{11}u_{12})$ and (u_1u_2, u_2u_3) . Clearly, in every extension of a locally irregular 2-edge-colouring of G^* to G' , the two new output edges u_3u_4 and u_4u_5 of G' are coloured with the thin colour. Hence, we may also spread the thin colour as well.

The generator gadget $G_F(S)$ of G_F is obtained by combining the previous remarks, that is by connecting several copies of G^* and P_5 .

From now on, we suppose that ϕ is the locally irregular 2-edge-colouring of G_F mentioned before. This colouring is initiated with $G_F(S)$ in such a way that its input is 1-coloured. According to our terminology, $G_F(S)$ has a finite number of outputs which are either *positive* or *negative*, depending on whether they are 1- or 0-coloured via ϕ , respectively. The number of these outputs will be clarified later.

In what follows, a (k_1, k_2) -vertex a of G_F for some $k_1, k_2 \geq 1$ is a vertex of degree $k_1 + k_2$ resulting from the identification of k_1 distinct positive outputs of $G_F(S)$ and k_2 distinct negative outputs of $G_F(S)$. Note that this resulting vertex a has degree k_1 (resp. k_2) in the 1- (resp. 0-) subgraph of $G_F(S)$. According to our terminology, we have $d_{\phi,1}(a) = k_1$ and $d_{\phi,0}(a) = k_2$. Besides, it should be understood that a subgraph of G_F containing a (k_1, k_2) -vertex is implicitly connected to $G_F(S)$.

We now introduce the clause gadgets $G_F(C_1), G_F(C_2), \dots, G_F(C_m)$ that are connected to some outputs of $G_F(S)$. Depending on the number $c_i \in$

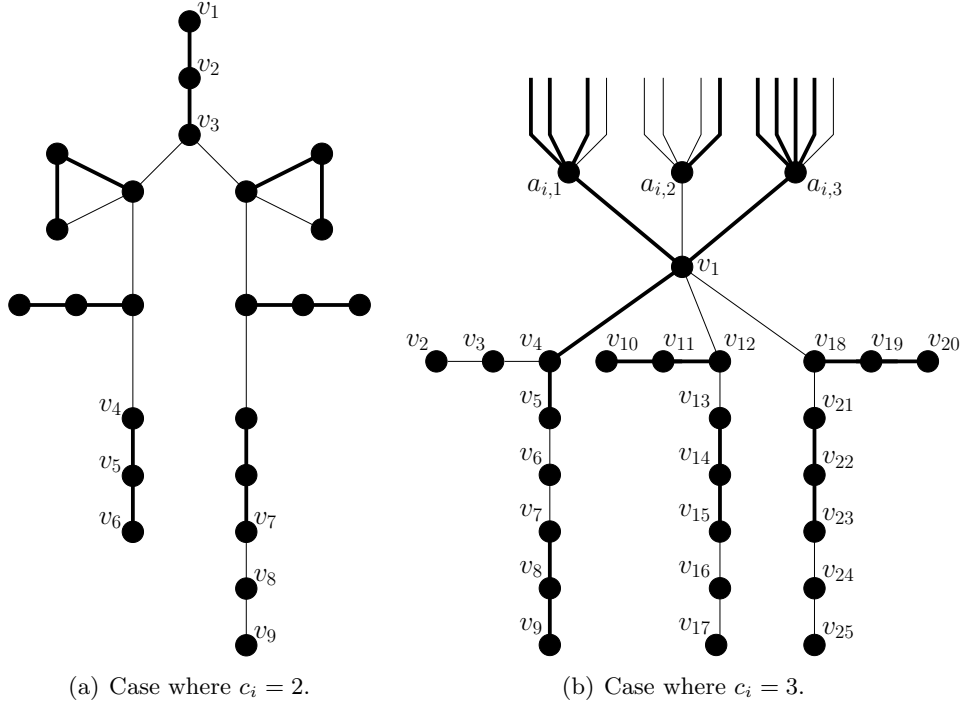


Figure 5: The clause gadget $G_F(C_i)$ and locally irregular 2-edge-colourings of $G_F(C_i)$.

$\{2, 3\}$ of distinct literals in C_i (recall that $c_i \neq 1$), the clause gadget $G_F(C_i)$ can be of two different forms.

- If $c_i = 2$, then $G_F(C_i)$ is the graph depicted in Figure 5.a, obtained by connecting a copy of G^* and P_5 in such a way that $G_F(C_i)$ has one positive output and one negative output. The graph $G_F(C_i)$ is connected to $G_F(S)$ along one positive output of $G_F(S)$ and (v_1v_2, v_2v_3) , where (v_1v_2, v_2v_3) is the input of $G_F(C_i)$.
- If $c_i = 3$, then $G_F(C_i)$ is a copy of the graph depicted in Figure 5.b, where $a_{i,1}, a_{i,2}$ and $a_{i,3}$ are $(3, 1)$ -, $(1, 3)$ - and $(4, 1)$ -vertices, respectively, and (v_7v_8, v_8v_9) , $(v_{15}v_{16}, v_{16}v_{17})$ and $(v_{23}v_{24}, v_{24}v_{25})$ are its three outputs.

When $c_i = 3$, the clause gadget $G_F(C_i)$ has the following property.

Lemma 4.2. *Suppose $c_i = 3$. In any extension of ϕ to $G_F(C_i)$, exactly one output of $G_F(C_i)$ is 1-coloured.*

Proof. Since $a_{i,1}$ is adjacent to a vertex of $G_F(S)$ with degree 2 in the 0-subgraph, we must set $\phi(a_{i,1}v_1) = 1$. Similarly, we have to set $\phi(a_{i,2}v_1) =$

0 and $\phi(a_{i,3}v_1) = 1$. We thus get $d_{\phi,1}(a_{i,1}) = d_{\phi,0}(a_{i,2}) = 4$ and $d_{\phi,1}(a_{i,3}) = 5$. If the number of 1-coloured edges in $\{v_1v_4, v_1v_{12}, v_1v_{18}\}$ were exactly 0, 1 or 2, then we would have $d_{\phi,0}(v_1) = d_{\phi,0}(a_{i,2})$, $d_{\phi,1}(v_1) = d_{\phi,1}(a_{i,1})$ or $d_{\phi,1}(v_1) = d_{\phi,1}(a_{i,3})$, respectively. Therefore, exactly one such edge is 1-coloured.

Let us suppose $\phi(v_1v_4) = 1$ without loss of generality. Observe that $d_{\phi,1}(v_1) = d_{\phi,0}(v_1) = 3$. Again, the edges v_2v_3 and v_3v_4 have to be coloured with the same colour, but this colour cannot be 1. Indeed, if $\phi(v_2v_3) = \phi(v_3v_4) = 1$, then we would have $d_{\phi,1}(v_3) = d_{\phi,1}(v_4) = 2$. Thus, we have to set $\phi(v_4v_5) = 1$ but then we get $d_{\phi,1}(v_4) = d_{\phi,1}(v_1) = 3$. Hence, we necessarily have $\phi(v_2v_3) = \phi(v_3v_4) = 0$, and $\phi(v_4v_5) = 1$ since otherwise we would get $d_{\phi,0}(v_3) = d_{\phi,0}(v_4) = 2$. Now, because $d_{\phi,1}(v_4) = 2$, we need $\phi(v_5v_6) = \phi(v_6v_7) = 0$. Analogously, we have $\phi(v_7v_8) = \phi(v_8v_9) = 1$.

Repeating the same arguments towards v_{17} and v_{25} , and considering that $\phi(v_1v_{12}) = \phi(v_1v_{18}) = 0$, we get $\phi(v_{15}v_{16}) = \phi(v_{16}v_{17}) = 0$ and $\phi(v_{23}v_{24}) = \phi(v_{24}v_{25}) = 0$. Finally, observe that this colouring of $G_F(C_i)$ via ϕ is not unique (it depends on which edge from $\{v_1v_4, v_1v_{12}, v_1v_{18}\}$ is 1-coloured). But in every extension of ϕ to $G_F(C_i)$, exactly one input of $G_F(C_i)$ is 1-coloured. \square

We finally clarify the literal gadgets. Recall that for $i \in \{1, 2, \dots, 2n\}$, n_i denotes the number of distinct clauses of F that contains ℓ_i . Besides, we have $n_i \geq 1$ for every such integer by assumption.

The outputs of the clause gadgets $G_F(C_1), G_F(C_2), \dots, G_F(C_m)$ are now connected with the literal gadgets $G_F(\ell_1), G_F(\ell_2), \dots, G_F(\ell_{2n})$ of G_F as follows. The literal gadget $G_F(\ell_i)$ has one output and exactly n_i inputs connected to n_i clause gadgets of G_F , according to which clauses of F contain ℓ_i . More precisely, if ℓ_i is contained at least once in C_j , then we connect $G_F(C_j)$ and $G_F(\ell_i)$ along exactly one output of $G_F(C_j)$ and one input of $G_F(\ell_i)$. The output of $G_F(C_j)$ used for the connection is chosen arbitrarily, except in the case where $c_j = 2$. Recall that, in this case, C_j has two distinct literals x_i and x'_i that are forced to false and true, respectively. To model this constraint, we force one input of $G_F(x_i)$ to be 0-coloured while one input of $G_F(x'_i)$ is 1-coloured. This is done by using a negative and a positive output of $G_F(C_j)$, respectively, for the connection of $G_F(C_j)$ and $G_F(x_i)$, and $G_F(C_j)$ and $G_F(x'_i)$.

The structure of the literal gadget $G_F(\ell_i)$ depends on the value of n_i :

- if $n_i = 1$, then $G_F(\ell_i)$ is a copy of $P_7 = u_1u_2 \dots u_6u_7$ and has input (u_1u_2, u_2u_3) and output (u_5u_6, u_6u_7) ,
- if $n_i = 2$, then $G_F(\ell_i)$ is a copy of the graph depicted in Figure 6.a, with inputs (w_1w_2, w_2w_5) and (w_3w_4, w_4w_5) , and output $(w_{11}w_{12}, w_{12}w_{13})$,

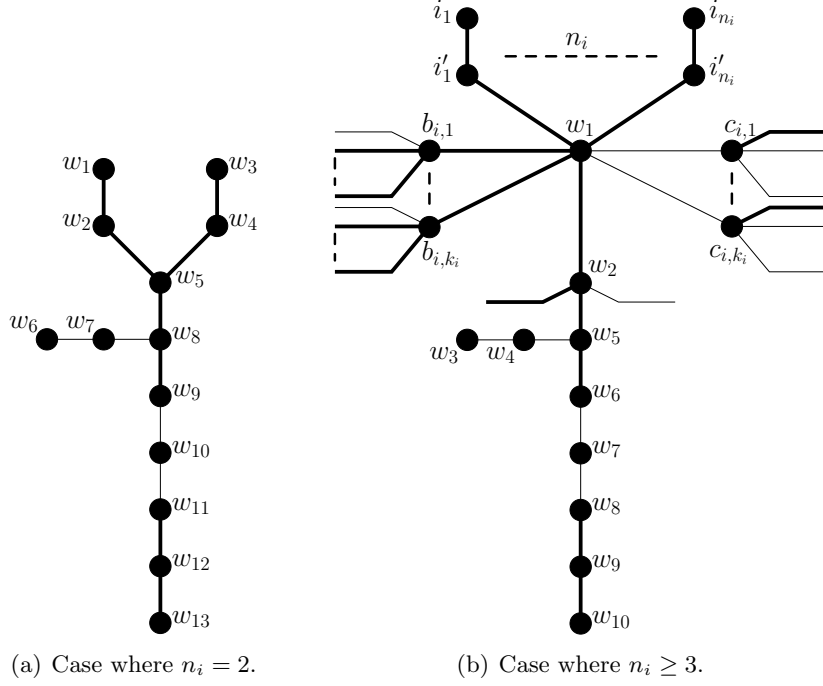


Figure 6: The literal gadget $G_F(\ell_i)$ in the cases where $n_i \geq 2$ and locally irregular 2-edge-colourings of $G_F(\ell_i)$.

- if $n_i \geq 3$, then $G_F(\ell_i)$ is a copy of the graph depicted in Figure 6.b, with inputs $(i_1 i'_1, i'_1 w_1), \dots, (i_{n_i} i'_{n_i}, i'_{n_i} w_1)$ and output $(w_8 w_9, w_9 w_{10})$. In this gadget, the vertices $b_{i,1}, b_{i,2}, \dots, b_{i,k_i}$ (resp. $c_{i,1}, c_{i,2}, \dots, c_{i,k_i}$), where $k_i = \lfloor \frac{n_i}{2} \rfloor + 1$, are $(n_i, 1)-, (n_i+1, 1)-, \dots, (n_i + \lfloor \frac{n_i}{2} \rfloor, 1)-$ vertices (resp. $(1, n_i)-, (1, n_i+1)-, \dots, (1, n_i + \lfloor \frac{n_i}{2} \rfloor)-$ vertices), respectively. Besides, the vertex w_2 results from the identification of a positive output and a negative output of $G_F(S)$.

When $n_i = 2$ or $n_i \geq 3$, the important property of $G_F(\ell_i)$ is that ϕ can be propagated to $G_F(\ell_i)$ if and only if all of its inputs have the same colour. This is proved via the following two lemmas.

Lemma 4.3. *Suppose $n_i = 2$. The colouring ϕ is extendible to $G_F(\ell_i)$ if and only if all the inputs of $G_F(\ell_i)$ have the same colour. Moreover, in any extension of ϕ to $G_F(\ell_i)$, the output of $G_F(\ell_i)$ is coloured with the input colour of $G_F(\ell_i)$.*

Proof. Suppose $\phi(w_1 w_2) = \phi(w_2 w_5) = 1$ and $\phi(w_3 w_4) = \phi(w_4 w_5) = 0$, without loss of generality. If $\phi(w_5 w_8) = 1$, then $d_{\phi,1}(w_2) = d_{\phi,1}(w_5) = 2$. Similarly, if $\phi(w_5 w_8) = 0$, then $d_{\phi,0}(w_4) = d_{\phi,0}(w_5) = 2$. Thus ϕ is only extendible to $G_F(\ell_i)$ when $\phi(w_1 w_2) = \phi(w_2 w_5) = \phi(w_3 w_4) = \phi(w_4 w_5)$.

Let us thus suppose $\phi(w_1w_2) = \phi(w_2w_5) = \phi(w_3w_4) = \phi(w_4w_5) = 1$, without loss of generality. Then, we have to set $\phi(w_5w_8) = 1$ since otherwise we would have $d_{\phi,1}(w_2) = d_{\phi,1}(w_5) = d_{\phi,1}(w_4) = 2$. For the same reasons as before, we necessarily have $\phi(w_6w_7) = \phi(w_7w_8)$. If this colour is 1, then we need to set $\phi(w_8w_9) = 1$ to distinguish w_7 and w_8 in the 1-subgraph, but then $d_{\phi,1}(w_5) = d_{\phi,1}(w_8) = 3$. So, $\phi(w_6w_7) = \phi(w_7w_8) = 0$ and we need to set $\phi(w_8w_9) = 1$ since otherwise the 0-subgraph would have two adjacent vertices with degree 2. Since $d_{\phi,1}(w_8) = 2$, the colouring ϕ is propagated alternatively along the path $w_9w_{10}w_{11}w_{12}w_{13}$ in such a way that $\phi(w_9w_{10}) = \phi(w_{10}w_{11}) = 0$ and $\phi(w_{11}w_{12}) = \phi(w_{12}w_{13}) = 1$. \square

We show that the same property holds for every $n_i \geq 3$.

Lemma 4.4. *Suppose $n_i \geq 3$. The colouring ϕ is extendible to $G_F(\ell_i)$ if and only if all the inputs of $G_F(\ell_i)$ have the same colour. Moreover, in any extension of ϕ to $G_F(\ell_i)$, the output of $G_F(\ell_i)$ is coloured with the input colour of $G_F(\ell_i)$.*

Proof. For the same reasons as before, the edges $b_{i,1}w_1, b_{i,2}w_1, \dots, b_{i,k_i}w_1$ (resp. $c_{i,1}w_1, c_{i,2}w_1, \dots, c_{i,k_i}w_1$) have to be 1-coloured (resp. 0-coloured) in any extension of ϕ to $G_F(\ell_i)$. Therefore, the vertex w_1 is adjacent to vertices with degree $n_i + 1, n_i + 2, \dots, n_i + \lfloor \frac{n_i}{2} \rfloor + 1$ in both the 1- and the 0-subgraphs.

Suppose now that y inputs of $G_F(\ell_i)$ are 1-coloured via ϕ , where $\lceil \frac{n_i}{2} \rceil \leq y \leq n_i - 1$. Then w_1 has degree $y + k_i$ (if $\phi(w_1w_4) = 0$) or $y + k_i + 1$ (otherwise) in the 1-subgraph, but these two values belong to the set $\{n_i + 1, n_i + 2, \dots, n_i + \lfloor \frac{n_i}{2} \rfloor + 1\}$. Therefore, all the inputs of $G_F(\ell_i)$ must have the same colour via ϕ . Assume that this colour is 1, without loss of generality. Then, we necessarily have $\phi(w_1w_2) = 1$ since otherwise we would have $d_{\phi,1}(w_1) = d_{\phi,1}(b_{i,k_i}) = n_i + \lfloor \frac{n_i}{2} \rfloor + 1$. Observe that so far, the 0-subgraph is locally irregular since $k_i < n_i + 1$ for every $n_i \geq 3$.

Since w_2 has degree 2 in the 1-subgraph and is already adjacent to a vertex with degree 2 in the 1-subgraph, we need to set $\phi(w_2w_5) = 1$. If we set $\phi(w_3w_4) = \phi(w_4w_5) = 1$, then we need to have $\phi(w_5w_6) = 1$ so that w_4 and w_5 do not have the same degree in the 1-subgraph. But then we have $d_{\phi,1}(w_5) = d_{\phi,1}(w_2) = 3$. Thus, $\phi(w_3w_4) = \phi(w_4w_5) = 0$ and $\phi(w_5w_6) = 1$ since otherwise w_4 and w_5 would have degree 2 in the 0-subgraph. Now, w_5 has degree 2 in the 1-subgraph and, thus, the colouring must alternate along $w_6w_7w_8w_9w_{10}$. For similar reasons as before, this is done in such a way that $\phi(w_8w_9) = \phi(w_9w_{10}) = 1$. \square

Finally, for each variable x_i in F , we identify the outputs of the literal gadgets $G_F(x_i)$ and $G_F(\bar{x}_i)$. Observe that ϕ is a locally irregular 2-edge-colouring of G_F if and only if the output of $G_F(x_i)$ is coloured with a colour

different from the one used to colour the output of $G_F(\bar{x}_i)$. Indeed, if this is not the case, then vertices with degree 2 would be adjacent in either the 1- or the 0-subgraph.

To show that this reduction may be achieved in polynomial time, we now determine the number of vertices of G_F . The number of outputs of $G_F(S)$ may be computed as follows.

- In the worst case, a clause C_i has three distinct literals and, in this special case, the clause gadget $G_F(C_i)$ contains a $(3, 1)$ -, a $(1, 3)$ - and a $(4, 1)$ -vertex. Thus, we need $\mathcal{O}(m)$ distinct outputs of $G_F(S)$ to construct the clause gadgets of G_F .
- We have $n_i \leq m$ for every $i \in \{1, 2, \dots, 2n\}$. Thus, in the worst case, the literal gadget $G_F(\ell_i)$ needs $2k_i$ special vertices constructed thanks to $\mathcal{O}(m^2)$ dedicated outputs of $G_F(S)$. Since there are $2n$ literals in F , we get that $\mathcal{O}(nm^2)$ outputs of $G_F(S)$ may be necessary to construct the literal gadgets of G_F .

Thus, $\mathcal{O}(nm^2)$ outputs of $G_F(S)$ are necessary to ensure that ϕ is propagated correctly along G_F . These are obtained by connecting $\mathcal{O}(nm^2)$ copies of G^* , while G^* has a constant number of vertices. Roughly, omitting that some outputs of $G_F(S)$ are negative, we get that the order of $G_F(S)$ is $\mathcal{O}(nm^2)$.

Finally, the number of vertices of a clause or literal gadget of G_F that do not belong to the generator gadget is clearly upper bounded by $\mathcal{O}(m + n)$. Therefore, the number of new vertices needed to construct the clause and literal gadgets of G_F , that is vertices that do not belong to $G_F(S)$, is irrelevant compared to $\mathcal{O}(nm^2)$.

According to the previous arguments, this reduction is achieved in polynomial time regarding the size of F .

5 Discussion

In this paper, we investigated the computational complexity of LOCALLY-IRREGULAR k -EDGE-COLOURING regarding both the specific case of trees and the general case.

Regarding trees, we proposed a linear-time algorithm for determining the irregular chromatic index of these graphs, see Theorem 3.8. As a side result, Theorem 3.3 gives a sufficient condition for a tree to have irregular chromatic index at most 2 which is easy to check. As mentioned in Section 3.3, trees with irregular chromatic index 3 have a predictable structure made up of

“bad pieces”, namely those given in Table 1. By carefully studying how these pieces must be connected, we can exhibit additional sufficient easy conditions for a tree to have irregular chromatic index 3. These conditions mainly concern the location of nodes with degree 3 or 4 and the way they are organized in such trees.

Observe, for example, that no bad signature includes $\{1\}$ whenever $p \geq 2$. This means that if a node with degree at least 3 of T is connected to a hanging path with odd length, then T has irregular chromatic index at most 2. Moreover, if the colouring procedure from Section 3.2 fails on T_r for some bad vertex r with degree $\deg(r) = \Delta(T) = 4$, then r necessarily has a neighbour with degree 4 since one of the $T_r[r, i]$'s is a 4-bad shrub. Therefore, if T has a node r' with degree 4 which has no neighbour with degree 4, then r' is not bad and T has irregular chromatic index at most 2 by Corollary 3.7.

Regarding the general case, we proved that LOCALLY-IRREGULAR 2-EDGE-COLOURING is NP-complete in general, see Theorem 1.3. It is worth mentioning that this result implies that there should not exist any FPT algorithm for LOCALLY-IRREGULAR k -EDGE-COLOURING parameterised by k . Indeed, such an algorithm applied to the case $k = 2$ would yield a polynomial-time algorithm for deciding whether the irregular chromatic index of a graph is 2, contradicting Theorem 1.3 unless $\mathbf{P} = \mathbf{NP}$.

Note that the only unavoidable edge crossings in a reduced graph G_F only concern edges between clause and literal gadgets. We can directly get rid of these crossings by assuming that F is a planar formula. Since 1-IN-3 SATISFIABILITY is known to remain NP-complete when restricted to planar formula, we get that LOCALLY-IRREGULAR 2-EDGE-COLOURING remains NP-complete when restricted to planar graphs.

In G_F , the vertices with the largest degrees are those from the literal gadgets whose associated literals appear in at least 3 clauses. Note that we could replace such gadgets by several copies of the gadget depicted in Figure 6.a, so that literal gadgets all have maximum degree 3. Proceed as follows for a gadget $G_F(\ell_i)$. Recall that n_i outputs $(o_1, o'_1), (o_2, o'_2), \dots, (o_{n_i}, o'_{n_i})$ of the clause gadgets must propagate the colouring to $G_F(\ell_i)$. First take one copy G_1 of the graph of Figure 6.a, and connected (o_1, o'_1) and (o_2, o'_2) with the inputs of G_1 , respectively. According to Lemma 4.3, the colouring is propagated to the output of G_1 , say $o(G_1)$ and $o'(G_1)$, if and only if (o_1, o'_1) and (o_2, o'_2) have the same colour. Besides, if the colouring is propagated, then $o(G_1)$ and $o'(G_1)$ have the same colour as the input edges. Now, take another copy G_2 of the graph of Figure 6.a, and connect its inputs with the output of G_1 and (o_3, o'_3) , respectively. Once again, the colouring is propagated if and only if the two input colours are the same. And so on.

Using the gadget obtained in this way by using $n_i - 1$ copies of the gadget

from Figure 6.a is thus equivalent to using the gadget from Figure 6.b. If any two of the outputs $(o_1, o'_1), (o_2, o'_2), \dots, (o_{n_i}, o'_{n_i})$ have distinct colours, then the colouring cannot be propagated. Therefore, the literal gadgets of G_F can all have maximum degree 3. Under this assumption, the maximum degree of any reduced graph G_F is at most 6, which is the maximum degree of a clause gadget whose associated clause of F have three distinct literals. We thus get that **LOCALLY-IRREGULAR 2-EDGE-COLOURING** remains **NP-hard** when restricted to graphs with maximum degree at most 6.

Observe that, in the proof of Theorem 1.3, the resulting graph G_F is not bipartite mainly because of the induced triangles of the gadget G^* . The existence of a bipartite gadget with the same properties as G^* would be a first step towards a proof that the problem **LOCALLY-IRREGULAR 2-EDGE-COLOURING** remains **NP-complete** when restricted to bipartite graphs. But we did not manage to find such a gadget so far. So we ask the following.

Question 5.1. *Is **LOCALLY-IRREGULAR 2-EDGE-COLOURING** **NP-complete** when restricted to bipartite graphs?*

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