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The Bogoliubov c-Number Approximation for Random Boson Systems

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We justify the Bogoliubov c-number approximation for the case of interacting Bose-gas in a homogeneous random media. To this aim we take into account occurrence of generalised (extended/fragmented) Bose-Einstein condensation in an infinitesimal band of low kinetic-energy modes, to generalise the c-number substitution procedure for this band of low-momenta modes.

1 Introduction

One of the key developments in the theory of the Bose gas, especially the theory of the low density gases currently at the forefront of experiment, is Bogoliubov’s 1947 analysis [2], [3] of the many-body Hamiltonian by means of a c-number substitution for the most relevant operators in the problem. These are the zero-momentum mode boson operators, namely $b_0 \to z$, $b_0^* \to z^*$. Later this idea triggered a more general The Approximating Hamiltonian Method [6]. Naturally, the appropriate value of $z$ has to be determined by some sort of consistency or variational principle, which might be complicated, but the concern is whether this sort of substitution is legitimate, i.e., error free.
The rigorous justification for this substitution, as far as calculating the pressure of interacting (superstable) boson gas is concerned, was done for the first time in the paper by Ginibre [10]. Later it was revised and essentially improved by Lieb-Seiringer-Yngvason (LSY) [15], [16]. In textbooks it is often said, for instance, that it is tied to the imputed "fact" that the expectation value of the zero-mode particle number operator $\hat{N}_0 = \hat{n}_0\hat{b}_0$ is of order $V$ = volume. This was the second Bogoliubov ansatz: the Bose-Einstein condensation (BEC) justifies the substitution [26].

As Ginibre pointed out, however, that BEC has nothing to do with it. The $z$ substitution still gives the right answer for any value of the Gibbs average of the operator $\hat{N}_0$. On the other hand, the zero-mode translation invariant condensation (the first Bogoliubov ansatz) plays a distinguished role in the Bogoliubov Weakly Imperfect Bose theory [26].

The problem of justification becomes delicate in a (bona fide) homogeneous random external potential; first of all because of the translation invariance breaking and secondly because of the problem with nature of the generalised BEC for this case even for the perfect Bose-gas [11], [12]. The aim of the present note is to elucidate this problem for interacting boson gas in a homogeneous random potential following the LSY method. The later allows to simplify and make more transparent the arguments of [13] versus the generalised condensation à la Van den Berg-Lewis-Pulé [24].

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2 The Bogoliubov c-Number Approximation

2.1 Imperfect Bose-Gas

Let interacting bosons of mass $m$ be enclosed in a cubic box $\Lambda = L \times L \times L \subset \mathbb{R}^3$ of the volume $V \equiv |\Lambda| = L^3$, with (for simplicity) periodic boundary conditions on $\partial \Lambda$: $t_\Lambda := (-\hbar^2\Delta/2m)_{p.b.c.}$. Let $u(x)$ be isotropic two-body
interaction with non-negative Fourier transformation:

\[ v(q) = \int_{\mathbb{R}^3} d^3x \ u(x)e^{-iq\cdot x}, \ u \in L^1(\mathbb{R}^3) \]

The second-quantized Hamiltonian of the imperfect Bose-gas acting as operator in the boson Fock space \( \mathcal{F} := \mathcal{F}_{\text{boson}}(\mathcal{H} = L^2(\Lambda)) \) is

\[
H_\Lambda = \sum_{k \in \Lambda^*} \varepsilon_k b_k^* b_k + \frac{1}{2} \sum_{k_1, k_2, q \in \Lambda^*} v(q) b_{k_1+q}^* b_{k_2-q}^* b_{k_2} b_{k_1}
\]

where (dual) set \( \Lambda^* = \{ k \in \mathbb{R}^3 : k_\alpha = 2\pi n_\alpha / L \text{ et } n_\alpha \in \mathbb{Z}, \alpha = 1, 2, 3 \} \) and \( \{ \varepsilon_k \}_{k \in \Lambda^*} = \text{Spec}(t_\Lambda) \). Here \( \{ \varepsilon_k = \hbar^2 k^2 / 2m \geq 0 \}_{k \in \Lambda^*} \) is the one-particle excitations spectrum. The perfect Bose-gas Hamiltonian and particle-number operators are

\[
T_\Lambda := \sum_{k \in \Lambda^*} \varepsilon_k b_k^* b_k , \quad N_k := b_k^* b_k , \quad N_\Lambda := \sum_{k \in \Lambda^*} N_k .
\]

Here \( \{ b_k^*, b_k \}_{k \in \Lambda^*} \) are boson creation and annihilation operators in the one-particle eigenstates (kinetic-energy modes) verifying the CCR

\[
[b_k, b_q^*] = \delta_{k,q} : \psi_k(x) = \frac{1}{\sqrt{V}} e^{ikx} \chi_\Lambda(x) \in \mathcal{H}, \ k \in \Lambda^* \]

\[
b_k := b(\psi_k) = \int_\Lambda dx \overline{\psi_k(x)} b(x), \ b_k^* = (b(\psi_k))^* \)

Here \( b^\#(x) \) are boson-field operators in the Fock space over \( \mathcal{H} \).

### 2.2 Grand-Canonical \((\beta, \mu)\)-Ensemble

Recall that the grand-canonical \((\beta, \mu)\)-state generated by \( H_\Lambda \) on algebra of observables \( \mathcal{A}(\mathcal{F}) \) [20], is define by

\[
(A)_{H_\Lambda} := \text{Tr}_{\mathcal{F}}(e^{-\beta(H_\Lambda-\mu N_\Lambda)} A) / \text{Tr}_{\mathcal{F}}e^{-\beta(H_\Lambda-\mu N_\Lambda)} , \ A \in \mathcal{A}(\mathcal{F}) .
\]

The grand-canonical pressure: \( p[H_\Lambda](\beta, \mu) := (\beta V)^{-1} \ln \text{Tr}_{\mathcal{F}}e^{-\beta(H_\Lambda-\mu N_\Lambda)} \) corresponds to the temperature \( \beta^{-1} \) and to the chemical potential \( \mu \).
Example 1. For the perfect Bose-gas $T_\Lambda$ one must put $\mu < 0$, then the expectation value of the particle number in mode $k$ is

$$\langle b_k^* b_k \rangle_{T_\Lambda} := \frac{1}{e^{\beta (\varepsilon_k - \mu)} - 1} , \quad \varepsilon_k \geq 0 .$$

The expectation value of the total density of bosons in $\Lambda$ is

$$\rho_\Lambda(\beta, \mu) := \frac{1}{V} \langle b_0^* b_0 \rangle_{T_\Lambda} + \frac{1}{V} \sum_{k \in \Lambda^* \setminus \{0\}} \langle b_k^* b_k \rangle_{T_\Lambda} = \frac{1}{V} \frac{1}{e^{-\beta \mu} - 1} + \rho_\Lambda(\beta, \mu)$$

Then the critical density (if finite) is defined by the limit:

$$\rho_c(\beta) := \lim_{\mu \uparrow 0} \lim_{\Lambda \uparrow \mathbb{R}^3} \rho_\Lambda(\beta, \mu) < \infty .$$

2.3 Conventional Bose-Einstein Condensation

For a fixed density $\rho$, let $\mu_\Lambda(\beta, \rho)$ be solution of the equation

$$\rho = \rho_\Lambda(\beta, \mu) \quad \Rightarrow \quad \rho \equiv \rho_\Lambda(\beta, \mu_\Lambda(\rho)) \quad (always \ exists!).$$

- low density: $\lim_\Lambda \mu_\Lambda(\rho < \rho_c(\beta)) = \mu_\Lambda(\rho) < 0$
- high density: $\lim_\Lambda \mu_\Lambda(\rho \geq \rho_c(\beta)) = 0$, and

$$\rho_0(\beta) = \rho - \rho_c(\beta) = \lim_\Lambda \frac{1}{V} \left\{ e^{-\beta \mu_\Lambda(\rho \geq \rho_c(\beta))} - 1 \right\}^{-1} \Rightarrow$$

$$\mu_\Lambda(\rho \geq \rho_c(\beta)) = - \frac{1}{V} \frac{1}{\beta (\rho - \rho_c(\beta))} + o(1/V) .$$

- Since $\varepsilon_k = \hbar^2 \sum_{j=1}^d (2\pi n_j/V^{1/3})^2/2m$, the BEC is in $k=0$-mode:

$$\lim_\Lambda \frac{1}{V} \left\{ e^{\beta (\varepsilon_{k=0} - \mu_\Lambda(\rho))} - 1 \right\}^{-1} = 0 ,$$

This type of condensation based on the concept of the one-level macroscopic occupation is known as the conventional zero-mode (or type I) BEC [9], [17].
2.4 Generalised Bose-Einstein Condensation

This type of condensation was predicted by Casimir [8] and elucidated by Van den Berg-Lewis-Pulé in [22], [23], [24].

Let \( \Lambda = L_1 \times L_2 \times L_3 = V^{\alpha_1} \times V^{\alpha_2} \times V^{\alpha_3} \), \( \alpha_1 \geq \alpha_2 \geq \alpha_3 > 0 \), and \( \alpha_1 + \alpha_2 + \alpha_3 = 1 \).

- The *Casimir box* (1968): Let \( \alpha_1 = 1/2 \), i.e. \( \alpha_{2,3} < 1/2 \).

Since \( \varepsilon_{k_{1,0,0}} = \hbar^2 (2\pi n_1/V^{1/2})^2 / 2m \sim 1/V \), then again the asymptotics of solution:

\[
\rho = \rho_{\Lambda}(\beta, \mu_{\Lambda}(\rho)) \Rightarrow \mu_{\Lambda}(\rho \geq \rho_c(\beta)) = -A/V + o(1/V), \ A \geq 0
\]

\[
\lim_{\Lambda} \left\{ \frac{1}{V} \left[ e^{-\beta \mu_{\Lambda}(\rho)} - 1 + \sum_{\varepsilon \in \{\Lambda^*: n_1 \neq 0, n_2 = n_3 = 0\}} \frac{1}{e^{\beta(\varepsilon_k - \mu_{\Lambda}(\rho))} - 1} \right] \right\} = \rho - \rho_c(\beta) > 0, \ \lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k_{0,0}} - \mu_{\Lambda}(\rho))} - 1 \right\}^{-1} \neq 0, \varepsilon_{k_{0,0}} = \varepsilon_{k_{1,0,0}}
\]

\[
\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_{k_{0,0}} - \mu_{\Lambda}(\rho))} - 1 \right\}^{-1} = 0, \varepsilon_{0,k_{2,3} \neq 0} = \hbar^2 (2\pi n_{2,3}/V^{3/2})^2 / 2m
\]

The generalised *type II* BEC [23]:

\[
\rho - \rho_c(\beta) = \lim_{L \to \infty} \frac{1}{V} \sum_{n_1 \in \mathbb{Z}} \left\{ e^{\beta((2\pi n_1/V^{1/2})^2 / 2m - \mu_{\Lambda}(\rho))} - 1 \right\}^{-1}
\]

\[
= \sum_{n_1 \in \mathbb{Z}} \frac{\beta^{-1}}{\hbar^2 (2\pi n_1)^2 / 2m + A} \Rightarrow \lim_{\Lambda} \frac{1}{V} \langle b_0^* b_0 \rangle_{T_n(\mu_{\Lambda}(\rho))} < \rho - \rho_c(\beta).
\]

Here \( A \geq 0 \) is a unique *not* of the above equation. Note that BEC in the zero mode is less than the total amount of the condensation density.

For \( \alpha_1 = 1/2 \) the BEC is still mode by mode macroscopic, but it is infinitely *fragmented*. This type of BEC is also known as *quasi-condensate* and it was observed in the *rotating condensate* (2000) and in the condensate with *chaotic* phases (2008), [18].

- The *Van den Berg box* (1982): \( \alpha_1 > 1/2 \).

**Proposition 1.** There is no macroscopic occupation of any kinetic-energy mode:

\[
\lim_{\Lambda} \frac{1}{V} \left\{ e^{\beta(\varepsilon_k - \mu_{\Lambda}(\rho))} - 1 \right\}^{-1} = 0.
\]
This is the generalised BEC of type III [Van den Berg-Lewis-Pulé (1978)]. It occurs one-direction anisotropy $\alpha_1 > 1/2$ i.e. $\alpha_2 + \alpha_3 < 1/2$. Since $\varepsilon_{k_1,0,0} = (2\pi n_1/V^{\alpha_1})^2/2 \sim 1/V^{2\alpha_1}$, $2\alpha_1 > 1$, then the solution $\mu_\Lambda(\rho)$ has a new asymptotics:

$$\mu_\Lambda(\rho \geq \rho_c(\beta)) = -B/V^4 + o(1/V^4), \quad B \geq 0, \quad \delta = 2(1 - \alpha_1) < 1,$$

$$0 < \rho - \rho_c(\beta) = (2\pi\beta)^{-1/2} \int_0^\infty d\xi \, e^{-\beta B\xi} \xi^{-1/2}.$$

Here parameter $B = B(\beta, \rho) > 0$ is a unique root of the equation:

$$\rho - \rho_c(\beta) = \frac{1}{\sqrt{2\beta^2 B(\beta, \rho)}}.$$

The generalised BEC of type III yields for the one-mode particle occupation

$$\lim_\Lambda \frac{1}{V} \langle N_k \rangle_{\Lambda, \beta} (\beta, \mu_\Lambda (\rho > \rho_c(\beta))) = 0 \quad \text{for all } k \in \{\Lambda^*\}.$$

For the "renormalised" $k_1$-modes occupation "density" one obtains:

$$\lim_\Lambda \frac{1}{V^{1-\epsilon}} \langle N_k \rangle_{\Lambda, \beta} (\beta, \mu_\Lambda (\rho > \rho_c(\beta))) = 2\beta (\rho - \rho_c(\beta))^2,$$

where $k \in \{\Lambda^* : (n_1, 0, 0)\}$ and $1 - \epsilon = \delta < 1$.

**Definition 1.** [24] In kinetic-energy modes the amount of the generalised BEC is defined as

$$\rho - \rho_c(\beta) := \lim_{\eta \to 0} \frac{1}{V} \sum_{k \in \Lambda^*, \|k\| \leq \eta} \left\{ e^{\beta(\varepsilon_k - \mu_\Lambda(\beta, \rho))} - 1 \right\}^{-1}.$$

**Remark 1.** [22],[24] Saturation and $\rho_m$-problem: it is possible that there is a new critical density $\rho_m$ such that $\rho_c \leq \rho_m \leq \infty$ and the type III (or II) condensation transforms into conventional type I BEC when $\rho \geq \rho_m$? The answer is positive. Recently the second critical density $\rho_m$ was discovered for a cigar-type harmonic anisotropy [1]. There it was also proved that the type I and the type III condensations may coexist.
2.5 The Bogoliubov Theory and the Zero-Mode c-Number Substitution

The first Bogoliubov ansatz. If one expects that the Bose-Einstein condensation, which occurs in the mode $k = 0$ for the perfect Bose-gas, persists for a weak two-body interaction $u(x)$, then one can to truncate Hamiltonian: $H_{\Lambda} \rightarrow H_{\Lambda}^B$, and to keep in $H_{\Lambda}^B$ only the most important condensate terms, in which at least two zero-mode operators $b_0^\dagger$, $b_0$ are involved. This approximation gives the Bogoliubov Weakly Imperfect Bose-Gas (WIBG) Hamiltonian $H_{\Lambda}^B$ [26].

The second Bogoliubov ansatz. Since for a large volume (thermodynamic limit) the condensate operators $b_0^\dagger/\sqrt{V}$, $b_0/\sqrt{V}$ almost commute: $[b_0^\dagger/\sqrt{V}, b_0/\sqrt{V}] = 1/V$, one may use substitutions:

$$b_0/\sqrt{V} \rightarrow c \cdot I, \quad b_0^\dagger/\sqrt{V} \rightarrow c^* \cdot I, \quad c \in \mathbb{C},$$

in the truncated grand-canonical WIBG Hamiltonian $H_{\Lambda}^B(\mu) := H_{\Lambda}^B - \mu N_{\Lambda} \rightarrow H_{\Lambda}^B(c, \mu)$ to produce a diagonalizable bilinear operator form.

2.6 The Zero-Mode c-Number Approximation

For the periodic boundary conditions on $\partial \Lambda$, let $\mathfrak{F}_0 := \mathfrak{F}_{\text{boson}}(\mathcal{H}_0)$ be the boson Fock space constructed on the one-dimensional Hilbert space $\mathcal{H}_0$ spanned by $\psi_{k=0}(x) = \chi_\Lambda(x)/\sqrt{V}$.

Let $\mathfrak{F}_0^\perp := \mathfrak{F}_{\text{boson}}(\mathcal{H}_0^\perp)$ be the Fock space constructed on the orthogonal complement $\mathcal{H}_0^\perp$. Then $\mathfrak{F}_{\text{boson}}(\mathcal{H}) = \mathfrak{F}_{\text{boson}}(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp)$ is isomorphic to the tensor product:

$$\mathfrak{F}_{\text{boson}}(\mathcal{H}_0 \oplus \mathcal{H}_0^\perp) \approx \mathfrak{F}_{\text{boson}}(\mathcal{H}_0) \otimes \mathfrak{F}_{\text{boson}}(\mathcal{H}_0^\perp) = \mathfrak{F}_0 \otimes \mathfrak{F}_0^\perp.$$

For any complex number $c \in \mathbb{C}$ the coherent vector in $\mathfrak{F}_0$ is

$$\psi_{\Lambda}(c) := e^{-V|c|^2/2} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sqrt{V}c\right)^k (b_0^\dagger)^k \Omega_0 = e^{-V|c|^2/2 + \sqrt{V}c b_0^\dagger} \Omega_0,$$

where $\Omega_0$ is the vacuum of $\mathfrak{F}$. Notice that

$$\frac{b_0}{\sqrt{V}} \psi_{\Lambda}(c) = c \psi_{\Lambda}(c) \equiv c \cdot \mathbb{I} \psi_{\Lambda}(c).$$
Definition 2. The \(e\)-number Bogoliubov approximation of the grand-canonical Hamiltonian \((N_\Lambda := \sum_{k \in \Lambda} b_0^* b_k := b_0^* b_0 + N'_\Lambda)\)

\[
H_\Lambda (\mu) := H_\Lambda - \mu N_\Lambda , \quad \text{dom}(H_\Lambda (\mu)) \subset \mathfrak{g} \approx \mathfrak{g}_{boson}(\mathcal{H}_0) \otimes \mathfrak{g}_{boson}(\mathcal{H}_0^+) \]

is a self-adjoint operator \(H_\Lambda (c, \mu)\) defined in \(\mathfrak{g}'_0 = \mathfrak{g}_{boson}(\mathcal{H}_0^+)\), for any fixed vector \(\psi_{0\Lambda} (c)\), by the closable sesquilinear form:

\[
(\psi'_1, H_\Lambda (c, \mu) \psi'_2)_{\mathfrak{g}'_0} \equiv (\psi_{0\Lambda} (c) \otimes \psi'_1, H_\Lambda (\mu) \psi_{0\Lambda} (c) \otimes \psi'_2)_{\mathfrak{g}} ,
\]

for vectors \((\psi_{0\Lambda} (c) \otimes \psi'_1, 2) \in \text{form-domain} \) of the operator \(H_\Lambda (\mu)\).

Remark 2. Since \((b_0 + \sqrt{V}) \psi_{0\Lambda} (c) = c \cdot \mathbb{I} \psi_{0\Lambda} (c)\), the \(e\)-number approximation is equivalent to substitutions:

\[
b_0 + \sqrt{V} \rightarrow c \cdot \mathbb{I} , \quad b_0^* + \sqrt{V} \rightarrow c^* \cdot \mathbb{I}
\]

in the Hamiltonian \(H_\Lambda (\mu) \rightarrow H_\Lambda (c, \mu) =: H'_\Lambda (z) - \mu(|z|^2 \mathbb{I} + N'_\Lambda) , \quad z := c \sqrt{V}.

2.7 Exactness of the \(e\)-Number Approximation

Definition 3. The grand-canonical pressure for Hamiltonian \(H_\Lambda (\mu)\) and for its \(e\)-number Bogoliubov approximation \(H'_\Lambda (z, \mu)\), are defined by:

\[
p_\Lambda (\mu) := \frac{1}{\beta V} \ln \text{Tr}_{\mathfrak{g}} \exp[-\beta H_\Lambda (\mu)]
\]

\[
p'_\Lambda (\mu) := \frac{1}{\beta V} \ln \int_C d^2 z \text{Tr}_{\mathfrak{g}_0} \exp[-\beta H'_\Lambda (z, \mu)]
\]

Proposition 2. (Variational Principle) \([10], [15]\).

\[
e^{\beta V p_\Lambda (\mu)} \geq \int_C d^2 z \text{Tr}_{\mathfrak{g}_0} \exp[-\beta H'_\Lambda (z, \mu)] \geq \sup_\zeta \text{Tr}_{\mathfrak{g}_0} \exp[-\beta H'_\Lambda (\zeta, \mu)] =: e^{\beta V p_\Lambda_{\text{max}} (\mu)}
\]

Proposition 3.

\[
\lim_{\Lambda} p_\Lambda (\mu) = \lim_{\Lambda} p'_\Lambda (\mu) = \lim_{\Lambda} p_{\Lambda, \text{max}} (\mu) ,
\]

with the rate of convergence:

\[
0 \leq p_\Lambda (\mu) - p_{\Lambda, \text{max}} (\mu) \leq O((\ln V)/V),
\]
see [15]. The rate of convergence proved by the **Approximating Hamiltonian Method** (AHM) is

\[ 0 \leq p_\Lambda(\mu) - p_{\Lambda,\text{max}}(\mu) \leq O(1/\sqrt{V}), \]

see [10], [26].

**Remark 2.** Although in [10] and in [15] the use of coherent states is essential, the method of the last paper efficiently exploits the Peierls-Bogoliubov and Berezin-Lieb inequalities instead of the AHM. To be more flexible, this method covers also the case of infinitely many \( k \)-modes, provided the card\( \{k : k \in I_\Lambda \subset \Lambda^*\}\) < \( c V^{1-\gamma} \), \( \gamma > 0 \), and it gives also more accurate estimates. The Bogoliubov \( c \)-Number Approximation is *exact* on the thermodynamic level (AHM) [6],.

### 2.8 The \( c \)-Number Approximation for Ideal Bose-Gas

The \( c \)-number substitution in the grand-canonical Hamiltonian \( T_\Lambda(\mu) := T_\Lambda - \mu N_\Lambda \) is

\[ T_\Lambda(\mu) \rightarrow T_\Lambda(c,\mu) = \sum_{k \in \Lambda^* \setminus \{0\}} (\varepsilon_k - \mu)b_k^* b_k - V\mu |c|^2 \]

Then one gets for the pressures (note that \( \mu < 0 \) and \( \varepsilon_k = 0 \)):

\[ p[T_\Lambda(\mu)] = \frac{1}{\beta V} \ln \text{Tr} \exp[-\beta T_\Lambda(\mu)] = \frac{1}{\beta V} \sum_{k \in \Lambda^*} \ln(1 - e^{-\beta(\varepsilon_k - \mu)})^{-1} \]

\[ p[T_\Lambda(c,\mu)] = \frac{1}{\beta V} \sum_{k \in \Lambda^* \setminus \{0\}} \ln(1 - e^{-\beta(\varepsilon_k - \mu)})^{-1} + \mu |c|^2 \]

\[ 0 \leq p[T_\Lambda(\mu)] - p[T_\Lambda(c,\mu)] = \frac{1}{\beta V} \ln(1 - e^{\beta\mu})^{-1} - \mu |c|^2 =: \Delta_\Lambda(c,\mu) \]

**Variational Principle:** \( \{c : \inf_c \lim_\Lambda \Delta_\Lambda(c,\mu)\} = \{c_*(\mu)\} \Rightarrow c_*(\mu < 0) = 0 \lor (\mu c_*(\mu)) |_{\mu=0} = 0 \). Hence, the BEC density is not defined.

### 2.9 Gauge Invariance and Bogoliubov Quasi-Averages

Since \([H_\Lambda, N_\Lambda] = 0\) (total particle number conservation law),

\[ H_\Lambda = e^{i \varphi N_\Lambda} H_\Lambda e^{-i \varphi N_\Lambda}, \quad U(\varphi) := e^{i \varphi N_\Lambda}, \]

\[ \text{where } \varphi \text{ is a phase parameter.} \]
\( H_\Lambda \) is invariant w.r.t. gauge transformations \( U(\varphi) \).

**Corollary 1.** The grand-canonical expectation value:

\[
\langle \frac{b_0}{\sqrt{V}} \rangle_{H_\Lambda} (\beta, \mu) = 0.
\]

Let \( H_{\Lambda, \nu}(\mu) := H_\Lambda(\mu) - \sqrt{V}(\nu b_0^* + \nu^* b_0) \), \( \nu \in \mathbb{C} \). Then

\[
\langle \frac{b_0}{\sqrt{V}} \rangle_{H_{\Lambda, \nu}} (\beta, \mu) \neq 0, \langle \frac{b_{k\neq 0}}{\sqrt{V}} \rangle_{H_{\Lambda, \nu}} (\beta, \mu) = 0.
\]

**Remark 4.** Whether the limit: \( \lim_{\nu \to 0} \lim_{\Lambda} \langle \frac{b_0}{\sqrt{V}} \rangle_{H_{\Lambda, \nu}} (\beta, \mu) =: c_0 \neq 0 \)?

If it is the case this yields a spontaneous breaking of the gauge symmetry. Here \( c_0 \) is the Bogoliubov quasi-average \([4], [5]\). The idea of quasi-averages allowed Bogoliubov to prove his famous \( 1/q^2 \)-Theorem for interacting Bose-gas as well as to advance later in elucidating the c-Number Approximation, see \([15], [16], [21], [26]\).

**Example 2. (Ideal Bose-Gas)** The gauge-breaking sources imply

\[
T_{\Lambda, \nu}(\mu) := T_\Lambda(\mu) - \sqrt{V}(\nu b_0^* + \nu^* b_0) = \\
- \mu(b_0^* + \sqrt{V}/\mu)(b_0 + \sqrt{V}/\nu/\mu) + T^{(k\neq 0)}_\Lambda(\mu) + V|\nu|^2/\mu.
\]

The \( c \)-number substitution gives:

\[
T_{\Lambda, \nu}(\mu) \to T_{\Lambda, \nu}(c, \mu) = - \mu V(\bar{c} + \bar{\nu}/\mu)(c + \nu/\mu) + T^{(k\neq 0)}_\Lambda(\mu) + V|\nu|^2/\mu
\]

One gets for the pressure (note that \( \mu < 0 \) and \( \varepsilon_{k=0} = 0 \)):

\[
p[T_{\Lambda, \nu}(\mu)] = p[T_\Lambda(\mu)] - |\nu|^2/\mu ,
\]

\[
p[T_{\Lambda, \nu}(c, \mu)] = p[T^{(k\neq 0)}_\Lambda(\mu)] + \mu V(\bar{c} + \bar{\nu}/\mu)(c + \nu/\mu) - |\nu|^2/\mu ,
\]

\[
0 \leq p[T_{\Lambda, \nu}(\mu)] - p[T_{\Lambda, \nu}(c, \mu)] = \\
\frac{1}{\beta V} \ln(1 - e^{\beta \mu})^{-1} - \mu|c + \eta/\mu|^2 =: \Delta_{\Lambda, \nu}(c, \mu).
\]

The Variational Principle: \( \{c : \inf_{c} \lim_{\Lambda} \Delta_{\Lambda, \nu}(c, \mu) \} = \{c_*(\mu, \nu) = -\nu/\mu \} \) implies that the variational BEC density \( \rho_{0*} \) is defined by the limit

\[
\rho_{0*} := \lim_{\nu \to 0} \lim_{\mu \to \mu(\nu) \to 0} |c_*(\mu, \nu)|^2 = \lim_{\nu \to 0} \lim_{\mu \to \mu(\nu) \to 0} \langle \frac{b_0^*}{\sqrt{V}} \rangle_{T_{\Lambda, \nu}(\mu)} \langle \frac{b_0}{\sqrt{V}} \rangle_{T_{\Lambda, \nu}(\mu)}.
\]
The relation of BEC versus the quasi-average BEC and the maximizer \( \rho_{0*} \) takes the form:

\[
\text{zero-mode BEC } \rho_0 \Rightarrow \frac{1}{V} \langle b_0^* b_0 \rangle_{T,\nu=0(\mu)} = \frac{1}{V} \frac{1}{e^{-\beta \mu} - 1} \leq \\
\frac{|\nu|^2}{\mu^2} + \frac{1}{V} \frac{1}{e^{-\beta \mu} - 1} = \frac{1}{V} \langle b_0^* b_0 \rangle_{T,\nu(\mu)} \Rightarrow \text{quasi-average BEC}.
\]

Then by the Variational Principle for the \( \xi \)-Number Approximation one obtains:

\[
\lim_{\nu \to 0} \lim_{\mu \to \mu(\nu)} \frac{1}{V} \langle b_0^* b_0 \rangle_{T,\nu(\mu)} = \lim_{\nu \to 0} \frac{\langle b_0^* b_0 \rangle_{T,\nu(\mu)}}{\sqrt{V}} \Rightarrow \text{gauge-symmetry breaking BEC} = \lim_{\mu \to \mu(\nu)} |c_\mu(\mu, \nu)|^2 = \rho_{0*}.
\]

\textbf{Remark 5.} Is it possible that \( \rho_0 < \rho_{0*} \)? The answer is positive: one can prove this inequality for the \textit{ideal} as well as for an \textit{interacting} Bose-gas \cite{7} if they manifest \textit{generalised} BEC of the type II or III.

\textbf{Proposition 4} \cite{15}, \cite{16}. The \( k = 0 \) mode BEC \( \Rightarrow \) quasi-average BEC \( \Leftrightarrow \) spontaneous gauge-symmetry breaking BEC \( \Leftrightarrow \) non-zero \( \xi \)-number approximation for the mode \( k = 0 \).

The proof is based on Griffith’s arguments and on the following two Propositions:

\textbf{Proposition 5} For a real \( \nu \) one gets equality between the limits:

\[
\lim_{\Lambda} p_\Lambda(\mu; \nu) = \lim_{\Lambda} p_\Lambda'(\mu; \nu) = \lim_{\Lambda} p_{\Lambda,\text{max}}(\mu; \nu),
\]

which are convex in \( \nu \).

\textbf{Proposition 6} (Gauge-Symmetry Breaking and BEC)

\[
\lim_{|\nu| \to 0, \arg(\nu)} \lim_{\Lambda} \left( \frac{b_0^*}{\sqrt{V}} \right)_{H,\nu(\mu)} = \\
\lim_{|\nu| \to 0, \arg(\nu)} \lim_{\Lambda} |z_{\Lambda,\text{max}}(\nu)| e^{\arg(\nu)/\sqrt{V}} =: c_0.
\]

Here by the Variational Principle:

\[
\sup_{\zeta} \text{Tr}_{\mathcal{G}_0} \exp[-\beta H_\Lambda(\zeta, \mu; \nu)] = \text{Tr}_{\mathcal{G}_0} \exp[-\beta H_\Lambda(z_{\Lambda,\text{max}}(\nu), \mu; \nu)] \\
= \exp \left[ \beta V p_{\Lambda,\text{max}}(\mu; \nu) \right] =: \exp \left[ \beta V p_{\Lambda,\text{max}}(\mu; \nu) \right],
\]
and \( z_{\Lambda, \text{max}}(0) = \| z_{\Lambda, \text{max}}(0) \| e^{i\phi}, \) \( p_{\Lambda, z_{\Lambda, \text{max}}(\nu)}(\mu; \nu) \|_{\nu=0} = p_{\Lambda, \text{max}}(\mu). \)

**Corollary 2.** One obtains for the quasi-average condensate density and for the condensate density equation:

\[
\rho_0(\beta, \mu) = \lim_{|\nu| \to 0} \lim_{\Lambda} \left( \begin{array}{c} b_0^* b_0 \end{array} \right)_{H_{\Lambda, \nu}} (\beta, \mu) = \lim_{\Lambda} |c_{0, \Lambda, \text{max}}(\nu)|^2 (\beta, \mu).
\]

where \( c_{0, \Lambda, \text{max}} \) is a maximizer of the *variational problem*:

\[
\sup_{c_0} \text{Tr}_{\mathcal{F}_0} \exp[-\beta H_0'(c_0 \sqrt{V}, \mu)] = \text{Tr}_{\mathcal{F}_0} \exp[-\beta H_0'(c_{0, \Lambda, \text{max}} \sqrt{V}, \mu)]
\]

### 3 Random Homogeneous (Ergodic) External Potentials.

#### 3.1 Random and Kinetic-Energy Eigenfunctions

For the *almost surely* (a.s.) self-adjoint random Schrödinger operator in \( \Lambda \subset \mathbb{R}^d \) one has:

\[
h_{\Lambda} \phi_j^\omega = \left( t_{\Lambda} + \nu \omega \right) \phi_j^\omega = E_j^\omega \phi_j^\omega,
\]

for almost all (a.a.) \( \omega \in \Omega \),

where \( \{ \phi_j^\omega \}_{j \geq 1} \) are the *random* eigenfunctions. In the limit \( \Lambda \uparrow \mathbb{R}^d \) the spectrum \( \sigma(h_\omega) \) of this operator is a.s. nonrandom [19].

Let \( N_\Lambda(\phi_j^\omega) \) be particle-number operator in the eigenstate \( \phi_j^\omega \).

\[
N_\Lambda := \sum_{j \geq 1} N_{\Lambda}(\phi_j^\omega) := \sum_{j \geq 1} b^*(\phi_j^\omega)b(\phi_j^\omega)
\]

is the *total* number operator in the boson Fock space \( \mathcal{F}(\mathcal{L}^2(\Lambda)) \), \( b(\phi_j^\omega) := \int_{\Lambda} dx \phi_j^\omega(x) b(x) \), and \( \{ \phi_j^\omega \}_{j \geq 1} \) is a.s. a (random) basis in \( \mathcal{H} = \mathcal{L}^2(\Lambda) \).

Let \( t_{\Lambda} \psi_k = \varepsilon_k \psi_k \) be the kinetic-energy operator eigenfunctions \( \{ \psi_k \}_{k \in \Lambda^*} \) with eigenvalues \( \varepsilon_k = \hbar^2 k^2/2m \). Recall that one of the *key hypothesis* of the conventional Bogoliubov Theory is the existence of translation-invariant *ground-state* (i.e. the zero-mode \( \psi_{k=0} \)) Bose condensation.

Random Hamiltonian \( H_\Lambda^\omega \) of interacting Bosons in \( \mathcal{F}(\mathcal{H}) \):

\[
H_\Lambda^\omega := T_\Lambda^\omega + U_\Lambda = \text{random Schrödinger operator + interaction},
\]
where the kinetic-energy operator has two forms:

$$d\Gamma(h^\omega_A) := T^\omega_A = \sum_{j \geq 1} E_j^\omega \, b^*(\phi^\omega_j)b(\phi^\omega_j) = \sum_{k_1, k_2 \in \Lambda^*} (\psi_{k_1}, (t^\omega + v^\omega)\psi_{k_2})_H b_{k_1}^* b_{k_2},$$

Note that there are also two faces for the second-quantised two-body interaction $u(x - y)$ in $\mathfrak{F}(H)$:

$$U_A := \frac{1}{2} \sum_{j_1, j_2} (\phi^\omega_{j_1} \otimes \phi^\omega_{j_2}, u \phi^\omega_{j_3} \otimes \phi^\omega_{j_4})_H b^*(\phi^\omega_{j_1})b^*(\phi^\omega_{j_2}) b(\phi^\omega_{j_3})b(\phi^\omega_{j_4})$$

$$= \frac{1}{2V} \sum_{k_1, k_2 \in \Lambda^*} v(q) b_{k_1}^* b_{k_2}^* q b_{k_2} b_{k_1}.$$

**Remark 6.** Our aim is to elucidate the status and in particularly exactness of the Bogoliubov c-Number Approximation for the random interacting boson gas. For example to answer the questions concerning the (generalised) BEC:

$$\sum_{j: E_j^\omega \leq \delta} \langle N_A(\phi^\omega_j) \rangle_H^\omega / V \to c ? \text{ or } \sum_{k: \varepsilon_k \leq \gamma} \langle N_A(\psi_k) \rangle_H^\omega / V \to c ?$$

### 3.2 Random versus Kinetic-Energy Condensation

**Proposition 7** [11] Let $H^\omega_A := T^\omega_A + U_A$ be many-body Hamiltonian of interacting bosons in random external potential $V^\omega_A$. If the particle interaction $U_A$ commutes with any of number operators $N_A(\phi^\omega_j)$ (local gauge invariance), then

$$a.s. - \lim_{\delta \downarrow 0} \lim_{\Lambda} \sum_{j: E_j^\omega \leq \delta} \frac{1}{V} \langle \langle N_A(\phi^\omega_j) \rangle_H^\omega \rangle > 0 \iff$$

$$\iff a.s. - \lim_{\gamma \downarrow 0} \lim_{\Lambda} \sum_{k: \varepsilon_k \leq \gamma} \frac{1}{V} \langle \langle N_A(\psi_k) \rangle_H^\omega \rangle > 0,$$

and: $\lim_{\gamma \downarrow 0} \lim_{\Lambda} \sum_{k: \varepsilon_k \geq \gamma} \langle N_A(\psi_k) \rangle_H^\omega / V = 0$. Here $\langle \rangle_H^\omega$ is quantum Gibbs expectation with random Hamiltonian $H^\omega_A$.

**Remark 7** If a many-body interaction satisfies the local gauge invariance:

$$[U_A, N_A(\phi_j)] = 0,$$

then $U_A$ is a function of the occupation number operators $\{N_A(\phi_j)\}_{j \geq 1}$. For this reason it is called a "diagonal interaction".
Corollary 3 A random localized generalised (of a yet unknown type) boson condensation occurs if and only if there is a generalised (type II/III) condensation in the extended (kinetic-energy) eigenstates. This is a possible way to save the Bogoliubov theory in the case of non-translation invariant, but homogeneous random external potential.

3.3 Amounts of Random and of Kinetic-Energy Condensates

Let for any $A \subset \mathbb{R}_+$ the particle occupation measures $m_A$ and $\tilde{m}_A$ are defined for the perfect Bose-gas by:

$$m_A(A) := \frac{1}{V} \sum_{j : E_j \in A} \langle N_A(\phi_j^\omega) \rangle_{\mathcal{T}_N^\omega} , \quad \tilde{m}_A(A) := \frac{1}{V} \sum_{k : \epsilon_k \in A} \langle N_A(\psi_k) \rangle_{\mathcal{T}_N^\omega} .$$

Proposition 8 [11] For the perfect Bose-gas amounts of random and kinetic-energy condensates coincide:

$$m(dE) = \begin{cases} (\bar{\rho} - \rho_c) \delta_0(dE) + (e^{\beta E} - 1)^{-1} \mathcal{N}(dE) & \text{if } \bar{\rho} \geq \rho_c , \\ (e^{\beta(E - \mu_c)} - 1)^{-1} \mathcal{N}(dE) & \text{if } \bar{\rho} < \rho_c , \end{cases}$$

$$\tilde{m}(d\epsilon) = \begin{cases} (\bar{\rho} - \rho_c) \delta_0(d\epsilon) + F(\epsilon)d\epsilon & \text{if } \bar{\rho} \geq \rho_c , \\ F(\epsilon)d\epsilon & \text{if } \bar{\rho} < \rho_c . \end{cases}$$

with explicitly defined density $F(\epsilon)$. For models with diagonal interactions: $m_A(A) \leq \tilde{m}_A(A)$.

3.4 BEC in One-Dimensional Random Potential. Poisson Point-Impurities

For $d = 1$ and for repulsive Poisson point-impurities with density $\tau$ and $a > 0$, the homogeneous ergodic random external potential has the form:

$$v^\omega(x) := \int_{\mathbb{R}^1} \mu_\tau^\omega(dy) a \delta(x - y) = \sum_j a \delta(x - y_j^\omega)$$

$$\mathbb{P}\{\omega : \mu_\tau^\omega(\Lambda) = s\} = \frac{|\Lambda|^s}{s!} e^{-\tau|\Lambda|} , \quad \mathbb{E}(\mu_\tau^\omega(\Lambda)) = \tau|\Lambda| , \quad \Lambda \subset \mathbb{R}^1 .$$
The Bogoliubov $\epsilon$-Number Approximation

**Proposition 9** [14] Let $a = +\infty$. Then $\sigma(h^\omega)$ is a.s. nonrandom, dense pure-point spectrum such that the closure $\sigma_{p.p.}(h^\omega) = [0, +\infty)$, with the Integrated Density of States

$$
\mathcal{N}(E) = \tau \frac{e^{-\pi \sqrt{2E}}}{1 - e^{-\pi \sqrt{2E}}} \sim \tau e^{-\pi \sqrt{2E}}, \ E \downarrow 0, \text{ (Lifshitz tail)}. 
$$

One gets for the spectrum:

$$(a.s.) - \sigma(h^\omega) = \bigcup_j \left\{ \frac{\pi^2 s^2}{2(L_j^\omega)^2} \right\}_s^\infty,$$

where intervals $L_j^\omega = y_j^\omega - y_{j-1}^\omega$ are independent identically distributed random variables:

$$dP_{\tau, j_1, \ldots, j_k}(L_{j_1}, \ldots, L_{j_k}) = \tau^k \prod_{s=1}^k e^{-\tau L_{j_s}} dL_{j_s}.$$

The eigenfunctions: for a.a. $\omega \in \Omega$ the one-particle localized quantum states $\{ \phi_{\omega}^j \}_{j \geq 1}$, give a basis in $L^2(\Lambda)$.

4 Generalized $\epsilon$-numbers approximation

4.1 Existence of the approximating pressure

Since randomness implies fragmented (or generalized type II/III) condensation, following the Bogoliubov approximation philosophy, we want to replace all creation/annihilation operators in the momentum states $\psi_k$ with kinetic energy less than some $\delta > 0$ by $\epsilon$-numbers. Let $I_\delta \subset \Lambda^*$ be the set of all replaceable modes

$$I_\delta := \{ k \in \Lambda^* : h^2 k^2 / 2m \leq \delta \},$$

and we denote $n_\delta := \text{card}\{ k : k \in I_\delta \}$.

**Remark 8** The number of quantum states $n_\delta$ is of the order $V_1$ since by definition of the Integrated Density of States: $n_\delta = V \mathcal{N}_A(\delta)$. To use the Lieb-Seiringer-Yngvason method we consider $n_\delta = O(V^{1-\gamma})$, $0 < \gamma < 1$. Why it is possible? See Corollary 4, and [25] for details.
4.2 Generalised BEC of type III: one-mode particle occupations

**Definition 4** [12] We call eigenfunctions: \( \{ \phi_j^\omega \}_{j \geq 1} \) *weakly* localised if

\[
\lim_{\Lambda} \frac{1}{\sqrt{\Lambda}} \int dx \, |\phi_j^\omega(x)| = 0 \quad \text{for a.a. } \omega \in \Omega .
\]

**Proposition 10** [12],[13] Let all \( \{ \phi_j^\omega \}_{j \geq 1} \) be localised. Then for models \( H_\Lambda^\omega \) with *diagonal interactions*

\[
\lim_{\Lambda} \frac{1}{\sqrt{\Lambda}} \langle N_\Lambda(\psi_k) \rangle_{H_\Lambda^\omega} = 0 \quad \text{for all } k \in \Lambda^*
\]

This implies that any possible generalised kinetic-energy BEC in these models is of *type III*.

**Corollary 4** The number of *condensed* kinetic-modes is at most \( O(V^{1-\gamma}) \), \( 0 < \gamma < 1 \), and in this case one can use the LSY method for the modes:

\[
\lim_{\Lambda} \frac{1}{\sqrt{\Lambda}} \langle N_\Lambda(\psi_k) \rangle_{H_\Lambda^\omega} \neq 0 \quad \text{for } k \in I_{\delta \Lambda}, \gamma = 1 - \epsilon
\]

Let \( \mathcal{H}^\delta \) be the subspace of \( \mathcal{H} \) spanned by the set of \( \psi_k \) with \( k \in I_{\delta \Lambda} \), and \( P_\delta \) be orthogonal projector onto this subspace. Hence, we have a natural decomposition of the total space \( \mathcal{H} \) and the corresponding representation for the associated symmetrised Fock space:

\[
\mathcal{H} = \mathcal{H}^\delta \oplus \mathcal{H}' , \quad \mathfrak{g} \approx \mathfrak{g}^\delta \otimes \mathfrak{g}'.
\]

Then we proceed with the Bogoliubov substitution \( b_k \rightarrow c_k \) and \( b_k^* \rightarrow \bar{c}_k \) for all \( k \in I_{\delta \Lambda} \), which provides an *approximating* (for the initial) Hamiltonian, that we denote by \( H_\Lambda^{\text{low}}(\mu, \{ c_k \}) \).

The partition function and the corresponding pressure for this *approximating* Hamiltonian have the form:

\[
\Xi_\Lambda^{\text{low}}(\mu, \{ c_k \}) = \text{Tr} \mathfrak{g}^\delta e^{-\beta H_\Lambda^{\text{low}}(\mu, \{ c_k \})},
\]

\[
p^{\text{low}}_\Lambda(\mu, \{ c_k \}) = \frac{1}{V} \ln \Xi_\Lambda^{\text{low}}(\mu, \{ c_k \}).
\]

**Proposition 10** [13], [25] The c-numbers substitution for all operators in the energy-band \( I_{\delta \Lambda} \), \( \text{card}\{k : k \in I_{\delta \Lambda}\} = O(V^{1-\gamma}) \), does not affect the original pressure in the following sense:

\[
a.s. - \lim_{\Lambda} [P_\Lambda(\beta, \mu) - \{ \max_{\{ c_k \}} P^{\text{low}}_{\Lambda,(\delta \Lambda)}(\mu, \{ c_k \}) \}] = 0
\]
Remark 9 Besides the type III condensation the last statement covers the one-mode case. For the case of eventual type II condensation the arguments are similar, but with a volume-dependent cut-off of the converging sum over modes [25].

References


