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# A Shrinkage-Thresholding Metropolis adjusted Langevin algorithm for Bayesian Variable Selection

Amandine Schreck\*, Gersende Fort\*, Sylvain Le Corff†, Eric Moulines\*

May 5, 2015

## Abstract

This paper introduces a new Markov Chain Monte Carlo method for Bayesian variable selection in high dimensional settings. The algorithm is a Hastings-Metropolis sampler with a proposal mechanism which combines a Metropolis Adjusted Langevin (MALA) step to propose local moves associated with a shrinkage-thresholding step allowing to propose new models. The geometric ergodicity of this new trans-dimensional Markov Chain Monte Carlo sampler is established. An extensive numerical experiment, on simulated and real data, is presented to evidence the superior behavior of the proposed algorithm in comparison with some more classical trans-dimensional algorithms.

## 1 Introduction

An important problem in Statistics is the choice of an appropriate model from a collection of plausible models. In this paper, we focus on variable selection in the familiar regression problem, where the objective is to explain a response variable with a (possibly very large) number of explanatory variables, which can be either discrete or continuous. In such a setting, the model selection problem reduces to the choice of a typically small subset of explanatory variables. In many applications, it is known that only a small fraction of explanatory variables explains a large fraction of the observations, and using this information is crucial for inference. The degree of sparsity and the penalty for *false* detection is very problem specific.

Variable selection is particularly challenging in high dimensional settings, the number of possible subsets of regressors being huge. A variety of algorithms to explore the collection of models and criteria for selecting among competing models have been proposed. In the Bayesian framework, the variable selection problem is transformed into posterior inference: rather than searching a highly hypothetical "best" model, Bayesian analysis attempts to estimate the joint posterior distribution of the collection of all subsets of parameters. In high dimension, this aim is often overly ambitious: estimating the marginal posterior probability that a variable should be included in the model is already a difficult problem.

In the last three decades, Markov Chain Monte Carlo (MCMC) methods are the most commonly used computational procedures to sample posterior distributions [20]. Intensive research efforts have been spent to implement variable selection methods that fit into the MCMC framework. An early attempt to perform variable selection is the Reversible Jump MCMC (RJMCMC) introduced in [10], which is one of the main breakthrough in the MCMC literature. RJMCMC is a trans-dimensional sampler which produces a Markov chain evolving between spaces of different dimensions. The dimension of the sample varies at each iteration as active parameters are added or discarded from the model. Each new sample is accepted or rejected using a Metropolis-Hastings step where the acceptance probability is adjusted to the trans-dimensional moves. RJMCMC requires ingenuity in designing appropriate jumping rules to produce computationally efficient and theoretically effective methods. Despite many attempts [4, 13], this algorithm is prone to fail when the dimension of the parameter space is large (as illustrated in our numerical section).

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[6] considers another setting that encompasses all the models jointly: at each iteration, pseudo-prior distributions are used to jointly sample regression parameters associated with all models. For high dimensional statistical problems, sampling jointly all models is of course out of reach. A more efficient algorithm, the Metropolized Carlin and Chib (MCC), simultaneously proposed by [9, 8] and later improved by [19], does not require to sample from the whole collection of models and therefore can be implemented in practice. The mixing of this algorithm depends critically on the specification of pseudo-priors, which requires also a fair amount of tuning.

Other MCMC approaches for Bayesian variable selection define a posterior distribution on the model space, where a model is a binary vector locating the active (non null) components of the regression space. The objective is then to explore this posterior distribution, which is equivalent to estimate probabilities of activation for each regression parameter. In [5] for example, this exploration is performed with a Gibbs sampler. Variants and adaptive versions of the Gibbs sampler for this problem have been proposed in [18, 14]. Samples from the posterior distribution of the models are obtained in [24] and in [23] with particle filters.

In this paper, we introduce a novel algorithm, the Shrinkage-Thresholding Metropolis-Adjusted Langevin Algorithm (STMALA) to perform sparse regression in high dimensional models. This algorithm might be seen as a trans-dimensional MCMC method relying on the MALA algorithm (see [22]). The proposal distribution in the STMALA algorithm goes as follows.

- compute a noisy gradient step of the logarithm of the smooth part of the target distribution;
- apply a shrinkage-thresholding operator to ensure sparsity and shrink values of the regression parameters toward zero.
- use an accept-reject step to guarantee the convergence to the correct target distribution.

Each iteration of the STMALA algorithm may be seen as a randomized version of the Shrinkage-Thresholding algorithm (see [2]) to guide variable selection. The Shrinkage-Thresholding algorithm (and its accelerated version FISTA) is one of the most effective method to solve sparse inverse problems with sparsity inducing penalty. Our intuition is that a single iteration of the Shrinkage-Thresholding algorithm (with some additional noise added to ensure irreducibility) is a sensible way to visit the collection of models.

This intuition is supported both by very promising experimental results obtained in a variety of challenging situations and by some theoretical results. In particular, we have established the geometric ergodicity of the STMALA algorithm for a large class of target distributions. To our best knowledge, it is the first result providing a rate of convergence for a trans-dimensional MCMC algorithm (like RJMCMC and MCC); usually, only Harris recurrence is proved, see [21].

This paper is organized as follows. STMALA and its application to Bayesian variable selection is described in Section 2. The geometric ergodicity of the STMALA algorithm is addressed in Section 3. Numerical experiments on simulated and real data sets to assess the performance of STMALA are given in Section 4. All the proofs are postponed to Section 5.

## 2 The Shrinkage-Thresholding MALA algorithm

This section introduces the Shrinkage-Thresholding MALA (STMALA) algorithm which is designed to sample from a target distribution defined on  $\mathbb{R}^p$ ,  $p \in \mathbb{N}^*$ , under the sparsity assumption that only a small fraction of the explanatory variables is active. Denote by  $\mathcal{M} \stackrel{\text{def}}{=} \{0, 1\}^p$  the set of binary vectors. For any  $m = (m_1, \dots, m_p) \in \mathcal{M}$ , set

$$I_m \stackrel{\text{def}}{=} \{i \in \{1, \dots, p\}; m_i = 1\}, \quad (1)$$

the family of active variables. We consider a target distribution  $\pi$  on  $\mathbb{R}^p$  absolutely continuous with respect to the positive measure  $\nu$  given for any Borel set  $A \subset \mathbb{R}^p$  by

$$\nu(A) \stackrel{\text{def}}{=} \sum_{m \in \mathcal{M}} \int \cdots \int \left( \prod_{i \notin I_m} \delta_0(dx_i) \right) \left( \prod_{i \in I_m} dx_i \right) \mathbf{1}_A(x_1, \dots, x_p). \quad (2)$$

For any  $m \in \mathcal{M}$ , denote by  $S_m$  the subset of  $\mathbb{R}^p$  defined by

$$S_m \stackrel{\text{def}}{=} \{z \in \mathbb{R}^p, z_i \neq 0, i \in I_m, z_j = 0, j \notin I_m\} \quad (3)$$

and for  $m \in \mathcal{M}$ , by  $|m| \stackrel{\text{def}}{=} \sum_{i=1}^p m_i$  the number of non-zero components in  $m$ .  $\{S_m, m \in \mathcal{M}\}$  is a partition of  $\mathbb{R}^p$  and we assume that the target distribution may be written as

$$\pi(x) = \sum_{m \in \mathcal{M}} \omega_m \pi_m(x) \mathbb{1}_{S_m}(x), \quad (4)$$

where  $\{\omega_m, m \in \mathcal{M}\}$  is the prior probability of the models and  $\pi_m(x)$  is the joint distribution of the observations for model  $m \in \mathcal{M}$ . For each  $m \in \mathcal{M}$ , we assume that  $\pi_m(x) \propto \exp(-U_m(x) - V_m(x))$  where  $U_m$  is continuously differentiable and  $V_m$  is a possibly non-smooth (penalization) term.

Two different shrinkage-thresholding operators are considered to sample sparse vectors, namely the Proximal one (Prox)  $\Psi_1$  and the soft thresholding operator with vanishing shrinkage (STVS)  $\Psi_2$ : for any  $\gamma > 0$ ,  $1 \leq i \leq p$  and  $u = (u_1, \dots, u_p) \in \mathbb{R}^p$ ,

$$(\Psi_1(u))_i \stackrel{\text{def}}{=} u_i \left(1 - \frac{\gamma}{|u_i|}\right)_+, \quad (5)$$

$$(\Psi_2(u))_i \stackrel{\text{def}}{=} u_i \left(1 - \frac{\gamma^2}{|u_i|^2}\right)_+, \quad (6)$$

where for  $a \in \mathbb{R}$ ,  $a_+$  denotes the positive part of  $a$ :  $a_+ \stackrel{\text{def}}{=} \max(a, 0)$ . Lemma 2.1 shows that the

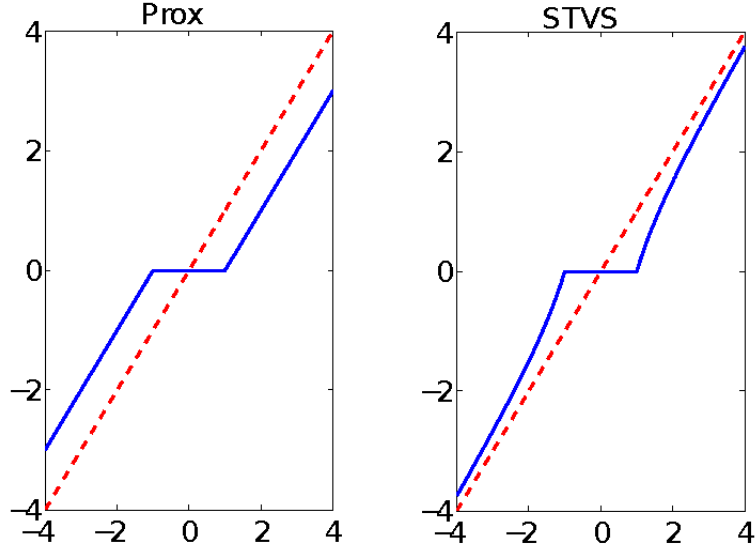


Figure 1: Shrinkage-Thresholding functions associated with Prox (left) and STVS (right) in one dimension.

soft thresholding operator with vanishing shrinkage  $\Psi_2$ , known as the empirical Wiener operator (see [25]), compromises between minimizing a (non-convex) function  $h$  and being close to  $u$ .

**Lemma 2.1.** For any  $\gamma > 0$  and  $u \in \mathbb{R}^p$ ,

$$\Psi_2(u) = \operatorname{argmin}_{x \in \mathbb{R}^p} \left( h(x) + \frac{1}{2} \|x - u\|^2 \right),$$

where the function  $h : \mathbb{R}^p \rightarrow \mathbb{R}$  is given by

$$h(x) = \gamma^2 [\operatorname{asinh}(\|x\|/(2\gamma)) - (1/2) \exp(-2 \operatorname{asinh}(\|x\|/(2\gamma)))] .$$

*Proof.* The proof of Lemma 2.1 is in Section 5.2. □

From a current state  $X^n \in \mathbb{R}^p$  the algorithm proposes a new point  $Z$  defined by

$$Z = \Psi \left( X^n + \frac{\sigma^2}{2} h(X^n) + \sigma \xi^{n+1} \right), \quad (7)$$

where  $\sigma > 0$ ,  $\xi^{n+1} \sim \mathcal{N}_p(0, I)$  and for all  $x \in \mathbb{R}^p$  and  $D > 0$ ,

$$h(x) \stackrel{\text{def}}{=} \frac{D \sum_{m \in \mathcal{M}} \nabla U_m(x) \mathbf{1}_{S_m}(x)}{D \vee \left\| \sum_{m \in \mathcal{M}} \nabla U_m(x) \mathbf{1}_{S_m}(x) \right\|}, \quad (8)$$

with  $a \vee b = \max(a, b)$ . Define

$$\mu(x) \stackrel{\text{def}}{=} x + \sigma^2 h(x) / 2. \quad (9)$$

**Lemma 2.2.** *Let  $x \in \mathbb{R}^p$ ,  $D, \gamma, \sigma > 0$ . The random vector  $\Psi_1(\mu(x) + \sigma \xi)$  where  $\xi \sim \mathcal{N}_p(0, I)$  has a density with respect to  $\nu$  (see (2)) given by  $z \mapsto q_{\Psi_1}(x, z)$  with*

$$q_{\Psi_1}(x, z) = \sum_{m \in \mathcal{M}} \left( \prod_{i \notin I_m} \rho(\mu_i(x)) \right) \left( \prod_{i \in I_m} f(\mu_i(x), z_i) \right) \mathbf{1}_{S_m}(z); \quad (10)$$

for any  $c \in \mathbb{R}$  and  $z \in \mathbb{R}^*$

$$\begin{aligned} \rho(c) &\stackrel{\text{def}}{=} \mathbb{P} \{ |c + \sigma \zeta| \leq \gamma \}, \text{ with } \zeta \sim \mathcal{N}(0, 1), \\ f(c, z) &\stackrel{\text{def}}{=} (2\pi\sigma^2)^{-1/2} \exp \left\{ -|(1 + \gamma|z|^{-1})z - c|^2 / (2\sigma^2) \right\}. \end{aligned}$$

This proposal distribution is equivalent to sampling a new binary vector  $m' \in \mathcal{M}$  conditionally to  $x$ ; and then sampling a new vector with non null components in  $\mathbb{R}^{|m'|}$  conditionally to  $(m', x)$  as follows:

- (i) sample independently the components  $m'_i$ ,  $1 \leq i \leq p$ , such that  $m'_i$  is a  $\{0, 1\}$ -Bernoulli random variable with success parameter

$$1 - \mathbb{P}(|\mu_i(x) + \sigma \zeta| \leq \gamma),$$

where  $\zeta \sim \mathcal{N}(0, 1)$ ;

- (ii) for  $i \notin I_{m'}$ , set  $z_i = 0$ ; conditionally to  $(m', x)$ , sample independent components such that for any  $i \in I_{m'}$ , the distribution of  $z_i$  on  $\mathbb{R}^*$  is proportional to

$$\exp \left( -|(1 + \gamma/|z_i|)z_i - \mu_i(x)|^2 / (2\sigma^2) \right).$$

**Lemma 2.3.** *Let  $x \in \mathbb{R}^p$ ,  $D, \gamma, \sigma > 0$ . The random vector  $\Psi_2(\mu(x) + \sigma \xi)$  where  $\xi \sim \mathcal{N}_p(0, I)$  has a density with respect to  $\nu$  (see (2)) given by  $z \mapsto q_{\Psi_2}(x, z)$  with*

$$q_{\Psi_2}(x, z) = \sum_{m \in \mathcal{M}} \left( \prod_{i \notin I_m} \rho(\mu_i) \right) \left( \prod_{i \in I_m} f_{st}(\mu_i(x), z_i) \right) \mathbf{1}_{S_m}(z);$$

for any  $c \in \mathbb{R}$ ,  $z \in \mathbb{R}^*$ ,  $u > 0$

$$f_{st}(c, z) \stackrel{\text{def}}{=} (2\pi\sigma^2)^{-1/2} g(\gamma^2|z|^{-2}) \tilde{g}(\gamma^2|z|^{-2}) \exp \left( -|g(\gamma^2|z|^{-2})z - c|^2 / (2\sigma^2) \right),$$

with

$$g(u) \stackrel{\text{def}}{=} 1 + \frac{2u}{1 + \sqrt{1 + 4u}}, \quad \tilde{g}(u) \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + 4u}},$$

and  $c \mapsto \rho(c)$  is given by Lemma 2.2.

The proposal (7) is then accepted with probability  $\alpha(X^n, Z)$  given by

$$\alpha_{\Psi}(x, z) \stackrel{\text{def}}{=} 1 \wedge \frac{\pi(z) q_{\Psi}(z, x)}{\pi(x) q_{\Psi}(x, z)}. \quad (11)$$

In high dimensional settings, STMALA may encounter some difficulties to accept the proposed moves. Following [17], we introduce a variant of the algorithm in which only a fixed proportion of components of  $X^n$  is updated at each iteration  $n$ . This is achieved by combining STMALA and a Gibbs sampler in a STMALA-within-Gibbs algorithm, called block-STMALA. This algorithm depends on a new parameter  $\eta \in \{1, \dots, p\}$  which specifies the number of components to be updated at each iteration of the algorithm.

### 3 $V$ -Geometric ergodicity of the $L_1$ proximal STMALA

In this section, we address the  $V$ -geometric ergodicity of the STMALA chain  $(X^n)_{n \geq 0}$  where, at iteration  $n$ , the candidate  $Z$  is given by

$$Z = \Psi_1 \left( X^n + \frac{\sigma^2}{2} h(X^n) + \sigma \xi^{n+1} \right); \quad (12)$$

$(\xi^n, n \geq 1)$  is a i.i.d. sequence of  $\mathcal{N}_p(0, I)$  random variables,  $\Psi_1$  is given by (5), and  $h$  is given by (8). Hereafter,  $\gamma$  (in the definition of  $\Psi_1$ ) and  $D$  (in the definition of  $h$ ) are fixed positive constants. The convergence of this STMALA is addressed under the following assumptions on the target density  $\pi$ .

- A1** (i) For any  $m \in \mathcal{M}$  and  $x \in S_m$ ,  $\omega_m > 0$  and  $\pi_m(x) > 0$ .
- (ii) For any  $m \in \mathcal{M}$ ,  $\pi_m$  is continuous on  $S_m$ .
- (iii)  $\pi(x) \rightarrow 0$  when  $\|x\| \rightarrow \infty$ .

**A2** For any  $s > 0$ ,  $m \in \mathcal{M}$ ,

$$\lim_{r \rightarrow \infty} \sup_{x \in S_m, \|x\| \geq r} \pi_m(x + s n(x)) / \pi_m(x) = 0,$$

where  $n(x) \stackrel{\text{def}}{=} x / \|x\|$ .

When for any  $m \in \mathcal{M}$ , the restriction  $\pi_m$  of  $\pi$  to the subset  $S_m$  is differentiable, A2 is satisfied if (see [12, Section 4] for details), for all  $m \in \mathcal{M}$ ,

$$\lim_{x \in S_m, \|x\| \rightarrow \infty} \langle n(x), \nabla \log(\pi_m(x)) \rangle = -\infty. \quad (13)$$

Let  $b, \epsilon > 0$  and  $u \in (0, b)$ . For any  $m \in \mathcal{M}$  and  $x \in S_m$ , define

$$W_m(x) \stackrel{\text{def}}{=} \{(\|x\| - u)n(x) - s\zeta : s \in (0, b - u), \zeta \in S_m, \|\zeta\| = 1, \|\zeta - n(x)\| \leq \epsilon\}. \quad (14)$$

$W_m(x)$  is the cone of  $S_m$  with apex  $x - u n(x)$  and aperture  $2\epsilon$ . We will prove (see Lemma 5.5) that A3 guarantees that, the probability to accept a move from  $x$  to any point of  $W_m(x)$  converges to one as  $\|x\|$  goes to infinity.

- A3** There exist  $b, R, \epsilon > 0$  and  $u \in (0, b)$  such that for any  $m \in \mathcal{M}$ , for any  $x \in S_m \cap \{\|x\| \geq R\}$ , for all  $y \in S_m \cap W_m(x)$ :  $\pi_m(x - u n(x)) \leq \pi_m(y)$ .

An example of target density satisfying these assumptions is presented in the Appendix.

Let  $P$  denote the transition kernel associated to the Hastings-Metropolis move with proposal (12) and acceptance-rejection (11) in the case  $\Psi = \Psi_1$ .

**Theorem 3.1.** *Assume A1-3 hold. Then, for any  $\beta \in (0, 1)$ , there exist  $C > 0$  and  $\lambda \in (0, 1)$  such that for any  $n \geq 0$  and any  $x \in \mathbb{R}^p$ ,*

$$\|P^n(x, \cdot) - \pi\|_V \leq C V(x) \lambda^n, \quad (15)$$

where  $V(x) \propto \pi(x)^{-\beta}$  and for any signed measure  $\eta$ ,  $\|\eta\|_V \stackrel{\text{def}}{=} \sup_{f, |f| \leq V} |\eta(f)|$ .

*Proof.* By definition of the acceptance probability,  $\pi$  is invariant with respect to  $P$ . The  $V$ -uniform geometric ergodicity follows from Proposition 5.3 and Proposition 5.7 given in Section 5.4: Proposition 5.3 establishes that the chain is psi-irreducible and aperiodic and shows that any Borel set  $C \subset \mathbb{R}^p$  such that  $C \cap S_m$  is a compact subset of  $S_m$  is a small set for  $P$ ; Proposition 5.7 shows that there exists an accessible small set  $C \subset \mathbb{R}^p$  and constants  $c_1 \in (0, 1)$  and  $c_2 < \infty$  such that for any  $x \in \mathbb{R}^p$ ,  $PV(x) \leq c_1 V(x) + c_2 \mathbb{1}_C(x)$ . The proof is then concluded by [16, Theorem 15.0.2].  $\square$

## 4 Numerical illustrations

In this section, STMALA<sup>1</sup> is compared to the reversible jump Markov chain Monte Carlo (RJMCMC) algorithm. For any  $\ell \times \ell'$  matrix  $A$  and any  $1 \leq j \leq \ell$ ,  $1 \leq k \leq \ell'$ ,  $A_{.,k}$  (resp.  $A_{j,.}$ ) denotes the  $k$ -th column (resp. the  $j$ -th row) of  $A$ .

### 4.1 Logistic regression

In this section, the components of  $m \in \mathcal{M}$  are i.i.d. Bernoulli random variables with parameter  $\theta_*$ . Given  $m \in \mathcal{M}$ , the prior distribution of the nonzero components of  $X$  is  $\mathcal{N}(0, c(G'_m G_m)^{-1})$ , where  $G$  is a known  $N \times p$  design matrix,  $c$  is a known scaling parameter, and  $G_m$  denotes the matrix given by the columns of  $G$  corresponding to the nonzero components of  $X$ .  $c$  is chosen large enough so that the prior distribution contains little information about  $X$ . Given  $m$  and  $X$ , the observations  $(Y_1, \dots, Y_N)$  are independent and, for all  $1 \leq i \leq N$ ,  $Y_i$  is distributed as a Bernoulli random variable with parameter  $\exp(G_{i,.} X) / (1 + \exp(G_{i,.} X))$ . In this experiment, we choose  $p = 50$  and  $N = 100$  to assess the performance of STMALA in a simple framework with the soft thresholding operator  $\Psi = \Psi_2$ . The components of  $G$  are i.i.d.  $\mathcal{N}(0, 1)$  and  $\theta_* = 0.05$ .

The algorithm is run with  $c = 100$ ,  $\sigma = 0.3$  and only 10% of the components of  $X$  are updated at each iteration. The choice of the threshold is crucial (if  $\gamma$  is too large, few nonzero samples are proposed and the algorithm converges slowly and if  $\gamma$  is too small, the algorithm proposes non-sparse solutions that are not likely to be accepted).  $\gamma$  is set to 0.4 to get a mean acceptance rate of around 20%.

STMALA is used to estimate the posterior probabilities of activation of the components of  $X$ , defined for all  $1 \leq i \leq p$  as the conditional probability of the event  $\{X_i \neq 0\}$  given the observations  $Y$ . The estimation is given by  $\sum_{n=B}^{N_{it}+B} \mathbb{1}_{\{X_i^n \neq 0\}} / N_{it}$  where  $N_{it}$  is the number of iterations of the algorithm and  $B$  denotes the number of iterations discarded as a burn-in period. Figure 2 (top) provides the true regression vector, the mean regression vector given by STMALA (the posterior mean) and the estimated activation probabilities over 100 independent Monte Carlo runs with  $N_{it} = 50.000$  and  $B = 10.000$ . This experiment highlights the ability of STMALA to choose the good model (the 3 nonzero components of  $X$  are recovered) and to get high posterior probabilities of activation for the selected components of  $X$ .

### 4.2 A sparse spike and slab model

The model for the observations  $Y \in \mathbb{R}^N$  is assumed to be

$$Y = GX + \tau^{-1/2} E,$$

where  $G$  is a  $N \times p$  (known) design matrix,  $E$  is a Gaussian random vector with i.i.d. standard entries and  $\vartheta$  is the (known) precision. We want to find the subset of nonzero covariate parameters from  $X \in \mathbb{R}^p$ . We consider the following sparse spike and slab hierarchical model.

<sup>1</sup>MATLAB codes for STMALA are available at the address <http://www.math.u-psud.fr/~lecorff/software.html>

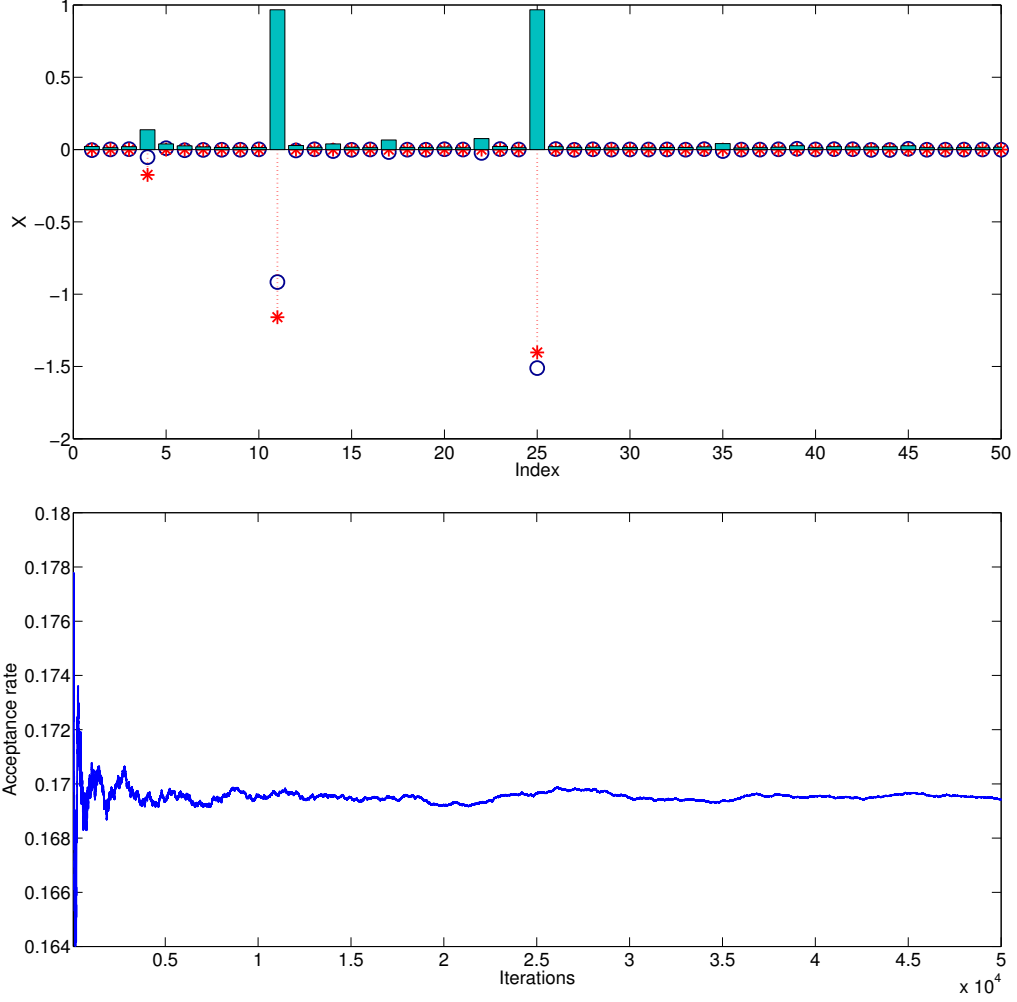


Figure 2: (top) True regression vector (stars), mean regression vector (circles) and estimated activation probabilities (bars) when  $\Psi = \Psi_2$ . (bottom) Mean acceptance rate as a function of the number of iterations.

- Given  $m = (m_1, \dots, m_p) \in \mathcal{M}$  and positive precisions  $(\vartheta_1, \dots, \vartheta_p)$ , the entries of the covariate vector  $X = (X_1, \dots, X_p)$  are independent with distribution

$$X_k | m, \vartheta_1, \dots, \vartheta_p \sim \begin{cases} \delta_0(X_k) & \text{if } m_k = 0, \\ \mathcal{N}(0, 1/\vartheta_k) & \text{if } m_k = 1. \end{cases}$$

- The precision parameters  $\vartheta = (\vartheta_1, \dots, \vartheta_p)$  are i.i.d. with Gamma distribution  $\text{Ga}(a, aK)$ , where  $a, K$  are fixed.
- The components of  $m \in \mathcal{M}$  are i.i.d. Bernoulli random variables with parameter  $\theta_*$ .

Under this model, the posterior density  $\pi(X, m, \vartheta | Y)$  can be derived analytically:

$$\begin{aligned} \pi(X, m, \vartheta | Y) \propto & \exp\left(-\frac{\tau}{2} \|Y - GX\|^2\right) \theta_*^{|m|} (1 - \theta_*)^{p - |m|} \prod_{\ell=1}^p \vartheta_\ell^{a-1} \exp(-aK\vartheta_\ell) \mathbf{1}_{\{\mathbb{R}^+\}}(\vartheta_\ell) \\ & \times \left\{ \exp\left(-\frac{\vartheta_\ell}{2} X_\ell^2\right) \vartheta_\ell^{1/2} \mathbf{1}_{\{m_\ell=1\}} + \delta_0(X_\ell) \mathbf{1}_{\{m_\ell=0\}} \right\}. \end{aligned}$$



By integrating out, we obtain the posterior density of  $(X, m)$ :

$$\pi(X, m|Y) \propto \exp\left(-\frac{\tau}{2}\|Y - GX\|^2\right) \theta_*^{|m|} (1 - \theta_*)^{p-|m|} \times \prod_{\ell=1}^p \left\{ \left(1 + \frac{X_\ell^2}{2aK}\right)^{-(a+1/2)} \mathbb{1}_{\{m_\ell=1\}} + \delta_0(X_\ell) \mathbb{1}_{\{m_\ell=0\}} \right\}.$$

The performance of block-STMALA with STVS operator is illustrated with the model introduced in [3] and presented in [11, Section 8]. It is assumed that  $N = 100$ ,  $p = 200$ ,  $\theta_* = 0.1$  and that  $\tau$  is known and fixed to  $\tau = 1$ . The covariates  $(G_{\cdot,1}, \dots, G_{\cdot,p})$  are sampled from a Gaussian distribution with  $\mathbb{E}[G_{\cdot,i}] = 0$  and  $\mathbb{E}[G_{ji}G_{ki}] = \rho^{|j-k|}$ . In the example below,  $\rho$  is given by  $\rho = 0.3$ . The nonzero coefficients of  $X$  are in 4 clusters of 5 adjacent variables such that, for all  $k \in \{1, 2, 3, 4\}$  and all  $j \in \{1, 2, 3, 4, 5\}$ ,  $X_{50*(k-1)+j} = (-1)^{k+1} j^{1/k}$ .  $a = 2$  and  $K = 0.08$  are chosen so that the Gamma distribution with parameters  $a$  and  $aK$  has a mode at  $\vartheta_*$  such that  $\vartheta_*^{-1/2} = \max(|X|)/2$ . The other parameters are given by  $\sigma = \sqrt{2/L_g}$ ,  $\eta = 20$  and  $\gamma = 0.35$ . The standard deviation of the RJMCMC proposal is  $\sigma_{\text{RJ}} = 0.01$  and chosen so that block-STMALA and RJMCMC have similar acceptance rates (between 15% and 20%). The computations are made over 50 independent trajectories of  $10^6$  iterations. As the dimension is high and RJMCMC modifies only one component uniformly chosen at each iteration, its autocorrelation function is expected to converge more slowly than the autocorrelation function of block-STMALA. This is illustrated in figure 3 (top) which displays the two mean autocorrelation functions estimated over the 50 independent trajectories of length  $10^6$  iterations (10% of these iterations are discarded as a burn-in period). Figure 3 (bottom) shows the true regression vector  $X$  and its estimates by block-STMALA and RJMCMC: it shows that block-STMALA provides a sparse estimation while RJMCMC needs a lot of components to explain the observations. This is probably because RJMCMC is more or less equivalent to test each model in turn, which yields slow convergence in high dimensional settings. This slow convergence is also illustrated in Figure 4. Figure 4 (top) shows the evolution of the mean number of active components  $|m|$ . RJMCMC has not converged after the 300.000 iterations while the mean number of active components of block-STMALA is stable after few iterations. Figure 4 (bottom) displays the boxplots of the estimation of the first component  $X_1$  estimated by block-STMALA and RJMCMC as a function of the number of iterations.

Figure 5 (top) shows the signal  $G\hat{X}$  estimated by block-STMALA and RJMCMC as a function of the actual emitted signal  $GX$  (blue circles), where  $\hat{X}$  is the mean regression vector over a trajectory. To highlight over fitting effects, a test sample  $Y_{\text{test}} = G_{\text{test}}X + \tau^{-1/2}E_{\text{test}}$ , where  $G_{\text{test}} \in \mathbb{R}^{100 \times 200}$  and  $E_{\text{test}} \in \mathbb{R}^{100}$  are generated exactly as  $G$  and  $E$ , is also used. With green circles,  $G_{\text{test}}\hat{X}$  as a function of  $G_{\text{test}}X$  are displayed. This test data set is also used to compute a test error, which is given by

$$\mathcal{E}_{\text{test}} \stackrel{\text{def}}{=} \frac{\|G_{\text{test}}\hat{X} - G_{\text{test}}X\|^2}{100}.$$

The evolution of the mean test error  $\epsilon_{\text{test}}$  over 100 independent runs, is displayed in Figure 5 (bottom). Both figures show that RJMCMC is subject to some over fitting, which is not the case of block-STMALA.

### 4.3 Regression for spectroscopy data

We use the biscuits data set composed of near infrared absorbance spectra of 70 cookies with different water, fat, flour and sugar contents studied in [5] and [7]. The data are divided into a training data set containing measurements for  $N = 39$  cookies, and a test data set containing measurements for 31 cookies. The observation model is given by

$$Y = GX + \tau^{1/2}E,$$

where  $G$  is the design matrix,  $X$  is the unknown regression vector and  $E \sim \mathcal{N}(0, I)$  is the measurement noise. Each row of the design matrix consists of absorbance measurements for  $p = 300$  different wavelengths from 1202 nm to 2400 nm with gaps of 4 nm. We compare the results obtained by block-STMALA with those obtained by RJMCMC for the prediction of fat content. To

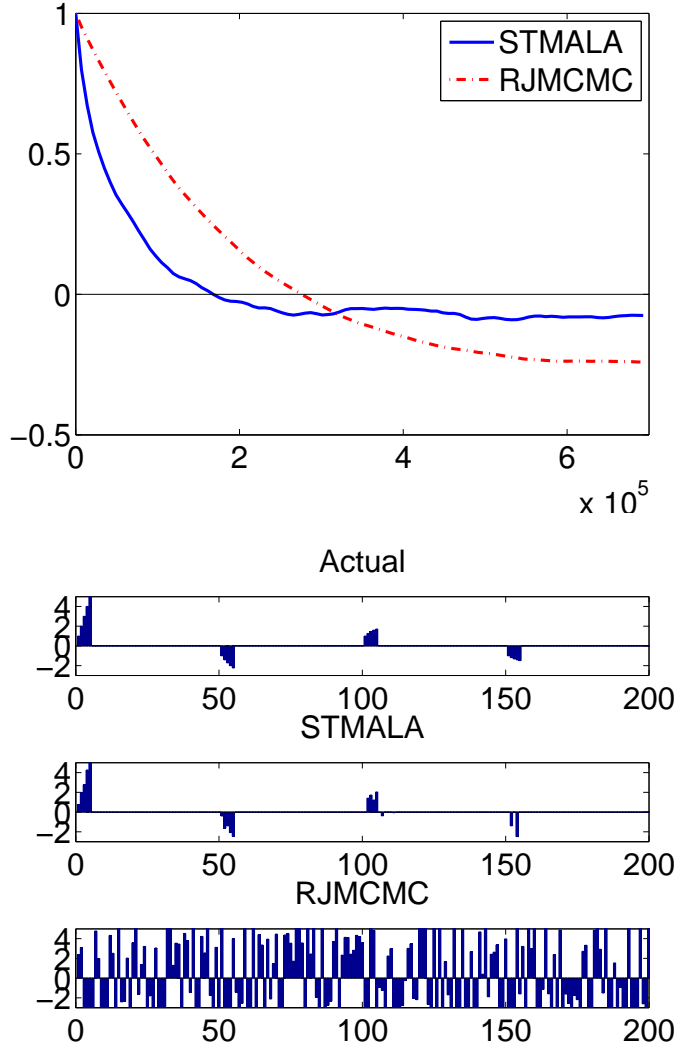


Figure 3: (top) Mean autocorrelation function of  $X_1$  for block-STMALA and RJMCMC. (bottom) Regression vectors estimated by block-STMALA and RJMCMC.

improve the stability of the algorithm, the columns of the matrix  $G$  containing the measurements are centered and a column with each entry being equal to one is added.

The parameters of the algorithms are given by  $\tau = 0.5$ ,  $\sigma = 2\sqrt{2/L_g}$ ,  $\eta = 15$ ,  $\gamma = 0.35$  for block-STMALA, and by  $\sigma_{RJ} = 0.9$  for RJMCMC. The gradient in block-STMALA is truncated as suggested in [22], so that the norm of the truncated gradient does not exceed 0.7. The computations are made over 100 independent trajectories of  $N_{it} = 2.10^6$  iterations, with  $B = 10^5$ . We choose the parameters so that the two algorithms have similar acceptance-rejection ratios (the final ratios are about 45% for block-STMALA and 42% for RJMCMC). Figure 6 shows the regression vectors  $\hat{X}$  estimated by block-STMALA and RJMCMC after one trajectory (left) and the mean regression vector estimated by block-STMALA and RJMCMC over the 100 independent trajectories (right).

The regression vector estimated by block-STMALA with STVS operator has a spike around 1726 nm, which is known to be in a fat absorbance region (see [5, 7]), in almost all the trajectories.

Figure 7 displays the boxplots of the 100 independent values of the components of the regression vectors estimated by block-STMALA and RJMCMC associated to 9 wavelengths close to 1726 nm.

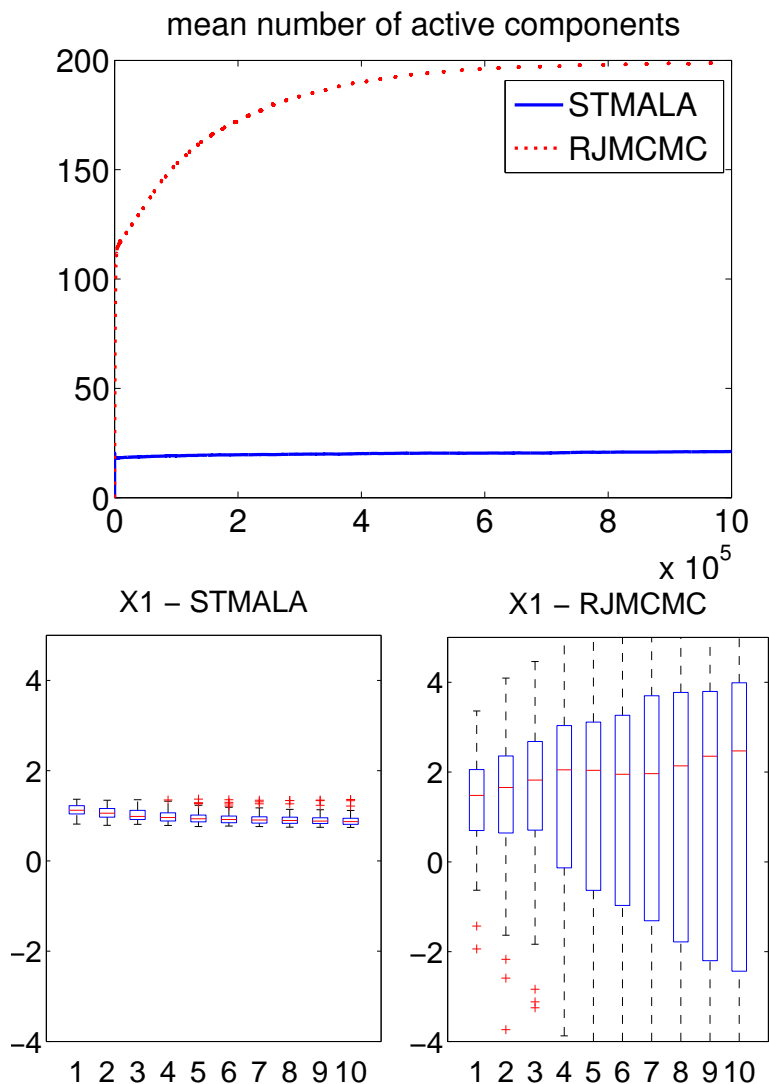


Figure 4: (top) Evolution of the mean number of active components for STMALA and RJMCMC. (bottom) Evolution of the estimation of  $X_1$  (mean over iterations) for block-STMALA and RJMCMC.

It illustrates that the location of the spike retrieved by RJMCMC is not stable, while block-STMALA retrieves a spike centered at 1726 nm in almost every trajectory.

Figure 8 (top) shows the emitted signal  $G\hat{X}$  estimated over one trajectory by block-STMALA and RJMCMC as a function of the observations  $Y$ . In this numerical experiment, block-STMALA provides better results than RJMCMC for both the training set and the test set. This is confirmed by Figure 8 (bottom) which displays the evolution of the mean square error (MSE) on the test dataset, defined by

$$\text{MSE} = \frac{\|G_{\text{test}}\hat{X} - Y_{\text{test}}\|^2}{31},$$

as a function of the number of iterations (mean over 100 independent trajectories). The mean MSE after  $2 \cdot 10^6$  iterations is about 0.75 for block-STMALA and about 1.6 times greater for RJMCMC.

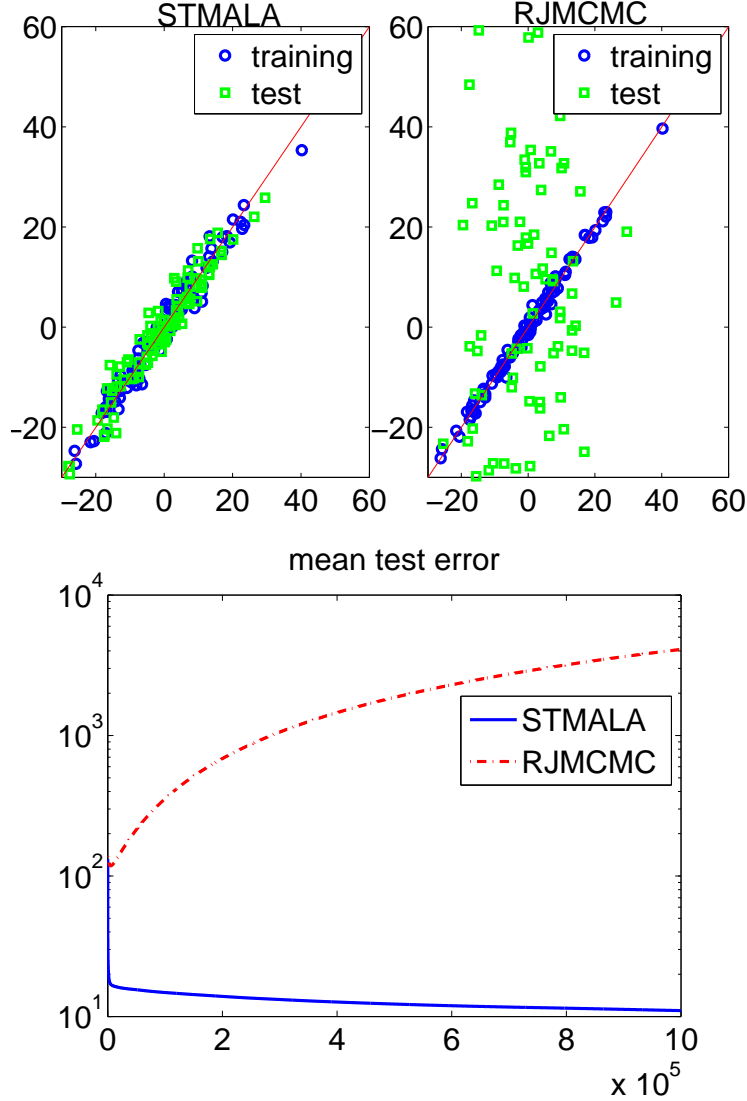


Figure 5: (top) Emitted signal  $G\hat{X}$  estimated by block-STMALA and RJMCMC versus actual emitted signal  $GX$ . (bottom) Evolution of the mean test error for RJMCMC and block-STMALA.

## 5 Proofs

For all  $m \in \mathcal{M}$ , define  $\kappa^{[m]} = (\kappa_1^{[m]}, \dots, \kappa_p^{[m]})$  by  $\kappa_1^{[m]} \stackrel{\text{def}}{=} \inf\{1 \leq i \leq p ; m_i = 1\}$  and for  $2 \leq j \leq |m|$ ,  $\kappa_j^{[m]} \stackrel{\text{def}}{=} \inf\{i > \kappa_{j-1}^{[m]} : m_i = 1\}$ . Then, for all  $x \in \mathbb{R}^{|m|}$ , let  $x^{[m]}$  be the vector of  $\mathbb{R}^p$  such that for all  $1 \leq i \leq |m|$ ,  $x_{\kappa_i^{[m]}}^{[m]} = x_i$  and for all  $i \notin \{\kappa_1^{[m]}, \dots, \kappa_{|m|}^{[m]}\}$ ,  $x_i^{[m]} = 0$ . For all  $y \in \mathbb{R}^p$  and all  $m \in \mathcal{M}$ , let  $y_{[m]}$  be the vector of  $\mathbb{R}^{|m|}$  such that for all  $1 \leq i \leq |m|$ ,  $(y_{[m]})_i = y_{\kappa_i^{[m]}}$ .

### 5.1 Proof of Lemma 2.1

Consider first the case  $p = 1$ . We first compute the derivative of  $h$  on  $(0, \infty)$  (note that  $h$  is symmetric). For any  $x \in (0, \infty)$ ,

$$h'(x) = \gamma^2 \left[ (x^2 + 4\gamma^2)^{-1/2} + (x^2 + 4\gamma^2)^{-1/2} \exp(-2 \operatorname{asinh}(x/(2\gamma))) \right].$$

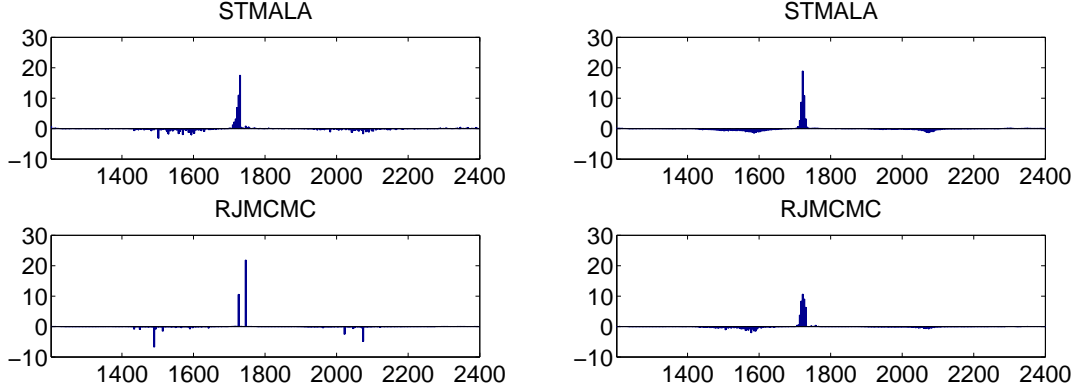


Figure 6: (left) Regression vectors estimated by block-STMALA and RJMCMC after one trajectory. (right) Mean regression vectors estimated by block-STMALA and RJMCMC over 100 independent trajectories.

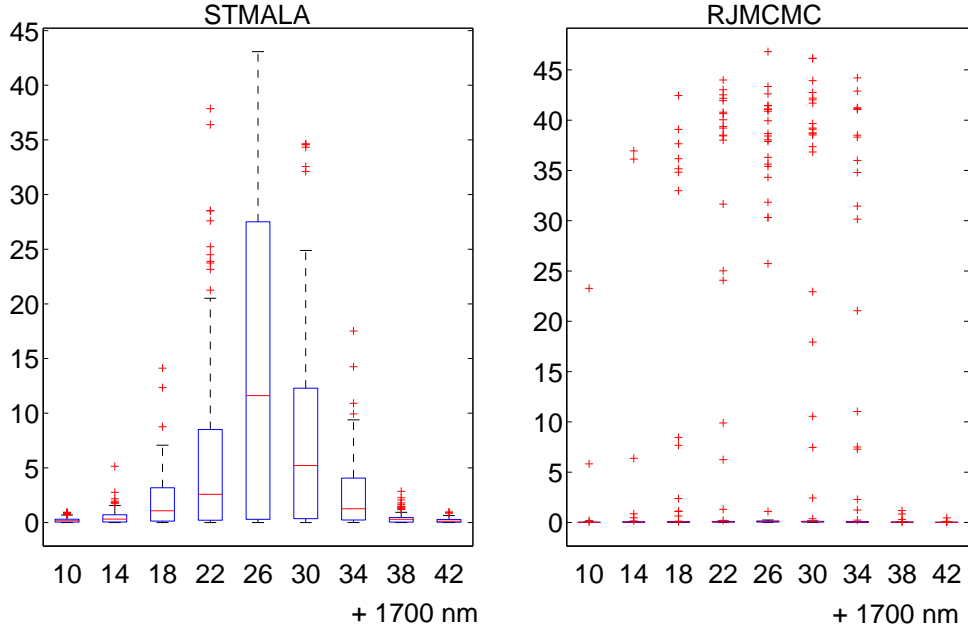


Figure 7: Boxplots of the 100 independent values of the components of the regression vectors estimated by block-STMALA and RJMCMC associated with 9 wavelengths close to 1726 nm.

Using straightforward computations, we get

$$h'(x) = (-x + \text{sign}(x)\sqrt{x^2 + 4\gamma^2})/2.$$

Set  $\psi_u(x) \stackrel{\text{def}}{=} h(x) + (x - u)^2/2$ . Since we have  $\psi_{-u}(x) = \psi_u(-x)$ , we only have to consider the case when  $u \geq 0$ . Hereafter,  $u \geq 0$ . It is easily proved that on  $(0, \infty)$ , the derivative  $\psi'_u$  is strictly increasing to infinity, and a solution to the equation  $\psi'_u(x) = 0$  exists on  $(0, \infty)$  if and only if  $u > \gamma$ . In this case, this solution is  $u - \gamma^2/u$ , and  $\psi_u(u - \gamma^2/u) \leq \psi_u(0)$ . When  $u \in [0, \gamma)$ ,  $\inf_{x>0} \psi_u(x) = \psi_u(0)$ . Moreover, it can be proved that  $\psi'_u(x) = 0$  has no solution on  $(-\infty, 0)$ , and therefore that  $\inf_{x<0} \psi_u(x) = \psi_u(0)$  whatever  $u > 0$  is. Hence, the minimum is reached at 0 if  $u \in [0, \gamma)$  and at  $u - \gamma^2/u$  if  $u > \gamma$ . This concludes the proof.

Consider now the case  $p > 1$ . Set  $x \in \mathbb{R}^p$  of the form  $x = r\xi$  where  $r > 0$  and  $\xi$  is on the

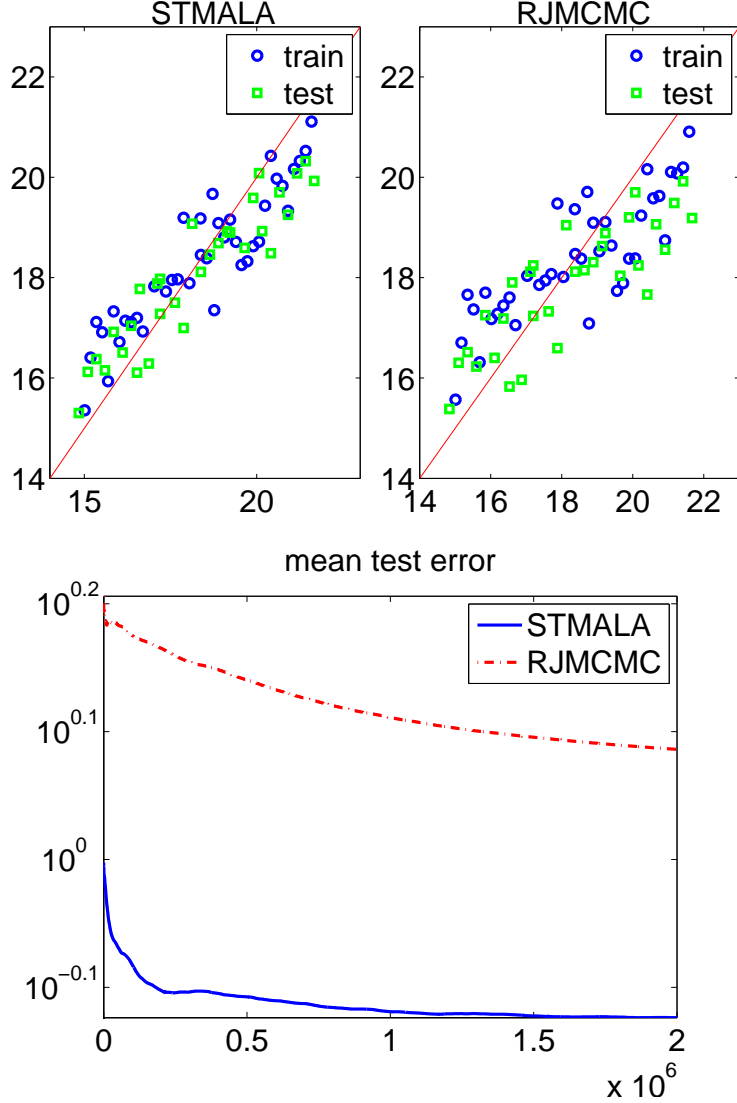


Figure 8: (top) Emitted signal  $G\hat{X}$  estimated by block-STMALA and RJMCMC versus the observations  $Y$ . (bottom) Evolution of the mean MSE (over 100 independent trajectories) on the test data set for RJMCMC and block-STMALA.

unit sphere of  $\mathbb{R}^p$ . Since the function  $h$  only depends on the radius  $r$ , the minimum over  $\mathbb{R}^p$  of  $x \mapsto h(x) + \|x - u\|^2/2$  is reached in the direction  $\xi_\star = u/\|u\|$ . Then, finding the minimum in this direction is equivalent to find the minimum of the function  $\psi_u$  on  $\mathbb{R}^+$ , which yields  $r_\star = 0$  if  $\|u\| \leq \gamma$  and  $r_\star = (1 - \gamma^2/\|u\|^2)$  otherwise. This concludes the proof.

## 5.2 Proof of Lemma 2.2

Let  $\varphi$  be a bounded continuous function on  $\mathbb{R}^p$ . Then,

$$\mathbb{E}[\varphi(Z)] = (2\pi\sigma^2)^{-p/2} \int_{\mathbb{R}^p} \varphi(\Psi_1(y)) \prod_{i=1}^p \exp\left(-\frac{|y_i - \mu_i(x)|^2}{2\sigma^2}\right) dy .$$

For  $m \in \mathcal{M}$  and  $y \in \mathbb{R}^{|m|}$ , set  $\bar{y} = (\bar{y}_1, \dots, \bar{y}_{|m|})$  where  $\bar{y}_i \stackrel{\text{def}}{=} y_i (1 - \gamma/|y_i|)$ . Fubini's theorem yields

$$\mathbb{E}[\varphi(Z)] = (2\pi\sigma^2)^{-|m|/2} \sum_{m \in \mathcal{M}} \prod_{i \notin I_m} \rho(\mu_i(x)) \int_{\mathbb{R}^{|m|}} \varphi(\bar{y}^{[m]}) \left( \prod_{k=1}^{|m|} \mathbb{1}_{\{|y_k| > \gamma\}} \right) \times \exp\left(-\frac{\|y - \mu(x)_{[m]}\|^2}{2\sigma^2}\right) dy.$$

It is sufficient to compute integrals of the form

$$I(\tilde{\varphi}) = \int_{\mathbb{R}} \tilde{\varphi}\left(v \left(1 - \frac{\gamma}{|v|}\right)\right) \mathbb{1}_{\{|v| > \gamma\}} \exp\left(-\frac{|v - \mu|^2}{2\sigma^2}\right) dv,$$

for a generic function  $\tilde{\varphi}$ . Consider the change of variable  $\mathbb{R} \setminus [-\gamma, \gamma] \rightarrow \mathbb{R}^*$ :  $z = v \left(1 - \frac{\gamma}{|v|}\right)$ . Note that  $|z| = |v| - \gamma$  and  $v = \psi(z)$ , where for any  $z \in \mathbb{R}^*$ ,  $\psi(z) \stackrel{\text{def}}{=} (1 + \gamma/|z|)z$ . Then,

$$I(\tilde{\varphi}) = \int_{\mathbb{R}^*} \tilde{\varphi}(v) \exp\left(-\frac{|\psi(v) - \mu|^2}{2\sigma^2}\right) dv.$$

This concludes the proof.

### 5.3 Proof of Lemma 2.3

The proof of Lemma 2.3 follows the same lines as the proof of Lemma 2.2, with the function  $\psi$  replaced by  $\tilde{\psi}(z) = g(\gamma^2/\|z\|^2)z$ .

### 5.4 Proof of Theorem 3.1

For ease of notations, we denote by  $q$  the proposal distribution. Lemma 2.2 shows that for any  $m \in \mathcal{M}$  and  $y \in S_m$

$$q(x, y) = \prod_{i \notin I_m} \rho(\mu_i(x)) \prod_{i \in I_m} f(\mu_i(x), y_i), \quad (16)$$

where  $\rho$  and  $f$  are given by Lemma 2.2 and  $\mu(x) = (\mu_1(x), \dots, \mu_p(x))$  is given by (9). We start with a preliminary lemma which will be fundamental for the proofs since it allows to compare the proposal distribution  $q$  to Gaussian proposals.

**Lemma 5.1.** *Denote by  $g_\sigma$  the one-dimensional centered Gaussian density with standard deviation  $\sigma$ . Set  $\sigma_1 \stackrel{\text{def}}{=} \sigma^2/2$ ,  $\sigma_2 \stackrel{\text{def}}{=} 2\sigma^2$  and*

$$k_1 \stackrel{\text{def}}{=} \exp\left(-(\gamma/\sigma^2 + D/2)^2\right) / \sqrt{2},$$

$$k_2 \stackrel{\text{def}}{=} \exp\left((\gamma/(2\sigma^2) + D/4)^2\right) \sqrt{2}.$$

(i) *For any  $x, y \in \mathbb{R}^p$  and any  $1 \leq i \leq p$ ,*

$$k_1 g_{\sigma_1}(y_i - x_i) \leq f(\mu_i(x), y_i) \leq k_2 g_{\sigma_2}(y_i - x_i).$$

(ii) *For any  $x \in \mathbb{R}^p$  and  $y \in S_m$ ,  $q(x, y) \leq k_2^{|m|} \prod_{i \in I_m} g_{\sigma_2}(y_i - x_i)$ . Therefore, there exists a constant  $C > 0$  such that for any  $x, y \in \mathbb{R}^p$ ,  $q(x, y) \leq C$ .*

*Proof.* Let  $x, y \in \mathbb{R}^p$  and  $i \in \{1, \dots, p\}$ . By definition of  $\mu$  (see (9)), we have  $|\mu_i(x) - x_i| \leq \|\mu(x) - x\| \leq D\sigma^2/2$ . Thus,

$$|y_i - x_i| \leq |y_i + \gamma \text{sign}(y_i) - \mu_i(x)| + \gamma + \frac{D\sigma^2}{2},$$

which implies  $|y_i + \gamma \operatorname{sign}(y_i) - \mu_i(x)|^2 \geq \frac{1}{2} |y_i - x_i|^2 - (\gamma + D\sigma^2/2)^2$ . Similarly,

$$|y_i + \gamma \operatorname{sign}(y_i) - \mu_i(x)|^2 \leq 2 |y_i - x_i|^2 + 2 (\gamma + D\sigma^2/2)^2 .$$

This concludes the proof of (i). The second statement follows from (16) since  $\rho(\mu_i(x)) \leq 1$ .  $\square$

The proof of Theorem 3.1 also requires a lower bound on the probability that a component of the proposed point will be set to zero. Such a bound is given in Lemma 5.2.

**Lemma 5.2.** *Let  $\rho$  and  $\mu$  be given by Lemma 2.2 and (9). It holds*

$$\inf_{m \in \mathcal{M}} \inf_{z \in S_m} \inf_{i \in I_m} \rho(\mu_i(z)) > 0 .$$

*Proof.* For any  $z \in \mathbb{R}^p$ ,  $\|z - \mu(z)\| \leq D\sigma^2/2$  by definition of  $\mu(z)$  (see (9)). Then,  $|\mu_i(z)| \leq D\sigma^2/2$  for any  $i$  such that  $z_i = 0$ . Hence, there exists a constant  $C > 0$  such that

$$\inf_{z \in \mathbb{R}^p} \min_{i: z_i=0} \mathbb{P}(|\mu_i(z) + \xi| \leq \gamma) \geq C , \quad (17)$$

with  $\xi \sim \mathcal{N}(0, 1)$ .  $\square$

**Proposition 5.3.** (i) *Let  $C$  be a Borel set of  $\mathbb{R}^p$  such that for any  $m \in \mathcal{M}$ ,  $C \cap S_m$  is a compact set of  $S_m$ . Then,  $C$  is a one-small set for the kernel  $P$ : there exists a positive measure  $\tilde{\nu}$  on  $\mathbb{R}^p$  such that  $P(x, A) \geq \tilde{\nu}(A) \mathbf{1}_C(x)$ .*

(ii) *The Markov kernel  $P$  is psi-irreducible and aperiodic.*

*Proof.* (i): Let  $C$  and  $K$  be two Borel sets of  $\mathbb{R}^p$  such that  $\nu(K) > 0$  and for any  $m \in \mathcal{M}$ ,  $C \cap S_m$  and  $K \cap S_m$  are compact subsets of  $S_m$ . Since  $\mathbb{R}^p = \bigcup_{m \in \mathcal{M}} S_m$ , we have

$$\inf_{x \in C} P(x, A) = \inf_{m \in \mathcal{M}} \inf_{x \in C \cap S_m} P(x, A) ,$$

so that it is enough to establish a minorization on the kernel for any  $x \in C \cap S_{m_\star}$  whatever  $m_\star \in \mathcal{M}$ . Let  $m_\star \in \mathcal{M}$ . By definition of  $P$ ,  $q$  (see (16)) and  $\nu$  (see (2))

$$P(x, A) \geq \int_{A \cap K} \alpha(x, y) q(x, y) d\nu(y)$$

where, for any  $x \in S_{m_\star}$  and  $y \in S_m$ , we have

$$q(x, y) = \prod_{i \notin I_m} p(\mu_i(x)) \prod_{i \in I_m} f(\mu_i(x), y_i) .$$

The latter inequality implies

$$P(x, A) \geq \sum_{m \in \mathcal{M}} k_1^{|m|} \prod_{i \notin I_m} \rho(\mu_i(x)) \int_{A \cap K \cap S_m} \alpha(x, y) \prod_{i \in I_m} g_{\sigma_1}(x_i - y_i) dy_i ,$$

where the last inequality follows from Lemma 5.1(i). For any  $x \in S_{m_\star}$  and  $y \in S_m$ , we have

$$\alpha(x, y) = 1 \wedge \frac{\omega_m \pi_m(y) q(y, x)}{\omega_{m_\star} \pi_{m_\star}(x) q(x, y)} .$$

There exists a compact set of  $\mathbb{R}$  such that for any  $x \in C \cap S_{m_\star}$  and  $y \in K \cap S_m$ ,  $\mu_i(x)$  and  $\mu_i(y)$  are in this compact for any  $i$ . Hence, A1(i)-(ii) and Lemmas 5.1(i) and 5.2 imply that there exists  $\varepsilon_m > 0$  such that for any  $x \in C \cap S_{m_\star}$  and  $y \in K \cap S_m$ ,

$$\alpha(x, y) \geq \varepsilon_m , \quad \inf_{i \in I_m} g_{\sigma_1}(x_i - y_i) \geq \varepsilon_m .$$

This yields for any  $x \in C \cap S_{m_\star}$ ,  $P(x, A) \geq (\inf_{m \in \mathcal{M}} \varepsilon_m) \int_A \mathbf{1}_K(y) d\nu(y)$ , thus concluding the proof.

(ii): By [15, Lemma 1.1], the Markov chain  $(X^n)_{n \geq 0}$  is psi-irreducible since for any  $x, y \in \mathbb{R}^p$ ,  $q(x, y) > 0$  as a consequence of Lemma 5.1(i) and strongly aperiodic since by Proposition 5.3(i) it possesses an accessible 1-small set.  $\square$



For any measurable function  $f : \mathbb{R}^p \rightarrow \mathbb{R}^+$ ,  $Pf : \mathbb{R}^p \rightarrow \mathbb{R}^+$  denotes  $Pf(x) = \int P(x, dz)f(z)$ . Fix  $\beta \in (0, 1)$  and set  $V : \mathbb{R}^p \rightarrow [1, \infty)$ ,  $x \mapsto c_\beta \pi^{-\beta}(x)$ . Define the possible rejection region  $R(x)$  by

$$R(x) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^p : \pi(x)q(x, y) > \pi(y)q(y, x)\}.$$

We have

$$\frac{PV(x)}{V(x)} \leq \sum_{m \in \mathcal{M}} T_m(x) + \int_{R(x)} q(x, y) d\nu(y), \quad (18)$$

where

$$T_m(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^{|m|}} \alpha(x, z^{[m]}) \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) dz. \quad (19)$$

**Lemma 5.4.** For any  $m \in \mathcal{M}$ ,  $\limsup_{\|x\| \rightarrow \infty} T_m(x) = 0$ .

*Proof.* The proof is adapted from [12] and [1]. Let  $m \in \mathcal{M}$  be fixed. Define

$$\begin{aligned} \mathcal{B}_m(x, a) &\stackrel{\text{def}}{=} \{z \in \mathbb{R}^{|m|}, \|z - x_{[m]}\| \leq a\}, \\ \mathcal{C}_m(x) &\stackrel{\text{def}}{=} \{z \in \mathbb{R}^{|m|}, \pi(z^{[m]}) = \pi(x)\}, \\ \mathcal{C}_m(x, u) &\stackrel{\text{def}}{=} \{z + sn(z), |s| \leq u, z \in \mathcal{C}_m(x)\}, \\ R_m(x) &\stackrel{\text{def}}{=} \mathbb{R}^{|m|} \setminus A_m(x), \end{aligned}$$

where

$$A_m(x) \stackrel{\text{def}}{=} \{z \in \mathbb{R}^{|m|}, \pi(z^{[m]})q(z^{[m]}, x) \geq \pi(x)q(x, z^{[m]})\}.$$

We decompose as follows

$$T_m(x) \leq T_{m,1}(x, a) + \sum_{j=2}^4 T_{m,j}(x, a, u),$$

where

$$\begin{aligned} T_{m,1}(x, a) &\stackrel{\text{def}}{=} \int_{\mathcal{B}_m^c(x, a)} \alpha(x, z^{[m]}) \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) dz, \\ T_{m,2}(x, a, u) &\stackrel{\text{def}}{=} \int_{\mathcal{B}_m(x, a) \cap \mathcal{C}_m(x, u)} \alpha(x, z^{[m]}) \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) dz, \\ T_{m,3}(x, a, u) &\stackrel{\text{def}}{=} \int_{A_m(x) \cap \mathcal{B}_m(x, a) \cap \mathcal{C}_m^c(x, u)} \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) dz, \\ T_{m,4}(x, a, u) &\stackrel{\text{def}}{=} \int_{R_m(x) \cap \mathcal{B}_m(x, a) \cap \mathcal{C}_m^c(x, u)} \frac{\pi^{1-\beta}(z^{[m]})}{\pi^{1-\beta}(x)} q(z^{[m]}, x) dz. \end{aligned}$$

We prove that we may choose the constant  $C > 0$  large enough so that for any  $\epsilon > 0$  there exists  $M > 0$  such that  $\sup_{\|x\| \geq M} T_m(x) \leq C\epsilon$ . Since  $\epsilon$  is arbitrarily small, this yields the lemma. Note that for any  $z \in \mathbb{R}^{|m|}$ ,

$$\alpha(x, z^{[m]}) \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} \leq \left( \frac{q(z^{[m]}, x)}{q(x, z^{[m]})} \right)^\beta. \quad (20)$$

**Control of  $T_{m,1}$**  By (20),  $T_{m,1}(x, a) \leq \int_{\mathcal{B}_m^c(x, a)} q(x, z^{[m]})^{1-\beta} q(z^{[m]}, x)^\beta dz$ . By (16) and Lemma 5.1, there exists a constant  $C > 0$  such that

$$\begin{aligned} T_{m,1}(x, a) &\leq Ck_2^{|m|(1-\beta)} \int_{\mathcal{B}_m^c(x, a)} \prod_i g_{\sigma_2}((x_{[m]})_i - y_i)^{1-\beta} dy_i \\ &\leq Ck_2^{|m|(1-\beta)} \int_{\mathcal{B}_m^c(0, a)} \prod_i g_{\sigma_2}(y_i)^{1-\beta} dy_i. \end{aligned}$$

Therefore, for any  $\epsilon > 0$ , there exists  $a > 0$  such that  $\sup_{x \in \mathbb{R}^p} T_{m,1}(x, a) \leq \epsilon$ .

**Control of  $T_{m,2}$**  By (20),  $T_{m,2}(x, a, u) \leq \int_{\mathcal{B}_m(x,a) \cap \mathcal{C}_m(x,u)} q(x, z^{[m]})^{1-\beta} q(z^{[m]}, x)^\beta dz$ . By A2, the Lebesgue measure of  $\mathcal{B}_m(x, a) \cap \mathcal{C}_m(x, u)$  can be made arbitrarily small - independently of  $x \in \mathbb{R}^p$  - when  $u$  is small enough (see [12, Proof of Theorem 4.1] for details). Therefore, since  $q$  is bounded (see Lemma 5.1(ii)), for any  $\epsilon > 0$ , there exists  $u > 0$  such that for any  $a > 0$ :  $\sup_{x \in \mathbb{R}^p} T_{m,2}(x, a, u) \leq \epsilon$ .

**Control of  $T_{m,3}$**  Set  $d_r(u) \stackrel{\text{def}}{=} \sup_{\|x\| \geq r} \pi(x + u n(x)) / \pi(x)$ . By A2, for any  $\epsilon, u > 0$ , there exists  $r > 0$  large enough so that  $(d_{r-u}(u))^{1-\beta} \vee (d_r(u))^{1-\beta} \leq \epsilon$ . By A1,  $\sup_{z \in \mathcal{B}_m(0,r)} \pi(z^{[m]})^{-\beta} < \infty$ , so that by Lemma 5.1(ii)

$$\sup_{x \in \mathbb{R}^p} \int_{\mathcal{I}_m(x,a,u,r)} q(x, z^{[m]}) \pi^{-\beta}(z^{[m]}) dz < \infty,$$

where

$$\mathcal{I}_m(x, a, u, r) \stackrel{\text{def}}{=} A_m(x) \cap \mathcal{B}_m(x, a) \cap \mathcal{C}_m^c(x, u) \cap \mathcal{B}_m(0, r).$$

A1(iii) implies that

$$\limsup_{\|x\| \rightarrow \infty} \int_{\mathcal{I}_m(x,a,u,r)} \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) dz = 0.$$

Moreover, by definition of  $A_m(x)$ , for any  $z \in A_m(x)$  it holds

$$\frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) \leq \frac{\pi^{1-\beta}(z^{[m]})}{\pi^{1-\beta}(x)} q(z^{[m]}, x);$$

by Lemma 5.1(ii), there exists a constant  $C$  such that for any  $x \in \mathbb{R}^p$  and  $z \in A_m(x)$

$$\frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) \leq C \left( \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} \wedge \frac{\pi^{1-\beta}(z^{[m]})}{\pi^{1-\beta}(x)} \right).$$

This yields there exists  $C_\star$  such that for any  $a, u, r > 0$ ,

$$\begin{aligned} & \int_{A_m(x) \cap \mathcal{B}_m(x,a) \cap \mathcal{J}_m(x,u,r)} \frac{\pi^{-\beta}(z^{[m]})}{\pi^{-\beta}(x)} q(x, z^{[m]}) dz \\ & \leq C_\star \left( \sup_{z \in \mathcal{J}_m(x,u,r)} \frac{\pi^\beta(x)}{\pi^\beta(z^{[m]})} \wedge \sup_{z \in \mathcal{J}_m(x,u,r)} \frac{\pi^{1-\beta}(z^{[m]})}{\pi^{1-\beta}(x)} \right), \end{aligned}$$

where

$$\mathcal{J}_m(x, u, r) \stackrel{\text{def}}{=} \mathcal{C}_m^c(x, u) \cap \mathcal{B}_m^c(0, r).$$

Let  $z \in \mathcal{C}_m^c(x, u) \cap \{z : \pi(z^{[m]}) < \pi(x)\}$ . By A1(ii),  $h : s \mapsto \pi(z^{[m]} - s n(z^{[m]}) - \pi(x))$  is continuous, and by definition of  $\mathcal{C}_m^c(x, u)$ ,  $h(s) \neq 0$  for any  $0 \leq s \leq u$ . Since  $h(0) < 0$  (we assumed that  $\pi(z^{[m]}) < \pi(x)$ ), this implies that  $h(u) < 0$  i.e.  $\pi(z^{[m]} - u n(z^{[m]})) \leq \pi(x)$ . Then,

$$\sup_{z \in \mathcal{C}_m^c(x,u) \cap \mathcal{B}_m^c(0,r)} \frac{\pi(z^{[m]})}{\pi(x)} \leq \frac{\pi(z^{[m]})}{\pi(z^{[m]} - u n(z^{[m]}))} \leq d_{r-u}(u).$$

If  $z \in \mathcal{C}_m^c(x, u) \cap \{z : \pi(z^{[m]}) \geq \pi(x)\}$ , we obtain similarly that  $\pi(x) / \pi(z^{[m]}) \leq d_r(u)$ . Hence, we established that

$$\sup_{z \in \mathcal{C}_m^c(x,u) \cap \mathcal{B}_m^c(0,r)} \frac{\pi(z^{[m]})}{\pi(x)} \leq d_r(u) \vee d_{r-u}(u).$$

As a conclusion, there exists  $C_\star > 0$  and for any  $\epsilon, a, u > 0$ , there exists  $M > 0$  such that  $\sup_{\|x\| \geq M} T_{m,3}(x, a, u) \leq C_\star \epsilon$ .

**Control of  $T_{m,4}$**  Following the same lines as for the control of  $T_{m,3}(x, a, u)$ , it can be shown that there exists  $C_\star > 0$  and for any  $\epsilon, a, u > 0$ , there exists  $M > 0$  such that  $\sup_{\|x\| \geq M} T_{m,4}(x, a, u) \leq C_\star \epsilon$ .  $\square$

**Lemma 5.5.** *Let  $u, b, \epsilon, R$  be given by A3 and  $W_m(x)$  be defined by (14). There exists  $r > R$  such that for any  $m \in \mathcal{M}$  and  $x \in S_m \cap \{\|x\| \geq r\}$ ,  $W_m(x) \subset \{y \in S_m, \alpha(x, y) = 1\}$ .*

*Proof.* The proof is adapted from [12]. Let  $m \in \mathcal{M}$  and  $x \in S_m$  such that  $\|x\| \geq r$  for some  $r > R$  to be fixed later (the constant  $R$  is given by A3). We first prove that there exists a positive constant  $C_b$  such that

$$\frac{\pi(x)}{\pi(x - un(x))} \leq C_b \leq \inf_{z \in \mathcal{B}_m(x, b)} \frac{q(z^{[m]}, x)}{q(x, z^{[m]})}. \quad (21)$$

By (16), Lemma 5.1(i) and Lemma 5.2, there exist  $C, C_b > 0$  - independent of  $x \in S_m$  - such that

$$\inf_{z \in \mathcal{B}_m(x, b)} \frac{q(z^{[m]}, x)}{q(x, z^{[m]})} \geq C^{p-|m|} k_1^{|m|} k_2^{-|m|} \inf_{z \in \mathcal{B}_m(x, b)} \prod_{i \in I_m} \frac{g_{\sigma_1}(x_i - z_i)}{g_{\sigma_2}(x_i - z_i)} \geq C_b.$$

By A2, we can choose  $r$  large enough so that for all  $\|x\| \geq r$ ,  $\pi(x)/\pi(x - un(x)) \leq C_b$ . This yields (21). Let  $z \in W_m(x)$ . Then,  $\|z - x\| \leq b$  so that  $z \in \mathcal{B}_m(x, b)$ . Hence, by (21),  $q(z^{[m]}, x)/q(x, z^{[m]}) \geq C_b$ . In addition,

$$\frac{\pi(z^{[m]})}{\pi(x)} = \frac{\pi(z^{[m]})}{\pi(x - un(x))} \frac{\pi(x - un(x))}{\pi(x)} \geq \frac{\pi(z^{[m]})}{\pi(x - un(x))} \frac{1}{C_b} \geq \frac{1}{C_b},$$

where in the last inequality we used A3. Hence,

$$\frac{\pi(z^{[m]})}{\pi(x)} \frac{q(z^{[m]}, x)}{q(x, z^{[m]})} \geq 1,$$

and  $\alpha(x, z^{[m]}) = 1$  thus showing the lemma.  $\square$

**Lemma 5.6.**  $\limsup_{\|x\|_2 \rightarrow \infty} \int_{R(x)} q(x, y) d\nu(y) < 1$ .

*Proof.* Let  $x \in S_{m_\star}$ . By definition of  $d\nu$ , by Lemma 5.1(i) and by Lemma 5.2, there exists a constant  $C > 0$  such that

$$\begin{aligned} 1 - \int_{R(x)} q(x, y) d\nu(y) &= \sum_{m \in \mathcal{M}} \int_{A_m(x)} q(x, z^{[m]}) dz \geq \sum_{m \in \mathcal{M}} k_1^{|m|} \mathcal{G}_m(x) \prod_{i \notin I_m} \rho(\mu_i(x)) \\ &\geq k_1^{|m_\star|} \mathcal{G}_{m_\star}(x) \prod_{i \notin I_{m_\star}} \rho(\mu_i(x)), \\ &\geq C k_1^{|m_\star|} \mathcal{G}_{m_\star}(x), \end{aligned}$$

where

$$\mathcal{G}_m(x) \stackrel{\text{def}}{=} \int_{A_m(x)} \prod_{i \in I_m} g_{\sigma_1}(x_i - y_i) dy_i.$$

By Lemma 5.5, for any  $x \in S_{m_\star}$  large enough,

$$1 - \int_{R(x)} q(x, y) d\nu(y) \geq C k_1^{|m_\star|} I_{m_\star}(x)$$

where, denoting  $A - x \stackrel{\text{def}}{=} \{z, z + x \in A\}$ ,

$$I_{m_\star}(x) = \int_{W_{m_\star}(x) - x} \left( \prod_{i \in I_{m_\star}} g_{\sigma_1}(y_i) dy_i \right) \left( \prod_{i \notin I_{m_\star}} \delta_0(dy) \right). \quad (22)$$

Note that

$$W_{m_\star}(x) - x = \{-un(x) - s\xi; 0 < s < b - u, \zeta \in S_{m_\star}, \|\zeta\| = 1, \|\zeta - n(x)\| \leq \epsilon\},$$

so that the integrals in (22) depend on  $x$  only through  $m_\star$ . Since  $\mathcal{M}$  is finite, there exists a constant  $C' > 0$  independent of  $x$  such that for any  $m \in \mathcal{M}$  and  $x \in S_m$ ,

$$\int_{W_m(x)} \left( \prod_{i \in I_m} g_{\sigma_1}(x_i - y_i) dy_i \right) \left( \prod_{i \notin I_m} \delta_0(dy) \right) \geq C'.$$

□

**Proposition 5.7.**  $\limsup_{\|x\| \rightarrow \infty} PV(x)/V(x) < 1$ .

*Proof.* The result follows from (18) and Lemmas 5.4 and 5.6. □

## 6 Appendix

In this section  $\|\cdot\|$  denotes the Euclidean norm and  $\|\cdot\|_1$  is the  $L^1$ -norm. Let  $v > 0$  and  $\lambda \geq 0$ . We prove that the density  $\pi$  defined on  $\mathbb{R}^p$  by

$$\pi(x) \propto w_m c_\lambda^{-|m|} \exp(-U(x) + V(x)), \quad (23)$$

with, for  $m \in \mathcal{M}$  and  $x \in S_m$ ,

$$\begin{aligned} U(x) &= 1/(2\tau) \|Y - Gx\|^2 \\ V(x) &= \lambda \|x\|_1 + v \|x\|^2 - \log(w_m c_\lambda^{-|m|}), \end{aligned}$$

satisfies the conditions A1-3. For any  $m \in \mathcal{M}$ , the restriction of  $\pi_m$  is continuous. Moreover, for any  $x \in \mathbb{R}^p$ ,

$$\pi(x) \leq C \max_{|m| \in \{1, \dots, p\}} c_\lambda^{-|m|} \exp(-\lambda \|x\|_1 - v \|x\|^2)$$

and as  $\lambda \geq 0$  and  $v > 0$ , the RHS tends to zero when  $\|x\| \rightarrow \infty$ . Therefore, A1 is satisfied. A2 and A3 are proved in Lemma 6.1 and Lemma 6.2.

**Lemma 6.1.** *Let  $\pi$  be given by (23). Then,  $\lim_{r \rightarrow \infty} \sup_{\|x\| \geq r} \pi_m(x + sn(x))/\pi_m(x) = 0$  for any  $m \in \mathcal{M}$  and  $s > 0$ .*

*Proof.* Let  $m \in \mathcal{M}$ ,  $s > 0$  and  $x \in S_m$ . Since  $x + sn(x) \in S_m$ , we have

$$\begin{aligned} \frac{\pi_m(x + sn(x))}{\pi_m(x)} &= \exp\left(\frac{1}{2\tau} (\|Y - Gx\|^2 - \|Y - G(x + sn(x))\|^2)\right) \\ &\quad \times \exp(\lambda (\|x\|_1 - \|x + sn(x)\|_1)) \exp(v (\|x\|^2 - \|x + sn(x)\|^2)). \end{aligned}$$

First,  $\|x\|_1 - \|x + sn(x)\|_1 = -s\|x\|_1/\|x\| \leq -s$ . Moreover,

$$\|Y - Gx\|^2 - \|Y - G(x + sn(x))\|^2 = -s^2 \|Gn(x)\|^2 + 2s \langle Y - Gx, Gn(x) \rangle.$$

And finally,

$$\|x\|^2 - \|x + sn(x)\|^2 = -2s \langle x, n(x) \rangle - s^2 \|n(x)\|^2 \leq -2s \|x\|_2.$$

This implies that for any  $x$ ,

$$\begin{aligned} \frac{\pi_m(x + sn(x))}{\pi_m(x)} &\leq \exp\left(\frac{s}{\tau} \langle Y - Gx, Gn(x) \rangle\right) \exp(-2sv \|x\|), \\ &\leq \exp\left(\frac{s}{\tau} \sup_{\|z\|=1} \langle Y, Gz \rangle\right) \exp(-2sv \|x\|). \end{aligned}$$

As  $v > 0$ , the RHS tends to zero as  $\|x\| \rightarrow \infty$ , uniformly for  $\|x\| \geq r$ . □

**Lemma 6.2.** *The target density  $\pi$  defined by (23) satisfies the condition A3.*

*Proof.* Let  $b > 0$  and  $u \in (0, b)$  be fixed. Let  $\epsilon > 0$  be such that

$$\epsilon \leq \inf_{z, \|z\|=1} \|Gz\|^2 / \sup_{z, \|z\|=1} \|G'Gz\| \quad \text{and} \quad \epsilon < 1. \quad (24)$$

Let  $m \in \mathcal{M}$  and  $x \in S_m$  such that  $\|x\| \geq b$ . Set  $x_\star \stackrel{\text{def}}{=} x - u n(x)$ , and  $y \stackrel{\text{def}}{=} x_\star - s\xi \in S_m$  such that  $y \in W_m(x)$ , where by definition of  $W_m(x)$  in (14),  $s \in (0, b - u)$  and  $\xi \in S_m$  is such that  $\|\xi\| = 1$  and  $\|\xi - n(x)\| \leq \epsilon$ . First, note that when  $\|x\| > b$ ,  $x_\star \in S_m$ , and

$$\begin{aligned} \frac{\pi_m(y)}{\pi_m(x_\star)} &= \exp\left(- (2\tau)^{-1} (\|Y - G(x_\star - s\xi)\|^2 - \|Y - Gx_\star\|^2)\right) \\ &\quad \times \exp\left(-\lambda (\|x_\star - s\xi\|_1 - \|x_\star\|_1)\right) \exp\left(-v (\|x_\star - s\xi\|^2 - \|x_\star\|^2)\right). \end{aligned}$$

Moreover,

$$\begin{aligned} \|x_\star - s\xi\|_1 - \|x_\star\|_1 &\leq \|x_\star\|_1 \left(1 - \frac{s}{\|x_\star\|}\right) + s\|n(x_\star) - \xi\|_1 - \|x_\star\|_1, \\ &\leq -s \frac{\|x_\star\|_1}{\|x_\star\|} + s\epsilon \frac{\|n(x_\star) - \xi\|_1}{\|n(x_\star) - \xi\|} \end{aligned}$$

since  $y \in W_m(x)$ . By equivalence of the norm, and as  $\epsilon < 1$ , there exists a constant  $C$  such that

$$\sup_{x \in S_m} \|x_\star - s\xi\|_1 - \|x_\star\|_1 \leq C.$$

Define  $\phi : s \mapsto \|Y - G(x_\star - s\xi)\|^2$ ;  $\phi$  is differentiable on  $\mathbb{R}$ , and

$$\begin{aligned} \phi'(s) &= 2 \langle Y - Gx_\star + sG\xi, G\xi \rangle = 2 \langle Y, G\xi \rangle - 2 \langle G\xi, Gx_\star \rangle + 2s\|G\xi\|^2, \\ &= 2 \langle Y, G\xi \rangle + 2s\|G\xi\|^2 \\ &\quad - 2\|x_\star\| \langle \xi - n(x_\star), G'Gn(x_\star) \rangle \|Gn(x_\star)\|^2. \end{aligned}$$

By definition of  $\phi$ ,  $\|Y - G(x_\star - s\xi)\|^2 - \|Y - Gx_\star\|^2 = \phi(s) - \phi(0)$ . By the mean value theorem, there exists  $\tau \in [0, s]$  such that

$$\phi(s) - \phi(0) \leq 2s\tau\|G\xi\|^2 + 2s \langle Y, G\xi \rangle - 2s\|x_\star\| \left( \inf_{z, \|z\|=1} \|Gz\|^2 - \epsilon \sup_{z, \|z\|=1} \|G'Gz\| \right).$$

Therefore, by (24), there exists a constant  $C > 0$  independent of  $x$  such that,

$$\sup_{x \in S_m} \|Y - G(x_\star - s\xi)\|^2 - \|Y - Gx_\star\|^2 \leq C.$$

And finally,

$$\|x_\star - s\xi\|^2 - \|x_\star\|^2 \leq s^2 - 2s\|x_\star\| (1 - \epsilon).$$

Therefore, as  $\epsilon < 1$ ,  $\|x_\star - s\xi\|^2 - \|x_\star\|^2 \xrightarrow{\|x\| \rightarrow \infty} -\infty$ ; and there exists  $\epsilon > 0$  and  $R > 0$  such that for any  $\|x\| \geq R$  and any  $y \in W_m(x)$ ,  $\pi_m(y)/\pi_m(x_\star) \geq 1$ .  $\square$

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