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A DYNAMICAL SYSTEM APPROACH TO HEISENBERG UNIQUENESS PAIRS

PHILIPPE JAMING & KARIM KELLAY

Abstract. Let Λ be a set of lines in $\mathbb{R}^2$ that intersect at the origin. For $\Gamma \subset \mathbb{R}^2$ a smooth curve, we denote by $\mathcal{AC}(\Gamma)$ the subset of finite measures on Γ that are absolutely continuous with respect to arc length on Γ. For such a μ, $\widehat{\mu}$ denotes the Fourier transform of μ.

Following Hendenmalm and Montes-Rodríguez, we will say that (Γ, Λ) is a Heisenberg Uniqueness Pair if $\mu \in \mathcal{AC}(\Gamma)$ is such that $\widehat{\mu} = 0$ on Λ, then $\mu = 0$.

The aim of this paper is to provide new tools to establish this property. To do so, we will reformulate the fact that $\widehat{\mu}$ vanishes on Λ in terms of an invariance property of μ induced by Λ. This leads us to a dynamical system on Γ generated by Λ. The investigation of this dynamical system allows us to establish that (Γ, Λ) is a Heisenberg Uniqueness Pair. This way we both unify proofs of known cases (circle, parabola, hyperbola) and obtain many new examples. This method also allows to have a better geometric intuition on why (Γ, Λ) is a Heisenberg Uniqueness Pair.

1. Introduction

The aim of this paper is to contribute to the study of Fourier uniqueness sets of measures supported on planar curve. More precisely, in the terminology introduced by Hendenmalm and Montes-Rodríguez in [HMR], we will provide new tools for proving that a piecewise smooth curve Γ and a set Λ of lines through the origin form a *Heisenberg Uniqueness Pairs* (HUP).

The concept of HUP is an extension of the notion of annihilating pairs for the Fourier transform on $L^2(\mathbb{R})$ to the setting of measures *see e.g.* [AB, Be], Havin and Jöricke's book [HJ] or the survey [FS]. Its original motivations comes from sets of uniqueness of PDEs (in particular for the Klein-Gordon equation). We will show that the problem can be reformulated in terms of a dynamical system on Γ. This will allow us to find new proofs for many existing results as well as to find many new cases that seemed out of reach with the methods used so far.

Let us now be more precise. If μ denotes a finite complex-valued Borel measure in the plane $\mathbb{R}^2$. The Fourier transform of μ is defined by

$$\widehat{\mu}(x, y) = \int_{\mathbb{R}^2} e^{-i(xs+yt)} \, \mathrm{d}\mu(s, t).$$

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For $\Gamma \subset \mathbb{R}^2$ that is the finite union of smooth curves that are disjoint (except possibly for the endpoints), denote by $\mathcal{M}(\Gamma)$ the set of finite complex-valued Borel measures supported in Γ. Moreover, we denote by $\mathcal{AC}(\Gamma)$ the subset of $\mathcal{M}(\Gamma)$ that consists of measures that are absolutely continuous with respect to arc length on Γ.

Definition. *Let $\Lambda \subset \mathbb{R}^2$ and Γ a finite union of smooth disjoint curves. Then (Γ, Λ) is a* Heisenberg Uniqueness Pair *if $\mu \in \mathcal{AC}(\Gamma)$ and $\widehat{\mu}\Big|_\Lambda = 0$ implies $\Lambda = 0$.*

Clearly, some of the invariance properties of the Fourier transform transfer to HUPs, namely:

[Inv 1] Fix $(s_0, t_0), (x_0, y_0) \in \mathbb{R}^2$. Then $\big(\Gamma, \Lambda\big)$ is a HUP if and only if $\big(\Gamma - (s_0, t_0), \Lambda - (x_0, y_0)\big)$ is a HUP.

[Inv 2] Fix T a linear invertible transformation $\mathbb{R}^2 \to \mathbb{R}^2$ and denote by T^* its adjoint. Then $\big(\Gamma, \Lambda\big)$ is a HUP if and only if $\big(T^{-1}(\Gamma), T^*(\Lambda)\big)$ is a HUP.

This notion was introduced by Hendenmalm and Montes-Rodríguez [HMR] who considered the case where Γ is a hyperbola $\{(x, y \in \mathbb{R}^2 : xy = 1\}$ and $\Lambda = \alpha\mathbb{Z} \times \{0\} \cup \{0\} \times \beta\mathbb{Z}$ is the lattice cross *i.e.* a discrete set included in two lines. The case of Γ an ellipse and Λ two lines was soon after settled independently by Sjölin [Sj] and Lev [Le]. Finally Sjölin [Sj] considered the case where Γ is a parabola, thus completing the study of quadratic curves.

Our aim here is to give more geometric proofs of the results of Sjölin and Lev that allow us to extend their results to case where Γ is a rather general curve and Λ is a union of two intersecting lines. According to the invariance properties we can assume that the lines intersect at the origin and write $\ell_\theta = \{(t\cos\theta, t\sin\theta),\ t \in \mathbb{R}\}$ for $\theta \in [0, \pi)$.

Our starting point was Sjölin's proof that parabolas and two well chosen lines form an HUP. In particular, Sjölin used a simple change of variable that directly reformulates as Lemma 2.1-Corollary 2.2 in our case. These results show that if $\mu \in \mathcal{AC}(\Gamma)$ and $\widehat{\mu}\Big|_\Lambda = 0$ then there is a mapping $\Phi : \Gamma \to \Gamma$ that leaves μ invariant and this mapping a simple geometric interpretation. We will then be able to deduce from the properties of the dynamical system generated by Φ (existence of a wandering set, existence of attractive points and ergodicity) that (Γ, Λ) form a Heseinberg Uniqueness Pair. Note that dynamical systems already play a crucial role in [HMR, CMHMR].

Let us here summarize our main results:

Main Theorem. *Let Γ be any of the following curves:*

(i) *the graph of $\psi(t) = |t|^\alpha$, $\alpha > 0$;*
(ii) *a hyperbola;*
(iii) *a polygon;*
(iv) *an ellipse.*

Then there exists a set $E \subset (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ of positive measure such that, if $\theta_1 \neq \theta_2 \in E$, $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.

The actual results are both more general and more precise. For i, the case $\alpha \geq 1$ is covered by Corollary 3.3 while the case $0 < \alpha < 1$ is covered by Lemma 3.4. We refer

to Proposition 3.6 for iii). To prove those results we show that the dynamical system generated by Φ has many wandering sets. We prove ii) in Theorem 3.5 by first transferring the problem to the circle (using a simple transform from projective geometry) in order to prove that here too the dynamical system generated by Φ has many wandering sets. Finally, the case iv is proved in Theorem 4.1 using ergodic theory. In this case, the map Φ is an irrational rotation. Our technique shows that the same result holds if Γ is any smooth convex closed curve such that the map Φ has irrational rotation number. However we are also able to construct an example of a smooth convex closed curve and a set of two lines that form a HUP though the map Φ has rational rotation number.

Let us conclude this introduction by explaining how our results apply to PDEs.

Let p be a polynomial of two variables and let $\Gamma = \{(s,t) \in \mathbb{R}^2 \, : p(s,t) = 0\}$. Then

$$p(i\partial_x, i\partial_y)\widehat{\mu}(x,y) = \int_{\mathbb{R}^2} e^{-i(xs+yt)} p(s,t) \, \mathrm{d}\mu(s,t). \tag{1.1}$$

Therefore, if $\mu \in \mathcal{AC}(\Gamma)$ then $F = \widehat{\mu}$ solves the PDE

$$p(i\partial_x, i\partial_y)F = 0. \tag{1.2}$$

Now $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness pair if and only if for every solution F of (1.2) such that $F = \widehat{\mu}$ with $\mu \in \mathcal{AC}(\Gamma)$, $F(x, x\operatorname{cotan}\theta_1) = F(x, x\operatorname{cotan}\theta_2) = 0$ for every x implies $F = 0$.

We can than reformulate our results in terms of solutions of certain PDEs (and more generally for certain pseudo-differential equations). We obtain the following

Theorem. *Let $\theta_1 \neq \theta_2 \in (0,\pi)$, $a_j = \operatorname{cotan}\theta_j$ and $\alpha > 0$. Assume that $F \in \mathcal{C}^2(\mathbb{R}^2)$ satisfy one of the following equations:*

(i) Shrödinger Equation
$$i\partial_x F \pm |\Delta_y|^{\alpha/2} F = 0$$
then denote $\Gamma = \{(t, |t|^\alpha), t \in \mathbb{R}\}$;

(ii) Helmholtz equation
$$\partial_x^2 F + \partial_y^2 F = -\alpha^2 F$$
then denote $\Gamma = \{(x,y) \in \mathbb{R}^2 \, : x^2 + y^2 = \alpha^2\}$ and further assume that $\dfrac{\theta_1 - \theta_2}{\pi} \notin \mathbb{Q}$;

(iii) Klein-Gordon equation
$$\partial_x^2 F - \partial_y^2 F = \alpha^2 F.$$
then denote $\Gamma = \{(x,y) \in \mathbb{R}^2 \, : x^2 - y^2 = \alpha^2\}$ and further assume that $|\theta_1 - \theta_2| \neq \pi/2$.

If $F = \widehat{\mu}$ with $\mu \in \mathcal{AC}(\Gamma)$ and $F(x, a_1 x) = F(x, a_2 x) = 0$ for all $x \in \mathbb{R}$, then $F = 0$.

One would of course like to relax the condition $F = \widehat{\mu}$ with $\mu \in \mathcal{AC}(\Gamma)$ to $F = \widehat{\mu}$ with μ a bounded measure on $\mathbb{R}^2$ (which would then necessarily be supported in Γ). It would be natural to say that $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a *strong* Heisenberg Uniqueness Pair in that case.

The remaining of the paper is organized as follows. The following section is devoted to the technical lemmas we will need. In particular, Section 2.3 contains the three technical

lemmas linking HUPs and properties of the dynamical system generated by Φ. Section 3 is then devoted to cases where the dynamical system has many wandering sets, in particular establishing **??**) to iii of the Main theorem in four consecutive subsections. The last section is devoted to closed curves when the map Φ has a rotation number.

2. Technical Lemmas

2.1. **Notation and key lemma.** Throughout this paper, I will be a finite union of disjoint intervals and $\Gamma = \{\gamma(s), s \in I\}$ will be a smooth curve in the plane parametrized by a function $\gamma : I \to \mathbb{R}^2$ that is assumed to be one-to-one (except possibly for the end points of I).

For $\theta \in \mathbb{S}^1$ the unit circle of $\mathbb{R}^2$ denote by $\theta^\perp$ be the vector in $\mathbb{S}^1$ directly orthogonal to θ. We will use the common abuse of notation by identifying θ with its the angle with the horizontal axes, $\theta = (\cos\theta, \sin\theta)$. Let $\ell_\theta = \{t\theta, t \in \mathbb{R}\}$ be the line spanned by θ and define $\pi_\theta x = \langle x, \theta\rangle$ so that $x \to \pi_\theta(x)\theta$ is the orthogonal projection of x on ℓ_θ.

Given $\mu \in \mathcal{AC}(\Gamma)$ *i.e.* a measure that is absolutely continuous with respect to arc length on Γ we write $\mu(s) = g_\mu(s)\|\gamma'(s)\|\,\mathrm{d}s = f_\mu(s)\,\mathrm{d}s$, with $f_\mu \in L^1(I)$.

We are now in position to prove the following simple but key lemma:

Lemma 2.1. *Assume that Γ, θ are such that there exists a finite partition of $I = \bigcup_{k=1}^N I_k$ of intervals that are disjoint (up to the endpoints) such that $s \to \pi_\theta\gamma(s)$ is one-to-one on each I_k.*

Let $\mu \in \mathcal{AC}(\Gamma)$. Then $\hat\mu(\xi) = 0$ for $\xi \in \ell_\theta$ if and only if, for almost every $\zeta \in \mathbb{R}$

$$\sum_{s\in\pi_\theta\gamma^{-1}(\zeta)} \frac{f_\mu(s)}{\pi_\theta\gamma'(s)} = 0. \tag{2.3}$$

Proof. Note that $(\pi_\theta\gamma)' = \pi_\theta\gamma'$. Then

$$\begin{aligned}
\hat\mu(t\theta) &= \int_I g_\mu(s)\|\gamma'(s)\| e^{-it\langle\gamma(s),\theta\rangle}\,\mathrm{d}s \\
&= \sum_{k=1}^N \int_{I_k} g_\mu(s)\|\gamma'(s)\| e^{-it\pi_\theta\gamma(s)}\,\mathrm{d}s \\
&= \sum_{k=1}^N \int_{\pi_\theta\gamma^{-1}(I_k)} g_\mu\big(\pi_\theta\gamma^{-1}(\zeta)\big)\big\|\gamma'\big(\pi_\theta\gamma^{-1}(\zeta)\big)\big\| e^{-it\zeta} \frac{\mathrm{d}\zeta}{\pi_\theta\gamma'\big(\pi_\theta\gamma^{-1}(\zeta)\big)}
\end{aligned}$$

with the change of variable $s = \pi_\theta\gamma^{-1}(\zeta)$ on each I_k. It follows that

$$\begin{aligned}
\hat\mu(t\theta) &= \int_{\mathbb{R}} \sum_{k=1}^N 1_{\pi_\theta\gamma^{-1}(I_k)}(u) \frac{f_\mu\big(\pi_\theta\gamma^{-1}(\zeta)\big)}{\pi_\theta\gamma'\big(\pi_\theta\gamma^{-1}(\zeta)\big)} e^{-it\zeta}\,\mathrm{d}\zeta \\
&= \int_{\mathbb{R}} \sum_{s\in\pi_\theta\gamma^{-1}(\zeta)} \frac{f_\mu(s)}{\pi_\theta\gamma'(s)} e^{-it\zeta}\,\mathrm{d}\zeta.
\end{aligned}$$

This is now an ordinary Fourier transform so that $\hat{\mu}(t\theta) = 0$ for every t if and only if (2.3) is satisfied. □

Remark.
— If γ is contained in a half place $\{\langle x, \theta^\perp\rangle \geq \alpha\}$ or $\{\langle x, \theta^\perp\rangle \leq \alpha\}$, then it is enough to assume that $\widehat{\mu}(t\theta) = 0$ for $t \in E$ a set of finite positive measure for (2.3) to hold.

This follows immediately from the previous proof and the well known fact (*see e.g.* [HJ, Page 36]) that if $f \in L^1(\mathbb{R})$ is such that $\operatorname{supp} f \subset [0, +\infty)$ and if

$$\int_{\mathbb{R}} \frac{\log|\widehat{f}(\xi)|}{1+|\xi|^2}\,\mathrm{d}\xi = -\infty$$

(in particular if $\widehat{f}$ is compactly supported) then $f = 0$.
— Further, if Γ is contained in a strip $\{-\alpha \leq \langle x, \theta^\perp\rangle \leq \alpha\}$ then we may further restrict E to be a discrete set of density $\geq \frac{\alpha}{2\pi}$.

From now on, we will restrict our attention to curves for which $(\pi_\theta\gamma)^{-1}(\zeta)$ contains at most two points. More precisely, the following is a direct reformulation of Lemma 2.1:

Corollary 2.2. *Let $\gamma : I \to \mathbb{R}^2$ be a piecewise smooth function and $\theta \in [0, 2\pi)$. Assume that we may split $I = I_0 \cup I_- \cup I_+$ in such a way that*

(i) *$\pi_\theta\gamma$ is one-to-one on each interval I_0, I_+, I_-.*
(ii) *let $\sigma \in I$ and $\zeta = \pi_\theta\gamma(\sigma)$ and consider the equation $\pi_\theta\gamma(s) = \zeta$. Then*
  – *if $\sigma \in I_0$ this equation has as unique solution $s = \sigma$;*
  – *if $\sigma \in I_-$ (resp. I_+) this equation has two solutions $\sigma_\pm$ with $\sigma_- = \sigma \in I_-$ and $\sigma_+ \in I_+$ (resp. $\sigma_+ = \sigma \in I_+$ and $\sigma_- \in I_-$). In this case, we denote $\pi_\theta\gamma_\pm^{-1}(\zeta) = \sigma_\pm$*

Let $\mu \in \mathcal{AC}(\Gamma)$. Then $\hat{\mu}(\xi) = 0$ for $\xi \in \ell_\theta$ if and only if,

(i) *$f_\mu = 0$ on I_0*
(ii) *for every $s_- \in I_-$, $s_+ \in I_+$, with $\pi_\theta\gamma(s_-) = \pi_\theta\gamma(s_+)$ —that is $s_+ = \pi_\theta\gamma_+^{-1}\big(\pi_\theta\gamma(s_-)\big)$ and $s_- = \pi_\theta\gamma_-^{-1}\big(\pi_\theta\gamma(s_+)\big)$—*

$$\frac{f_\mu(s_+)}{\pi_\theta\gamma'(s_+)} = -\frac{f_\mu(s_-)}{\pi_\theta\gamma'(s_-)}. \tag{2.4}$$

Moreover, if $\alpha_+, \beta_+ \in I_+$ and $\alpha_-, \beta_- \in I_-$ are such that $\pi_\theta\gamma(\alpha_+) = \pi_\theta\gamma(\alpha_-)$, $\pi_\theta\gamma(\beta_+) = \pi_\theta\gamma(\beta_-)$ then

$$\int_{\alpha_-}^{\beta_-} f_\mu(s_-)\,ds_- = -\int_{\alpha_+}^{\beta_+} f_\mu(s_+)\,ds_+. \tag{2.5}$$

and

$$\left|\int_{\alpha_-}^{\beta_-} |f_\mu(s_-)|\,ds_-\right| = \left|\int_{\alpha_+}^{\beta_+} |f_\mu(s_+)|\,ds_+\right|. \tag{2.6}$$

Note that (2.5)-(2.6) follows directly from (2.4) if we change variable $s_+ = \pi_\theta\gamma_-^{-1}(s_-)$ in the second integral.

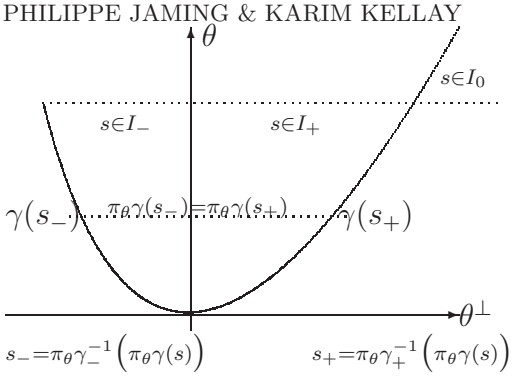


FIGURE 1. The notation of Corollary 2.2

Notation : If γ, θ satisfy the hypothesis of Corollary 2.2 we may define the map

$$\Phi_\theta : I_- \cup I_+ \to I_- \cup I_+, \quad \Phi_\theta(s_\pm) = s_\mp.$$

This map has a nice geometric interpretation: Consider a point $\gamma(s)$ with $s \in I$ and draw a line orthogonal to θ starting at $\gamma(s)$. This line will intersect Γ again in $\gamma\big(\Phi_\theta(s)\big)$.

Let us now give a first application:

Proposition 2.3. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous piecewise $\mathcal{C}^1$ function, such that ψ is concave on $\mathbb{R}^-$ and convex on $\mathbb{R}^+$ and that ψ has a left and a right derivative in 0 Let $\Gamma = \big\{\big(s, \psi(s)\big), s \in \mathbb{R}\big\}$ be the graph of ψ. Then there is a θ_0 such that, if $0 \le \theta < \theta_0$, (Γ, ℓ_θ) is a Heisenberg Uniqueness Pair.*

Proof. Let $\gamma(s) = \big(s, \psi(s)\big)$, $\theta \in (-\pi/2, \pi/2]$. Then $\varphi(s) := \pi_\theta\gamma(s) = s\cos\theta + \psi(s)\sin\theta$. We have to show that φ is one-to-one, but $\varphi'(s) = \cos\theta + \psi'(s)\sin\theta$. As $\sin\theta \ge 0$, the convexity hypothesis on ψ implies that $\varphi'(s) \ge \cos\theta + \psi'(0^-)\sin\theta$ for $s < 0$ and $\varphi'(s) \ge \cos\theta + \psi'(0^+)\sin\theta$ for $s > 0$. Thus, if θ is small enough, $\varphi'(s) > 0$ for all s thus $\pi_\theta\gamma = \varphi$ is one-to-one.

In the notation of Corollary 2.2, $I_0 = \mathbb{R}$. The result follows. □

Example. Let $\alpha > 0$, $\Gamma = \{(s, \mathrm{sign}(s)|s|^\alpha), s \in \mathbb{R}\}$. If $\alpha \ge 1$ and $\theta \in [0, \pi/2]$ or $0 < \alpha \le 1$ and $\theta \in [-\pi/2, 0]$, then (Γ, ℓ_θ) is a Heisenberg Uniqueness Pair.

2.2. **The regularity of Φ_θ.** The aim of this section is to establish the regularity of the map Φ_θ. This is only needed when we investigate closed curves and the reader may skip this section when first reading the paper.

We will devote this section to prove the following proposition. The result may be extended to more general curves but, as we will not need it, we will refrain from doing so.

Proposition 2.4. *Let Γ be a closed convex curve with non vanishing curvature and let $\gamma : [0, 1) \to \mathbb{R}^2$ be a parametrization of Γ such that γ' does not vanish. Assume that γ is of class $\mathcal{C}^k$. For every $\theta \in \mathbb{R}$, let $\Phi_\theta : [0, 1] \to [0, 1)$ be defined by the fact that, for every $s \in [0, 1]$, $\pi_\theta\gamma^{-1}\big(\gamma(s)\big) = \{s, \Phi_\theta(s)\}$ with the understanding that $\Phi_\theta(s) = s$ if $\pi_\theta\gamma^{-1}\big(\gamma(s)\big)$ is a singleton.*

Then Φ_θ is well defined, one-to-one on $[0,1)$, of class $\mathcal{C}^{k-1}$ in s and θ.

Proof. In this proof it will be convenient to consider θ as being the unit vector $(\cos\theta, \sin\theta)$.

First note that, as Γ is a convex curve, a line through a point $\gamma(s) \in \Gamma$, orthogonal to θ, either intersects Γ in two distinct points and the second point is $\gamma\big(\Phi_\theta(s)\big)$ or this line tangent to Γ. In this case, as the curvature of Γ does not vanish, the tangent to Γ at $\gamma(s)$ intersects Γ only in $\gamma(s)$ and $\Phi_\theta(s) = s$. Note that Φ_θ is an involution, $\Phi_\theta\big(\Phi_\theta(s)\big) = s$.

Now fix θ_0 and $s_-^0 \in [0,1]$ and let $s_+^0 = \Phi_{\theta_0}(s_-^0)$. Then Φ_θ is defined implicitly as follows: let $F(s,t,\theta) := \pi_\theta\gamma(s) - \pi_\theta\gamma(t)$ then

$$\begin{cases} \Phi_{\theta_0}(s_-^0) = s_+^0 \\ F(s, \Phi_\theta(s), \theta) = 0 \end{cases}.$$

Note that $\dfrac{\partial F}{\partial t} = -\pi_\theta\gamma'(t)$ and $\dfrac{\partial F}{\partial \theta} = F(s,t,\theta^\perp)$. Further $\pi_{\theta_0}\gamma'(s_+^0) = 0$ means that θ_0 (resp. $\theta_0^\perp$) is normal (resp. tangent) to Γ at $\gamma(s_+^0)$. The previous discussion then shows that $\gamma(s_+^0) = \gamma(s_-^0)$ *i.e* $s_+^0 = s_-^0$. Next, $F(s_0, \Phi_{\theta_0}(s_0), \theta_0^\perp) = 0$ means that $\pi_{\theta_0^\perp}\gamma(s_-^0) = \pi_{\theta_0^\perp}\gamma(s_+^0)$. As $\pi_{\theta_0}\gamma(s_-^0) = \pi_{\theta_0}\gamma(s_+^0)$, this again implies that $\gamma(s_+^0) = \gamma(s_-^0)$ thus θ_0 is not normal to Γ at $\gamma(s_-^0)$.

Using the Implicit Function Theorem, we deduce that, if θ_0 is not normal to Γ at $\gamma(s_-^0)$, there exists a neighborhood $V \times W$ of (s_-^0, θ_0) on which $(s,\theta) \to \Phi_\theta(s)$ is of class $\mathcal{C}^k$ and

$$\frac{\partial \Phi_{\theta_0}}{\partial s}(s_-) = \frac{\pi_{\theta_0}\gamma'(s_-)}{\pi_{\theta_0}\gamma'(s_+)}. \tag{2.7}$$

while

$$\frac{\partial \Phi_\theta}{\partial \theta}(s_-^0) = -\frac{\pi_\theta\gamma'(s_-^0)}{F(s_-^0, \Phi_\theta(s_-^0), \theta^\perp)}. \tag{2.8}$$

Let us now explore what happens in a neighborhood of a point where θ_0 is normal to Γ.

First, given $\theta \in \mathbb{S}^1$, the curvature asumption implies that there is exactly one point $\gamma(s)$ in Γ such that $\langle \theta, \gamma'(s)\rangle = 0$ and $\langle \theta, \gamma''(s)\rangle > 0$. Denote by $s(\theta)$ this point. As the curvature at the point $\gamma(s)$ is given by $\kappa(s) = \dfrac{|\det[\gamma'(s), \gamma''(s)]|}{\|\gamma'(s)\|^3}$ and as $\langle \theta, \gamma'(s)\rangle = 0$, $\langle \theta, \gamma''(s)\rangle \neq 0$. It follows from the implicit function theorem implies that $\theta \to s(\theta)$ is of class $\mathcal{C}^{k-1}$ with

$$\frac{\partial s(\theta)}{\partial \theta} = -\frac{\big\langle \gamma'(s), \theta^\perp \big\rangle}{\langle \gamma''(s), \theta\rangle}.$$

It remains to prove that $s \to \Phi_\theta(s)$ is of class $\mathcal{C}^{k-1}$ in the neighbourhood of $s(\theta)$. To do so, we will appeal to the following simple lemma. The proof is a classical exercise on Taylor expansions:

Lemma 2.5. *Let $\psi : V \to \mathbb{R}^2$ be of class $\mathcal{C}^k$ in a neighborhood V of 0. Assume that the Taylor expansion of ψ is of the form $\psi(s) = (a_0 + a_1 s + a_2 s^2 + \cdots + a_k s^k, b_0 + b_2 s^2 + \cdots + b_k s^k) + o(s^k)$. Then there is a neighborhood W of 0 such that ψ is two-to-one on W: if $s \in W$, there is exactly one $\varphi(s) \in W$ such that $\varphi(s) \neq s$ and $\psi\big(\varphi(s)\big) = \psi(s)$. Moreover, the map $s \to \varphi(s)$ is of class $\mathcal{C}^{k-1}$ with $\varphi(s) = -s + o(s)$.*

Write

$$\gamma\big(s(\theta)+s\big)=\gamma\big(s(\theta)\big)+s\pi_{\theta^\perp}\gamma'\big(s(\theta)\big)+\frac{s^2}{2}\big[\pi_{\theta^\perp}\gamma'\big(s(\theta)\big)+\pi_\theta\gamma'\big(s(\theta)\big)\big]+o(s^2)$$

where the $o(s^2)=s^2\chi_\theta(s)$ with $\chi_\theta(s)\to 0$ uniformly in θ (since $s(\theta)$ is smooth). Applying Lemma 2.5 (in the basis $\theta,\theta^\perp$ and at the point $s(\theta)$ instead of the standard basis and the point 0), we obtain that $\Phi_\theta\big(s(\theta)+s\big)=s(\theta)-s+o(s)$ with a $o(s)$ that is uniform in θ. Therefore, Φ_θ is of class $\mathcal{C}^1$ both in s and θ. If we use the fact that γ has a Taylor expansion of order k, Lemma 2.5 implies that Φ_θ is of class $\mathcal{C}^{k-1}$ both in s and θ. □

2.3. **Two lines: a dynamical system approach.** We will now consider Heisenberg Uniqueness Pairs of the form $(\Gamma,\ell_{\theta_1}\cup\ell_{\theta_2})$ where $\Gamma=\{\gamma(s),s\in I\}$ is a piecewise smooth curve, and $\theta_1\neq\theta_2\in[-\pi/2,\pi/2)$ are two angles. Assume that for both angles, Γ is as in Corollary 2.2. We thus have two splittings $I=I_0^1\cup I_+^1\cup I_-^1=I_0^2\cup I_+^2\cup I_-^2$ and two maps $\Phi_j=\Phi_{\theta_j}$, $j=1,2$. Write Γ_ε^j, $j=1,2$, $\varepsilon=0,+,-$ for the corresponding parts of Γ.

Let $\mu\in\mathcal{AC}(\Gamma)$ and assume that $\widehat{\mu}(\xi)=0$ for $\xi\in\ell_{\theta_1}\cup\ell_{\theta_2}$. According to Corollary 2.2, $f_\mu=0$ on $I_0^1\cup I_0^2$. It follows that $\mu\in\mathcal{AC}\big(\Gamma\setminus(\Gamma_0^1\cup\Gamma_0^2)\big)$. Without loss of generality, we may now assume that $I_0^1=I_0^2=\emptyset$.

From (2.6) we deduce that, for every interval $J\subset I$,

$$\int_{\Phi_j(J)}|f_\mu(s)|\,\mathrm{d}s=\int_J|f_\mu(s)|\,\mathrm{d}s. \tag{2.9}$$

This is direct if $J\subset I_\pm^j$ and otherwise, we write J as a disjoint union $J=J_+\cup J_-$ with $J_\pm=J\cap I_\pm^j$ and write (2.6) for each of $J_\pm$. As Φ_j is one-to-one, $\Phi_j(J_-)$ and $\Phi_j(J_+)$ are disjoint so that (2.9) is just the sum of (2.6) for J_- and J_+. Note that $\Phi_j(J)$ is also an interval.

The fact that $(\Gamma,\ell_{\theta_1}\cup\ell_{\theta_2})$ is a Heisenberg Uniqueness Pair will depend on the properties of the dynamical system generated by the map $\Phi=\Phi_2\circ\Phi_1$. We will denote by Φ^n the n-th iterate of Φ.

We will now prove three lemmas that will allow us to establish Heisenberg Uniqueness.

Lemma 2.6. *Let $J\subset I$ be an interval and assume that J is* wandering *for $\Phi:=\Phi_2\circ\Phi_1$, that is for every $j\geq 1$ $\Phi^j(J)\cap J=\emptyset$ (up to a set of measure 0). If $\mu\in\mathcal{AC}(\Gamma)$ is such that $\widehat{\mu}=0$ on $\ell_{\theta_1}\cup\ell_{\theta_2}$ then $f_\mu=0$ on $\bigcup_{j=0}^{\infty}\Phi^j(J)$.*

Proof. According to (2.9), for every interval J

$$\int_{\Phi_2\left(\Phi_1(J)\right)}|f_\mu(s)|\,\mathrm{d}s=\int_{\Phi_1(J)}|f_\mu(s)|\,\mathrm{d}s=\int_J|f_\mu(s)|\,\mathrm{d}s$$

and more generally, for every $k\geq 0$

$$\int_{\Phi^k(J)}|f_\mu(s)|\,\mathrm{d}s=\int_J|f_\mu(s)|\,\mathrm{d}s.$$

But as the interval J is wandering and $f_\mu \in L^1(I)$

$$\begin{aligned} +\infty \;&>\; \int_{\bigcup_{j=0}^{\infty}\Phi^j(J)} |f_\mu(s)|\,\mathrm{d}s = \sum_{j=0}^{\infty}\int_{\Phi^j(J)} |f_\mu(s)|\,\mathrm{d}s \\ &=\; \sum_{j=0}^{\infty}\int_{J} |f_\mu(s)|\,\mathrm{d}s \end{aligned}$$

so that $0 = \displaystyle\int_J |f_\mu(s)|\,\mathrm{d}s = \int_{\Phi^k(J)} |f_\mu(s)|\,\mathrm{d}s$ thus $f_\mu = 0$ on $\Phi^k(J)$. □

Lemma 2.7. *Let $J \subset I$ be an interval and assume that J is* attractive *for $\Phi := \Phi_2 \circ \Phi_1$, that is, there exists k such that $\Phi^k(J) \subset J$. If $\mu \in \mathcal{AC}(\Gamma)$ is such that $\widehat{\mu} = 0$ on $\ell_{\theta_1} \cup \ell_{\theta_2}$ then supp $f_\mu \cap J \subset \displaystyle\bigcap_{n\geq 1} \Phi^{nk}(J)$.*

Proof. As in the previous proof

$$\int_J |f_\mu(s)|\,\mathrm{d}s = \int_{\Phi^k(J)} |f_\mu(s)|\,\mathrm{d}s$$

so that, if $\Phi^k(J) \subset J$, $f_\mu = 0$ on $J \setminus \Phi^k(J)$. The result follows by noting that $\Phi^{(n+1)k}(J) \subset \Phi^{nk}(J)$. □

The last lemma only applies to closed curves. In this case, we can parametrize γ with a function $\gamma : [0,1] \to \mathbb{R}^2$ with $\gamma(1) = \gamma(0)$ and γ is one-to-one on $[0,1)$. A rotation of angle α is then the map $t \to t + \alpha \bmod 1$.

Lemma 2.8. *Assume further that Γ is a closed curve. Assume that there is a diffeomorphism h such that Φ is conjugated by h to a rotation R_α with $\alpha \in \mathbb{R}\setminus\mathbb{Q}$: $\Phi = h^{-1} \circ R_\alpha \circ h$. Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.*

Proof. As previously, but using (2.5) instead of (2.6), for every interval I,

$$\int_{\Phi(I)} f_\mu(s)\,\mathrm{d}s = \int_I f_\mu(s)\,\mathrm{d}s$$

thus, changing variable $s = h^{-1}(t)$ in both integrals we get

$$\int_{h^{-1}(I)} \frac{f_\mu\big(h^{-1}(R_{-\alpha}t)\big)}{h'\big(h^{-1}(R_{-\alpha}t)\big)}\,\mathrm{d}t = \int_{R_\alpha h^{-1}(I)} \frac{f_\mu\big(h^{-1}(t)\big)}{h'\big(h^{-1}(t)\big)}\,\mathrm{d}t = \int_{h^{-1}(I)} \frac{f_\mu\big(h^{-1}(t)\big)}{h'\big(h^{-1}(t)\big)}\,\mathrm{d}t.$$

As this holds for every I,

$$\frac{f_\mu\big(h^{-1}(R_{-\alpha}t)\big)}{h'\big(h^{-1}(R_{-\alpha}t)\big)} = \frac{f_\mu\big(h^{-1}(t)\big)}{h'\big(h^{-1}(t)\big)} \quad a.e. \tag{2.10}$$

But then

$$\frac{f_\mu\big(h^{-1}(t)\big)}{h'\big(h^{-1}(t)\big)} = \frac{1}{n}\sum_{k=1}^{n} \frac{f_\mu\big(h^{-1}(R_{-\alpha}^k t)\big)}{h'\big(h^{-1}(R_{-\alpha}^k t)\big)} \to \int_0^1 \frac{f_\mu\big(h^{-1}(t)\big)}{h'\big(h^{-1}(t)\big)}\,\mathrm{d}t$$

for almost every t, according to Birkhoff's Ergodic Theorem. In particular, $\frac{f_\mu\left(h^{-1}(t)\right)}{h'\left(h^{-1}(t)\right)}$ is a constant, but (2.4) shows that it is sign changing, thus $f_\mu = 0$.

□

3. Heisenberg Uniqueness pairs obtained with the help of wandering sets

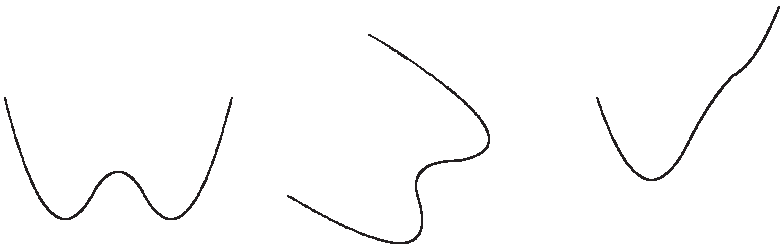
3.1. Graphs of functions that go to $+\infty$ in $\pm\infty$.

Theorem 3.1. *Let ψ be a piecewise smooth function and let $\theta_1 \neq \theta_2 \in (0,\pi)$ be such that*
– $\psi(s)\sin\theta_i + s\cos\theta_i \to +\infty$ when $t \to \pm\infty$
– $\psi(s)\sin\theta_i + s\cos\theta_i \to +\infty$ has a unique local minimum.
Let $\Gamma = \big\{\big(s,\psi(s)\big), s \in \mathbb{R}\big\}$ be the graph of ψ. Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg uniqueness pair.

Before proving the theorem, Let us make a few comments on the hypothesis on ψ.
First, (Γ, ℓ_0) is a Heisenberg uniqueness pair according to corollary 2.2.
Next, the requirements of Theorem 3.1 are commonly met as shows the following lemma:

Lemma 3.2. *Let χ be a smooth function such that $|t|^{-1}\chi(t) \to +\infty$ when $t \to \pm\infty$. Then there exists $-\pi/2 < \alpha < \beta < \pi/2$ such that, for $\theta \notin (\alpha,\beta)$, the function ψ defined by $\psi(t) = \chi(t)\sin\theta + t\cos\theta$ is such that $\psi(t) \to +\infty$ when $t \to \pm\infty$ and ψ has a unique local minimum.*

Proof of Lemma 3.2. The first part of the theorem is obvious and it follows that ψ has at least one local minimum. To see that this minimum is unique, it is enough to check that ψ' has a unique zero. But, the mean value theorem implies that $\chi'(t) \to +\infty$ when $t \to +\infty$ and $\chi'(t) \to -\infty$ when $t \to -\infty$ thus there is an interval $[\alpha,\beta]$ such that $\psi'(t) := \chi'(t)\sin\theta + \cos\theta = 0$ has no solution if θ is outside $[\alpha,\beta]$. □



a) The graph of $\chi(t) = t^2(t^2-1)$ | b) The rotation of c) the graph of t | The graph of $\chi(t) + t$

Figure 2. In a), an horizontal line may cross the graph more than twice while in b) and c) it crosses the graph at most twice.

Remark. This lemma has a geometric meaning namely, if Γ is the graph of a function χ that goes sufficiently fast to $+\infty$ in $\pm\infty$ then there is an interval I such that, if $\theta \in I$, one

may rotate by θ the graph in such a way that an horizontal line intersects the graph at most twice. The following figure illustrates this in the case $\chi(t) = t^2(t^2-1)$ and $\theta = \pi/4$.

We are now in position to prove the theorem.

Proof of Theorem 3.1. As noticed above, the result is trivial if $\theta_1 = 0$ or $\theta_2 = 0$. We will thus exclude this case.

Let $\gamma(s) = \big(s, \psi(s)\big)$ and let σ_0 be the unique local minimum of $\langle \gamma(s), \theta_1\rangle = s\cos\theta_1 + \psi(s)\sin\theta_1$. Note that if γ is smooth this is the unique point such that θ_1 is normal to Γ thus $\theta_1^\perp$ is tangent to Γ. Without loss of generality, using the invariance property (Inv1), we may assume that $\sigma_0 = 0$ and $\gamma(0) = (0,0)$. Using (Inv2) we may further assume that $\theta_1 = -\pi/2$ so that $\theta_1^\perp = \vec{i} := (1,0)$.

The hypothesis on Γ and θ_1 ensure that we may apply Corollary 2.2. In the notation of Section 2.3, $I_0^1 = \emptyset$, $I_-^1 = (-\infty, 0]$ and $I_+^1 = [0, +\infty)$ and the map $\Phi_1 : \mathbb{R} \to \mathbb{R}$ is the map such that, for every $s \neq 0$, $s\Phi_1(s) < 0$ and $\psi\big(\Phi_1(s)\big) = \psi(s)$. Note that ψ is decreasing on I_-^1 and increasing on I_+^1.

Now $\langle \gamma(s), \theta_2\rangle = s\cos\theta_2 + \gamma(s)\sin\theta_2$ has also a unique local minimum at s_2. Up to a symmetry $T : (x,y) \to (-x,y)$, the invariance property (Inv2) shows that we may assume that $s_2 \geq 0$ (note that this implies that $0 < \theta_2^\perp < \pi/2$). Thus in the notation of Section 2.3, $I_0^2 = \emptyset$, $I_-^2 = (-\infty, s_2]$ and $I_+^2 = [s_2, +\infty)$ and the map $\Phi_2 : \mathbb{R} \to \mathbb{R}$ is such that $\Phi_2(I_\pm^2) = I_\mp^2$ and $\Gamma \cap \big(\gamma(s) + \mathbb{R}\theta_2^\perp\big) = \big\{\gamma(s), \gamma\big(\Phi_2(s)\big)\big\}$. Note that $\psi(s) + s\sin\theta_2$ is decreasing on I_-^2 and increasing for I_+^2.

Let us first assume that $s_2 > 0$ and let $\sigma_0 = 0$.

Next, define $\sigma_1 = \Phi_2(\sigma) > s_2 > 0$ and, for $k \geq 1$, $\sigma_{k+1} = \Phi_2\big(\Phi_1(\sigma_k)\big)$. We assert that $[\sigma_1, \sigma_2]$ is a wandering set for $\Phi = \Phi_2 \circ \Phi_1$ and $\bigcup_{k\geq 1}[\sigma_k, \sigma_{k+1}] = [\sigma_1, +\infty)$.

Before proving this assertion, let us show that the conclusion of Theorem 3.1 follows. Indeed, according to Lemma 2.6, $f_\mu = 0$ on $[\sigma_1, +\infty)$. Appealing to Corollary 2.2 for Φ_2, (2.4) reduces to $f_\mu = 0$ on $(-\infty, 0] = \Phi_2^{-1}\big([\sigma_1, +\infty)\big)$ and then, appealing to Corollary 2.2 for Φ_1, (2.4) reduces to $f_\mu = 0$ on $[0, +\infty)$ as well.

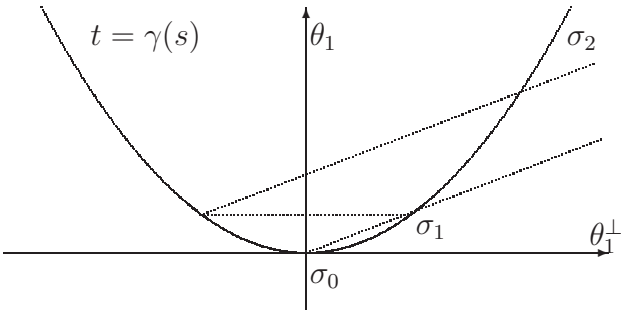


FIGURE 3. The construction of σ_k

Let us now show that σ_k is strictly increasing. This follows from a simple geometric consideration: since $0 < \theta_2^\perp < \pi/2$, if we start at a point $A \in \Gamma$ in the plane, moving

horizontally to the left till we reach Γ again in some point B and then to the right in direction $\theta_2^\perp$, we are moving upward and can therefore only reach Γ again on the right of A.

More precisely, let $s_k := \Phi_1(\sigma_k) < 0 < \sigma_k$ so that $\psi(s_k) = \psi(\sigma_k)$. Then, for $t > 0$ if $s = s_k + t(\sigma_k - s_k)$,

$$\psi(s) \begin{cases} < \psi(s_k) & \text{if } 0 < t < 1 \\ = \psi(\sigma_k) & \text{if } t = 1 \\ > \psi(s_k) & \text{if } t > 1 \end{cases}$$

since ψ decreases on $[s_k, 0]$ and increases on $\mathbb{R}^+$. On the other hand, for $t > 0$ $\psi(s_k) + t(\sigma_k - s_k)\sin\theta_2 > \psi(s_k) = \psi(\sigma_k)$. Thus if $\psi(s_k) + t(\sigma_k - s_k)\sin\theta_2 = \psi(s)$ then $t > 1$ that is $s > \sigma_k$. But, by definition, $s = \Phi_2(s_k) = \Phi_2\big(\Phi(\sigma_k)\big) = \sigma_{k+1}$.

Finally, the only possible finite limit of σ_k is a fix point of $\Phi_2 \circ \Phi_1$ that is 0. As $\sigma_k > \sigma_1 > s_2 \geq 0$, this is not possible.

In the case $s_2 = 0$, it is enough to take $\sigma_0 < 0$ and then $\sigma_1 = \Phi_2(\sigma_0) > 0$. The same reasonning works and shows that $f_\mu = 0$ on $\mathbb{R} \setminus [\sigma_0, \sigma_1]$. But as σ_0 is arbitrary, we let $\sigma_0 \to 0$ and $s_2 = 0$ implies $\sigma_1 \to 0$ as well. □

Corollary 3.3. *Let $p \geq 1$ and $\Gamma = \{(s, |s|^p),\ s \in \mathbb{R}\}$. Let $\theta_1 \neq \theta_2 \in [-\pi/2, \pi)$. Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.*

The case $p > 1$ is covered by Theorem 3.1. The case $p = 2$ is due to P. Sjölin [Sj] and the previous proof is inspired by his work.

Proof. It remains to prove the case $p = 1$. For $|\theta - \pi/2| > \pi/4$, Corollary (2.2) shows that (Γ, ℓ_θ) is a Heisenberg Uniqueness Pair since then $I_0 = \mathbb{R}$.

If $|\theta_1 - \pi/2|, |\theta_2 - \pi/2| < \pi/4$, then we may again apply Theorem 3.1.

It remains to consider the case $\theta_1 = \pi/4$ or $3\pi/4$ and $|\theta_2 - \pi/2| \leq \pi/4$. We will only consider the case $\theta_1 = \pi/4$, the other case being similar.

Let $\gamma(s) = |s|$, and $\mu = f_\mu \, \mathrm{d}s$. Write $f_\mu^\pm$ for the restriction of f_μ to $\mathbb{R}^\pm$. If $\widehat{\mu} = 0$ on $\ell_{\pi/4}$ then, for every $t \in \mathbb{R}$

$$0 = \int_{\mathbb{R}} f_\mu(s) e^{-i(s+|s|)t/\sqrt{2}} \, \mathrm{d}t = \int_{\mathbb{R}^-} f(s) \, \mathrm{d}s + \widehat{f_\mu^+}(\sqrt{2}t).$$

Thus $\displaystyle\int_{\mathbb{R}^-} f(s) \, \mathrm{d}s = -\widehat{f_\mu^+}(\sqrt{2}t)$. Riemann-Lebesgue's Lemma then implies that $\displaystyle\int_{\mathbb{R}^-} f(s) \, \mathrm{d}s = 0$ thus $\widehat{f_\mu^+}(\sqrt{2}t) = 0$ thus $f_\mu^+ = 0$. Now, if $\theta_2 \neq \pi/4$ and $\widehat{\mu} = 0$ on ℓ_{θ_2} then

$$0 = \int_{\mathbb{R}^-} f_\mu(s) e^{-i(\cos\theta_2 - \sin\theta_2)st} \, \mathrm{d}s = \widehat{f_\mu}\big((\cos\theta_2 - \sin\theta_2)t\big)$$

for every $t \in \mathbb{R}$ and as $\cos\theta_2 - \sin\theta_2 \neq 0$, $f_\mu = 0$. □

3.2. **Cusps.**

Lemma 3.4. *Let $\gamma_- : (-\infty, 0] \to \mathbb{R}$ be a smooth strictly convex function such that $\gamma_-(0) = 0$, $\gamma_-(t) \to +\infty$ when $t \to -\infty$ and $\frac{\gamma(t)}{t} \to -\infty$ when $t \to 0$ and let $\gamma_+ : [0, +\infty) \to \mathbb{R}$ be a smooth strictly concave function such that $\gamma_+(0) = 0$, $\gamma_+(t) \to +\infty$ when $t \to +\infty$ and $\frac{\gamma(t)}{t} \to +\infty$ when $t \to 0$. Finally let $\gamma(t) = \begin{cases} \big(t, \gamma_-(t)\big) & \text{if } t < 0 \\ \big(t, \gamma_+(t)\big) & \text{if } t > 0 \end{cases}$ and $\Gamma = \{\gamma(t), t \in \mathbb{R}\}$.*

Let $\theta_1 \neq \theta_2 \in [-\pi/2, \pi/2)$, then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.

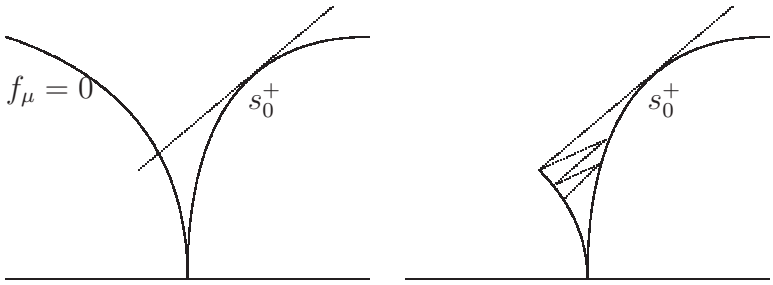


FIGURE 4. The wandering set in case of a cusp

Proof. This curve does not exactly fit the hypothesis of Section 2.3

First note that if $\theta_1 = 0$ then Γ satisfies the hypothesis of Corollary 2.2 with $I_0 = \mathbb{R}$ so that (Γ, ℓ_0) is a Heisenberg Uniqueness Pair.

Let $\Gamma_- = \big\{\big(s, \gamma_-(s)\big), s \leq 0\big\}$ and $\Gamma_+ = \big\{\big(s, \gamma_+(s)\big), s \geq 0\big\}$

Assume first that $-\frac{\pi}{2} \leq \theta_2 < \theta_1 < 0$. Observe that $\theta_j^\perp$ is transverse to Γ_- and more precisely, for every $s > 0$, there exists a unique $s_j^- < 0$ such that $\big(\gamma(s) + \mathbb{R}\theta_j^\perp\big) \cap \Gamma_- = \{\gamma(s_j^-)\}$. Denote by Φ_j^- the map $s \to s_j^-$.

Let s_0^+ be the unique $s > 0$ such that $\langle \gamma'(s), \theta_1 \rangle = 0$ *i.e.* the tangent to Γ at $\gamma(s)$ is directed by θ_1. Note that s_0^+ always exists if $\gamma'(s) \to 0$. If this s does not exist, we set $s_0^+ = +\infty$. Let $s_0^- = \Phi_1^-(s_0^+)$. Now, for $s < s_0^-$, $s \notin \Phi_j^-(\mathbb{R}^+)$ so that $s \in I_0^1$ and $f_\mu(s) = 0$. Observe now that, since $\theta_2 < \theta_1$, if $s > s_0^+$ $\Phi_2^-(s) < s_0^-$ so that (2.4) reduces to $f_\mu(s) = 0$.

If we write $\Gamma_0 = \{\gamma(s), s \in [s_0^-, s_0^+]\}$ then $\mu \in \mathcal{AC}(\Gamma_0)$. Note that Γ_0 now satisfies the conditions of Section 2.3. Moreover, define the map $\Phi_2^+ : [s_0^-, 0] \to [0, s_0^+]$ as Φ_2^+ being the unique $s \in [0, s_0^+]$ such that $\big(\gamma(s) + \mathbb{R}\theta_2^\perp\big) \cap \Gamma_+ = \{\gamma(s)\}$. Define $\begin{cases} s_{k+1}^+ &= \Phi_2^+(s_k^-) \\ s_{k+1}^- &= \Phi_1^-(s_k^+) \end{cases}$. Note that the assumption on θ_1 implies that $\gamma_-\big(\Phi_1^-(s)\big) < \gamma(s)$ and $\gamma_+\big(\Phi_2^+(s)\big) < \gamma_-(s)$. It follows that $\gamma_+(s_{k+1}^+) < \gamma_+(s_k^+)$ thus $s_{k+1}^+ < s_k^+$. This implies that s_k^+ converges to the unique fixed point of $\Phi_2^+ \circ \Phi_1^-$, that is 0. Similarily, s_k^- is strictly decreasing and converges to 0. As a consequence $[s_{k+1}^+, s_k^+]$ is wandering for $\Phi_2^+ \circ \Phi_1^-$ and $[s_k^-, s_{k+1}^-]$ is wandering for $\Phi_1^- \circ \Phi_2^+$. According to Lemma 2.6, $f_\mu = 0$ on $\bigcup [s_{k+1}^+, s_k^+] = (0, s_0^+)$ and on $\bigcup [s_k^-, s_{k+1}^-] = [s_0^-, 0]$.

The case $0 < \theta_2 < \theta_1 \leq \frac{\pi}{2}$ is similar, exchanging the roles of Γ_+ and Γ_-. It remains to treat the case $-\pi/2 < \theta_1 < 0 < \theta_2 < \pi/2$. Define $s_0^\pm$ as previously and again f_μ is supported in $[s_0^-, s_0^+]$. Define $\tilde{s}_0^-$ to be the unique $s < 0$ such that $\langle \gamma'(s), \theta_1\rangle = 0$ (if it exists and $s_0^- = -\infty$ otherwise) and for $s < 0$ define $\Phi_2^+(s)$ as the unique $s_+ > 0$ such that $\big(\gamma(s) + \mathbb{R}\theta_2^\perp\big) \cap \Gamma_+ = \{\gamma(s_+)\}$. Let $\tilde{s}_0^+ = \Phi_2^+(\tilde{s}_0^-)$. As previously, f_μ is supported in $[\tilde{s}_0^-, \tilde{s}_0^+]$ thus in $[\max(s_0^-, \tilde{s}_0^-), \min(s_0^+, \tilde{s}_0^+)]$. The remaining of the proof is similar. □

Example. Let $0 < \alpha < 1$ and $\Gamma = \{(s, |s|^\alpha, s \in \mathbb{R}\}$. Let $\theta_1 \neq \theta_2 \in (-\pi/2, \pi/2]$. Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.

3.3. **Hyperbolas.** Let Γ be the hyperbola

$$\Gamma = \{(u, v) \in \mathbb{R}^2 \,:\, v^2 - u^2 = 1\}.$$

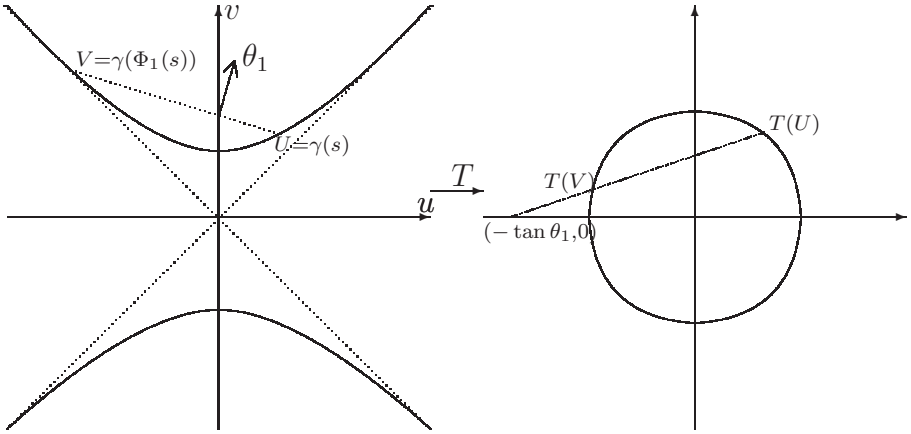
Let $I = (0, 1/2) \cup (1/2, 1)$ and $\gamma : I \to \mathbb{R}^2$ be a parametrization of Γ given by

$$\gamma(s) = (\operatorname{cotan}(2\pi s), 1/\sin(2\pi s)).$$

Theorem 3.5. *Let Γ be the hyperbola $\Gamma = \{\gamma(s), s \in I\}$. Then $(\Gamma, \ell_{\pm\pi/4})$ is a Heisenberg Uniqueness Pair. Moreover if $\theta_1 \neq \theta_2 \in (-\pi/2, \pi/2)$. Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair if and only if $\theta_1 \not\perp \theta_2$.*

Remark. In [HMR], the authors give a necessary and sufficient conditions for a lattice cross Λ in $\ell_{+\pi/4} \cup \ell_{-\pi/4}$ to form a Heisenberg uniqueness pair (Γ, Λ).

Proof. If $\theta = \pm\dfrac{\pi}{4}$ than any line orthogonal to θ intersects Γ in at most one point. That is, in the notation of Corollary 2.2 $I_0 = I$ and the theorem follows.



Let $\Phi_j = \Phi_{\theta_j}$, $j = 1, 2$ be the maps defined in section 2.3. Consider the transformation[1] $T : (u, v) \to (u/v, 1/v)$. Notice that T is a one-to-one map from Γ onto the circle $\mathbb{T}_* = \mathbb{S}^1 \backslash \{(-1, 0), (1, 0)\}$ and moreover the image of any line orthogonal to θ is a line L through the point $(-\tan\theta, 0)$.

[1]This transformation has a natural interpretation in projective geometry.

Let $\widetilde{\Phi}_j : \mathbb{T}_* \to \mathbb{T}_*$ be a map defined as follows: $\widetilde{\Phi}_j(\alpha)$ is the unique $\beta \in \mathbb{T}_* \backslash \{\alpha\}$ such that the line $L_{(\alpha,\beta)}$ joining α and β contains the point $(-\tan\theta_j, 0)$. Note that

$$T(\gamma(\Phi_j(s))) = \widetilde{\Phi}_j(T(\gamma(s))), \qquad j = 1, 2.$$

This transformation allows to transfer the dynamical system generated by $\Phi = \Phi_1 \circ \Phi_2$ on Γ to a new dynamical system on $\mathbb{T}_*$ generated by $\widetilde{\Phi} = \widetilde{\Phi}_1 \circ \widetilde{\Phi}_2$. In particular we will cover $\mathbb{T}_*$ by wandering sets for $\widetilde{\Phi}$. As T is a bijection, I will thus be covered by wandering sets for Φ. The theorem then follows from lemma 2.6.

We distinguish two cases:

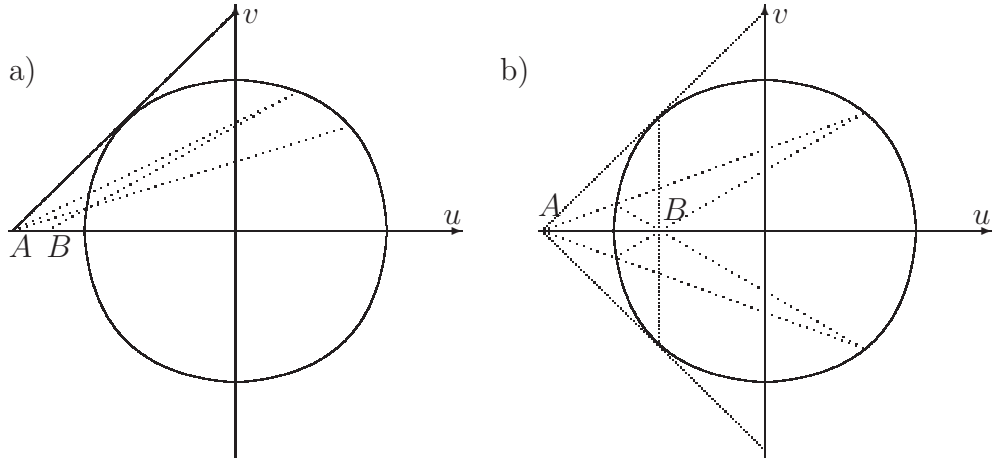


FIGURE 5. a) $\theta_1 > \pi/4$ and b) $0 < \theta_1 < \pi/4$,$\theta_2 = \theta_1 - \frac{\pi}{2}$

First case. $\theta_1 \notin (-\pi/4, \pi/4)$

Using the invariance property (Inv2), we assume without loss of generality that $\theta_1 \in (\pi/4, \pi/2)$ and that $\theta_2 < \theta_1$. Thus $\tan\theta_1 > 1$ and $\tan\theta_2 < \tan\theta_2$. Let $A = (-\tan\theta_1, 0)$ and note that, since $|\tan\theta_1| > 1$, A is in the "exterior" of $\mathbb{T}_*$. Let

$$\mathbb{T}_\pm = \{z \in \mathbb{T}_* : \ \pm \operatorname{Im} z > 0\}$$

and $\alpha_\pm \in \mathbb{T}_\pm$ the unique point $\alpha \in \mathbb{T}_\pm$ such that the line $L_{(\alpha,A)}$ is tangent to $\mathbb{T}$. Note that $\operatorname{Re}\alpha_\pm = -1/\tan\theta_1$. For $\epsilon_1, \epsilon_2 = \pm 1$, let

$$\mathbb{T}_{\epsilon_1,\epsilon_2} = \{z \in \mathbb{T}_{\epsilon_2} : \ \epsilon_1 \operatorname{Re} z > -1/\tan\theta_1\}.$$

Note that $\widetilde{\Phi}_1$ is a bijection for $\mathbb{T}_{\epsilon_1,\epsilon_2}$ onto $\mathbb{T}_{-\epsilon_1,\epsilon_2}$ while $\widetilde{\Phi}_2$ is a bijection for $\mathbb{T}_\epsilon$ onto $\mathbb{T}_{-\epsilon}$.

We need the following observation. Let $\alpha \in \mathbb{T}_{-,+}$ and $\beta = \widetilde{\Phi}(\alpha)$ and let φ —resp. ψ— be the angle between the real axis and the line $L_{(A,\alpha)}$ —resp. $L_{(B,\alpha)}$. Then :

- if $-\tan\theta_1 < -\tan\theta_2 < -1/\tan\theta_1$, then $|\varphi| > |\psi|$;

- if $-\tan\theta_2 = -1/\tan\theta_1$, hence $\theta_1 \perp \theta_2$ and then $\varphi = -\psi$;
- if $\tan\theta_2 > -1/\tan\theta_1$ then $|\varphi| < |\psi|$.

So if $\theta_1 \not\perp \theta_2$, then the absolute value of the angles between the real axis and $L_{(A,\widetilde{\Phi}^k(\alpha))}$ is strictly monotonic. Therefore $[\alpha, \widetilde{\Phi}(\alpha)]$ is wandering.

If $\theta_1 \perp \theta_2$ the $\widetilde{\Phi}^k(\alpha)$ is 2–periodic. Since $\theta_1 \in (\pi/4, \pi/2)$, we can define

$$x_{\theta_1} = \frac{1}{\sqrt{\sin^2\theta_1 - \cos^2\theta_1}}(-\cos\theta_1, \sin\theta_1) \in \Gamma.$$

Note that this point is defined by $\alpha_+ = T(x_{\theta_1})$. Let $s_0 \in [0, 1/2)$ be such that $x_{\theta_1} = \gamma(s_0)$ and $\Gamma_0 = \{\gamma(s),\ s \in (0, s_0]\}$. Let f be any function $f \in L^1(\Gamma_0)$. We will now extend f to $L^1(\Gamma)$ as follows:

– first, for $s^1_+ \in (s_0, 1/2)$, there is a unique $s^1_- \in (0, s_0)$ such that the line joining $\gamma(s^1_-)$ to $\gamma(s^1_+)$ is orthogonal to θ_1 and we define $f(s^1_+)$ via Equation 2.4 *for* θ_1:

$$\frac{f(s^1_+)}{\pi_{\theta_1}\gamma'(s^1_+)} = \frac{f(s^1_-)}{\pi_{\theta_1}\gamma'(s^1_-)}. \tag{3.11}$$

– Next, for every $s^2_+ \in (1/2, 1)$ there is a unique $s^2_- \in (0, 1/2)$ such that the line joining $\gamma(s^2_-)$ to $\gamma(s^2_+)$ is orthogonal to θ_2 and we define $f(s^2_+)$ via Equation 2.4 *for* θ_2:

$$\frac{f(s^2_+)}{\pi_{\theta_2}\gamma'(s^2_+)} = \frac{f(s^2_-)}{\pi_{\theta_2}\gamma'(s^2_-)}. \tag{3.12}$$

We will denote by $s_1 = (s_0) - +^2$.

– Finally, for $s^1_+ \in (1/2, s_1)$, there is a unique $s^1_- \in (s_1, 1/2)$ such that the line joining $\gamma(s^1_-)$ to $\gamma(s^1_+)$ is orthogonal to θ_1 and one easily checks that (3.11) is satifyed.

Let $\mu = f\,\mathrm{d}s \in \mathcal{AC}(\Gamma)$. According to Corollary 2.2, $\widehat{\mu} = 0$ on $\ell_{\theta_1} \cup \ell_{\theta_2}$. Moreover, every $\mu \in \mathcal{AC}(\Gamma)$ such that $\widehat{\mu} = 0$ on $\ell_{\theta_1} \cup \ell_{\theta_2}$ can be constructed this way.

Second case. $\theta_1, \theta_2 \in (-\pi/4, \pi/4)$.

Without loss of generality, we assume that $\theta_2 < \theta_1$ thus $-1 < \tan\theta_2 < \tan\theta_1 < 1$. Let

$$\{\alpha_\pm\} = \{z \in \mathbb{T}_\pm : \operatorname{Re} z = -\tan\theta_1\}$$

be two point at the vertical of A and define $\mathbb{T}_{\pm,\pm}$ as previously. Let $\alpha \in \mathbb{T}_{+,+}$ and let φ (resp. ψ) be the angle of $L_{(\alpha,A)}$ (resp. $L_{(\widetilde{\Phi}(\alpha),A)}$) with the real axis, then $\psi > \varphi$. Again this implies that $[\alpha, \widetilde{\Phi}(\alpha)]$ is wandering. The other case are similar.

□

3.4. **Closed convex curves with a corner point.** Let $\Gamma = \{\gamma(s), s \in [0, 1]\}$ be a closed *convex* curve and assume that γ is piecewise smooth 1-periodic and that γ' has a jump singularity at 0 *i.e.* Γ has a corner point at $\gamma(0)$. Without loss of generality $\gamma(0) = 0$ and let $\gamma'_\pm(0)$ the vectors defining the two half tangents to Γ at 0. Let H_0 be a supporting hyperplane of Γ at 0. As 0 is a corner point of Γ, this supporting hyperplane is not unique and we may assume that $H_0 \cap \Gamma = \{0\}$. Up to a rotation, we may assume that H_0 is the

vertical axes. Up to a symmetry, we may also assume that γ covers Γ in counter-clockwise order.

The fact that 0 is a corner point implies that H_0 and $\gamma'_\pm(0)$ define two positive open cones $\mathcal{C}_\pm$ with $\mathcal{C}_+$ in the upper half right quadrant and $\mathcal{C}_-$ in the lower half right quadrant. Let $\mathcal{C}_\pm^*$ be the dual cone of $\mathcal{C}_\pm$ (*i.e.* $\theta^\perp \in \mathcal{C}_+$ if and only if $\theta \in \mathcal{C}_+^*$.

Let $\theta_1, \theta_2 \in \mathcal{C}_-^* \cup \mathcal{C}_+^*$and assume that Γ does not contain a face normal to θ_1 nor to θ_2, so that Γ, θ_i satisfy the hypothesis of Corollary 2.2. We will first treat the case $\theta_1 \in \mathcal{C}_+^*$ and $\theta_2 \in \mathcal{C}_-^*$ and $\theta_1 < \theta_2 \in \mathcal{C}_+^*$. The case $\theta_1 \neq \theta_2 \in \mathcal{C}_-^*$ is obtained by a symmetry with respect to the horizontal axis.

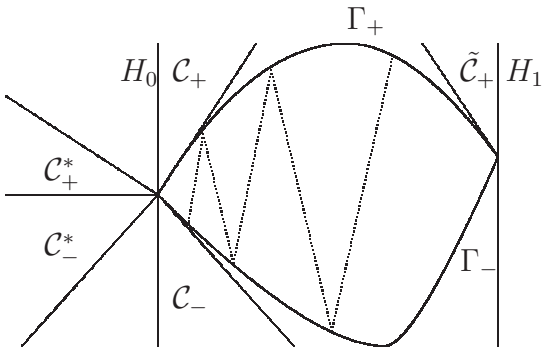


FIGURE 6. Closed curve with a corner point

Now, there is a unique $s \in (0,1)$ that we denote by s_* such that the line through $\gamma(s)$ directed by $\theta_1^\perp$ is a supporting line for Γ. Define $\Gamma_+ = \{\gamma(s), s \in (0, s_*)\}$ and $\Gamma_- = \{\gamma(s), s \in (s_*, 1)\}$. Observe that, due to the convexity of Γ, every line issued from a point $A \in \Gamma_+$ directed by $\theta_1^\perp$ will intersect Γ again in a point $B \in \Gamma_-$. Further, a line through B directed by $\theta_2^\perp$ will then intersect Γ again in a point $C \in \Gamma_+$. The assumption on the angles imply that we go from A to C along Γ clockwise. In the language of Section 2.3, the mapping Φ is strictly decreasing on $(0, s_*)$. But then, for every $s \in (0, s_*)$ the interval $[\Phi(s), s]$ is wandering for Φ. According to Lemma 2.6, if $\mu \in \mathcal{AC}(\Gamma)$ is such that $\widehat{\mu} = 0$ on $\ell_{\theta_1} \cup \ell_{\theta_2}$, then $f_\mu = 0$ on $[\Phi(s), s]$. As s is arbitrary in $(0, s_*)$, $f_\mu = 0$ on $(0, s_*)$. Using the fact that Φ is strictly increasing on $(s_*, 1)$ we obtain that $f_\mu = 0$ on $(s_*, 1)$ as well.

If θ_1, θ_2 are both in $\mathcal{C}_+$, a slight adaptation is needed. Without loss of $\theta_1 < \theta_2$. Then the same geometric argument shows that the map Φ is still strictly decreasing and again $f_\mu = 0$ on $(0, s_*)$. Let s_* be defined $\big(\gamma(1/2) + \mathbb{R}\theta_-\big) \cap \Gamma = \{\gamma(1/2), \gamma(s_*)\}$ (note that we might have $s_* = 1/2$). Corollary 2.2-(2.4) for θ_- shows that $f_\mu = 0$ on $(s_*, 1)$. Let s_+ be the

We have thus proved the following:

Proposition 3.6. *With the above notation, if $\theta_1 \in \mathcal{C}_+^*$ and $\theta_2 \in \mathcal{C}_-^*$ assume that Γ has no face normal to θ_1 nor to θ_2. Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.*

Remark. In the case $\gamma(1/2)$ is also a corner point, the previous proof may easily be extended to prove the following: Let again $\mathcal{C}_\pm$ be the previous cones and define $\tilde{\mathcal{C}}_\pm$ to be the analogous cones for $\gamma(1/2)$, translated to have there summit at the origin. $\tilde{\mathcal{C}}_+$ is in the upper left quadrant and $\tilde{\mathcal{C}}_-$ in the lower left one.

Then, if $\theta_1 \not\parallel \theta_2$ are in $\mathcal{C}_+ \cup \mathcal{C}_- \cup \tilde{\mathcal{C}}_+ \cup \tilde{\mathcal{C}}_-$, $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.

Example. If Γ is a convex polygon, then for almost every θ_1, there is an open interval of θ_2's such that $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair. In the case of a regular n-gon, this interval has length π/n.

4. Heisenberg Uniqueness Pairs and rotation numbers

4.1. **The ellipse revisited.** Let Γ be an ellipse. According to the invariance properties (Inv1)-(Inv2) there is no loss of generality in assuming that Γ is the circle centered at 0 of radius 1, $\Gamma = \{\gamma(t) = (\cos 2\pi t, \sin 2\pi t), t \in [0,1)\}$.[2] Let $\theta_1 \neq \theta_2$ be two angles. Without loss of generality $\theta_1 = 0$ and $\theta_2 \in [0, \pi)$. Let Φ_1, Φ_2 be the maps associated to them as in Section 2.3. It is easy to see that Φ_j is the orthogonal symmetry with respect to the line through 0 directed by θ_j, in particular $\Phi_1(s) = -s \bmod 1$ while $\Phi_2(s) = \frac{\theta_2}{\pi} - s \bmod 1$. Throughout the remaining of Section 4.1 all functions on Γ will be lifted as 1-periodic functions on $\mathbb{R}$.

Then, according to Corollary 2.2, $\mu \in \mathcal{AC}(\Gamma)$ is such that $\widehat{\mu} = 0$ on $\ell_0 \cup \ell_{\theta_2}$ if and only if $f_\mu(-s) = -f_\mu(s)$ and $f_\mu(\theta_2/\pi - s) = -f_\mu(s)$. In particular, $f_\mu(\theta_2/\pi + s) = f_\mu(s)$. Note that conversely, if $f_\mu(\theta_2/\pi + s) = f_\mu(s)$ and $f_\mu(-s) = -f\mu(s)$, then $f_\mu(\theta_2/\pi - s) = -f_\mu(s)$.

According to Lemma 2.8, $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair if $\theta_2 \notin \mathbb{Q}$. Assume now that $\dfrac{\theta_2}{\pi} = \dfrac{p}{q}$, $p, q \in \mathbb{N}$, p, q coprime. Then every integer j may be written in the form $j = kp + \ell q$ so that, if f_μ is both 1-periodic and p/q-periodic, then $f_\mu(s + j/q) = f_\mu(s + kp/q + \ell) = f_\mu(s + kp/q) = f_\mu(s)$ *i.e.* f_μ is also $1/q$-periodic. The converse is trivial. Thus $\widehat{\mu} = 0$ on $\ell_0 \cup \ell_{\theta_2}$ if and only if f_μ is both odd and $1/q$-periodic. Such functions are all constructed in the following way: take any f_μ on $(0, 1/2q)$, extend it into an odd function on $(-1/2q, 1/2q)$ and then to a $1/q$-periodic function on $\mathbb{R}$ (thus also to a 1-periodic function).

This gives a more geometric and constructive proof of the following result:

Theorem 4.1 (Lev [Le] and Sjolin [Sj])**.** *Let Γ be a circle and let $\theta_1, \theta_2 \in \mathbb{R}$ Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair if and only if $\dfrac{1}{\pi}(\theta_2 - \theta_1) \notin \mathbb{Q}$.*

For a general ellipse the condition is a bit more complicated. First let a and b the major and minor semi-axes of the ellipse So that, if we denote by $L(x, y) = (x, ay/b)$ then there is a rotation R_θ such that $LR_\theta\Gamma$ is a circle $\mathcal{C}$ of radius a. According to the invariance properties, $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg uniqueness pair if and only if $(\mathcal{C}, \ell_{\varphi_1} \cup \ell_{\varphi_2})$ with $\ell_{\varphi_j} = (R_\theta^{-1})^*(L^{-1})^*\ell_{\theta_j}$. It follows that $\varphi_2 - \varphi_1 = \arcsin \frac{b \sin\theta_2}{\sqrt{a^2+b^2}} - \arcsin \frac{b \sin\theta_1}{\sqrt{a^2+b^2}}$.

[2]This parametrization has been chosen to be coherent with the usual definition of rotation numbers in the next section.

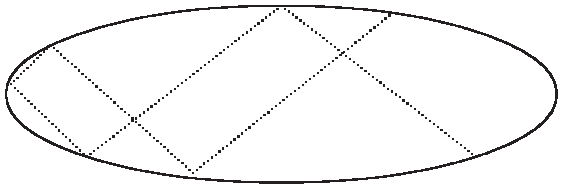


FIGURE 7. Ellipse

4.2. **An extension.** Let $-\pi/2 < \theta_1 \not< 0 < \theta_2 < \pi/2$ be two angles and let $\ell = -2\tan\theta_1 + 2\tan\theta_2$. Define $\Gamma = \{\gamma(t), t \in [0,1]\}$ as follows:

$$\gamma(t) = \begin{cases} \big(\ell + \cos 4\pi(t-1/8), \sin 4\pi(t-1/8)\big) & \text{for } t \in [0,1/4] \\ (\ell(2-4t), 1) & \text{for } t \in [1/4,1/2] \\ \big(\cos 4\pi(t-3/8), \sin 4\pi(t-3/8)\big) & \text{for } t \in [1/2,3/4] \\ (\ell(-3+4t), -1) & \text{for } t \in [3/4,1] \end{cases}$$

and write $\Gamma = \mathcal{C}_+ \cup \mathcal{S}_+ \cup \mathcal{C}_- \cup \mathcal{S}_-$ for the four corresponding pieces of Γ.

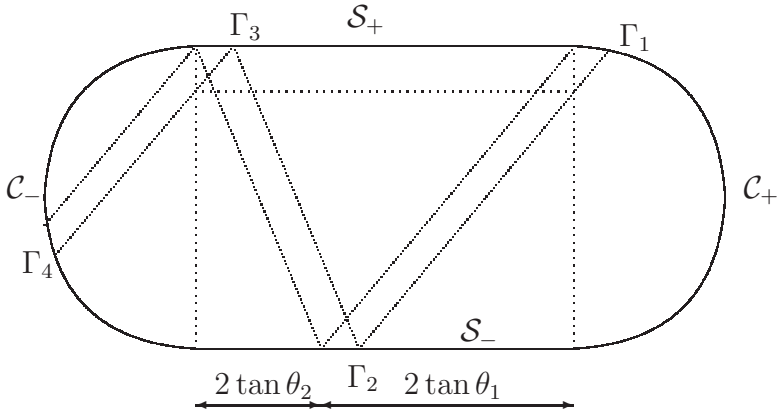


FIGURE 8. The domain Γ

In other words, Γ is a circle of radius 1, cut into two halves, the two halves are then separated by a distance ℓ and glued together by a rectangle of length ℓ and height 2. This length is chosen so that the following holds

– take a point Γ_1 in $\mathcal{C}_+$, and draw a line orthogonal to θ_1 and assume this line intersects $\mathcal{S}_-$ in a point Γ_2. (Otherwise it intersects $\mathcal{C}_+$ in a point $\tilde{\Gamma}_1$)

– From Γ_2, draw a line orthogonal to θ_2. This line will intersect $\mathcal{S}_+$ in a point Γ_3.

– From Γ_3, draw a line orthogonal to θ_1. This line will intersect $\mathcal{C}_-$ in a point Γ_4.

Then Γ_4 is the translate by $(-\ell, 0)$ of the point $\tilde{\Gamma}_4$ that is the intersection of $\mathcal{C}_- + (\ell, 0)$ with the line orthogonal to θ_1 starting at Γ_1.

We may of course exchange $\mathcal{C}_+$ and $\mathcal{C}_-$ that is, to go backwards in the above argument. Moreover, we can replace θ_1 by θ_2 (this is needed if, at the first step, we go from Γ_1 to $\tilde{\Gamma}_1$).

Define s_j, $j = 1, \dots, 4$ so that $\gamma(s_j) = \Gamma_j$.

Now let $\mu \in \mathcal{AC}(\Gamma)$ be such that $\widehat{\mu} = 0$ on $\ell_{\theta_1} \cup \ell_{\theta_2}$. Then (2.3) for θ_1 in (4.13), then for θ_2 in (4.14) and for θ_1 again in (4.15) shows that:

$$\frac{f_\mu(s_1)}{\pi_{\theta_1}\gamma'(s_1)} = -\frac{f_\mu(s_2)}{\pi_{\theta_1}\gamma'(s_2)} = -\frac{f_\mu(s_2)}{\pi_{\theta_2}(4\ell, 0)}\frac{\pi_{\theta_2}(4\ell, 0)}{\pi_{\theta_1}(4\ell, 0)} \tag{4.13}$$

$$= \frac{f_\mu(s_3)}{\pi_{\theta_2}(-4\ell, 0)}\frac{\pi_{\theta_2}(4\ell, 0)}{\pi_{\theta_1}(4\ell, 0)} = \frac{f_\mu(s_3)}{\pi_{\theta_1}(-4\ell, 0)} \tag{4.14}$$

$$= \frac{f_\mu(s_4)}{\pi_{\theta_1}\gamma'(s_4)}. \tag{4.15}$$

A similar identity holds if we replace θ_1 by θ_2.

Let us now define ν a measure on the unit sphere $\{(\cos 2\pi t, \sin 2\pi t), t \in [-1/4, 3/4]\}$ by

$$f_\nu(t) = \begin{cases} f_\mu(t/2 + 1/8) & \text{for } t \in [-1/4, 1/4] \\ f_\mu(t/2 + 3.8) & \text{for } t \in [1/4, 3/4] \end{cases}.$$

In other words, ν is μ restricted to the two half-circles (when glued back together).

From the discussion above, we see that (4.15) is (2.3) for f_ν and θ_1. The same holds for θ_2. Therefore, $\widehat{\nu} = 0$ on $\ell_{\theta_1} \cup \ell_{\theta_2}$. But, according to Theorem 4.1, $\nu = 0$, that is $f_\mu = 0$ on $[0, 1/4] \cup [1/2, 3/4]$. It follows from (4.13)-(4.14) that $f_\mu = 0$ on $[1/4, 1/2] \cup [3/4, 1]$.

We have thus proved:

Proposition 4.2. *Let θ_1, θ_2, Γ be as above. Then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.*

4.3. **Rotation numbers.** Till the end of section 4.3, we will assume that Γ is a $\mathcal{C}^k$-smooth, $k \geq 4$ closed curve with non vanishing curvature. We parametrize $\Gamma = \{\gamma(s), s \in \mathbb{R}\}$ where γ is one-to-one and 1-periodic. Let $\theta_1 \neq \theta_2$ be two angles and assume that Γ satisfies the hypothesis of Corollary 2.2 for both θ_1 and θ_2. Let $\Phi_j = \Phi_{\theta_j}$ be the corresponding maps and write $\Phi_{\theta_1,\theta_2,\Gamma}$ or simply Φ for $\Phi = \Phi_2 \circ \Phi_1$.

Note that Φ is of class $\mathcal{C}^{k-1}$ and, as Φ_1 and Φ_2 are orientation reverting, Φ is orientation preserving. We denote by $\tilde{\Phi}$ a $\mathcal{C}^{k-1}$ lifting of Φ as a map from $\mathbb{R} \to \mathbb{R}$.

We need a bit more notation. All results mentioned in here are standard facts in the theory of dynamical systems and can be found in [He1, Yo] which also give precise references for them.

The rotation number of $\tilde{\Phi}$ is defined as $\rho(\tilde{\Phi}) = \lim \frac{\tilde{\Phi}^n(x)-x}{n}$. As is well known, this limit exists and does not depend on x. Moreover, we define $\rho(\Phi) = \rho(\tilde{\Phi}) \bmod 1$ and this number does not depend on the choice of lifting $\tilde{\Phi}$.

Notation : We will write $\rho(\Gamma; \theta_1, \theta_2) = \rho(\Phi)$ to stress the dependence on θ_1, θ_2 and Γ.

Recall that $\rho(\Phi)$ is rational if and only if Φ has a periodic orbit. On the other hand if $\alpha = \rho(\Phi)$ is irrational, it is known that Φ is conjugated to the rotation of angle α (for this we only need Φ to be of class $\mathcal{C}^2$ but $\mathcal{C}^1$ may not suffice). However this conjugation may not be regular, even though Φ is of class $\mathcal{C}^\infty$. In order to obtain a regular map, we need more. Recall that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is called diophantian of order β (in $\alpha \in \mathcal{C}_\beta$) if there exists $C > 0$ such that $|\alpha - p/q| \geq C/q^{2+\beta}$ for every $p/q \in \mathbb{Q}$. Note that $\bigcup_{\beta \geq 0} \mathcal{C}_\beta$ has full Lebesgue measure. We will use the following theorem:

Theorem 4.3 (Yoccoz [Yo])**.** *If Φ is of class $\mathcal{C}^{k-1}$, $k \geq 4$, and assume that $\alpha := \rho(\Phi) \in \mathcal{C}_\beta$ with $k > 2(\beta + 1)$. Then there exists a diffeomorphism h of class $\mathcal{C}^{k-\beta-2-\varepsilon}$ for every $\varepsilon > 0$ such that $\Phi = h^{-1} \circ R_\alpha \circ h$ where R_α is the rotation of angle α, $R_\alpha(t) = t + \alpha \ mod 1$.*

Together with Lemma 2.8 we obtain the following

Corollary 4.4. *Let $\beta \geq 0$, $k \geq \min(4, \beta + 3, 2\beta + 2)$. Let Γ be a $\mathcal{C}^k$ smooth closed convex curve with non-vanishing curvature and θ_1, θ_2 be two angles. Assume that $\rho(\Gamma; \theta_1, \theta_2) \in \mathcal{C}_\beta$ then $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.*

Unfortunately, computing the rotation number $\rho(\Gamma; \theta_1, \theta_2)$ is practically impossible. Nevertheless, if we assume that

$$a \leq \tilde{\Phi}(x) - x \leq b \tag{4.16}$$

i.e. if we bound the "displacement" of Φ then

$$na \leq \tilde{\Phi}^n(x) - x = \sum_{j=1}^{n} \tilde{\Phi}^j(x) - \tilde{\Phi}^{j-1}(x) \leq nb$$

thus $a \leq \rho(\Gamma; \theta_1, \theta_2) \leq b$. Note that it is enough to obtain the bound (4.16) for $x \in [0, 1]$. Write $t = \widetilde{\Phi}_1(x)$ thus $x = \widetilde{\Phi}_1(t)$, since $\widetilde{\Phi}_1^{-1} = \widetilde{\Phi}_1$, so that (4.16) is equivalent to

$$a \leq \widetilde{\Phi}_2(t) - \widetilde{\Phi}_1(t) \leq b.$$

Now, from this, it is obvious that $\rho(\Gamma; \theta_1, \theta_2) \to 0$ when $\theta_2 \to \theta_1$. On the other hand, $\rho(\Gamma; \theta_1, \theta_2) \neq 0$ since $\Phi_1(x) \neq \Phi_2(x)$ (otherwise $\theta_1 \neq \theta_2$ would both be normal to Γ) thus $\min_{[0,1]}(\widetilde{\Phi}_1(x) - \widetilde{\Phi}_2(x)) > 0$ by continuity of Φ_1, Φ_2. In particular $\rho(\Gamma; \theta_1, \theta_2)$ is not a constant function of θ_1 nor of θ_2.

We will appeal to the following

Theorem 4.5 (Herman [He2]). *Let Ψ_t be a family of diffeomorphisms of $[0,1)$ of class $\mathcal{C}^3$ such that the dependence in the parameter t is of class $\mathcal{C}^1$. Then either the rotation number $\rho(\Phi_t)$ does not depend on t or there exists a set E of positive Lebesgue measure such that, for every $t \in E$, Φ_t is conjugated by a $\mathcal{C}^1$ diffeomorphisms to a rotation with irrational angle.*

The above comment with the regularity of Φ_1 and Φ_2 established in Proposition 2.4 imply the following result:

Corollary 4.6. *Let Γ be a $\mathcal{C}^4$ smooth closed convex curve with non-vanishing curvature. Then there exists a set of positive Lebesgue measure $E \subset (-\pi/2, \pi/2)^2$ such that, for almost every $(\theta_1, \theta_2) \in E$, $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.*

4.4. **Rational rotation number is compatible with Heisenberg Uniqueness.** Let us conclude with an example of a smooth curve Γ and angles θ_1, θ_2 such that $\rho(\Phi)$ is rational but such that $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is *not* a Heisenberg uniqueness pair.

First, let χ be a 1-periodic $\mathcal{C}^\infty$ function on $\mathbb{R}$ such that $\operatorname{supp}\chi = [0, 1/4] + \mathbb{Z}$, $0 < \chi(s) < 1/2$ on $(0, 1/8)$ and $\chi(s) < 0$ on $(1/8, 1/4)$. Let $\gamma(s) = \big(1 + \chi(s)\big)(\cos 2\pi s, \sin 2\pi s)$. Note that $\gamma : [0,1] \to \mathbb{R}^2$ has the following properties

(1) γ is $\mathcal{C}^\infty$-smooth,
(2) $\gamma(s) = (\cos 2\pi s, \sin 2\pi s)$ for $s \in [1/4, 1]$ *i.e.* Γ contains 3/4 of the circle $\mathcal{C}$ centered at 0 and radius 1.
(3) $|\gamma(s)| < 1$ for $s \in (0, 1/8)$ and $|\gamma(s)| > 1$ for $s \in (1/8, 1/4)$. In other words the part of Γ in the first quadrant is inside the disc below the diagonal and outside the disc above the diagonal.
(4) If $\big(1 + \chi(s)\big)^2 + \chi'(s)^2 - 2\big(1 + \chi(s)\big)\chi''(s) \geq 0$ then Γ is convex.

Let $\theta_1 = 0$ and $\theta_2 = \dfrac{\pi}{2}$ and consider the associated maps Φ_1, Φ_2 and Φ as in Section 2.3. Note that $k/8$, $k \in \mathbb{Z}$ are 2-periodic points of Φ thus Φ has rotation number 1/2. Then, for $s \in [1/4, 3/4]$, $\Phi(s) = s + 1/2$. For $s \in (3/4, 7/8)$, $s - 1/2 < \Phi(s) < 3/8$ and for $s \in (7/8, 1)$, $\Phi(s) < s - 1/2$. As a consequence, if $a \in (1/4, 3/8)$ $\Phi^{2k}(a)$ is increasing and bounded, therefore it converges. The limit is a fixed point of Φ and the only possible one is 3/8. Similarly, if $b \in (3/8, 1/2)$, $\Phi^{2k}(b)$ is decreasing and bounded and converges to 3/8 as well. It follows that (a, b) is attractive for Φ.

According to Lemma 2.7, if $\mu \in \mathcal{AC}(\Gamma)$ is such that $\widehat{\mu} = 0$ on $\ell_0 \cup \ell_{\pi/2}$, then $\operatorname{supp} f_\mu \cap (a, b) \subset \bigcap_{n \geq 0} [\Phi^{2k}(a), \Phi^{2k}(b)] = \{3/8\}$. As a is arbitrary in $(1/4, 3/8)$ and b is arbitrary in $(3/8, 1/2)$, $f_\mu = 0$ on $(1/4, 1/2)$. Using Corollary 2.2-(2.4) for Φ_1, Φ_2, we deduce that $f_\mu = 0$ on $(0, 3/2)$ and using it again for Φ_1 or Φ_2 we deduce that $f_\mu = 0$.

We have thus proved the following:

Proposition 4.7. *There exists a smooth closed curve and two angles θ_1, θ_2 such that $\rho(\Gamma; \theta_1, \theta_2)$ is rational and $(\Gamma, \ell_{\theta_1} \cup \ell_{\theta_2})$ is a Heisenberg Uniqueness Pair.*

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