c-regular cyclically ordered groups

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[1]

Abstract

We define and we characterize regular and c-regular cyclically ordered abelian groups. We prove that every dense c-regular cyclically ordered abelian group is elementarily equivalent to some cyclically ordered group of unimodular complex numbers, that every discrete c-regular cyclically ordered abelian group is elementarily equivalent to some ultraproduct of finite cyclic groups, and that the discrete regular non-c-regular cyclically ordered abelian groups are elementarily equivalent to the linearly cyclically ordered group $\mathbb{Z}$.

Keywords: abelian cyclically ordered groups, regular, cyclic groups, groups of unimodular complex numbers, first order theory.

1 Introduction.

Unless otherwise stated the groups are abelian groups. The definitions and basic properties of cyclically ordered groups can be found for example in [1], [2] or [4]. For the reader's convenience we recall them in Section 2. Being c-archimedean (i.e. isomorphic to a subgroup of the cyclically ordered group $\mathbb{K}$ of the complex numbers of module 1) for a cyclically ordered group is not a first order property. In this paper we define the c-regular cyclically ordered groups, and we prove that the class of all c-regular cyclically ordered groups is the lowest elementary class containing all the c-archimedean cyclically ordered groups. We also define the regular cyclically ordered groups, and we prove that the class of all regular cyclically ordered groups is the union of the class of all c-regular cyclically ordered groups and of the class of all linearly cyclically ordered groups.

We prove that an abelian cyclically ordered group G which is not c-archimedean is c-regular if and only if its linear part $l(G)$ is a regular linearly ordered group and the cyclically ordered group $K(G) = (G/l(G))$ is divisible and its torsion group is isomorphic to the group of all roots of 1 in the field of all complex numbers. Another characterization is that its unwound is a regular linearly ordered group.

Every regular cyclically ordered group is either dense or discrete. We prove that any two dense c-regular cyclically ordered groups are elementarily equivalent if and only if they are elementarily equivalent in the language of groups, and that any dense c-regular group is elementarily equivalent to some c-archimedean group. In the case of discrete groups, we prove that every discrete c-regular group is elementarily equivalent to an ultraproduct of finite cyclic groups. Every discrete c-regular group G which is not c-archimedean contains an elementary substructure C such that $K(C) \simeq \mathbb{U}$ and $l(C) \simeq \mathbb{Z}$. We define first order formulas $D_{p^n,k}$ (p a prime number, $n \in \mathbb{N}$ and $k \in \{0, \dots, p^n - 1\}$) such that any two discrete non-c-archimedean c-regular groups G and G' are elementarily equivalent if and only if G and G' satisfy the same formulas $D_{p^n,k}$. If C and C' are the elementary substructures of G and G', respectively, which we defined above, then $G \equiv G'$ if and only if $C \simeq C'$. By means of the formulas $D_{p^n,k}$ which are satisfied by G, we define a family of subsets of $\mathbb{N}^*$, which we will call the family of subsets of $\mathbb{N}^*$ characteristic of G, such that for every non principal ultrafilter U on $\mathbb{N}^*$, G is elementarily equivalent to the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo U if and only if U contains the family of subsets of $\mathbb{N}^*$ characteristic of G. Furthermore, for every non-c-archimedean c-regular cyclically ordered group, there

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exists an ultrafilter containing the family of subsets of $\mathbb{N}^*$ characteristic of this group. Remark that the class of all discrete regular cyclically ordered groups is the lowest elementary class which contains all the cyclically ordered groups generated by one element. Furthermore, this class is divided into two disjoint elementary subclasses, the class of discrete c-regulars cyclically ordered groups, and the class of discrete regular linearly cyclically ordered groups, that is, the groups which are elementarily equivalent to the linearly cyclically ordered group $\mathbb{Z}$.

In the last section, we deal with the same groups, wihtout a predicate for the cyclic order, to show which informations this predicate carries.

2 Cyclically ordered groups.

Recall that a *cyclically ordered group* is a group G (which is not necessarily abelian) together with a ternary relation R which satisfies:
(1) $\forall(a,b,c) \in G^3$, $R(a,b,c) \Rightarrow a \neq b \neq c \neq a$ (R is "strict"),
(2) $\forall(a,b,c) \in G^3$, $R(a,b,c) \Rightarrow R(b,c,a)$ (R is cyclic),
(3) for all $c \in G$, $R(c,\cdot,\cdot)$ defines a linear order on $G\backslash\{c\}$,
(4) $\forall(a,b,c,u,v) \in G^5$, $R(a,b,c) \Rightarrow R(uav,ubv,ucv)$ (R is compatible).

For example, let $\mathbb{K}$ be the group of all complex numbers of module 1, for $e^{i\theta_1}$, $e^{i\theta_2}$, $e^{i\theta_3}$ in $\mathbb{K}$ with $0 \leq \theta_i < 2\pi$, ($i \in \{1,2,3\}$) we let $R(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$ if and only if $\theta_1 < \theta_2 < \theta_3$. The subgroup $\mathbb{U}$ of torsion elements of $\mathbb{K}$, that is, the roots of 1 in the field $\mathbb{C}$ of all complex numbers, is a cyclically ordered group. On the other side, any linearly ordered group $(G,<)$ can be equipped with a cyclic order: we set $R(a,b,c)$ if and only if either $a<b<c$ or $b<c<a$ or $c<a<b$. In this case we say that (G,R) is a *linearly cyclically ordered group* (or sometimes *linear cyclically ordered goup*).

If g_0 is a positive, central and cofinal element of a linearly ordered group $(G,<)$, then the quotient group $G/\langle g_0\rangle$ (where $\langle g_0\rangle$ denotes the subgroup generated by g_0) can be cyclically ordered by setting $R(a\langle g_0\rangle, b\langle g_0\rangle, c\langle g_0\rangle)$ if and only if there exist $a' \in a\langle g_0\rangle$, $b' \in b\langle g_0\rangle$ and $c' \in c\langle g_0\rangle$ such that $1_G \leq a' < g_0$, $1_G \leq b' < g_0$, $1_G \leq c' < g_0$ and either $a' < b' < c'$ or $b' < c' < a'$ or $c' < a' < b'$. This cyclically ordered group is called the *wound-round* associated to G and g_0. The theorem of Rieger ([1], IV, 6, th. 21) states that every cyclically ordered group can be obtained in this way, that is for every cyclically ordered group (G,R), there exists a linearly ordered group $\mathrm{uw}(G)$ and an element z_G which is positive, central and cofinal in $\mathrm{uw}(G)$, such that (G,R) is isomorphic to the cyclically ordered group $\mathrm{uw}(G)/\langle z_G\rangle$. $\mathrm{uw}(G)$ is the set $\mathbb{Z}\times G$ together with the order $(n,c) < (n',c')$ if and only if either $n<n'$, or $n=n'$ and $R(1_G,c,c')$, or $n=n'$ and $c=1_G$. We set $z_G=(1,1_G)$, and the group law is given by:

$$(m,a)\cdot(n,b) = \begin{cases} (m+n,b) & if \quad a = 1_G \\ (m+n,a) & if \quad b = 1_G \\ (m+n,ab) & if \quad R(1_G,a,ab) \\ (m+n+1,ab) & if \quad R(1_G,ab,a) \\ (m+n+1,1_G) & if \quad ab = 1_G \neq a \end{cases}$$

$\mathrm{uw}(G)$ is called the *unwound* of (G,R). For example, the unwound of $\mathbb{K}$ is isomorphic to $(\mathbb{R},+)$.

If C is the greatest convex subgroup of the unwound $\mathrm{uw}(G)$ of a cyclically ordered group (G,R), then $z_G^{-1} < C < z_G$, hence C embeds trivially in $G \simeq \mathrm{uw}(G)/\langle z_G\rangle$, we denote by $l(G)$ its image and we call it the *linear part* of G. The restriction of R to $l(G)$ is a linearly cyclic order, where the order on $l(G)$ is induced by the order on $\mathrm{uw}(G)$.

We will denote by $K(G)$ the quotient group $G/l(G)$. The class modulo $l(G)$ of an element $a \in G$ will be denoted by $\bar{a}$. $K(G)$ is equipped with a structure of a cyclically ordered group, which is induced by the cyclic order on G, because if a, b, c in G belong to pairwise distinct classes, and $d \in l(G)$, then $R(a,b,c) \Leftrightarrow R(ad,b,c) \& R(da,b,c)$. We know that $K(G)$ embeds into the cyclically ordered group $\mathbb{K}$.

We let $R(x_1,\ldots,x_n)$, where $x_1,\ldots,$ x_n belong to G, stand for:

$$\bigcup_{1\leq i<j<k\leq n} R(x_i,x_j,x_k), \text{ which is equivalent to: } \bigcup_{1\leq i\leq n-2} R(x_i,x_{i+1},x_n)$$

In the following, (G, R) is an abelian cyclically ordered group. Since G is abelian, we will use the notation $+$, except in the case of subgroups of $\mathbb{K}$, because it is customary to use the notation $\cdot$.

One can prove the following property by induction.

2.1 Lemma. *Let $g_1, \ldots, g_n$ be elements of G and $\theta_1, \ldots, \theta_n$ be the elements of the interval $[0, 2\pi[$ of $\mathbb{R}$ such that, for $1 \le j \le n$, $\bar{g}_j = e^{i\theta_j}$. Let k be the integer part of $(\theta_1 + \cdots + \theta_n)/(2\pi)$ and $\theta = (\theta_1 + \cdots + \theta_n) - 2k\pi$. There exists g in G such that $\bar{g} = e^{i\theta}$ and we have: $(0, g_1) + \cdots + (0, g_n) = (k, g)$ within $uw(G)$.*

We will use this property in order in order to obtain a representation of $n(a, x) \in \mathrm{uw}(G)$ as a 2-uple from $\mathbb{Z} \times G$.

Recall that G (a priori non necessarily abelian) is said to be *c-archimedean* if $l(G) = \{0\}$. Świerczkowski ([Sw 59]) proved that G is c-archimedean if and only if for every x and y in G there exists an integer $n > 0$ such that $R(0, nx, y)$ does not hold. Note that this is not a first order property. If G is c-archimedean, then it embeds into $\mathbb{K}$, hence it is abelian. A *c-convex subgroup* of G is a subgroup H which satisfies for every a and b in G: $(b \in H \Rightarrow b \neq -b) \& (b \in H \,\&\, R(0, b, -b) \,\&\, R(0, a, b)) \Rightarrow a \in H$. There exists a canonical one-to-one correspondence between the set of proper c-convex subgroups of G and the set of convex subgroups of $l(G)$ which are different from $\{0\}$ (note that $l(G)$ is a c-convex subgroup). We see that G is c-archimedean if and only if it doesn't contain any proper c-convex subgroup. We will say that G is *c-n-divisible* if it is n-divisible and it contains a subgroup which is isomorphic to the subgroup of $(\mathbb{U}, \cdot)$ generated by a primitive n-th root of 1. We see that this is a first order property. Now, G contains a subgroup which is isomorphic to the subgroup of $(\mathbb{U}, \cdot)$ generated by a primitive n-th root of 1 if and only if z_G is n-divisible in the unwound $\mathrm{uw}(G)$. It follows that if $\mathrm{uw}(G)$ is n-divisible then G is c-n-divisible. Conversely, let $z \in \mathrm{uw}(G)$ be such that $z_G = nz$, then, since G is n-divisible, for every $x \in \mathrm{uw}(G)$ there exists $y \in \mathrm{uw}(G)$ such that $x - ny$ belongs to the subgroup generated by z_G, that is, there exists an integer k such that $x - ny = kz_G = nkz$, so $x = n(y + kz)$. Consequently: G is c-n-divisible if and only if $\mathrm{uw}(G)$ is n-divisible. We will say that G is *c-divisible* if it is divisible and it contains a subgroup which is isomorphic to $(\mathbb{U}, \cdot)$.

3 c-regular groups.

3.1 Definitions. Let $n \ge 2$ be an integer. We say that (G, R) is *n-regular* if for every $x_1, \ldots, x_n$ in G which satisfies $R(0, x_1, \ldots, x_n, -x_n)$, there exists $x \in G$ such that $(R(x_1, nx, x_n)$ or $nx = x_1$, or $nx = x_n)$ and $R(0, x, \ldots, (n-1)x, x_n)$. We say that (G, R) is *c-n-regular* if for every $x_1, \ldots, x_n$ in G satisfying $R(0, x_1, \ldots, x_n)$, there exists $x \in G$ such that $(nx = x_1$, or $nx = x_n$, or $R(x_1, nx, x_n))$ and $R(0, x, \ldots, (n-1)x, x_n)$.
We say that (G, R) is *regular* if it is n-regular for every $n \ge 2$, and we say that (G, R) is *c-regular* if it is c-n-regular for every $n \ge 2$.

Note that these are first order properties, and clearly if (G, R) is c-n-regular, then it is n-regular. Now from Lemma 3.5 it follows that a nontrivial linearly cyclically ordered group is not c-n-regular. One can prove that if (G, R) is a linearly cyclically ordered group, then it is a n-regular cyclically ordered group if and only if it is a n-regular linearly ordered group, that the subgroups of $\mathbb{K}$ are c-regular.

The proofs of the three following lemmas are left to the reader.

3.2 Lemma. *Assume that G is a nontrivial subgroup of $\mathbb{K}$ and let $n \ge 2$. Then G is n-divisible and contains a primitive n-th root of 1 if and only if for every $\theta \in [0, \pi]$, if $e^{i\theta} \in G$, then $e^{i\theta/n} \in G$.*

Note that the c-archimedean groups (i.e. the subgroups of $\mathbb{K}$) are c-archimedean but they need not satisfy the conditions of Lemma 3.2.

3.3 Lemma. *Let $n \ge 2$, assume that G is n-regular and is not c-archimedean. For every θ in $[0, \pi]$, if $e^{i\theta} \in K(G)$, then $e^{i\theta/n} \in K(G)$.*

3.4 Lemma. *Let $n \ge 2$ such that G is n-regular, then $l(G)$ is n-regular, where the compatible total order on $l(G)$ is derived from the definition of the positive cone of G. In particular, if G is regular, then $l(G)$ is regular.*

3.5 Lemma. *Let $n \geq 2$, assume that G is c-n-regular and is not c-archimedean, then $K(G)$ contains a primitive n-th root of 1.*

Proof. Assume that $h_1 < \cdots < h_n < 0$ are elements of $l(G)$ such that $2h_n < h_1$. In the case where $l(G)$ is dense, clearly one can find such a sequence, and if $l(G)$ is discrete with first positive element ϵ, let $h_n \in l(G)$ such that $h_n < -n\epsilon$, and set $h_1 = h_n - n\epsilon + \epsilon, \ldots, h_{n-1} = h_n - \epsilon$. Since G is c-n-regular, there exists $g \in G$ such that $ng = h_1$, or $ng = h_n$, or $R(h_1, ng, h_n)$, and $R(0, g, \ldots, (n-1)g, h_n)$. Note that $\bar{g}$ is a n-th root of 1 in $K(G)$, because $\overline{ng} = \bar{0}$. Now $l(G)$ is c-convex, so $ng \in l(G)$, and $h_1 \leq ng \leq h_n < 0$. Let k, $1 \leq k \leq n$, be such that $kg \in l(G)$, and d be the gcd of k and n. Let a and b be two integers such that $ak + bn = 1$, then $dg = akg + bng \in l(G)$, hence, if k is the lowest positive integer such that $kg \in l(G)$, then k divides n. If $k < n$, since $R(0, kg, h_n)$ holds, we have $kg < h_n < 0$. Furthermore $2k \leq n$, hence $ng \leq 2kg < 2h_n < h_1$: a contradiction. Consequently, $k = n$. Since for $1 \leq k \leq n-1$ $\overline{kg} \neq \bar{0}$, $\bar{g}$ is a primitive n-th root. □

3.6 Proposition. *Let $n \geq 2$, assume that $l(G)$ is n-regular. If $K(G)$ is n-divisible and contains a primitive n-th root of 1, then G is c-n-regular.*

Proof. $K(G)$ is nontrivial because it contains a primitive n-th root of 1; since it is n-divisible, it is infinite, hence it is a dense subgroup of $\mathbb{K}$.

Let g_1 and g_2 in G, assume: $R(0, g_1, g_2)$ and $\bar{g_1} \neq \bar{g_2}$. There exist $\theta_1 < \theta_2$ in $[0, 2\pi]$ such that $\bar{g_1} = e^{i\theta_1}$ and $\bar{g_2} = e^{i\theta_2}$. Since $K(G)$ is dense, there exist g in G and θ in $]\theta_1/n, \theta_2/n[$ such that $\bar{g} = e^{i\theta}$. Then $\overline{ng} = e^{in\theta}$, with $0 \leq \theta_1 < n\theta < \theta_2$, and $0 < \theta < \cdots < (n-1)\theta < \theta_2$, hence $R(g_1, ng, g_2)$ and $R(0, g, \ldots, (n-1)g, g_2)$.

Now, let $g_1, \ldots, g_n$ in G be such that $R(0, g_1, \ldots, g_n)$. If $\bar{g_1} \neq \bar{g_n}$, then by what has been proved above, there exists $g \in G$ such that $R(g_1, ng, g_n)$ and $R(0, g, \ldots, (n-1)g, g_n)$. If $\bar{g_1} = \bar{g_n} = \bar{0}$ and $g_1 \in l(G)_+$, $g_n \in -l(G)_+$, we set $\bar{g_1} = e^{i0}$ and $\bar{g_n} = e^{2i\pi}$, and we conclude in the same way. We assume now that $\bar{g_1} = \cdots = \bar{g_n}$, we set $\bar{g_1} = e^{i\theta}$, with $0 \leq \theta \leq 2\pi$, and if $\theta = 0$ we assume that either each of $g_1, \ldots, g_n$ belongs to $l(G)_+$ or each of $g_1, \ldots, g_n$ belongs to $-l(G)_+$. In the first case, we set $\theta = 0$, in the second case, we set $\theta = 2\pi$. Note that in any case $l(G)$ is nontrivial.

If $\theta = 0$, i.e. both of $g_1, \ldots, g_n$ belong to $l(G)_+$, then we have $0 < g_1 < \cdots < g_n$. Since $l(G)$ is n-regular, there exists $g \in l(G)$ such that $g_1 \leq ng \leq g_n$, and we have also $0 < g < \cdots < (n-1)g < g_n$, hence ($R(g_1, ng, g_n)$ or $ng = g_1$ or $ng = g_n$) and $R(0, g, \ldots, (n-1)g, g_n)$.

Now assume that $0 < \theta \leq 2\pi$. Since $K(G)$ is n-divisible and it contains a primitive n-th root of 1, there exists an element g of G such that $\bar{g} = e^{i\theta/n}$. Since $g_1 - ng, \ldots, g_n - ng$ belong to $l(G)$, it follows from the relation $R(0, g_1, \ldots, g_n)$ that $g_1 - ng < \cdots < g_n - ng$ in $l(G)$. After replacing g by $ng - g_1$, if necessary, we can assume that $0 < g_1 - ng < \cdots < g_n - ng$. Since $l(G)$ is n-regular, there exists $h \in l(G)$ such that $g_1 - ng \leq nh \leq g_n - ng$. It follows: $R(g_1 - ng, nh, g_n - ng)$, or $nh = g_1 - ng$, or $nh = g_n - ng$, then by compatibility $R(g_1, n(g+h), g_n)$ or $n(g+h) = g_1$ or $n(g+h) = g_n$. Furthermore, since $0 < \theta/n < \cdots < (n-1)\theta/n < \theta$, we have $R(0, g+h, \ldots, (n-1)(g+h), g_n)$.

This proves that G is c-n-regular. □

3.7 Theorem. *Assume that G is not c-archimedean and let $n \geq 2$. The following assertions are equivalent.*
1) G is n-regular and $K(G) \neq \{0\}$.
2) $l(G)$ is n-regular, $K(G) \neq \{0\}$ and for every $\theta \in [0, \pi]$ such that $e^{i\theta} \in K(G)$ we have: $e^{i\theta/n} \in K(G)$.
3) G is c-n-regular.
4) $l(G)$ is n-regular, $K(G)$ is n-divisible and contains a primitive n-th root of 1.

Proof. 1) $\Rightarrow$ 2). If G is n-regular, then by Lemma 3.3, for every $\theta \in [0, \pi]$ such that $e^{i\theta} \in K(G)$ we have $e^{i\theta/n} \in K(G)$, furthermore $K(G)$ is n-divisible, and by Lemma 3.4, $l(G)$ is n-regular. 4) $\Rightarrow$ 3) follows from Proposition 3.6. 3) $\Rightarrow$ 1) follows from the definition, and from Lemma 3.2 we deduce 2) $\Leftrightarrow$ 4). □

3.8 Theorem. *Let $n \geq 2$. G is n-regular if and only if one of the two following conditions is satisfied.*
1) G is linearly cyclically ordered, and is a n-regular linearly ordered group.
2) G is not linearly cyclically ordered, and is c-n-regular.
In other words, the class of n-regular cyclically ordered groups is the union of the class n-regular linearly cyclically ordered groups and of the class of c-n-regular cyclically ordered groups.

Proof. Straightforward. □

3.9 Lemma. *Let $n \geq 2$, assume that there exist $g \in G$ and $x \in uw(G)$ such that $\bar{g} = 1$ and $nx = (1, g)$, then $e^{2i\pi/n} \in K(G)$. It follows that if $l(G)$ is nontrivial and $uw(G)$ is n-regular, then $e^{2i\pi/n} \in K(G)$.*

Proof. Since $(1, g) > 0$, by compatibility we have: $0 < x < (1, g)$. By the definition of the order on uw(G), there exists $h \in G$ such that either $x = (0, h)$ or $x = (1, h)$. Now $n(1, h) \geq (n, nh) \geq (n, 0) > (1, g)$, hence $x = (0, h)$. Set $\bar{h} = e^{i\theta}$, with $0 \leq \theta < 2\pi$. According to Lemma 2.1, we have $n\theta = 2\pi$, which proves that $e^{2i\pi/n} \in K(G)$. Now assume that $l(G)$ is nontrivial and that uw(G) is n-regular. Let $g_1, \dots, g_n$ in $l(G)$, assume that $0 < g_1 < \cdots < g_n$, hence $(0, 0) < (1, g_1) < \cdots < (1, g_n)$. There exists $y \in$ uw(G) such that $(1, g_1) \leq ny \leq (1, g_n)$. Then ny admits a representation as $ny = (1, g)$ with $g \in G$. By the definition of the order on uw(G), $R(g_1, g, g_n)$ holds, and by c-convexity of $l(G)$, we have $g \in l(G)$. Finally, from what we just proved, $e^{2i\pi/n} \in K(G)$. □

3.10 Lemma. *Let $n \geq 2$, if $uw(G)$ is n-regular, then G is n-regular.*

Proof. Let $g_1, \dots, g_n$ in G, assume that $R(0, g_1, \dots, g_n)$ holds. Since uw(G) is n-regular, and $(0, 0) < (0, g_1) < \cdots < (0, g_n)$, there exists x in uw(G) such that $(0, g_1) \leq nx \leq (0, g_n)$. In the same way of in the proof of Lemma 3.9, nx admits a representation as $nx = (0, g)$ with $g \in G$. It follows that x admits a representation as $x = (0, h)$ with $nh = g$. We have $(0, g_1) \leq (0, nh) \leq (0, g_n)$, hence $R(g_1, nh, g_n)$ or $nh = g_1$, or $nh = g_n$, which proves that G is n-regular. □

3.11 Proposition. *Let $n \geq 2$, assume that $l(G)$ is n-regular, and that $K(G)$ is n-divisible and contains a primitive n-th root of 1. Then $uw(G)$ is n-regular.*

Proof. By assumption, $K(G)$ is nontrivial, hence G is not equal to $l(G)$. Hence if G is c-archimedean, then uw(G) is archimedean, hence it is regular. If G is not c-archimedean, then $l(G)$ is nontrivial, hence G is infinite. $K(G)$ is infinite because it is n-divisible. Let $(0, 0) < (m_1, g_1) < \cdots < (m_n, g_n)$ in uw(G). Let q and r be the integers such that $0 \leq r < n$ and $m_1 = nq + r$. If we can find $x \in$ uw(G) such that $(r, g_1) \leq nx \leq (m_n - nq, g_n)$, then we have $(m_1, g_1) \leq n(x + (q, 0)) \leq (m_n, g_n)$, which proves that we can assume that $0 \leq m_1 < n$. Since $K(G)$ contains a primitive n-th root of 1, we have: $e^{2i\pi/n} \in K(G)$. Let $g \in G$ such that $\bar{g} = e^{2i\pi/n}$, and set $g' = m_1 g$, then $\bar{g'} = e^{2im_1\pi/n}$, with $0 \leq 2m_1\pi/n < 2\pi$, and $n(2m_1\pi/n) = 2m_1\pi$. By Lemma 2.1, there exists $h \in l(G)$ such that $n(0, g') = (m_1, h)$. If $\bar{g_1} = e^{i\theta_1}$, since $K(G)$ is divisible, one can find an element whose class is $e^{i\theta_1/n}$ in $K(G)$. We add such an element to g', then, if necessary, we add an element of $l(G)$ in order to get g'' such that $n(0, g'') = (m_1, ng'')$ and $R(0, ng'', g_1)$ holds. Assume that $m_1 = \cdots = m_n$ and $\bar{g_1} = \cdots = \bar{g_n}$. By substracting $n(0, g'')$ to each term, we get a sequence $(0, 0) < (0, h_1) < \cdots < (0, h_n)$, where the h_k's belong to $l(G)$. By the definition of the order on uw(G), we have either $0 < h_1 < \cdots < h_n$, or $h_1 < \cdots < h_n < 0$, or $h_n < 0 < h_1$. In the first case, there exists h in $l(G)$ such that $h_1 \leq nh \leq h_n$. Hence $R(h_1, nh, h_n)$, or $nh = h_1$, or $nh = h_n$. In the second case we consider the sequence $0 < -h_n < \cdots < -h_1$, there exists $h \in l(G)$ such that $-h_n \leq -nh \leq -h_1$, then $h_1 \leq nh \leq h_n$, and we conclude in the same way as above. In the third case, we set for example $h = h_1$, then $h_n < 0 < h_1 < nh$, which implies $R(h_1, nh, h_n)$. In any case, we have proved that there exists $h \in l(G)$ such that $h_1 \leq nh \leq h_n$. Then $(0, h_1) \leq (0, nh) = n(0, h) \leq (0, h_n)$. Consequently $(m_1, g_1) \leq n((0, h) + (0, g'')) \leq (m_n, g_n)$. If the m_k's are not equal, note that we have $m_1 \leq \cdots \leq m_n$, and in this case $m_1 < m_n$. Let $0 < h_2 < \cdots < h_n$ in $l(G)$, then $0 < (m_1, g_1) < (m_1, g_1 + h_2) < \cdots < (m_1, g_1 + h_n)$. It follows that there exists $h \in G$ such that $(m_1, g_1) \leq n(0, h) \leq (m_1, g_1 + h_n)$, and since $(m_1, g_1 + h_n) < (m_n, g_n)$, h is the required element. If all of the m_k's are equal, but the g_k's are not, then $R(0, g_1, g_n)$ holds, take the element $(0, h)$ that we just exhibit above. □

3.12 Theorem. *Let $n \geq 2$. The following conditions are equivalent.*
(1) $uw(G)$ is n-regular
(2) G is c-n-regular
(3) either G is c-archimedean,
or $l(G)$ is n-regular, $K(G)$ is n-divisible and it contains a primitive n-th root of 1
(4) the quotient of G by every proper c-convex subgroup is c-n-divisible.

Proof. If G is c-archimedean, then it is either dense or finite, hence it is c-regular. Furthermore, $\mathrm{uw}(G)$ is archimedean, hence it is regular. Now, assume that G is not c-archimedean. The equivalence (2) $\Leftrightarrow$ (3) has been proved in Theorem 3.7. If $l(G)$ is n-regular, $K(G)$ is n-divisible and contains a primitive n-th root of 1, then by Proposition 3.11, $\mathrm{uw}(G)$ is n-regular. If $\mathrm{uw}(G)$ is n-regular, since it contains a convex subgroup which is isomorphic to $l(G)$, $l(G)$ is n-regular. According to Lemma 3.10, G is n-regular, and by Lemma 3.3, $K(G)$ is n divisible. Finally, by Lemma 3.9, $e^{2i\pi/n} \in K(G)$.

Let H be a proper c-convex subgroup (that is a convex subgroup of $l(G)$ distinct from $\{0\}$). In the construction of the unwound, we note that $\mathrm{uw}(G/H) = \mathrm{uw}(G)/H$. By means of (1) $\Leftrightarrow$ (2) we get:
G/H is c-n-divisible, for every c-convex subgroup H of G, if and only if $\mathrm{uw}(G)/H$ is n-divisible, for every c-convex subgroup H of G, if and only if $\mathrm{uw}(G)$ is n-regular, if and only if G is c-n-regular. □

4 Elementarily equivalent c-regular cyclically ordered groups.

As being dense and being c-regular are first order properties, every cyclically ordered group which is elementarily equivalent to an infinite subgroup of $\mathbb{K}$ is dense and c-regular. Every ultraproduct of finite cyclically ordered groups is discrete and c-regular, since each factor satisfies both of these first order properties.

If G is discrete and infinite then $l(G)$ is nontrivial, because every infinite and c-archimedean cyclically ordered group is dense. Furthermore, if G is c-regular, then by Lemma 3.3, $K(G)$ is divisible, hence, it is infinite.

4.1 Preliminaries.

If A is an abelian group and p is a prime, we define the p-th *prime invariant of Zakon* of A, denoted by $[p]A$, to be the maximum number of p-incongruent elements in A. In the infinite case, we set $[p]A = \infty$, without distinguishing between infinities of different cardinalities.

4.1 Lemma. *Let H be a subgroup of $\mathbb{Q}$, and, for every prime p, let $m_p \in \mathbb{N}$ be such that $[p]H \leq p^{m_p}$. Then there exists a countable subgroup M of $(\mathbb{R},+)$ which contains H, such that H is pure in M and, for every prime p, $[p]M = p^{m_p}$.*

Proof. Denote by $p_1 < p_2 < \cdots < p_n < \cdots$ the increasing sequence of all primes, and, for every $n \in \mathbb{N}^*$, denote by m'_n the integer which satisfies $p_n^{m'_n} = P_n^{m_{p_n}}/[p_n]H$. We pick a transcendent number e, and, for n and j in $\mathbb{N}^*$ such that $1 \leq j \leq m'_n$, we pick pairwise distinct elements k_{n_j} in $\mathbb{N}^*$. Denote by M_0 the direct sum of the $\mathbb{Z}_{(p_n)}e^{k_{n_j}}$'s, where $n \in \mathbb{N}^*$ and $1 \leq j \leq m'_n$, and $\mathbb{Z}_{(p_n)}$ denotes the localization of $\mathbb{Z}$ at the prime ideal (p_n). By [Za 61], for every $n \in \mathbb{N}^*$ we have: $[p_n]M_0 = p^{m'_n}$. Since $\mathbb{Q} \cap M_0 = \{0\}$, we have: $\mathbb{Q} + M_0 = \mathbb{Q} \oplus M_0$. We set: $M = H \oplus M_0$, then M contains H, H is pure in M, M is countable, and for every prime p we have: $[p]M = ([p]H)([p]M_0) = p^{m_p}$. □

4.2 Proposition. *Let A, B, T be linearly ordered abelian groups, where B is a convex subgroup of T. Then there exists an exact sequence of ordered groups $0 \to B \to T \to A \to 0$ if and only if there exists a mapping θ from $A \times A$ to B which satisfies:*
(*) $\forall a \in A,\ \theta(a,0) = 0$,
(**) $\forall (a_1,a_2) \in A \times A,\ \theta(a_1,a_2) = \theta(a_2,a_1)$,
(***) $\forall (a_1,a_2,a_3) \in A \times A \times A,\ \theta(a_1,a_2) + \theta(a_1+a_2,a_3) = \theta(a_1,a_2+a_3) + \theta(a_2,a_3)$ (in other words, θ is a 2-cocycle),
such that T is isomorphic to $A \times B$ lexicographically ordered and equipped with the operation $+_\theta$ defined by:

$$\forall (a_1,b_1,a_2,b_2) \in A \times B \times A \times B,\ (a_1,b_1) +_\theta (a_2,b_2) = (a_1+a_2, b_1+b_2+\theta(a_1,a_2)).$$

We will say that this exact sequence is an extension of B by A, we set $T = A \overrightarrow{\times}_\theta B$, and we will omit θ when $\theta \equiv 0$.

Proof. See [3]. We recall the definition of the mapping θ. For every $a \in A$, we pick some $r(a)$ in T whose image is a (a can be viewed as a class modulo B), by taking $r(0) = 0$, and we set, for every a_1, a_2 in A, $\theta(a_1, a_2) = r(a_1 + a_2) - r(a_1) - r(a_2)$. The condition (*) is satisfied because we set $0 = r(0)$, (**) follows from commutativity, and (***) follows from the associativity of addition. We get the isomorphism by defining the image of (a, b), where $a \in A$, $b \in B$, to be the element $r(a) + b$ of T. □

4.3 Proposition.
1) Let $A\overrightarrow{\times}_{\theta_1}B_1$ and $A\overrightarrow{\times}_{\theta_2}B_2$ be two extensions of linearly ordered abelian groups, $B = B_1\overrightarrow{\times}B_2$, $\theta = (\theta_1, \theta_2)$ from $A \times A$ to B, then $A\overrightarrow{\times}_{\theta}B$ is an extension of linearly ordered abelian groups. Furthermore $A\overrightarrow{\times}_{\theta_1}B_1$ and $A\overrightarrow{\times}_{\theta_2}B_2$ embed canonically into cofinal subgroups of $A\overrightarrow{\times}_{\theta}B$.
2) Let $A_1\overrightarrow{\times}_{\theta_1}B$ and $A_2\overrightarrow{\times}_{\theta_2}B$ be two extensions of linearly ordered abelian groups, $A = A_1\overrightarrow{\times}A_2$, $\theta = \theta_1+\theta_2$ from $A \times A$ to B, then $A\overrightarrow{\times}_{\theta}B$ is an extension of linearly ordered abelian groups. Furthermore $A_1\overrightarrow{\times}_{\theta_1}B$ and $A_2\overrightarrow{\times}_{\theta_2}B$ embed canonically into subgroups of $A\overrightarrow{\times}_{\theta}B$.
3) Let A and B' be two linearly ordered abelian groups, B be a subgroup of B', and $A\overrightarrow{\times}_{\theta}B$ be an extension of linearly ordered abelian groups. Then there exists an extension of linearly ordered abelian groups $A\overrightarrow{\times}_{\theta'}B'$, where θ' extends θ, $A\overrightarrow{\times}_{\theta}B$ is a subgroup of $A\overrightarrow{\times}_{\theta'}B'$, $\{0\}\overrightarrow{\times}_{\theta'}B'$ is a convex subgroup of $A\overrightarrow{\times}_{\theta'}B'$, and the quotient group is isomorphic to A.

Proof. The proof is left to the reader. □

Note that Proposition 4.2 remains true by assuming that A and T are cyclically ordered, and B is linearly ordered. Then analogues of 1) and 3) of Proposition 4.3 hold.

4.4 Proposition. *Let A' and B be two linearly ordered abelian groups, A be a subgroup of A', and $A\overrightarrow{\times}_{\theta}B$ be an extension of linearly ordered abelian groups. Then θ extends by induction to $A' \times A'$ by means of the following properties. Let $a' \in A'\backslash A$.*
1) If a' is rationally independent from A, we can extend θ to $\mathbb{Z}a' + A$ ($= \mathbb{Z}a' \oplus A$) in the following way:

$$\forall(n_1, a_1, n_2, a_2) \in \mathbb{Z} \times A \times \mathbb{Z} \times A,\ \theta(n_1a' + a_1, n_2a' + a_2) = \theta(a_1, a_2).$$

2) If there exists a prime p such that $pa' \in A$, then for every $b_0 \in B$, θ extends to the subgroup generated by a' and A in the following way. The elements of the group generated by a' and A are the $na' + a$, where $0 \leq n \leq p - 1$ and $a \in A$. Denote by $int(x)$ the integer part of a rational number x. The extension is defined by:

$$\forall(n_1, a_1, n_2, a_2) \in \{0, \dots, p - 1\} \times A \times \{0, \dots, p - 1\} \times A,$$

$$\theta(n_1a' + a_1, n_2a' + a_2) = \theta(a_1, a_2) + \theta(int(\frac{n_1 + n_2}{p})(pa'), a_1 + a_2) + int(\frac{n_1 + n_2}{p})b_0.$$

So we get $[p]B$ non isomorphic extensions of $(A \times B, +_\theta)$, each one depending on the class of b_0 modulo pB. $A' \times B$ is a subgroup of the divisible closure of $A\overrightarrow{\times}_{\theta}B$. If $b_0 = 0$, then $p(a', 0) = (pa', 0)$.

Proof. In case 1) as well as in case 2), one easily checks that the mapping θ that we define satisfies (*) and (**). In case 1), one can easily verify that (***) holds, it remains to prove (***) in case 2). In order to simplify the notations, we set:

$$\epsilon_{12} = \text{int}(\frac{n_1 + n_2}{p}),\ \epsilon_{23} = \text{int}(\frac{n_2 + n_3}{p}) \text{ and } \epsilon_{123} = \text{int}(\frac{n_1 + n_2 + n_3}{p}).$$

So let n_1, n_2, n_3 be in $\{0, \dots, p - 1\}$ and a_1, a_2, a_3 be in A. In order to compute $\theta(n_1a' + a_1 + n_2a' + a_2, n_3a' + a_3)$, we start from the representation of $n_1a' + a_1 + n_2a' + a_2$ as

$$(n_1 + n_2 - \epsilon_{12})(pa') + a_1 + a_2 + \epsilon_{12}(pa'),$$

and we get:

$$\theta(n_1a' + a_1 + n_2a' + a_2, n_3a' + a_3) = \theta(a_1 + a_2 + \epsilon_{12}(pa'), a_3)+$$
$$\theta(\text{int}((n_1 + n_2 - \epsilon_{12}p + n_3)/p)(pa'), a_1 + a_2 + a_3 + \epsilon_{12}(pa'))$$
$$+\text{int}((n_1 + n_2 - \epsilon_{12}p + n_3)/p)b_0 = \theta(a_1 + a_2 + \epsilon_{12}(pa'), a_3)+$$

$$\theta((\epsilon_{123}-\epsilon_{12})(pa'), a_1+a_2+a_3+\epsilon_{12}(pa')) + (\epsilon_{123}-\epsilon_{12})b_0.$$

In the same way,

$$\theta(n_1a'+a_, n_2a'+a_2+n_3a'+a_3) = \theta(a_1, a_2+a_3+\epsilon_{23}(pa'))+$$

$$\theta((\epsilon_{123}-\epsilon_{23})(pa'), a_1+a_2+a_3+\epsilon_{23}(pa')) + (\epsilon_{123}-\epsilon_{23})b_0.$$

So, proving (***) reduces to proving that the following $(E1)$ and $(E2)$ are equal:

$$(E1) = \theta(a_1+a_2+\epsilon_{12}(pa'), a_3) + \theta((\epsilon_{123}-\epsilon_{12})(pa'), a_1+a_2+a_3+\epsilon_{12}(pa'))+$$

$$(\epsilon_{123}-\epsilon_{12})b_0 + \theta(a_1, a_2) + \theta(\epsilon_{12}(pa'), a_1+a_2) + \epsilon_{12}b_0$$

$$\text{and } (E2) = \theta(a_2, a_3) + \theta(\epsilon_{23}(pa'), a_2+a_3) + \epsilon_{23}b_0 + \theta(a_1, a_2+a_3+\epsilon_{23}(pa'))+$$

$$\theta((\epsilon_{123}-\epsilon_{23})(pa'), a_1+a_2+a_3+\epsilon_{23}(pa')) + (\epsilon_{123}-\epsilon_{23})b_0.$$

One can check that we have:

$$(E1) = \theta(a_1, a_2) + \theta(a_1+a_2, \epsilon_{12}(pa')) + \theta(a_1+a_2+\epsilon_{12}(pa'), a_3)$$

$$+\theta(a_1+a_2+\epsilon_{12}(pa')+a_3, (\epsilon_{123}-\epsilon_{12})(pa')) + \epsilon_{123}b_0.$$

$$\text{and } (E2) = \theta(a_2, a_3) + \theta(a_2+a_3, \epsilon_{23})(pa')) + \theta(a_2+a_3+\epsilon_{23}(pa'), a_1)$$

$$+\theta(a_2+a_3+\epsilon_{23}(pa')+a_1, (\epsilon_{123}-\epsilon_{23})(pa')) + \epsilon_{123}b_0.$$

The operation $+_\theta$ being associative, we note that

$$[[[(a_1,0) +_\theta (a_2,0)] +_\theta (\epsilon_{12}(pa'),0)] +_\theta (a_3,0)] +_\theta ((\epsilon_{123}-\epsilon_{12})(pa'),0) =$$

$$(a_1+a_2+\epsilon_{12}(pa')+a_3+(\epsilon_{123}-\epsilon_{12})(pa'), (E1)), \text{ and that}$$

$$[[[(a_2,0) +_\theta (a_2,0)] +_\theta (\epsilon_{23}(pa'),0)] +_\theta (a_1,0)] +_\theta ((\epsilon_{123}-\epsilon_{23})(pa'),0) =$$

$$(a_2+a_3+\epsilon_{23}(pa')+a_1+(\epsilon_{123}-\epsilon_{23})(pa'), (E2)).$$

Now, calculations show that in order to prove that $(E1) = (E2)$ it is sufficient to establish:

$$(\epsilon_{12}(pa'),0) +_\theta (\epsilon_{123}-\epsilon_{12})(pa'),0) = (\epsilon_{23}(pa'),0) +_\theta (\epsilon_{123}-\epsilon_{23})(pa'),0).$$

The 3-tuple $(\epsilon_{123}, \epsilon_{12}, \epsilon_{23})$ can take the values

$$(0,0,0),\ (2,1,1),\ (1,0,0),\ (1,1,0),\ (1,0,1) \text{ or } (1,1,1),$$

in any case, calculations prove that the equality holds, the details are left to the reader.

If $n_1+n_2<p$, then $(n_1a',0)+_\theta(n_2a',0) = ((n_1+n_2)a',0)$, and $\theta((p-1)a',a') = \theta(0,0)+\theta(pa',0)+b_0 = b_0$. Hence $((p-1)a',0) +_\theta (a',0) = (pa',b_0)$ i.e. $p(a',0) = (pa',b_0)$. In particular, if $b_0 = 0$, then $p(a',0) = (pa',0)$, and $(pa',0)$ is divisible by p in the extension we obtained. In all cases, $(a',0)$ belongs to the divisible hull of $A\overrightarrow{\times}_\theta B$.

Now, the order relation on the quotient group is fixed, hence any morphism from $A'\times B$ onto $A'\times B$, equipped with another operation, takes an element into an element which belongs to the same class modulo B. In particular, the image of $(a',0)$ is (a',b_1), for some $b_1\in B_1$. Then $p(a',b_1) = p(a',0)+p(0,b_1) = (pa',b_0)+(0,pb_1) = (pa',b_0+pb_1)$. Consider two operations extending the one of $A\times B$, $+_\theta$ and $+_{\theta'}$, characterized respectively by elements of B, say b_0 and b_0'. So, if there exists a morphism between these two extensions, then b_0' can be expressed as b_0+pb_1, for some $b_1\in B_1$, and conversely, which proves that there are (at least) the same number of non isomorphic extensions of the structure of group as the number of classes modulo pB in B.

If $a'\in A'\backslash A$ satisfies some relation $na'\in A$, we consider n as factored into prime powers, and we proceed in the same way as in 2). So, θ extends to every element of $A'\backslash A$, and by Zorn's lemma we conclude that θ extends to A'. $A'\overrightarrow{\times}_\theta B$ is contained in the divisible hull of $A\overrightarrow{\times}_\theta B$, because this holds at each step. □

4.5 Definition. Let T be a discrete linearly ordered abelian group, with first positive element 1_T, and which contains a fixed element z_T that we join to the language. Since 1_T is definable, we can assume that it lies in the language. For every prime p, $n \in \mathbb{N}^*$ and $k \in \{0, \dots, p^n - 1\}$, we define the formula $DD_{p^n,k}$: $\exists x,\ p^n x = z_T + k1_T$.

4.6 Lemma. *Let T be a discrete linearly ordered abelian group, with first positive element 1_T, and containing a fixed cofinal positive element z_T, that we join to the language. Assume that $T/\langle 1_T\rangle$ is divisible.*
1) For every prime p and $n \in \mathbb{N}^*$, there exists exactly one integer $k \in \{0, \dots, p^n - 1\}$ such that $DD_{p^j,k}$ holds in T.
2) Let p be a prime, $n \in \mathbb{N}^*$, and $k \in \{0, \dots, p^n - 1\}$ such that $DD_{p^j,k}$ holds within T. We express k as a sum $a_0 + a_1 p + \cdots + a_{n-1}p^{n-1}$, where, for $1 \leq j \leq n$, $a_j \in \{0, \dots, p-1\}$, then for every $j \leq n$, and $k_j = a_0 + \cdots + a_{j-1}p^{j-1}$, DD_{p^j,k_j} holds within T.

Proof. 1) Since $T/\langle 1_T\rangle$ is divisible, there exists $x \in T$ such that the class of $p^n x$ modulo $\langle 1_T\rangle$ is equal to the class of z_T. Let $x_1 \in T$, the class of x_1 is equal to the class of x if and only if $x_1 - x \in \mathbb{Z}\cdot 1_T$, and if this holds then $p^n x_1 - p^n x \in p^n\mathbb{Z}\cdot 1_T$. So there exists a unique element of the class of x (that we still denote by x) such that $p^n x - z_T \in \{0, \dots, p^n - 1\}$, which proves 1).

2) If $p^n x = z_T + k1_T = z_T + (a_0 + a_1 p + \cdots + a_{n-1}p^{n-1})1_T$, then for $j \leq n$ we have $p^j(p^{n-j}x - (a_j + a_{j+1}p + \cdots + a_{n-1}p^{n-j-1})1_T) = z_T + (a_0 + \cdots + a_{j-1}p^{j-1})1_T$, the proposition follows. □

4.7 Proposition. *Let T_1 and T_2 be discrete linearly ordered abelian groups such that $T_1/\langle 1_{T_1}\rangle \simeq \mathbb{Q} \simeq T_2/\langle 1_{T_2}\rangle$, and containing an element z_1 and z_2 respectively, which is positive and cofinal (which is equivalent to saying that it is not contained in any proper convex subgroup). In the language of ordered groups together with a predicate for the distinguished element, the following conditions are equivalent.*
T_1 and T_2 are isomorphic
T_1 and T_2 satisfy the same formulas $DD_{p^j,k}$
$T_1 \equiv T_2$.

Proof. Clearly, if T_1 and T_2 are isomorphic, then they are elementarily equivalent, and if they are elementarily equivalent then they satisfy the same formulas $DD_{p^j,k}$, because these are first order formulas. Now, assume that they satisfy the same formulas $DD_{p^j,k}$. We are going to define cocycles θ_1 and θ_2 such that $T_1 \simeq \mathbb{Q}\overrightarrow{\times}_{\theta_1}\mathbb{Z}$ and $T_2 \simeq \mathbb{Q}\overrightarrow{\times}_{\theta_2}\mathbb{Z}$, then we will prove: $\theta_1 = \theta_2$. We know that every element of $\mathbb{Q}$ has a unique representation as a finite sum $r = n + \sum_i \sum_j m_{ij}p_i^{-j}$, where $n \in \mathbb{Z}$, the p_i's run over the increasing sequence of all primes, the m_{ij}'s belong to $\{0, \dots, p_i - 1\}$, and $j \in \mathbb{N}^*$. Consequently, a generating subset of the additive group $\mathbb{Q}$ is $\{1\} \cup \{1/p^j;\ p \text{ a prime and } j \in \mathbb{N}^*\}$. We let $z_1 \in T_1$ be such that its class in $T_1/\langle 1_{T_1}\rangle$ is 1, and for every $n \in \mathbb{Z}$, we let nz_1 be the representative of the class n. If x belongs to the class $1/p$ and $n \in \mathbb{Z}$, then there exists $k \in \mathbb{Z}$ such that $p(x + n1_{T_1}) = px + pn1_{T_1} = z_1 + k1_{T_1} + pn1_{T_1}$, so we see that we can choose x in order to have $0 \leq k \leq p-1$, we will denote this element by k_p. For $m \in \{2, \dots, p-1\}$, we let mx be the representative of m/p. Assume that $j \geq 1$ and that the representative x of the class $1/p^j$ is fixed, we let the representative of the class $1/p^{j+1}$ be the element y which satisfies $0 \leq py - x \leq (p-1)1_{T_1}$, we let $k_{p^{j+1}}$ be the integer such that $k_{p^{j+1}}1_{T_1} = py - x$. For $m \in \{2, \dots, p-1\}$, we let my be the representative of the class m/p^{j+1}. In the sum of any two rational numbers expressed as $r = n + \sum_i \sum_j m_{ij}p_i^{-j}$, the only interactions between the generators occur with $1/p^n$ and $1/p^{n-1}$ (p a prime, $n \in \mathbb{N}^*$). The mapping θ_1 is the isomorphism between T_1 and $\mathbb{Q}\overrightarrow{\times}_{\theta_1}\mathbb{Z}$. For every integers m and n, we have $\theta_1(m, n) = 0$, and θ_1 is defined by induction in the same way as in 2) of Proposition 4.4. We see that θ_1 is uniquely determined by the k_{p^j}'s. We define θ_2 in the same way. Now, the k_{p^j}'s are uniquely determined by the formulas $DD_{p^j,k}$, since $DD_{p^j,k}$ holds if and only if $k = k_p + k_{p^2}p + \cdots + k_{p^j}p^{j-1}$. It follows that $\theta_2 = \theta_1$, and we have the isomorphism we were looking for. Note that if p divides z_1, then the representative of the class of $1/p$ is the divisor of z_1, in this case, $k_p = 0$, the same holds if p^j divides z_1, hence the divisible hull of $\langle z_1\rangle$ in T_1 is contained in the set of representatives. □

4.2 Dense c-regular cyclically ordered abelian groups.

4.8 Lemma. *Let p be a prime. If G contains a p-torsion element, then $[p]uw(G) = [p]G$, otherwise, $[p]uw(G) = p[p]G$.*

Proof. Let $x_1, \ldots, x_n$ be elements of uw(G) such that $\overline{x_1}, \ldots, \overline{x_n}$ are pairwise p-incongruent (within G), then $x_1, \ldots, x_n$ are pairwise p-incongruent, hence $[p]\mathrm{uw}(G) \geq [p]G$. If $[p]G$ is infinite, then the proposition is trivial. So we assume that $[p]G$ is finite, and let $\overline{x_1}, \ldots, \overline{x_n}$ be a maximal family of p-incongruent elements. In particular, for every $i \neq j$, $x_i - x_j \notin \langle z_G \rangle$. Let $x \in \mathrm{uw}(G)$, there exists some x_i such that $\overline{x}$ is p-congruent to $\overline{x_i}$, i.e. there exists $y \in \mathrm{uw}(G)$ and $k \in \mathbb{Z}$ such that $x - x_i = py + kz_G$. If z_G is divisible by p (in other words, if G contains a p-torsion element), this implies that x and x_i are p-congruent (within uw(G)), hence $x_1, \ldots, x_n$ is a maximal p-incongruent family of elements of uw(G). It follows: $[p]\mathrm{uw}(G) = [p]G$. If z_G is not p-divisible, then the elements $x_i + kz_G$, $1 \leq i \leq n$, $0 \leq k \leq p-1$, are pairwise p-incongruent. Let $x \in \mathrm{uw}(G)$ and x_i such that $\overline{x} = \overline{x_i}$, then, by what we did above, x is p-congruent to one of the $x_i + kz_G$'s within uw(G), which proves that the family we defined is maximal, and that $[p]\mathrm{uw}(G) = p[p]G$. □

4.9 Proposition. *Let G_1 and G_2 be two dense c-regular cyclically ordered groups such that G_1 is a subgroup of G_2. Then G_1 is an elementary substructure of G_2 if and only if G_1 is pure in G_2, and, for every prime p, $[p]G_1 = [p]G_2$.*

Proof. Trivially, if $G_1 \prec G_2$, then G_1 is pure in G_2 and, for every prime p, $[p]G_1 = [p]G_2$, because these are first order properties. Conversely, we know that uw(G_1) embeds into uw(G_2). First we prove that uw(G_1) is pure in uw(G_2). Let (n, g) be an element of G_1 which is divisible by some prime p within uw(G_2). Since $p(1, 0) = (p, 0)$, we can assume that $0 \leq n < p$, hence a divisor of (n, g) can be expressed as $(0, h)$, for some $h \in G_2$, and $ph = g$. Since G_1 is pure in G_2, it contains some element h' such that $ph' = g$. If $h' \neq h$, then $h'h^{-1}$ is a p-torsion element within G_2, and since G_1 is pure in G_2, it contains a p-torsion element. We know that if a cyclically ordered group contains a p-torsion element, then its subgroup of p-torsion elements is cyclic of cardinal p (to see this, look at the subgroups of $\mathbb{K}$). Consequently G_1 contains all the p-torsion elements of G_2, and in particular, it contains $h'h^{-1}$, hence it contains h. Since any integer can be factored into prime powers, we deduce by induction that for every $m \in \mathbb{N}^*$, if m divides (n, g) within uw(G_2), then it divides (n, g) within uw(G_1) (note that the divisors are the same). We see also that for every prime p, G_2 contains a p-torsion element if and only if G_1 contains an p-torsion element, and by Lemma 4.8, we have $[p]\mathrm{uw}(G_1) = [p]\mathrm{uw}(G_2)$. By [Za 61], we have: $\mathrm{uw}(G_1) \prec \mathrm{uw}(G_2)$. Since $z_{G_2} = z_{G_1} \in \mathrm{uw}(G_1)$, uw$(G_1)$ still remains an elementary substructure of uw(G_2) in a language augmented with a predicate consisting of the constant z_G, and by Theorem 4.1 of [2], we have: $G_1 \prec G_2$. □

4.10 Proposition. *Let G_1 and G_2 be two dense countable c-regular cyclically ordered groups having isomorphic torsion groups and such that, for every prime p, $[p]G_1 = [p]G_2$. Then G_1 and G_2 are elementary substructures of the same cyclically ordered group.*

Proof. Since G_1 and G_2 have the same prime invariants of Zakon and the same torsion groups, their linear parts have the same prime invariants of Zakon, hence, by Note 6.2 and Theorem 6.3 of [Za 61], they are elementary substructures of the same regular group L where for every $u \in L$ there exists $n \in \mathbb{N}^*$ such that nu can be expressed as $g_1 + g_2$, for some $g_1 \in G_1$ and $g_2 \in G_2$, the order being the lexicographic one. $\mathrm{uw}(G_1)/l(G_1)$ and $\mathrm{uw}(G_2)/l(G_2)$ embed into $\mathbb{R}$, where the images of the classes of z_{G_1} and z_{G_2} are 1. Let R be the subgroup of $\mathbb{R}$ generated by the images of $\mathrm{uw}(G_1)/l(G_1)$ and $\mathrm{uw}(G_2)/l(G_2)$, R is divisible, because it is generated by two divisible subgroups. Let θ_1 and θ_2 be two mappings satisfying (*), (**) and (***) of Proposition 4.2, and such that $\mathrm{uw}(G_1) \simeq (\mathrm{uw}(G_1)/l(G_1)) \overrightarrow{\times}_{\theta_1} l(G_1)$ and $\mathrm{uw}(G_2) \simeq (\mathrm{uw}(G_2)/l(G_2)) \overrightarrow{\times}_{\theta_2} l(G_2)$. By Proposition 4.4, θ_1 and θ_2 extend to R. We set $\theta = (\theta_1, \theta_2)$, in the same way as in Proposition 4.3. According to the same proposition, we get a structure of linearly ordered group $R \overrightarrow{\times}_\theta L$ where uw(G_1) and uw(G_2) are cofinal subgroups. By properties of regular groups, since L is regular and the quotient of $R \overrightarrow{\times}_\theta L$ by L is divisible, the group $R \overrightarrow{\times}_\theta L$ is regular, and it has the same prime invariants of Zakon as L and as uw(G_1) and uw(G_2). Since the theory of dense regular groups with fixed family of prime invariants of Zakon is model complete, it follows that uw(G_1) and uw(G_2) are elementary substructures of $R\overrightarrow{\times}_\theta L$. Since these two subgroups contain $(1,0)$, they still remain elementary substructures in a language augmented with a predicate consiting of this element, and their wound-rounds G_1 and G_2 are elementary substructures of the wound-round associated to $R\overrightarrow{\times}_\theta L$. □

4.11 Proposition. *For any choice of the family of prime invariants of Zakon and any subgroup of $\mathbb{U}$, there exists a countable c-archimedean dense cyclically ordered group G with family of prime invariants of Zakon and with torsion group so chosen.*

Proof. Let T be the subroup of $\mathbb{U}$, the unwound $\mathrm{uw}(T)$ of T is a subgroup of $\mathbb{Q}$, and for every prime p, $n \in \mathbb{N}$, the element z_T is p^n-divisible if and only if $\zeta_{p^n} \in T$. In the same way as at the beginning of Section 4.1, $[p]\mathrm{uw}(T) = 1$ if $\mathrm{uw}(T)$ is p-divisible, and $[p]\mathrm{uw}(T) = p$ otherwise. Let $\mathcal{Z} = \{p_i^{n_i} \mid i \in \mathbb{N}^*\}$ be the given family of invariants of Zakon, where (p_i) is the increasing sequence of all primes and the n_i's belong to $\mathbb{N} \cup \{\infty\}$. By Lemma 4.1, there exists a countable subgroup M of $\mathbb{R}$ such that for every $i \in \mathbb{N}^*$, $[p_i]M = [p]\mathrm{uw}(T)p_i^{n_i}$ and $\mathrm{uw}(T)$ is a pure subgroup of M. We set $G = M/\langle z_T \rangle$. Since $\mathrm{uw}(T)$ is pure in M, the groups G and T have the same torsion subgroup, which is T, and by Lemma 4.8, we have, for every $i \in \mathbb{N}^*$, $[p_i]G = p_i^{n_i}$. If G is finite, hence it is contained in $\mathbb{U}$, it suffices to consider the subgroup $H = \{e^{ir} \mid r \in \mathbb{Q}\}$ of $\mathbb{K}$, where $\mathbb{Q}$ is the group of rational numbers, and for every prime p we have $[p]H = 1$. Hence $G \oplus H$ is infinite, so it is dense, it has the same torsion subgroup as G, and, for every prime p, $[p](G \oplus H) = [p]G \cdot [p]H = [p]G$. □

4.12 Theorem. *There exists an infinite c-archimedean cyclically ordered group which is elementarily equivalent to G if and only if G is c-regular and dense.*
Any two dense c-regular cyclically ordered groups are elementarily equivalent if and only if their torsion subgroups are isomorphic and they have the same family of prime invariants of Zakon.

These conditions depend on the first order theory of G by Lemma 4.8 and because the number of p-torsion elements of G depends on the first order theory of G.

Proof. If G is elementarily equivalent to some c-archimedean infinite cyclically ordered group, then it shares with this group the first order properties, in particular being dense and c-regular. Now, assume that G is c-regular and dense. By the theorem of Löwenheim-Skolem, there exists a countable elementary substructure G_1 of G. By Proposition 4.11, there exists a countable c-archimedean dense group G_2 having the same torsion group and the same family of prime invariants of Zakon as G_1. By Proposition 4.10, there exists a cyclically ordered group G' such that $G_1 \prec G'$ and $G_2 \prec G'$. Since $G_1 \prec G$, we have $G_2 \equiv G$.

Let G and G' be c-regular dense cyclically ordered groups. If they are elementarily equivalent, then their torsion subgroups are isomorphic and they have the same family of prime invariants of Zakon. Assume now that the torsion subgroups of G and G' are isomorphic and that they have the same family of prime invariants of Zakon. Let $G_1 \prec G$ and $G_1' \prec G'$ be countable. By Proposition 4.10, there exists a cyclically ordered group G'' such that $G_1 \prec G''$ and $G_1' \prec G''$. It follows $G \equiv G'$. □

4.13 Remark. *One can see, for example by Theorem 3.12, that every abelian c-divisible cyclically ordered group is c-regular, hence by Theorem 4.12 an abelian cyclically ordered group is c-divisible if and only if it is elementarily equivalent to $\mathbb{U}$.*

4.3 Discrete c-regular cyclically ordered abelian groups.

4.14 Proposition. *Let G_1 and G_2 be two discrete c-regular cyclically ordered groups such that G_1 is a subgroup of G_2. Then G_1 is an elementary substructure of G_2 if and only if G_1 is pure in G_2 and the positive cones of G_1 and G_2 have the same first positive element.*

Proof. Trivially, if $G_1 \prec G_2$, then G_1 is pure in G_2, and their positive cones have the same first positive element, since it is definable. Conversely, in the same way as in the proof of Proposition 4.9, we can show that $\mathrm{uw}(G_1)$ is pure in $\mathrm{uw}(G_2)$. Furthermore, $\mathrm{uw}(G_1)$ and $\mathrm{uw}(G_2)$ have the same lowest positive element, hence by [Za 61]: $\mathrm{uw}(G_1) \prec \mathrm{uw}(G_2)$. Since $z_{G_2} = z_{G_1} \in \mathrm{uw}(G_1)$, $\mathrm{uw}(G_1)$ still remains an elementary substructure of $\mathrm{uw}(G_2)$ in a language augmented with a predicate consisting of z_G, and by Theorem 4.1 of [2], we have: $G_1 \prec G_2$. □

4.15 Proposition. *Assume that G is c-regular discrete and is not c-archimedean. Then G contains a discrete c-regular pure subgroup H such that $K(H) = \mathbb{U}$ and $l(H) \simeq \mathbb{Z}$, and which is an elementary substructure of G.*

Proof. Since G is discrete, it contains a smallest nontrivial c-convex subgroup $\langle 1_G \rangle$, furthermore, $\langle 1_G \rangle$ is a pure subgroup of uw(G), it is the smallest nontrivial convex subgroup of uw(G). Since G is c-regular, uw(G) is regular and uw$(G)/\langle 1_G \rangle$ is divisible. Denote by W the divisible hull of $\langle z_G \rangle \oplus \langle 1_G \rangle$ within uw(G). W is a pure subgroup of uw(G) and $W/\langle 1_G \rangle$ is the divisible hull of $(\langle z_G \rangle \oplus \langle 1_G \rangle)/\langle 1_G \rangle$ within uw$(G)/\langle 1_G \rangle$ (which is divisible), hence $W/\langle 1_G \rangle \simeq \mathbb{Q}$. Furthermore, W contains a unique proper convex subgroup: $\langle 1_G \rangle$, it follows that W is a regular subgroup of uw(G). Since W is pure in uw(G) and it contains the same first positive element 1_G, W is an elementary substructure of uw(G), and it still remains an elementary substructure in a language augmented with a predicate consisting of the element z_G. Set $H = W/\langle z_G \rangle$, by Theorem 4.1 of [2], H is an elementary substructure of G. The linear part of H is $\langle 1_G \rangle$, which is isomorphic to $\mathbb{Z}$, and $K(H) = (W/\langle z_G \rangle)/\langle 1_G \rangle \simeq \mathbb{Q}/\langle 1 \rangle = \mathbb{U}$. □

The proof of this proposition shows that the first order theory of the infinite c-regular discrete cyclically ordered groups is equal to the first order theory of the abelian cyclically ordered groups whose linear part is isomorphic to $\mathbb{Z}$ and with quotient isomorphic to $\mathbb{U}$, in other words, to the first order theory of the abelian linearly ordered groups W which contain an unique convex subgroup, isomorphic to $\mathbb{Z}$, and such that the quotient group is isomorphic to the ordered group $\mathbb{Q}$ and where the classe of 1 is fixed. Note that W is isomorphic to a subgroup of $\mathbb{Q} \overrightarrow{\times} \mathbb{Q}$ which contains $\mathbb{Z} \overrightarrow{\times} \{0\}$ and such that $\{0\} \overrightarrow{\times} \mathbb{Z}$ is its only convex subgroup.

4.16 Definition. If G is discrete and not c-archimedean, then the first positive element 1_G of G is definable, we can assume that it lies in the language. For a prime p, $n \in \mathbb{N}^*$ and $k \in \{0, \dots, p^n - 1\}$, $D_{p^n,k}$ will be the formula:
$\exists x,\ R(0, x, 2x, \dots, (p^n - 1)x) \wedge p^n x = k1_G$.

4.17 Remark. *Note that G contains a p^k-torsion element whose class in $G/\langle 1_G \rangle$ is a p^k-th root of 1 if and only if G satisfies the formula $D_{p^k,0}$, hence it contains an element of torsion $n = p_1^{k_1} \cdots p_j^{k_j}$, whose class is a n-th root of 1 in $G/\langle 1_G \rangle$ if and only if it satisfies the formulas $D_{p_1^{k_1},0}, \dots, D_{p_j^{k_j},0}$. In particular, G contains a subgroup isomorphic to $\mathbb{U}$ if and only if it satisfies the formulas $D_{p^k,0}$ for every prime number p and $k \in \mathbb{N}^*$.*

4.18 Lemma. *Assume that G is c-regular discrete and infinite. For every p prime, $n \in \mathbb{N}^*$ and $k \in \{0, \dots, p^n - 1\}$, G satisfies the formula $D_{p^n,k}$ if and only if uw(G) satisfies the formula $DD_{p^n,k}$, where the fixed element of the language consists of z_G.*

Proof. The formula $R(0, x, 2x, \dots, (p^n - 1)x) \wedge p^n x = k1_G$ says that the jx's where $1 \leq j \leq p^n - 1$ "do not turn full circle", but $p^n x$ "turns full circle", and is equal to $k1_G$. This is equivalent to saying that the image of x in uw(G) satisfies $p^n x = z_G + k1_G$. □

4.19 Lemma. *Assume that G is discrete, is not c-archimedean, and that $G/\langle 1_G \rangle$ is c-divisible.*
1) For every p prime and $n \in \mathbb{N}^*$, there exists exactly one integer $k \in \{0, \dots, p^n - 1\}$ such that $D_{p^j,k}$ holds within G.
2) Let p be a prime, $n \in \mathbb{N}^*$, and $k \in \{0, \dots, p^n - 1\}$ such that $D_{p^j,k}$ holds in G. We express k as $a_0 + a_1 p + \dots + a_{n-1} p^{n-1}$, where, for $1 \leq j \leq n$, $a_j \in \{0, \dots, p - 1\}$. Then for every $j \leq n$, and $k_j = a_0 + \dots + a_{j-1} p^{j-1}$, D_{p^j,k_j} holds within G.
3) There exists an unique sequence (φ_p), where p runs over the increasing sequence of prime numbers and the φ_p's are mappings from $\mathbb{N}^*$ into $\{0, \dots, p - 1\}$ such that for every prime p, every $n \in \mathbb{N}^*$ and every $k \in \{0, \dots, p^n - 1\}$, $D_{p^n,k}$ holds within G if and only if $k = \varphi_p(1) + \varphi_p(2)p + \dots + \varphi_p(n)p^{n-1}$.

Proof. 1) and 2) follow from Lemma 4.18, Lemma 4.6, and from the fact that G is c-divisible if and only if uw(G) is divisible. 3) is a consequence of 1) and 2). □

4.20 Definition. Assume that G is discrete, non-c-archimedean, and that $G/\langle 1_G\rangle$ is divisible. For every prime p, let φ_p be the mapping from $\mathbb{N}^*$ into $\{0,\dots,p-1\}$, defined in 3) of Lemma 4.19, the sequence (φ_p), will be called the *characteristic sequence* of G.

4.21 Remark. *Note that G contains a subgroup which is isomorphic to $\mathbb{U}$ if and only if for every prime p the mapping φ_p is the 0 mapping.*

4.22 Proposition. *Let C_1 and C_2 be two non-c-archimedean discrete cyclically ordered groups such that $C_1/\langle 1_{C_1}\rangle \simeq \mathbb{U} \simeq C_2/\langle 1_{C_2}\rangle$. The following conditions are equivalent.*
C_1 and C_2 are isomorphic
C_1 and C_2 satisfy the same formulas $D_{p^n,k}$
$C_1 \equiv C_2$.
The characteristic sequence of C_1 is equal to the characteristic sequence of C_2.

Proof. Trivially, if C_1 and C_2 are isomorphic, then they are elementarily equivalent, and if they are are elementarily equivalent then they satisfy the same formulas $D_{p^n,k}$, since these are first order formulas. Now, assume that they satisfy the same formulas $D_{p^n,k}$. By Lemma 4.18, $\mathrm{uw}(C_1)$ and $\mathrm{uw}(C_2)$ satisfy the same formulas $DD_{p^n,k}$. Since $C_1/\langle 1_{C_1}\rangle \simeq \mathbb{U} \simeq C_2/\langle 1_{C_2}\rangle$, we have $\mathrm{uw}(C_1)/\langle 1_{C_1}\rangle \simeq \mathbb{Q} \simeq \mathrm{uw}(C_2)/\langle 1_{C_2}\rangle$, and by Proposition 4.7, $\mathrm{uw}(C_1)$ and $\mathrm{uw}(C_2)$ are isomorphic in a language containing the element z_C. Consequently, C_1 and C_2 are isomorphic. Finally, by Lemma 4.19 the last condition is equivalent to the second one. □

4.23 Theorem. *Any two non-c-archimedean c-regular discrete cyclically ordered groups are elementarily equivalent if and only if they satisfy the same formulas $D_{p^n,k}$.*

Proof. Since these formulas are first order formulas, if two c-regular discrete cyclically ordered groups are elementarily equivalent then they satisfy the same formulas $D_{p^n,k}$. Conversely, assume that G and G' are c-regular, discrete, and they satisfy the same formulas $D_{p^n,k}$. Let H be the subgroup of G and H' be the subgroup of G' defined in the proof of Proposition 4.15. Since they are elementary substructures, they satisfy the same formulas $D_{p^n,k}$, hence by Proposition 4.22, they are isomorphic. Hence G and G' have elementary substructures which are isomorphic, so they are elementarily equivalent. □

4.24 Corollary. *Any two c-regular discrete non-c-archimedean cyclically ordered groups are elementarily equivalent if and only if their characteristic sequences are equal.*

Proof. This is a consequence of Theorem 4.23 and of the equivalence between the second and the fourth condition of Proposition 4.22, since if G is discrete, then G is c-regular if and only if $G/\langle 1_G\rangle$ is divisible. □

4.25 Corollary. *If G is c-regular and discrete, then the first order theory of G is uniquely determined by the subgroup H defined in Proposition 4.15.*

Proof. By Theorem 4.23, the first order theory of G is uniquely determined by the formulas $D_{p^n,k}$, now apply Proposition 4.22. □

4.26 Proposition. *For every sequence (φ_p) of functions from $\mathbb{N}^*$ into $\{0,\dots,p-1\}$, where p runs over the increasing sequence of prime numbers, there exists a non-c-archimedean discrete c-regular cyclically ordered group H whose characteristic sequence is (φ_p).*

Proof. Let $(p_n)_{n\in\mathbb{N}^*}$ be the increasing sequence of prime numbers. In the linearly ordered group $\mathbb{Q}\overrightarrow{\times}\mathbb{Q}$ we set $z_H = (1,0)$ and $1_H = (0,1)$. For $n \in \mathbb{N}^*$ and $i \in \{1,\dots,n\}$, let $k_i = \varphi_{p_i}(1) + \varphi_{p_i}(2)p + \cdots + \varphi_{p_i}(n)p_i^{n-1}$. According to the chinese remainder theorem, there exists $k \in \{0,1,\dots,(p_1\cdots p_n)^n - 1\}$ such that for $i \in \{1,\dots,n\}$ k is congruent to k_i modulo p_i^n. Denote by W_n the subgroup of $\mathbb{Q}\overrightarrow{\times}\mathbb{Q}$ generated by $\left(\dfrac{1}{(p_1\cdots p_n)^n}, \dfrac{k}{(p_1\cdots p_n)^n}\right)$ and 1_H. Hence $(p_1\cdots p_{n-1})^n \left(\dfrac{1}{(p_1\cdots p_n)^n}, \dfrac{k}{(p_1\cdots p_n)^n}\right) = \left(\dfrac{1}{p_n^n}, \dfrac{k}{p_n^n}\right)$ is congruent to $\left(\dfrac{1}{p_n^n}, \dfrac{k_n}{p_n^n}\right)$ modulo 1_H, the same holds for the other $i \in \{1,\dots,n\}$. It follows that W_n

satisfies the formulas $DD_{p_i^n,k_i}$, where $i \in \{1, \dots, n\}$. The sequence $(W_n)_{n\in\mathbb{N}^*}$ is an increasing sequence of subgroups of $\mathbb{Q}\overrightarrow{\times}\mathbb{Q}$, denote by W the union of all the W_n's. One can prove that z_H is divisible modulo 1_H within W, hence $W/\langle z_H\rangle \simeq \mathbb{Q}$, and that W is discrete with first element 1_H. Consequently, $H = W/\langle z_H\rangle$ is a discrete c-regular subgroup such that $K(H) \simeq \mathbb{U}$, $l(H) \simeq \mathbb{Z}$, the sequence (φ_p) is its characteristic sequence. □

In Remark 4.21 we characterized the discrete c-regular cyclically ordered groups whose torsion subgroups are isomorphic to $\mathbb{U}$, we can also characterize those which are divisible. The following result proves that this case can be seen as the opposite case.

4.27 Proposition. *Let G be a discrete c-regular cyclically ordered group, p be a prime and φ_p be the mapping from $\mathbb{N}^*$ to $\{0, \dots, p-1\}$ defined in 4.20. The following conditions are equivalent.*
1) G is p-divisible.
2) 1_G is p-divisible.
3) $\varphi_p(1) \neq 0$.
4) G doesn't contain any p-torsion element.

Proof. We have trivially: 1) ⇒ 2) and 4) ⇒ 3).

Assume that $\varphi_p(1) \neq 0$. Hence, for every $n \in \mathbb{N}^*$, p doesn't divide k_{p^n}. By Bezout identity, there exist two integers u and v such that $up^n + vk_{p^n} = 1$. We know that there exists x in G such that $p^n x = k_{p^n} 1_G$, hence $p^n(vx) = vk_{p^n} 1_G = (1 - up^n)1_G$, and $p^n(vx + u)1_G = 1_G$, wich proves that 1_G is p-divisible. Let $y \in G$, since G is c-regular, the class of y modulo $\langle 1_G\rangle$ is p^n-divisible, consequently there exists $z \in G$ such that $p^n z - y \in \langle 1_G\rangle$, hence there exists an integer k such that $p^n z - k1_G = y$. Since 1_G is p^n-divisible, it follows that y is p^n-divisible. We proved: 3) ⇒ 2) ⇒ 1).

Assume that G contains a p-torsion element y, then $2y, \dots, (p-1)y$ are p-torsion elements. By taking the lowest element of the set $\{y, 2y, \dots, (p-1)y\}$ ordered by $<_0$ instead of y, we can assume: $R(0, y, 2y, \dots, (p-1)y)$, and $py = 0$, so, by the definition of k_p, we have: $k_p = 0$, that is, $\varphi_p(1) = 0$. We proved 3) ⇒ 4). Assume furthermore that 1_G is p-divisible, hence there exists z such that $pz = 1_G$. Let $j \in \{0, \dots, p-1\}$ be such that $R(jy, z, (j+1)y)$ holds (z is not a multiple of y because it is not p-torsion), by setting $z' = z - jy$ we have $R(0, z', y)$, wich implies $R(0, z', 2z', \dots, pz')$, hence $R(0, z', 1_G)$, a contradiction, because 1_G is minimal. It follows that 1_G is not p-divisible, hence G is not p-divisible. Consequently, 2) ⇒ 4), and the proposition is proved. □

4.28 Proposition. *Let G be a non-c-archimedean cyclically ordered group, then G is c-regular discrete divisible if and only if there exists a discrete cyclic order R on the group $\mathbb{Q}$ such that $(\mathbb{Q}, R)$ is an elementary subextension of G.*

Proof. First assume that G is c-regular divisible and discrete, then according to Proposition 4.15, there exists an elementary subextension H of G such that $l(H) \simeq \mathbb{Z}$ and $K(H) \simeq \mathbb{U}$. $\mathbb{Q} \cdot 1_H$ is a subgroup of H, because H is divisible. Let $x \in H$, since $H/\langle 1_H\rangle \simeq \mathbb{U}$, there exist integers n and k such that $nx = k1_H$. Since H is divisible and torsion-free, (by Proposition 4.27), we have: $x = \frac{k}{n}1_H$, which proves: $H = \mathbb{Q} \cdot 1_H$. In order to prove the other implication, it suffices to show that if $H = (\mathbb{Q}, R)$ where R is a discrete cyclic order, then H is c-regular. Now, one can prove that $l(H) = \langle 1_H\rangle$, hence it is regular, and that $K(H) \simeq \mathbb{U}$, consequently, by Theorem 3.7, H is c-regular. □

4.29 Definitions. For every prime p, let φ_p be a mapping from $\mathbb{N}^*$ into $\{0, \dots, p-1\}$. For every prime p and every $n \in \mathbb{N}^*$, denote by N_{p^n,φ_p} the set $p^n\mathbb{N}^* - (\sum_{k=1}^{n} p^{k-1}\varphi_p(k))$. The set of the N_{p^n,φ_p}'s will be called the *family of subsets of* $\mathbb{N}^*$ *characteristic of* (φ_p). The family of subsets of $\mathbb{N}^*$ characteristic of the characteristic sequence of G will be called the *family of subsets of* $\mathbb{N}^*$ *characteristic of* G.

4.30 Proposition.
1) For every ultrafilter U on $\mathbb{N}^*$, there exists one and only one sequence (φ_p), where p runs over the increasing sequence of all primes and φ_p is a mapping from $\mathbb{N}^*$ into $\{0, \dots, p-1\}$, such that U contains the family of subsets of $\mathbb{N}^*$ characteristic of (φ_p).
2) For every (φ_p), where p runs over the increasing sequence of prime numbers and φ_p is a mapping from $\mathbb{N}^*$ into $\{0, \dots, p-1\}$, there exists a non principal ultrafilter U on $\mathbb{N}^*$ containing the family of subsets of $\mathbb{N}^*$ characteristic of (φ_p).

Proof. 1) By a property of ultrafilters, if $A \in U$ and $A = A_1 \cup \cdots \cup A_n$ is a finite partition of A, then U contains exactly one of the subsets A_k, $1 \le k \le n$. Now, for p prime, $n \in \mathbb{N}^*$, and $a_1, \ldots, a_n$ in $\{0, \ldots, p-1\}$, $p^{n+1}\mathbb{N}^* - (\sum_{k=1}^n a_k p^{k-1} + jp^n)$, $(0 \le j \le p-1)$ is a finite partition of $p^n\mathbb{N}^* - (\sum_{k=1}^n a_k p^{k-1})$, the result follows by induction.

2) Let p be fixed and $0 < n_1 < n_2 < \cdots < n_k$, then $N_{p^{n_1},\varphi_p} \cap \cdots \cap N_{p^{n_k},\varphi_p} = N_{p^{n_k},\varphi_p}$, and if $1 < p_1 < p_2 < \cdots < p_k$ are prime, then by the chinese remainder theorem $N_{p_1^{n_1},\varphi_{p_1}} \cap \cdots \cap N_{p_k^{n_1},\varphi_{p_k}}$ is infinite. It follows that the intersection of every finite family of sets N_{p^n,φ_p} has infinite cardinal. We join the elements of the filter of cofinite subsets to this family, then the property of non-empty intersection still remains satisfied, hence there exists an ultrafilter containing all the sets N_{p^n,φ_p} and the filter of cofinite subsets. This ultrafilter is not principal, because it contains all the cofinite subsets. □

4.31 Definition. Let U be an ultrafilter on $\mathbb{N}^*$, the family of subsets defined in 1) of Proposition 4.30 will be called *the family of subsets of* $\mathbb{N}^*$ *defined by* U.

4.32 Theorem.
1) Let U be an non principal ultrafilter on $\mathbb{N}^*$, C be the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo U, p be a prime, $n \in \mathbb{N}^*$ and $k \in \{0, \ldots, p^n - 1\}$. Then C satisfies the formula $D_{p^n,k}$ if and only if $p^n\mathbb{N}^* - k \in U$.
2) Let U be a non principal ultrafilter on $\mathbb{N}^*$, C be the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo U. The family of subsets of $\mathbb{N}^*$ defined by U is equal to the family of subsets of $\mathbb{N}^*$ characteristic of C.
3) Let U_1 and U_2 be two non principal ultrafilters on $\mathbb{N}^*$, C_1 (resp. C_2) be the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo U_1 (resp. U_2). Then $C_1 \equiv C_2$ if and only if the family of subsets of $\mathbb{N}^*$ defined by U_1 is the same as the family of subsets of $\mathbb{N}^*$ defined by U_2.

Proof. 1) C satisfies $D_{p^n,k}$ if and only if the set of all integers m such that $\mathbb{Z}/m\mathbb{Z}$ satisfies $D_{p^n,k}$ belongs to U. Let $m \in \mathbb{N}^*$, there exists some $j \in \{0, \ldots, p^n - 1\}$ such that $m \in p^n\mathbb{N}^* - j$, then there exists $x \in \mathbb{N}^*$ such that $m = xp^n - j$, and we have $p^n x = m + j$, and $0 < x < 2x < \cdots < (p^n - 1)x < m$. In $\mathbb{Z}/m\mathbb{Z}$, this is equivalent to $R(0, x, 2x, \ldots, (p^n - 1)x)$ and $p^n x = j$, hence $\mathbb{Z}/m\mathbb{Z}$ satisfies $D_{p^n,j}$, and for $j' \neq j$, $0 \le j' \le p^n - 1$, $\mathbb{Z}/m\mathbb{Z}$ does not satisfy $D_{p^n,j'}$, since every discrete c-regular cyclically ordered group satisfies exactly one relation $D_{p^n,j}$, for fixed p and n. It follows that $\mathbb{Z}/m\mathbb{Z}$ satisfies $D_{p^n,k}$ if and only if $m \in p^n\mathbb{N}^* - k$. Consequently, C satisfies the formula $D_{p^n,k}$ if and only if $p^n\mathbb{N}^* - k \in U$.

2) Follows from 1) and from Lemma 4.19.

3) Follows from 2) and from Corollary 4.24. □

4.33 Corollary. *1) If G is infinite, c-regular and discrete, and U is a non principal ultrafilter on $\mathbb{N}^*$, then G is elementarily equivalent to the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo U if and only if the family of subsets of $\mathbb{N}^*$ characteristic of G is the same as the family of subsets of $\mathbb{N}^*$ defined by U.*
2) G is c-regular and discrete if and only if there exists an ultrafilter on $\mathbb{N}^*$ such that G is elementarily equivalent to the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo U.

Proof. 1) Follows from Theorem 4.32 and from Corollary 4.24.
2) If there exists an ultrafilter on $\mathbb{N}^*$ such that G is elementarily equivalent to the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo U, then it is c-regular and discrete, because these are first order properties. Conversely, assume that G is c-regular and discrete. If G is finite, then there exists $n_0 \in \mathbb{N}^*$ such that $G = \mathbb{Z}/n_0\mathbb{Z}$, then G is isomorphic to the ultraproduct of the cyclically ordered groups $\mathbb{Z}/n\mathbb{Z}$ modulo the principal ultrafilter generated by $\{n_0\}$. If G is infinite, the result follows from 1). □

4.34 Proposition. *For every subgroup S of $\mathbb{U}$, there exists an ultraproduct of finite cyclic groups whose torsion subgroup is S.*

Proof. This proposition is a consequence of Theorem 4.32. □

4.35 Theorem. *1) The class of all discrete c-regular cyclically ordered abelian groups is the smallest elementary class which contains all finite cyclic groups.*
2) The class of all discrete regular cyclically ordered abelian groups is the smallest elementary class which contains all cyclic groups. If G belongs to this class, then either G is a linearly cyclically ordered group which is elementarily equivalent to $\mathbb{Z}$, or G is c-regular and elementarily equivalent to an ultraproduct of finite cyclic groups.

Proof. Follows from what we just proved, from Theorem 3.8 and from the properties of regular linearly ordered abelian groups. □

4.36 Proposition. *Every linearly ordered abelian group T, having a smallest proper convex subgroup, embeds into a minimal regular group T' which is contained into its divisible hull. If T is dense, then T' is dense, if T is discrete, then T' is discrete. In particular, every discrete linearly ordered abelian group embeds into a minimal discrete regular group which is contained in its divisible hull.*

Proof. Denote by C the smallest proper convex subgroup of T, then there exists a cocycle θ such that $T \simeq (T/C)\overrightarrow{\times}_\theta C$. We embed (T/C) into its divisible hull and we extend T in the same way as in Proposition 4.4 (note that this embedding need not be unique) the details are left to the reader. □

4.37 Proposition. *Every dense cyclically ordered group having a smallest proper c-convex subgroup embeds into a minimal c-regular cyclically ordered group which is contained in its divisible hull.*

Proof. By Proposition 4.36, uw(G) embeds into a minimal regular group contained in its divisible hull, we consider the wound-round associated to its divisible hull and z_G. By Theorem 3.12, this extension is minimal. □

4.38 Proposition. *Assume that G is ω_1-saturated and $K(G)$ is infinite.*
1) Every class of G modulo $l(G)$ contains a divisible element.
2) Every class of uw(G) modulo $l(G)$ contains a divisible element.
3) Let p be a prime such that G is not p-divisible, then every class of G modulo $l(G)$ contains an element which is not p-divisible and which is n-divisible for all integers n such that n and p are coprime.
4) Let p be a prime such that G is not p-divisible, then every class of uw(G) modulo $l(G)$ contains an element which is not p-divisible and is n-divisible for all integers n such that n and p are coprime.

Proof. 1) Note that by Propositon 6.3 a) of [2], $K(G) \simeq \mathbb{K}$, since G is ω_1-saturated and $K(G)$ is infinite. Let $x_0 \in G$, assume first that $\overline{x_0} \notin \mathbb{U}$, say $\overline{x_0} = e^{i\alpha}$ for some irrational element $\alpha \in [0,1[$. For every $n \in \mathbb{N}^*$, let m_n be the integer such that $2\pi\frac{m_n}{n} \leq \alpha < 2\pi\frac{m_n+1}{n}$. For any $n \in \mathbb{N}^*$, $x \in G$ and $\theta \in [0,1[$ such that $\bar{x} = e^{i\theta}$, we have: $\frac{2\pi}{n} < \theta < \frac{2\pi}{n-1} \Rightarrow R(0,x,2x,\ldots,(n-1)x) \& \neg R(0,x,2x,\ldots,(n-1)x,nx)) \Rightarrow \frac{2\pi}{n} \leq \theta \leq \frac{2\pi}{n-1}$, hence one can see that $2\pi\frac{m_n}{n} < \theta < 2\pi\frac{m_n+1}{n}$ implies

$$[\forall y, (R(0,y,2y,\ldots,(n-1)y) \& \neg R(0,y,2y,\ldots,(n-1)y,ny)) \Rightarrow R(0,m_ny,x)]$$

$$[\exists y, R(0,y,2y,\ldots,(n-1)y) \& \neg R(0,y,2y,\ldots,(n-1)y,ny) \& R(0,x,(m_n+1)y)]$$

which implies $2\pi\frac{m_n}{n} \leq \theta \leq 2\pi\frac{m_n+1}{n}$. In any case, $\bar{x} = \bar{x_0}$ if and only if x satisfies all of these first order formulas. The element x is divisible if and only if $\exists y,\ ny = x$ holds for every $n \in \mathbb{N}^*$. We get a countable type. Take a finite subset of this type, and let m be lcm of all the n's that appear in some formula $\exists y,\ ny = x$. Then pick an element y of $\frac{1}{m}\overline{x_0}$, and $x = my$, x satisfies this finite set of formulas. By ω_1-saturation, the type is satisfied, and $\overline{x_0}$ contains a divisible element.

Now assume that $\overline{x_0} \in \mathbb{U}$, say $\overline{x_0} = e^{2ik\pi/n}$, where k and n are coprime, $k < n$. In $\mathbb{K}$, $\overline{x_0}$ is characterized by $\overline{x_0} \neq 0,\ 2\overline{x_0} \neq 0, \ldots, (n-1)\overline{x_0} \neq 0, n\overline{x_0} = 0$, and for $2 \leq l \leq n-1$, one of the relations $R(0,\overline{x_0},l\overline{x_0})$ or $R(0,l\overline{x_0},\overline{x_0})$. We are going to find sufficient conditions in order that these formulas be satisfied. For $n\overline{x} = 0$, i.e. $x \in l(G)$, we take all the formulas $R(0,nx,knx)$, for $k \geq 2$. For $l\overline{x} \neq 0$, i.e. $lx \notin l(G)$, we take the smallest integer k_l such that $R(0,k_l l\overline{x_0},l\overline{x_0})$ holds, and we take the formula $R(0,k_l lx,lx)$. x is divisible if and only if $\exists y,\ qy = x$ holds for every $q \in \mathbb{N}^*$. This type is countable. Take a finite subset of this type, let m be the lcm of the q's such that $\exists y,\ qy = x$ appear, and $y \in \frac{1}{m}\overline{x_0}$. Then $mny \in l(G)$. If

mny does not belong to the positive cone of G, we take $y' = y - mny$, then $mny' = mny - mnmny$ belongs to positive cone of G. Set $x = my'$, and the finite subset of formulas is satisfied. By ω_1-saturation, the type is satisfied.
2) The elements of $\mathrm{uw}(G)$ can be expressed as (n, x), for some $n \in \mathbb{Z}$ and $x \in G$. In order to prove that (n, x) is divisible in $\mathrm{uw}(G)$, we can assume that $n \geq 0$, and it is sufficient to prove that (n, x) is divisible by every integer $m \geq n$. In this case, the divisor is some $(0, y)$ with $y \in G$. Then $m(0, y) = (n, x)$ is equivalent to: the sequence $y, 2y, \ldots, my$ "travels n times around" and furthermore $my = x$. $(k+1)y$ "being in the round which follows the round" of y is equivalent to: $R(0, (k+1)y, ky)$. Since m and the class of x are fixed, we know where are the jumps, hence we have a succession of $R(0, (k+1)y, ky)$ and $R(0, ky, (k+1)y)$ which is well-defined. The class of (n, x) modulo $l(G)$ containing a divisible element can be rendered by: there exists z such that $z - x \in l(G)$ and (n, z) is divisible. Finally $z - x \in l(G)$ is equivalent to an infinite set of formulas $R(0, z - x, q(z - x))$, for $q \geq 2$. This type is countable. In order to get an element which satisfies a finite subset of this type, we consider the lcm of all the m's which appear in this subset (we still denote it by m), we pick $(0, t)$ in the class of $\frac{1}{m}(n, x)$ modulo $l(G)$ (which exists since $\mathrm{uw}(G)/l(G)$ is divisible), then $z = mt$, satisfies the finite subset of formulas. The only formulas which appear belong to the language of cyclically ordered groups. By ω_1-saturation, the type is satisfied, which gives us the required divisible element.
3) and 4) Let x_0 be an element of G which is not p-divisible, and we consider the countable type $\forall y,\ py \neq x$, and $\exists y,\ ny = x$, for every positive integer n prime to p. Consider a finite subset of this type, and let m be the lcm of all the n's prime to p which appear. Then $x = mx_0$ satisfies this finite subset. By ω_1-saturation, G contains an element x which is not divisible by p and which is divisible by every integer which is prime to p. By 1), there exists y which is divisible and such that $\bar{y} = \bar{x}$, hence $x - y \in l(G)$ is not p-divisible and is divisible by every integer which is prime to p. Now, we add to $x - y$ some divisible element of a fixed class of G or of $\mathrm{uw}(G)$. □

4.39 Proposition. *Assume that G is c-regular, dense and ω_1-saturated. Then G contains a countable elementary substructure which is a c-archimedean group.*

Proof. Denote by H the divisible hull of $\langle z_G \rangle$ in $\mathrm{uw}(G)$. Since H is a pure subgroup of $\mathrm{uw}(G)$, we have for every prime p: $[p]H \leq [p]\mathrm{uw}(G)$. By Lemma 4.1, there exists a countable subgroup M of $\mathbb{R}$ containing a pure subgroup isomorphic to H and such that for every prime p we have $[p]M = [p]\mathrm{uw}(G)$. We assume that $\mathrm{uw}(G)/l(G) = \mathbb{R}$ by letting 1 being the class of z_G; M is a subgroup of $\mathrm{uw}(G)/l(G)$ that we embed into $\mathrm{uw}(G)$ in the following way. The image of 1 is z_G. Every $e^{k_{n_j}}$ is a class modulo $l(G)$, its image is some representative which is not divisible by p_n and is divisible by every integer which is prime to p_n. This element exists by Proposition 4.38. So, every $e^{k_{n_j}}$ satisfies the same divisibility relations within M as its image within $\mathrm{uw}(G)$. By the definition of M its image is a pure sbgroup of $\mathrm{uw}(G)$. Now, this image is indeed an ordered subgroup, because the images of the $e^{k_{n_j}}$'s are the corresponding classes. M is a pure subgroup of $\mathrm{uw}(G)$ such that, for every prime p, $[p]M = [p]\mathrm{uw}(G)$, furthermore M is regular since it is archimedean. It follows that $M \prec \mathrm{uw}(G)$. Consequently, $M/\langle z_G \rangle$ is a countable elementary substructure of G, which is a c-archimedean subgroup. □

4.40 Proposition. *Every discrete cyclically ordered group embeds into a c-regular discrete cyclically ordered group contained in its divisible hull.*

Proof. By Proposition 4.36, $\mathrm{uw}(G)$ embeds into a minimal regular group contained in its divisible hull, we consider the wound-round associated to this divisible hull and z_G. The minimality follows from Theorem 3.12. □

5 With or without the cyclic order predicate?

It seems to be natural and usefull for studying $\mathbb{R}$ to consider its linear order, what about the study of $\mathbb{K}$ and its cyclic order? More generally, to determine the contribution of the cyclic order in the study of the c-regular cyclically ordered groups, we focus on the theory of those groups either in the language for groups without a predicate for the cyclic order or with the cyclic order predicate. In the strict language of groups, we know that any two abelian groups are elementarily equivalent if and only if they have the same invariants of Szmielew (cf. [Sz 55]). Here the torsion subgroups of our groups embed into $\mathbb{U}$, hence the Szmielew invariants reduce to the Zakon invariants and the torsion subgroup. We can reformulate the second part of Theorem 4.12 as follows.

5.1 Proposition. *Any two dense c-regular cyclically ordered abelian groups are elementarily equivalent if and only if they are elementarily equivalent groups.*

The same holds for any two dense regular linearly ordered groups.

With respect to the discrete c-regular cyclically ordered groups, all groups having a fixed torsion subgroup have the same Szmielew invariants, it follows:

5.2 Proposition. *Any two infinite discrete c-regular cyclically ordered abelian groups having a fixed torsion subgroup are elementarily equivalent in the language of groups.*
Two infinite discrete c-regular cyclically ordered abelian groups having a fixed torsion subgroup need not be elementarily equivalent in the language of cyclically ordered groups.

Indeed, by Theorem 4.23, they are elementarily equivalent in the language of cyclically ordered groups if and only if they satisfy the same formulas $D_{n,k}$. Now, by Lemma 4.19 and Proposition 4.26, they need not satisfy the same formulas $D_{n,k}$.

In the case of linearly ordered groups, we have the following.

5.3 Proposition. *If G is a torsion-free abelian group, then there exists a structure of regular linearly ordered group on G with a smallest convex proper subgroup if and only if for every prime p the cardinal of every maximal family of pairwise p-incongruent elements is at most equal to the cardinal of $\mathbb{R}$. Moreover, there exists a structure of discrete regular linearly ordered group on G if and only if, for every p prime, $[p]G = 1$ and there exists an element which is not divisible by any prime p.*

Proof. Assume that for every prime p the cardinal of every maximal family of pairwise p-incongruent elements is at most equal to the cardinal of $\mathbb{R}$ and for each prime p let I_p be such a family. Denote by H the divisible hull within G, of the subgroup generated by the union of all the subsets I_p, where p run over the set of all prime numbers. The cardinal of H is at most equal to the cardinal of $\mathbb{R}$, hence the cardinal of a basis of the $\mathbb{Q}$-vector space $\tilde{H}$, where $\tilde{H}$ is the divisible hull of H, is at most equal to the cardinal of $\mathbb{R}$. We embed this basis into a rationally independent subset of $\mathbb{R}$, so we get an embedding of the group $\tilde{H}$ into $\mathbb{R}$, hence we have an embedding of H into $\mathbb{R}$. This embedding gives rise to a linear order on H, such that H is an archimedean group. Now, the quotient group G/H is torsion-free, hence there is a compatible linear order on G/H. Let H' be a set of representatives of the equivalence classes of G modulo H, we deduce a one-to-one mapping from G onto $H' \times H$. The lexicographic order on $H' \times H$ induces a linear order on G, and in the same way as in [3], one can check that this order is compatible. Furthermore, G/H is divisible and H is archimedean, hence G is regular. Conversely, let C the smallest convex proper subgroup of G. Since G/C is divisible, for every prime p there is a maximal family I_p of pairwise p-incongruent elements which is contained in C. Now, C is archimedean, hence $\mathrm{card}(C) \leq \mathrm{card}(\mathbb{R})$, so $\mathrm{card}(I_p) \leq \mathrm{card}(\mathbb{R})$. By [Za 61, Corollary 1.8] every maximal family of pairwise p-incongruent elements has the same cardinal, hence the cardinal of every maximal family of pairwise p-incongruent elements is at most equal to the cardinal of $\mathbb{R}$.

If for every prime p we have $[p]G = 1$ and there exists an element g_0 which is not divisible by any prime, we let C be the subgroup generated by g_0, it is a pure subgroup and G/C is divisible. We conclude in the same way as above. The converse is trivial. □

Now assume that G is a torsion-free abelian group such that there exists a prime p and a maximal family I_p of elements which are pairwise p-incongruent and of cardinal greater than the cardinal of $\mathbb{R}$ and, for every prime $q \neq p$, G is q-divisible. Let H be the divisible hull within G of the subgroup generated by I_p. Let C be a pure subgroup of H distinct from H. Hence one of the elements of the family I_p does not belong to C, hence H/C is not divisible. Since the cardinal of H is greater than the cardinal of $\mathbb{R}$, there cannot exist a structure of archimedean linearly ordered group on H. Hence for every linear order H contains a proper convex subgroup C, now H/C is not divisible, so, there does not exist any structure of regular linearly ordered group on H. For the same reason, there does not exist any structure of c-regular cyclically ordered group.

Turning to the cyclically ordered case we get:

5.4 Proposition. *If G is a torsion-free abelian group, then there exists a structure of c-regular cyclically ordered group on G with a smallest convex proper subgroup if and only if for every prime p the cardinal of every maximal family of pairwise p-incongruent elements is at most equal to the cardinal of $\mathbb{R}$. Moreover, there exists a structure of discrete c-regular cyclically ordered group on G if and only if, for every p prime, $[p]G = 1$ and there exists an element which is not divisible by any prime p.*

Proof. Leg G be an abelian group such that for every prime p the cardinal of every maximal family of pairwise p-incongruent elements is at most equal to the cardinal of $\mathbb{R}$ and for each prime p let I_p be such a family. Let H be the divisible hull within G of the subgroup generated by the union of all the I_p's (hence the cardinal of H is at most equal to the cardinal of $\mathbb{R}$), let A be a $\mathbb{Q}$-vectorial subspace of the divisible group G/H, of cardinal at most equal to the cardinal of $\mathbb{R}$, and let B be a subspace such that $G/H = A \oplus B$. Let $a \in A \backslash B$, $L := \{x \in G \mid x + H \in B \oplus \mathbb{Z}a\}$. Set $A = \mathbb{Q}a \oplus A'$, hence: $G/L \simeq \mathbb{U} \oplus A'$. So G/L embeds into $\mathbb{K}$ (the restriction of this embedding to U being identity), we equip L with a regular linear order, and applying [2, Lemma 5.4] we get a structure of c-regular cyclically ordered group on G. The remainder of the proof is similar to the proof of Proposition 5.3. □

We know that an abelian group is orderable if and only if it is torsion-free, this is a consequence of its first order theory. According to [2, Theorem 5.8] being cyclically orderable is a consequence of the first order theory of a group. We are going to see that the class of infinite groups which can be equipped with a c-regular and discrete cyclic order and the class of infinite groups which can be equipped with a discrete regular linear order are not elementary.

If G is an infinite abelian group equipped with a structure of discrete c-regular cyclically ordered group, then its torsion subgroup embeds into $\mathbb{U}$, and G contains an element e which is not divisible by any prime p and such that the quotient group $G/\mathbb{Z}e$ is divisible. The fact that G contains an element which is not p-divisible for every prime p is not a first order property in the language of groups (the following examples will prove this assertion), while this is a consequence of a first order formula of the language of cyclically ordered group (the positive cone has a smallest element).

We give examples of groups which are elementarily equivalent to discrete c-regular cyclically ordered groups in the language of groups, but which cannot be equipped with a structure of discrete c-regular cyclically ordered group. Let a_n, $n \in \mathbb{N}^*$, be a rationally independent family of elements of $\mathbb{K}\backslash\mathbb{U}$, and G be the direct sum of the $\mathbb{Z}_{(p_n)}a_n$'s (where $(p_n)_{n\in\mathbb{N}^*}$ is the increasing sequence of all primes, and $\mathbb{Z}_{(p_n)}$ is the localization of $\mathbb{Z}$ at the ideal (p_n)). Every element of G is contained in a finite sum $\mathbb{Z}_{(p_{n_1})}a_{n_1} + \cdots + \mathbb{Z}_{(p_{n_k})}a_{n_k}$, hence it is p-divisible, for every prime p which is distinct from $p_{n_1}, \ldots, p_{n_k}$. Consequently, G does not contain any element which is not divisible by any prime, hence it cannot be equipped with a structure of discrete c-regular cyclically ordered group. However, as we recalled in Section 4.1, for every prime p, we have $[p]G = p$, hence G is elementarily equivalent in the language of groups to every discrete torsion-free c-regular cyclically ordered group. Note that G is also elementarily equivalent to $\mathbb{Z}$, but it cannot be equipped with a structure of discrete regular linearly ordered group.

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