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# The densest subgraph problem in sparse random graphs

Venkat Anantharam\* and Justin Salez†

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## Abstract

We determine the asymptotic behavior of the maximum subgraph density of large random graphs with a prescribed degree sequence. The result applies in particular to the Erdős-Rényi model, where it settles a conjecture of Hajek (1990). Our proof consists in extending the notion of balanced loads from finite graphs to their local weak limits, using unimodularity. This is a new illustration of the objective method described by Aldous and Steele (2004).

**Keywords:** maximum subgraph density; load balancing; local weak convergence; objective method; unimodularity; pairing model.

**2010 MSC:** 60C05, 05C80, 90B15.

## 1 Introduction

**Balanced allocations** Let  $G = (V, E)$  be a simple undirected locally finite graph. Write  $\vec{E}$  for the set of oriented edges, i.e. ordered pairs of adjacent vertices. An *allocation* on  $G$  is a map  $\theta: \vec{E} \rightarrow [0, 1]$  satisfying  $\theta(i, j) + \theta(j, i) = 1$  for every  $\{i, j\} \in E$ . The *load* induced by  $\theta$  at a vertex  $o \in V$  is

$$\partial\theta(o) := \sum_{i \sim o} \theta(i, o),$$

where  $\sim$  means adjacency in  $G$ .  $\theta$  is *balanced* if for every  $(i, j) \in \vec{E}$ ,

$$\partial\theta(i) < \partial\theta(j) \implies \theta(i, j) = 0. \tag{1}$$

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Intuitively, one may think of each edge as carrying a unit amount of load, which has to be distributed over its end-points in such a way that the total load is as much balanced as possible across the graph. In that respect, (1) is a *local optimality* criterion: modifying the allocation along an edge cannot further reduce the load imbalance between its end-points. When  $G$  is finite, this condition happens to guarantee *global optimality* in a very strong sense. Specifically, the following conditions are equivalent (see [16]).

- (i)  $\theta$  is balanced.
- (ii)  $\theta$  minimizes  $\sum_{o \in V} f(\partial\theta(o))$ , for some strictly convex  $f: [0, \infty) \rightarrow \mathbb{R}$ .
- (iii)  $\theta$  minimizes  $\sum_{o \in V} f(\partial\theta(o))$ , for every convex  $f: [0, \infty) \rightarrow \mathbb{R}$ .

In particular, balanced allocations exist on  $G$  and they all induce the same loads  $\partial\theta: V \rightarrow [0, \infty)$ .

**The densest subgraph problem.** The load  $\partial\theta(o)$  induced at a vertex  $o \in V$  by some (hence every) balanced allocation  $\theta$  on  $G$  has a remarkable graph-theoretical interpretation: it measures the *local density* of  $G$  at  $o$ . In particular, it was shown in [16] that the vertices receiving the highest load in  $G$  solve the classical densest subgraph problem: the value  $\max \partial\theta$  coincides with the *maximum subgraph density* of a subgraph in  $G$ ,

$$\varrho(G) := \max_{\emptyset \subsetneq H \subseteq V} \frac{|E(H)|}{|H|},$$

and the set  $H = \operatorname{argmax} \partial\theta$  is the largest set achieving this maximum. This surprising connection with a well-known and important graph parameter justifies a deeper study of balanced loads in large graphs. For this purpose, it is convenient to encode the various loads of  $G$  into a probability measure on  $\mathbb{R}$ , called the *empirical load distribution* of  $G$ :

$$\mathcal{L}_G = \frac{1}{|V|} \sum_{o \in V} \delta_{\partial\theta(o)}.$$

**Conjecture in the Erdős-Rényi case.** Motivated by the above connection, Hajek [16] studied the asymptotic behavior of  $\mathcal{L}_G$  on the popular Erdős-Rényi model, where the graph  $G = G_n$  is chosen uniformly at random among all graphs with  $m = \lfloor \alpha n \rfloor$  edges on  $V = \{1, \dots, n\}$ . In the regime where the density parameter  $\alpha \geq 0$  is kept fixed while  $n \rightarrow \infty$ , he conjectured that the

empirical load distribution  $\mathcal{L}_{G_n}$  should concentrate around a deterministic probability measure  $\mathcal{L} \in \mathcal{P}(\mathbb{R})$ , in the sense that

$$\mathcal{L}_{G_n} \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{R})} \mathcal{L}.$$

Coming back to the densest subgraph problem and despite the non-continuity of the essential supremum w.r.t. to weak convergence, Hajek conjectured that

$$\varrho(G_n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \varrho := \sup\{t \in \mathbb{R} : \mathcal{L}([0, t]) < 1\}.$$

Finally, using a non-rigorous analogy with the case of finite trees, Hajek proposed a description of  $\mathcal{L}$  and  $\varrho$  in terms of the solutions to a distributional fixed-point equation which will be given later. In this paper, we establish this triple conjecture together with its analogue for various other sparse random graphs, using the unifying framework of *local weak convergence*.

## 2 The framework of local weak convergence

This section gives a brief account of the framework of local weak convergence. For more details, we refer to the seminal paper [6] and to the surveys [3, 2].

**Rooted graphs.** A *rooted graph*  $(G, o)$  is a graph  $G = (V, E)$  together with a distinguished vertex  $o \in V$ , called the *root*. We let  $\mathcal{G}_\star$  denote the set of all locally finite connected rooted graphs considered up to *rooted isomorphism*, i.e.  $(G, o) \equiv (G', o')$  if there exists a bijection  $\gamma: V \rightarrow V'$  that preserves roots ( $\gamma(o) = o'$ ) and adjacency ( $\{i, j\} \in E \iff \{\gamma(i), \gamma(j)\} \in E'$ ). We write  $[G, o]_h$  for the (finite) rooted subgraph induced by the vertices lying at graph-distance at most  $h \in \mathbb{N}$  from  $o$ . The distance

$$\text{DIST}((G, o), (G', o')) := \frac{1}{1+r} \text{ where } r = \sup\{h \in \mathbb{N} : [G, o]_h \equiv [G', o']_h\},$$

turns  $\mathcal{G}_\star$  into a complete separable metric space, see [2].

**Local weak limits.** Let  $\mathcal{P}(\mathcal{G}_\star)$  denote the set of Borel probability measures on  $\mathcal{G}_\star$ , equipped with the usual topology of weak convergence (see e.g. [7]). Given a finite deterministic graph  $G = (V, E)$ , we construct a random element of  $\mathcal{G}_\star$  by choosing uniformly at random a vertex  $o \in V$  to be the root, and restricting  $G$  to the connected component of  $o$ . The resulting law is denoted by  $\mathcal{U}(G)$ . If  $\{G_n\}_{n \geq 1}$  is a sequence of finite graphs such that  $\{\mathcal{U}(G_n)\}_{n \geq 1}$  admits a weak limit  $\mu \in \mathcal{P}(\mathcal{G}_\star)$ , we call  $\mu$  the *local weak limit* of  $\{G_n\}_{n \geq 1}$ .

**Edge-rooted graphs.** Let  $\mathcal{G}_{**}$  denote the space of locally finite connected graphs with a distinguished oriented edge, taken up to the natural isomorphism relation and equipped with the natural distance, which turns it into a complete separable metric space. With any function  $f: \mathcal{G}_{**} \rightarrow \mathbb{R}$  is naturally associated a function  $\partial f: \mathcal{G}_* \rightarrow \mathbb{R}$ , defined by

$$\partial f(G, o) = \sum_{i \sim o} f(G, i, o).$$

Dually, with any measure  $\mu \in \mathcal{P}(\mathcal{G}_*)$  is naturally associated a non-negative measure  $\vec{\mu}$  on  $\mathcal{G}_{**}$ , defined as follows: for any Borel function  $f: \mathcal{G}_{**} \rightarrow [0, \infty)$ ,

$$\int_{\mathcal{G}_{**}} f d\vec{\mu} = \int_{\mathcal{G}_*} (\partial f) d\mu.$$

Note that the total mass  $\vec{\mu}(\mathcal{G}_{**})$  of the measure  $\vec{\mu}$  is precisely

$$\text{deg}(\mu) := \int_{\mathcal{G}_*} \text{deg}(G, o) d\mu(G, o).$$

**Unimodularity.** It was shown in [2] that any  $\mu \in \mathcal{P}(\mathcal{G}_*)$  arising as the local weak limit of some sequence of finite graphs satisfies

$$\int_{\mathcal{G}_{**}} f d\vec{\mu} = \int_{\mathcal{G}_{**}} f^* d\vec{\mu},$$

for any Borel  $f: \mathcal{G}_{**} \rightarrow [0, \infty)$ . Here,  $f^*: \mathcal{G}_{**} \rightarrow \mathbb{R}$  denotes the *reversal* of  $f$ :

$$f^*(G, i, o) = f(G, o, i).$$

A measure  $\mu \in \mathcal{P}(\mathcal{G}_*)$  satisfying this invariance is called *unimodular*, and the set of all unimodular probability measures on  $\mathcal{G}_*$  is denoted by  $\mathcal{U}$ .

**Marks on oriented edges.** It will sometimes be convenient to work with *networks*, i.e. graphs equipped with a map from  $\vec{E}$  to some fixed complete separable metric space  $\Xi$ . The above definitions extend naturally, see [2].

**Unimodular Galton-Watson trees.** Let  $\pi = \{\pi_n\}_{n \geq 0}$  be a probability distribution on  $\mathbb{N}$  with non-zero finite mean, and let  $\hat{\pi} = \{\hat{\pi}_n\}_{n \geq 0}$  denote its size-biased version:

$$\hat{\pi}_n = \frac{(n+1)\pi_{n+1}}{\sum_k k\pi_k} \quad (n \in \mathbb{N}) \quad (2)$$

A *unimodular Galton-Watson tree* with degree distribution  $\pi$  is a random rooted tree  $\mathbb{T}$  obtained by a Galton-Watson branching process where the root has offspring distribution  $\pi$  and all its descendants have offspring distribution  $\hat{\pi}$ . The law of  $\mathbb{T}$  is unimodular, and is denoted by  $\text{UGWT}(\pi)$ . Such trees play a distinguished role as they are the local weak limits of many natural sequences of random graphs, including those produced by the pairing model.

**The pairing model.** Given a sequence  $\mathbf{d} = \{d(i)\}_{1 \leq i \leq n}$  of non-negative integers whose sum is even, the pairing model [8, 19] generates a random graph  $\mathbb{G}[\mathbf{d}]$  on  $V = \{1, \dots, n\}$  as follows:  $d(i)$  *half-edges* are attached to each  $i \in V$ , and the  $2m = d(1) + \dots + d(n)$  half-edges are paired uniformly at random to form  $m$  edges. Loops and multiple edges are removed (a few variants exist, see [18], but they are equivalent for our purpose). Now, consider a degree sequence  $\mathbf{d}_n = \{d_n(i)\}_{1 \leq i \leq n}$  for each  $n \geq 1$  and assume that

$$\forall k \in \mathbb{N}, \quad \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{d_n(i)=k\}} \xrightarrow[n \rightarrow \infty]{} \pi_k, \quad (3)$$

for some probability distribution  $\pi = \{\pi_k\}_{k \in \mathbb{N}}$  on  $\mathbb{N}$  with finite, non-zero mean. Under the additional assumption that

$$\sup_{n \geq 1} \left\{ \frac{1}{n} \sum_{i=1}^n d_n^2(i) \right\} < \infty,$$

the local weak limit of  $\{\mathbb{G}[\mathbf{d}_n]\}_{n \geq 1}$  is  $\mu := \text{UGWT}(\pi)$  almost-surely, see [9].

### 3 Main results

**Balanced loads on unimodular random graphs.** Our first main result is that the notion of balanced allocations can be extended from finite graphs to their local weak limits, in such a way that the induced loads behave continuously with respect to local weak convergence. Define a *Borel allocation* as a measurable function  $\Theta: \mathcal{G}_{**} \rightarrow [0, 1]$  such that  $\Theta + \Theta^* = 1$ , and call it *balanced* on  $\mu \in \mathcal{U}$  if for  $\vec{\mu}$ -a-e  $(G, i, o) \in \mathcal{G}_{**}$ ,

$$\partial\Theta(G, i) < \partial\Theta(G, o) \implies \Theta(G, i, o) = 0.$$

This natural definition is the right analogue of (1) when finite graphs are replaced by unimodular measures, as demonstrated by the following result.

**Theorem 1.** *Let  $\mu \in \mathcal{U}$  be such that  $\deg(\mu) < \infty$ . Then,*

1. Existence. *There is a Borel allocation  $\Theta_0$  that is balanced on  $\mu$ .*
2. Optimality. *For any Borel allocation  $\Theta$ , the following are equivalent:*
  - (i)  $\Theta$  is balanced on  $\mu$ .
  - (ii)  $\Theta$  minimizes  $\int f \circ \partial\Theta d\mu$  for some strictly convex  $f: [0, \infty) \rightarrow \mathbb{R}$ .
  - (iii)  $\Theta$  minimizes  $\int f \circ \partial\Theta d\mu$  for every convex  $f: [0, \infty) \rightarrow \mathbb{R}$ .
  - (iv)  $\partial\Theta = \partial\Theta_0$ ,  $\mu$ -a-e.
3. Continuity. *For any sequence  $\{G_n\}_{n \geq 1}$  with local weak limit  $\mu$ ,*

$$\mathcal{L}_{G_n} \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{R})} \mathcal{L},$$

where  $\mathcal{L}$  denotes the law of the random variable  $\partial\Theta_0 \in L^1(\mathcal{G}_*, \mu)$ .

4. Variational characterization. *The mean-excess function of the random variable  $\partial\Theta_0$ , namely  $\Phi_\mu: t \mapsto \int_{\mathcal{G}_*} (\partial\Theta_0 - t)^+ d\mu$ , is given by*

$$\Phi_\mu(t) = \max_{\substack{f: \mathcal{G}_* \rightarrow [0, 1] \\ \text{Borel}}} \left\{ \frac{1}{2} \int_{\mathcal{G}_{**}} \widehat{f} d\vec{\mu} - t \int_{\mathcal{G}_*} f d\mu \right\}, \quad (t \in \mathbb{R})$$

where  $\widehat{f}(G, i, o) := f(G, o) \wedge f(G, i)$ .

**The special case of unimodular Galton-Watson trees.** Our second main result is an explicit resolution of the above variational problem in the important special case where  $\mu = \text{UGWT}(\pi)$ , for an arbitrary degree distribution  $\pi = \{\pi_n\}_{n \geq 0}$  on  $\mathbb{N}$  with finite, non-zero mean. Throughout the paper, we let  $[x]_0^1$  denote the closest point to  $x \in \mathbb{R}$  in the interval  $[0, 1]$ , i.e.

$$[x]_0^1 := \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \in [0, 1] \\ 1 & \text{if } x \geq 1. \end{cases}$$

Given  $t \in \mathbb{R}$  and  $Q \in \mathcal{P}([0, 1])$ , we let  $F_{\pi, t}(Q) \in \mathcal{P}([0, 1])$  denote the law of

$$[1 - t + \xi_1 + \cdots + \xi_{\widehat{D}}]_0^1,$$

where  $\widehat{D}$  follows the size-biased distribution  $\widehat{\pi}$  defined at (2), and where  $\{\xi_n\}_{n \geq 1}$  are IID with law  $Q$ , independent of  $\widehat{D}$ . As it turns out, the solutions to the distributional fixed-point equation  $Q = F_{\pi, t}(Q)$  determine  $\Phi_\mu(t)$ .

**Theorem 2.** *When  $\mu = \text{UGWT}(\pi)$ , we have for every  $t \in \mathbb{R}$ :*

$$\Phi_\mu(t) = \max_{Q=F_{\pi,t}(Q)} \left\{ \frac{\mathbb{E}[D]}{2} \mathbb{P}(\xi_1 + \xi_2 > 1) - t \mathbb{P}(\xi_1 + \dots + \xi_D > t) \right\},$$

where  $D \sim \pi$  and where  $\{\xi_n\}_{n \geq 1}$  are IID with law  $Q$ , independent of  $D$ . The maximum is over all choices of  $Q \in \mathcal{P}([0, 1])$  subject to  $Q = F_{\pi,t}(Q)$ .

**Back to the densest subgraph problem.** By analogy with the case of finite graphs, we define the maximum subgraph density of a measure  $\mu \in \mathcal{U}$  with  $\deg(\mu) < \infty$  as the essential supremum of the random variable  $\partial\Theta_0$  constructed in Theorem 1. In other words,

$$\varrho(\mu) := \sup\{t \in \mathbb{R} : \Phi_\mu(t) > 0\}.$$

In light of Theorem 1, it is natural to seek a continuity principle of the form

$$\left( G_n \xrightarrow[n \rightarrow \infty]{\text{LWC}} \mu \right) \implies \left( \varrho(G_n) \xrightarrow[n \rightarrow \infty]{} \varrho(\mu) \right). \quad (4)$$

However, a moment of thought shows that the graph parameter  $\varrho(G)$  is too sensitive to be captured by local weak convergence. Indeed, if  $|V(G_n)| \rightarrow \infty$ , then adding a large but fixed clique to  $G_n$  will arbitrarily boost the value of  $\varrho(G_n)$  without affecting the local weak limit of  $\{G_n\}_{n \geq 1}$ . Nevertheless, our third main result states that (4) holds for graphs produced by the pairing model, under a mild exponential moment assumption.

**Theorem 3.** *Consider a degree sequence  $\mathbf{d}_n = \{d_n(i)\}_{1 \leq i \leq n}$  for each  $n \geq 1$ . Assume that (3) holds for some  $\pi = \{\pi_k\}_{k \geq 1}$  with  $\pi_0 + \pi_1 < 1$ , and that*

$$\sup_{n \geq 1} \left\{ \frac{1}{n} \sum_{i=1}^n e^{\theta d_n(i)} \right\} < \infty, \quad (5)$$

for some  $\theta > 0$ . Then,  $\varrho(\mathbb{G}[\mathbf{d}_n]) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \varrho(\mu)$  with  $\mu = \text{UGWT}(\pi)$ .

Note that this result applies in particular to the Erdős-Rényi random graph  $\mathbb{G}_n$  with  $n$  vertices and  $m = \lfloor \alpha n \rfloor$  edges. Indeed, the conditional law of  $\mathbb{G}_n$  given its (random) degree sequence  $\mathbf{d}_n$  is precisely  $\mathbb{G}[\mathbf{d}_n]$ , and  $\{\mathbf{d}_n\}_{n \geq 1}$  satisfies a-s the conditions (3) and (5) with  $\pi = \text{Poisson}(2\alpha)$ . Therefore, Theorems 1, 2 and 3 settle and generalize the conjectures of Hajek [17]. Since a graph  $G$  is  $k$ -orientable ( $k \in \mathbb{N}$ ) if and only if  $\varrho(G) < k$ , Theorem 3 also extends recent results on the  $k$ -orientability of the Erdős-Rényi random graph [12, 11]. See also [15, 13, 22, 21] for various generalizations.



## 4 Proof outline and related work.

**The objective method.** This work is a new illustration of the general principles exposed in the *objective method* by Aldous and Steele [3]. The latter provides a powerful framework for the unified study of sparse random graphs and has already led to several remarkable results. Two prototypical examples are the celebrated  $\zeta(2)$  limit in the random assignment problem due to Aldous [4], and the asymptotic enumeration of spanning trees in large graphs by Lyons [24]. Since then, the method has been successfully applied to various other combinatorial enumeration/optimization problems on graphs, including (but not limited to) [28, 14, 26, 10, 22, 25, 21, 20].

**Lack of correlation decay.** In the problem considered here, there is a major obstacle to a straightforward application of the objective method: the balanced load at a vertex is *not* determined by the local environment around that vertex. For example, a ball of radius  $h$  in a  $d$ -regular graph with girth  $h$  is indistinguishable from that of the root in a  $d$ -regular tree with height  $h$ . However, the induced load is  $\frac{d}{2}$  in the first case and  $1 - \frac{1}{(d-1)^{h-1}d}$  in the second case. This long-range dependence gives rise to non-uniqueness issues when one tries to properly extend the notion of balanced loads from finite to infinite graphs. We refer the reader to [17] for a detailed study of this phenomenon, therein called *load percolation*, as well as several related questions.

**Relaxation.** To overcome the lack of correlation decay, we introduce a suitable relaxation of the balancing condition (1), which we call  $\varepsilon$ -balancing. Remarkably enough, any positive value of the perturbative parameter  $\varepsilon$  suffices to annihilate the long-range dependences described above. This allows us to define a unique  $\varepsilon$ -balanced Borel allocation  $\Theta_\varepsilon: \mathcal{G}_{**} \rightarrow [0, 1]$  and to establish the continuity of the induced load  $\partial\Theta_\varepsilon: \mathcal{G}_* \rightarrow [0, \infty)$  with respect to local convergence (Section 5). In the limit where  $\varepsilon$  tends to 0, we further prove that  $\Theta_\varepsilon$  converges in a certain sense, and that the limiting Borel allocation  $\Theta_0$  is balanced (Section 6). This quickly leads to a proof of Theorem 1 (Section 7). In spirit, the role of the perturbative parameter  $\varepsilon > 0$  is comparable to that of the temperature in [10], although no Gibbs-Boltzmann measure is involved in the present work.

**Recursion on trees.** As many other graph-theoretical problems, load balancing has a simple recursive structure when considered on trees. Indeed, once the value of the allocation along a given edge  $\{i, j\}$  has been fixed, the problem naturally decomposes into two independent sub-problems, cor-

responding to the two disjoint subtrees formed by removing  $\{i, j\}$ . Note, however, that the loads of  $i$  and  $j$  must be shifted by a suitable amount to take into account the contribution of the removed edge. The precise effect of this shift on the loads induced at  $i$  and  $j$  defines what we call the *response functions* of the two subtrees (Section 8). It is those response functions that satisfy a recursion (Section 9). Recursions on trees automatically give rise to distributional fixed-point equations when specialized to Galton-Watson trees. Such equations are a common ingredient in the objective method, see [5]. In our case this leads to the proof of Theorem 2 (Section 10).

**Dense subgraphs in the pairing model.** Finally, the proof of Theorem 3 (Section 11) relies on a property of random graphs with a prescribed degree sequence that might be of independent interest: under an exponential moment assumption, we use the first-moment method to prove that dense subgraphs are extensively large with high probability. See Proposition 12 for the precise statement, and [23, Lemma 6] for a result in the same direction.

## 5 $\varepsilon$ -balancing

In all this section,  $G = (V, E)$  is a locally finite graph and  $\varepsilon > 0$  is a fixed parameter. An allocation  $\theta$  on  $G$  is called  $\varepsilon$ -balanced if for every  $(i, j) \in \vec{E}$ ,

$$\theta(i, j) = \left[ \frac{1}{2} + \frac{\partial\theta(i) - \partial\theta(j)}{2\varepsilon} \right]_0^1. \quad (6)$$

This can be viewed as a relaxed version of (1). Its interest lies in the fact that it fixes the non-uniqueness issue on infinite graphs.

**Proposition 1** (Existence, uniqueness and monotony). *If  $G$  has bounded degrees, then there is a unique  $\varepsilon$ -balanced allocation  $\theta$  on  $G$ . If moreover  $E' \subseteq E$ , then the  $\varepsilon$ -balanced allocation  $\theta'$  on  $G' = (V, E')$  satisfies  $\partial\theta' \leq \partial\theta$ .*

*Proof.* Existence is a consequence of Schauder's fixed-point Theorem, see e.g. [1, Theorem 8.2]. Now, consider  $E' \subseteq E$  and let  $\theta, \theta'$  be  $\varepsilon$ -balanced allocations on  $G, G'$  respectively. Fix  $o \in V$  and set

$$I := \{i \in V : \{i, o\} \in E', \theta'(i, o) > \theta(i, o)\}.$$

Clearly,

$$\partial\theta'(o) - \partial\theta(o) \leq \sum_{i \in I} (\theta'(i, o) - \theta(i, o)).$$

On the other-hand, since the map  $x \mapsto \left[\frac{1}{2} + \frac{x}{2\varepsilon}\right]_0^1$  is non-decreasing and Lipschitz with constant  $\frac{1}{2\varepsilon}$ , our assumption on  $\theta, \theta'$  implies that for every  $i \in I$ ,

$$\theta'(i, o) - \theta(i, o) \leq \frac{1}{2\varepsilon} (\partial\theta'(i) - \partial\theta(i) - \partial\theta'(o) + \partial\theta(o)).$$

Injecting this into the above inequality and rearranging, we obtain

$$\begin{aligned} \partial\theta'(o) - \partial\theta(o) &\leq \frac{1}{|I| + 2\varepsilon} \sum_{i \in I} (\partial\theta'(i) - \partial\theta(i)) \\ &\leq \frac{\Delta}{\Delta + 2\varepsilon} \max_{i \in I} (\partial\theta'(i) - \partial\theta(i)), \end{aligned} \quad (7)$$

where  $\Delta$  denotes the maximum degree in  $G$ . Now, observe that  $\partial\theta, \partial\theta'$  are  $[0, \Delta]$ -valued, so that  $M := \sup_V (\partial\theta' - \partial\theta)$  is finite. Property (7) forces  $M \leq 0$ , which proves the monotony  $E' \subseteq E \implies \partial\theta' \leq \partial\theta$ . In particular,  $E' = E$  implies  $\partial\theta' = \partial\theta$ , which in turns forces  $\theta' = \theta$ , thanks to (6).  $\square$

We now remove the bounded-degree assumption as follows. Fix  $\Delta \in \mathbb{N}$ , and consider the truncated graph  $G^\Delta = (V, E^\Delta)$  obtained from  $G$  by isolating all nodes having degree more than  $\Delta$ , i.e.

$$E^\Delta = \{\{i, j\} \in E : \deg(G, i) \vee \deg(G, j) \leq \Delta\}.$$

By construction,  $G^\Delta$  has degree at most  $\Delta$ , and we let  $\Theta_\varepsilon^\Delta(G, i, j)$  denote the mass sent along  $(i, j) \in \vec{E}$  in the unique  $\varepsilon$ -balanced allocation on  $G^\Delta$ , with the understanding that  $\Theta_\varepsilon^\Delta(G, i, j) = 0$  if  $\{i, j\} \notin E^\Delta$ . By uniqueness, this quantity depends only on the isomorphism class of the edge-rooted graph  $(G, i, j)$ , so that we have a well-defined map  $\Theta_\varepsilon^\Delta: \mathcal{G}_{**} \rightarrow [0, 1]$ . By an immediate induction on  $r \in \mathbb{N}$ , the local contraction (7) yields

$$[G, o]_r \equiv [G', o']_r \implies |\partial\Theta_\varepsilon^\Delta(G, o) - \partial\Theta_\varepsilon^\Delta(G', o')| \leq \Delta \left(1 + \frac{2\varepsilon}{\Delta}\right)^{-r}.$$

Since the map  $x \mapsto \left[\frac{1}{2} + \frac{x}{2\varepsilon}\right]_0^1$  is Lipschitz with constant  $\frac{1}{2\varepsilon}$ , it follows that

$$[G, i, j]_r \equiv [G', i', j']_r \implies |\Theta_\varepsilon^\Delta(G, i, j) - \Theta_\varepsilon^\Delta(G', i', j')| \leq \frac{\Delta}{2\varepsilon} \left(1 + \frac{2\varepsilon}{\Delta}\right)^{-r}.$$

Thus, the map  $\Theta_\varepsilon^\Delta$  is equicontinuous. Now, the sequence of sets  $\{E_\Delta\}_{\Delta \geq 1}$  increases to  $E$ , so the monotony in Proposition 1 guarantees that  $\{\partial\Theta_\varepsilon^\Delta\}_{\Delta \geq 1}$  converges pointwise on  $\mathcal{G}_*$ . Moreover, any given  $\{i, j\} \in E$  belongs to  $E^\Delta$  for large enough  $\Delta$ , and the definition of  $\varepsilon$ -balancing yields

$$\Theta_\varepsilon^\Delta(G, i, j) = \left[\frac{1}{2} + \frac{\partial\Theta_\varepsilon^\Delta(G, i) - \partial\Theta_\varepsilon^\Delta(G, j)}{2\varepsilon}\right]_0^1.$$

Consequently, the pointwise limit  $\Theta_\varepsilon := \lim_{\Delta \rightarrow \infty} \Theta_\varepsilon^\Delta$  exists in  $[0, 1]^{\mathcal{G}_{**}}$ . It clearly satisfies  $\Theta_\varepsilon + \Theta_\varepsilon^* = 1$  and it is Borel as the pointwise limit of continuous maps. Thus, it is a Borel allocation. Moreover, letting  $\Delta \rightarrow \infty$  above yields

$$\Theta_\varepsilon(G, i, j) = \left[ \frac{1}{2} + \frac{\partial \Theta_\varepsilon(G, i) - \partial \Theta_\varepsilon(G, j)}{2\varepsilon} \right]_0^1. \quad (8)$$

## 6 The $\varepsilon \rightarrow 0$ limit

In this section, we fix  $\mu \in \mathcal{U}$  with  $\deg(\mu) < \infty$ . We write  $\|f\|_p$  for the norm in both  $L^p(\mu)$  and  $L^p(\bar{\mu})$ , as which one is meant should be clear from the context. Note that by unimodularity, we have for any Borel allocation  $\Theta$ ,

$$\|\Theta\|_1 = \int_{\mathcal{G}_{**}} \Theta d\bar{\mu} = \int_{\mathcal{G}_{**}} \frac{\Theta + \Theta^*}{2} d\bar{\mu} = \frac{\deg(\mu)}{2}. \quad (9)$$

**Proposition 2.** *The limit  $\Theta_0 := \lim_{\varepsilon \rightarrow 0} \Theta_\varepsilon$  exists in  $L^2(\bar{\mu})$  and is a balanced Borel allocation on  $\mu$ .*

*Proof.* We will establish the following Cauchy property: for  $0 < \varepsilon \leq \varepsilon'$ ,

$$\|\Theta_{\varepsilon'} - \Theta_\varepsilon\|_2^2 \leq \|\Theta_\varepsilon\|_2^2 - \|\Theta_{\varepsilon'}\|_2^2. \quad (10)$$

This guarantees the existence of  $\Theta_0 = \lim_{\varepsilon \rightarrow 0} \Theta_\varepsilon$  in  $L^2(\bar{\mu})$ . The rest of the claim follows, since Borel allocations are closed in  $L^2(\bar{\mu})$  and letting  $\varepsilon \rightarrow 0$  in (8) shows that  $\Theta_0$  is balanced on  $\mu$ . Let us first prove (10) under the extra assumption that  $\mu(\{(G, o) : \deg(G, o) \leq \Delta\}) = 1$  for some  $\Delta \in \mathbb{N}$ . This ensures that  $f \in L^2(\bar{\mu})$ , where

$$f(G, i, o) := \partial \Theta_\varepsilon(G, o) + \varepsilon \Theta_\varepsilon(G, i, o).$$

A straightforward manipulation of (8) shows that

$$f(G, i, o) > f(G, o, i) \implies \Theta_\varepsilon(G, i, o) = 0.$$

This implies  $\Theta_\varepsilon f + \Theta_\varepsilon^* f^* = f \wedge f^*$ . On the other hand,  $f \wedge f^* \leq \Theta_{\varepsilon'} f + \Theta_{\varepsilon'}^* f^*$  since  $\Theta_{\varepsilon'} + \Theta_{\varepsilon'}^* = 1$ . Thus,  $\Theta_\varepsilon f + \Theta_\varepsilon^* f^* \leq \Theta_{\varepsilon'} f + \Theta_{\varepsilon'}^* f^*$ . Integrating against  $\bar{\mu}$  and invoking unimodularity, we get  $\langle \Theta_\varepsilon - \Theta_{\varepsilon'}, f \rangle_{L^2(\bar{\mu})} \leq 0$  or more explicitly,

$$\langle \partial \Theta_\varepsilon - \partial \Theta_{\varepsilon'}, \partial \Theta_\varepsilon \rangle_{L^2(\mu)} + \varepsilon \langle \Theta_\varepsilon - \Theta_{\varepsilon'}, \Theta_\varepsilon \rangle_{L^2(\bar{\mu})} \leq 0.$$

But we have not yet used  $\varepsilon \leq \varepsilon'$ , so we may exchange  $\varepsilon, \varepsilon'$  to get

$$\langle \partial \Theta_{\varepsilon'} - \partial \Theta_\varepsilon, \partial \Theta_{\varepsilon'} \rangle_{L^2(\mu)} + \varepsilon' \langle \Theta_{\varepsilon'} - \Theta_\varepsilon, \Theta_{\varepsilon'} \rangle_{L^2(\bar{\mu})} \leq 0.$$

Adding-up those inequalities and rearranging, we finally arrive at

$$(\varepsilon' - \varepsilon)\langle \Theta_\varepsilon - \Theta_{\varepsilon'}, \Theta_{\varepsilon'} \rangle_{L^2(\bar{\mu})} \geq \|\partial\Theta_\varepsilon - \partial\Theta_{\varepsilon'}\|_2^2 + \varepsilon\|\Theta_\varepsilon - \Theta_{\varepsilon'}\|_2^2.$$

In particular,  $\langle \Theta_\varepsilon, \Theta_{\varepsilon'} \rangle_{L^2(\bar{\mu})} \geq \|\Theta_{\varepsilon'}\|_2^2$  and (10) follows since

$$\|\Theta_{\varepsilon'} - \Theta_\varepsilon\|_2^2 = \|\Theta_{\varepsilon'}\|_2^2 + \|\Theta_\varepsilon\|_2^2 - 2\langle \Theta_\varepsilon, \Theta_{\varepsilon'} \rangle_{L^2(\bar{\mu})}.$$

Finally, if our extra assumption is dropped, we may use (10) with  $\Theta_\varepsilon, \Theta_{\varepsilon'}$  replaced by  $\Theta_\varepsilon^\Delta, \Theta_{\varepsilon'}^\Delta$  and let then  $\Delta \rightarrow \infty$ . By construction,  $\Theta_\varepsilon^\Delta \rightarrow \Theta_\varepsilon$  and  $\Theta_{\varepsilon'}^\Delta \rightarrow \Theta_{\varepsilon'}$  pointwise, and (10) follows by dominated convergence.  $\square$

**Proposition 3.** *Let  $\{G_n\}_{n \geq 1}$  be finite graphs with local weak limit  $\mu$ . Then,*

$$\mathcal{L}_{G_n} \xrightarrow[n \rightarrow \infty]{\mathcal{P}(\mathbb{R})} \mathcal{L},$$

where  $\mathcal{L} = \mathcal{L}_\mu$  is the law of the random variable  $\partial\Theta_0 \in L^1(\mu)$ .

*Proof.* For each  $n \geq 1$  we let  $\widehat{G}_n$  denote the network obtained by encoding a balanced allocation  $\theta_n$  as  $[0, 1]$ -valued marks on the oriented edges of  $G_n$ . The sequence  $\{\mathcal{U}(\widehat{G}_n)\}_{n \geq 1}$  is tight, because  $\{\mathcal{U}(G_n)\}_{n \geq 1}$  converges weakly and the marks are  $[0, 1]$ -valued. Consider any subsequential weak limit  $(\mathbb{G}, o, \theta)$ . By construction,  $(\mathbb{G}, o)$  has law  $\mu$  and  $\theta$  is a-s a balanced allocation on  $\mathbb{G}$ . Our goal is to establish that a-s,  $\partial\theta(o) = \partial\Theta_0(\mathbb{G}, o)$ . Set  $\theta'(i, j) := \Theta_0(\mathbb{G}, i, j)$ . Note that the random rooted network  $(\mathbb{G}, o, \theta, \theta')$  is unimodular, since  $(\mathbb{G}, o, \theta)$  is a local weak limit of finite networks and  $\Theta_0$  is Borel. Now,

$$\begin{aligned} \mathbb{E} \left[ (\partial\theta(o) - \partial\theta'(o))^+ \right] &= \mathbb{E} \left[ \sum_{i \sim o} (\theta(i, o) - \theta'(i, o)) \mathbf{1}_{\partial\theta(o) > \partial\theta'(o)} \right] \\ &= \mathbb{E} \left[ \sum_{i \sim o} (\theta(o, i) - \theta'(o, i)) \mathbf{1}_{\partial\theta(o) > \partial\theta'(o)} \right] \\ &= \mathbb{E} \left[ \sum_{i \sim o} (\theta'(i, o) - \theta(i, o)) \mathbf{1}_{\partial\theta(o) > \partial\theta'(o)} \right], \end{aligned}$$

where the second equality follows from unimodularity and the third one from the identities  $\theta(o, i) = 1 - \theta(i, o)$  and  $\theta'(o, i) = 1 - \theta'(i, o)$ . Combining the first and last lines, we see that  $\mathbb{E} \left[ (\partial\theta(o) - \partial\theta'(o))^+ \right]$  equals

$$\frac{1}{2} \mathbb{E} \left[ \sum_{i \sim o} (\theta(i, o) - \theta'(i, o)) (\mathbf{1}_{\partial\theta(o) > \partial\theta'(o)} - \mathbf{1}_{\partial\theta(i) > \partial\theta'(i)}) \right].$$

The fact that  $\theta, \theta'$  are balanced on the edge  $\{i, o\}$  easily implies that  $\theta(i, o) - \theta'(i, o)$  and  $\mathbf{1}_{\partial\theta(o) > \partial\theta'(o)} - \mathbf{1}_{\partial\theta(i) > \partial\theta'(i)}$  can neither be simultaneously positive, nor simultaneously negative. Therefore,  $\mathbb{E} \left[ (\partial\theta(o) - \partial\theta'(o))^+ \right] \leq 0$ . Exchanging the roles of  $\theta, \theta'$  yields  $\partial\theta(o) = \partial\theta'(o)$  a-s, as desired.  $\square$

## 7 Proof of Theorem 1

**Proposition 4.** *Let  $\Theta$  be a Borel allocation. Then for all  $t \in \mathbb{R}$ ,*

$$\int_{\mathcal{G}_*} (\partial\Theta - t)^+ d\mu \geq \sup_{\substack{f: \mathcal{G}_* \rightarrow [0,1] \\ \text{Borel}}} \left\{ \frac{1}{2} \int_{\mathcal{G}_{**}} \widehat{f} d\bar{\mu} - t \int_{\mathcal{G}_*} f d\mu \right\},$$

*with equality for all  $t \in \mathbb{R}$  if and only if  $\Theta$  is balanced on  $\mu$ .*

*Proof.* Fix a Borel  $f: \mathcal{G}_* \rightarrow [0,1]$ . Clearly,  $(\partial\Theta - t)^+ \geq (\partial\Theta - t)f$  and hence

$$\int_{\mathcal{G}_*} (\partial\Theta - t)^+ d\mu \geq \int_{\mathcal{G}_*} f \partial\Theta d\mu - t \int_{\mathcal{G}_*} f d\mu. \quad (11)$$

Using the unimodularity of  $\mu$  and the identity  $\Theta + \Theta^* = 1$ , we have

$$\begin{aligned} \int_{\mathcal{G}_*} f \partial\Theta d\mu &= \frac{1}{2} \int_{\mathcal{G}_{**}} (f(G,o)\Theta(G,i,o) + f(G,i)\Theta(G,o,i)) d\bar{\mu}(G,i,o) \\ &\geq \frac{1}{2} \int_{\mathcal{G}_{**}} (f(G,o) \wedge f(G,i)) d\bar{\mu}(G,i,o). \end{aligned} \quad (12)$$

Combining (11) and (12) proves the inequality. Let us now examine the equality case. First, equality holds in (11) if and only if for  $\mu$ -a-e  $(G,o) \in \mathcal{G}_*$ ,

$$\begin{aligned} \partial\Theta(G,o) > t &\implies f(G,o) = 1 \\ \partial\Theta(G,o) < t &\implies f(G,o) = 0. \end{aligned}$$

Second, equality holds in (12) if and only if for  $\bar{\mu}$ -a-e  $(G,i,o) \in \mathcal{G}_{**}$ ,

$$f(G,i) < f(G,o) \implies \Theta(G,i,o) = 0.$$

If  $\Theta$  is balanced on  $\mu$ , then the choice  $f = \mathbf{1}_{\{\partial\Theta > t\}}$  clearly satisfies all those requirements, so that equality holds for each  $t \in \mathbb{R}$  in the Proposition. This proves the *if* part and shows that the supremum in Proposition 4 is attained, since at least one balanced allocation exists by Proposition 2. Now, for the *only if* part, suppose that equality is achieved in Proposition 4. Then the above requirements imply that for  $\bar{\mu}$ -a-e  $(G,i,o) \in \mathcal{G}_{**}$ ,

$$\partial\Theta(G,o) < t < \partial\Theta(G,i) \implies \Theta(G,o,i) = 1.$$

Since this must be true for all  $t \in \mathbb{Q}$ , it follows that  $\Theta$  is balanced on  $\mu$ .  $\square$

*Proof of Theorem 1.* The existence, the continuity and the variational characterization were established in Proposition 2, 3 and 4, respectively. Now, let  $\Theta, \Theta'$  be Borel allocations, and assume that  $\Theta$  is balanced. Applying Proposition 4 to  $\Theta$  and  $\Theta'$  shows that for all  $t \in \mathbb{R}$ ,

$$\int_{\mathcal{G}_*} (\partial\Theta - t)^+ d\mu \leq \int_{\mathcal{G}_*} (\partial\Theta' - t)^+ d\mu.$$

On the other-hand, (9) guarantees that  $\partial\Theta, \partial\Theta'$  have the same mean. Those two conditions together are well-known to be equivalent to the convex ordering  $\partial\Theta \preceq_{cx} \partial\Theta'$  (see e.g. [27]), meaning that for any convex  $f: [0, \infty) \rightarrow \mathbb{R}$ ,

$$\int_{\mathcal{G}_*} (f \circ \partial\Theta) d\mu \leq \int_{\mathcal{G}_*} (f \circ \partial\Theta') d\mu.$$

We have just proved (i)  $\implies$  (iii), and (iii)  $\implies$  (ii) is obvious. In particular,  $\Theta_0$  satisfies (ii) and (iii). The *only if* part of Proposition 4 shows that (iii)  $\implies$  (i). The implication (iv)  $\implies$  (iii) is obvious given that  $\Theta_0$  satisfies (iii). Thus, it only remains to prove (ii)  $\implies$  (iv). Assume that  $\Theta$  minimizes  $\int (f \circ \partial\Theta) d\mu$  for some strictly convex function  $f: [0, \infty) \rightarrow \mathbb{R}$ , and let  $m$  denote the value of this minimum. Since  $\Theta_0$  satisfies (ii), we also have  $\int (f \circ \partial\Theta_0) d\mu = m$ . But then  $\Theta' := (\Theta_0 + \Theta)/2$  is an allocation and by convexity,

$$\int_{\mathcal{G}_*} (f \circ \partial\Theta') d\mu \leq \int_{\mathcal{G}_*} \frac{(f \circ \partial\Theta) + (f \circ \partial\Theta_0)}{2} d\mu = m.$$

This inequality contradicts the definition of  $m$ , unless it is an equality. Since  $f$  is strictly convex, this forces  $\partial\Theta = \partial\Theta_0$   $\mu$ -a-e.  $\square$

## 8 Response functions

As many other graph-theoretical problems, load balancing has a simple recursive structure when specialized to trees. However, the exact formulation of this recursion requires the possibility to *condition* the allocation to take a certain value at a given edge, and we first need to give a proper meaning to this operation. Let  $G = (V, E)$  be a locally finite graph and  $b: V \rightarrow \mathbb{R}$  a function called the *baseload*. An allocation  $\theta$  is *balanced with respect to  $b$*  if

$$b(i) + \partial\theta(i) < b(j) + \partial\theta(j) \implies \theta(i, j) = 0,$$

for all  $(i, j) \in \vec{E}$ . This is precisely the definition of balancing, except that the load *felt* by each vertex  $i \in V$  is shifted by a certain amount  $b(i)$ . Similarly,

$\theta$  is  $\varepsilon$ -balanced with respect to  $b$  if for all  $(i, j) \in \vec{E}$ ,

$$\theta(i, j) = \left[ \frac{1}{2} + \frac{b(i) + \partial\theta(i) - b(j) - \partial\theta(j)}{2\varepsilon} \right]_0^1. \quad (13)$$

The arguments used in Proposition 1 are easily extended to this situation.

**Proposition 5** (Existence, uniqueness and monotony). *If  $G$  has bounded degree and if  $b$  is bounded, then there is a unique  $\varepsilon$ -balanced allocation with baseload  $b$ . Moreover, if  $b' \leq b$  is bounded and if  $E' \subseteq E$ , then the  $\varepsilon$ -balanced allocation  $\theta'$  on  $G' = (V, E')$  with baseload  $b'$  satisfies  $b' + \partial\theta' \leq b + \partial\theta$  on  $V$ .*

As in Section 5, we then define an  $\varepsilon$ -balanced allocation in the general case by considering the truncated graph  $G^\Delta$  with baseload the truncation of  $b$  to  $[-\Delta, \Delta]$ , and let then  $\Delta \rightarrow \infty$ . Monotony guarantees the existence of a limiting  $\varepsilon$ -balanced allocation. We shall need the following property.

**Proposition 6** (Non-expansion). *Let  $\theta, \theta'$  be the  $\varepsilon$ -balanced allocations with baseloads  $b, b' : V \rightarrow \mathbb{R}$ . Set  $f = \partial\theta + b$  and  $f' = \partial\theta' + b'$ . Then,*

$$\|f' - f\|_{\ell^1(V)} \leq \|b' - b\|_{\ell^1(V)}.$$

*Proof.* By considering  $b'' = b \wedge b'$  and using the triangle inequality, we may assume that  $b \leq b'$ . Note that this implies  $f \leq f'$ , thanks to Proposition 5. When  $G$  is finite, the claim trivially follows from conservation of mass:

$$\sum_{o \in V} (f'(o) - f(o)) = \sum_{o \in V} (b'(o) - b(o)).$$

This then extends to the case where  $G$  has bounded degrees with  $b, b'$  bounded as follows: choose finite subsets  $V_1 \subseteq V_2 \subseteq \dots$  such that  $\cup_{n \geq 1} V_n = V$ . For each  $n \geq 1$ , let  $\theta_n, \theta'_n$  denote the  $\varepsilon$ -balanced allocations on the subgraph induced by  $V_n$ , with baseloads the restrictions of  $b, b'$  to  $V_n$ . Then  $\theta_n \rightarrow \theta$  and  $\theta'_n \rightarrow \theta'$  pointwise, by compactness and uniqueness. Now, any finite  $K \subseteq V$  is contained in  $V_n$  for large enough  $n$ , and since  $V_n$  is finite we know that  $f_n := \partial\theta_n + b$  and  $f'_n := \partial\theta'_n + b'$  satisfy

$$\sum_{i \in K} |f'_n(i) - f_n(i)| \leq \sum_{i \in V_n} |b'(i) - b(i)|.$$

Letting  $n \rightarrow \infty$  yields the desired result, since  $K$  is arbitrary. Finally, for the general case, we may apply the result to the truncated graph  $G^\Delta$  with baseloads the truncation of  $b, b'$  to  $[-\Delta, \Delta]$ , and let then  $\Delta \rightarrow \infty$ .  $\square$



Although the uniqueness established in Proposition 5 does not extend to the  $\varepsilon = 0$  case, the following weaker result will be useful in the next Section.

**Proposition 7** (Weak uniqueness). *Assume that  $\theta, \theta'$  are balanced with respect to  $b$  and that  $\|\partial\theta - \partial\theta'\|_{\ell^1(V)} < \infty$ . Then,  $\partial\theta = \partial\theta'$ .*

*Proof.* Fix  $\delta > 0$ . Then the level set  $S := \{j \in V : \partial\theta'(j) - \partial\theta(j) > \delta\}$  must be finite. Therefore, it satisfies the conservation of mass :

$$\sum_{j \in S} \partial\theta'(j) - \partial\theta(j) = \sum_{(i,j) \in E(V-S,S)} \theta'(i,j) - \theta(i,j). \quad (14)$$

Now, if  $(i,j) \in E(V-S,S)$  then clearly,  $\partial\theta'(i) - \partial\theta(i) < \partial\theta'(j) - \partial\theta(j)$ . Consequently, at least one of the following inequalities must hold :

$$b(j) - b(i) < \partial\theta(i) - \partial\theta(j) \quad \text{or} \quad b(j) - b(i) > \partial\theta'(i) - \partial\theta'(j).$$

The first one implies  $\theta(i,j) = 1$  and the second  $\theta'(i,j) = 0$ , since  $\theta, \theta'$  are balanced with respect to  $b$ . In either case, we have  $\theta'(i,j) \leq \theta(i,j)$ . Thus, the right-hand side of (14) is non-positive, hence so must the left-hand side be. This contradicts the definition of  $S$  unless  $S = \emptyset$ , i.e.  $\partial\theta' \leq \partial\theta + \delta$ . Since  $\delta$  is arbitrary, we conclude that  $\partial\theta' \leq \partial\theta$ . Equality follows by symmetry.  $\square$

Given  $o \in V$  and  $x \in \mathbb{R}$ , we set  $f_{(G,o)}^\varepsilon(x) = x + \partial\theta(o)$  where  $\theta$  is the  $\varepsilon$ -balanced allocation with baseload  $x$  at  $o$  and 0 elsewhere. We call  $f_{(G,o)}^\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  the *response function* of the rooted graph  $(G, o)$ . Propositions 5 and 6 guarantee that  $f_{(G,o)}^\varepsilon$  is non-decreasing and non-expansive, i.e.

$$x \leq y \implies 0 \leq f_{(G,o)}^\varepsilon(y) - f_{(G,o)}^\varepsilon(x) \leq y - x. \quad (15)$$

Note for future use that the definition of  $f_{(G,o)}^\varepsilon(x)$  also implies

$$0 \leq f_{(G,o)}^\varepsilon(x) - x \leq \deg(G, o). \quad (16)$$

When  $G$  is a tree, response functions turn out to satisfy a simple recursion.

## 9 A recursion on trees

We are now ready to formulate the promised recursion. Fix a tree  $T = (V, E)$ . Deleting  $\{i, j\} \in E$  creates two disjoint subtrees, which we view as rooted at  $i$  and  $j$  and denote  $T_{i \rightarrow j}$  and  $T_{j \rightarrow i}$ , respectively.

**Proposition 8.** For any  $o \in V$ , the response function  $\mathfrak{f}_{(T,o)}^\varepsilon$  is invertible and

$$\{\mathfrak{f}_{(T,o)}^\varepsilon\}^{-1} = \text{Id} - \sum_{i \sim o} \left[ 1 - \{\mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon + \varepsilon(2\text{Id} - 1)\}^{-1} \right]_0^1. \quad (17)$$

*Proof.*  $\mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon + \varepsilon(2\text{Id} - 1)$  increases continuously from  $\mathbb{R}$  onto  $\mathbb{R}$ , so its inverse  $\{\mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon + \varepsilon(2\text{Id} - 1)\}^{-1}$  exists and increases continuously from  $\mathbb{R}$  onto  $\mathbb{R}$ . Consequently, the function  $g: \mathbb{R} \rightarrow \mathbb{R}$  appearing in the right-hand side of (17) is itself continuously increasing from  $\mathbb{R}$  onto  $\mathbb{R}$ , hence invertible. Given  $x \in \mathbb{R}$ , it now remains to prove that  $t := \mathfrak{f}_{(T,o)}^\varepsilon(x)$  satisfies  $g(t) = x$ . By definition,

$$t = x + \partial\theta(o), \quad (18)$$

where  $\theta$  denotes the  $\varepsilon$ -balanced allocation on  $T$  with baseload  $x$  at  $o$  and 0 elsewhere. Now fix  $i \sim o$ . The restriction of  $\theta$  to  $T_{i \rightarrow o}$  is clearly an  $\varepsilon$ -balanced allocation on  $T_{i \rightarrow o}$  with baseload  $\theta(o, i)$  at  $i$  and 0 elsewhere. This is precisely the allocation appearing in the definition of  $\mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon(\theta(o, i))$ , hence

$$\mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon(\theta(o, i)) = \partial\theta(i).$$

Thus, the fact that  $\theta$  is  $\varepsilon$ -balanced along  $(o, i)$  may now be rewritten as

$$\theta(o, i) = \left[ \frac{1}{2} + \frac{t - \mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon(\theta(o, i))}{2\varepsilon} \right]_0^1. \quad (19)$$

But by definition,  $x_i := \{\mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon + \varepsilon(2\text{Id} - 1)\}^{-1}(t)$  is the unique solution to

$$x_i = \frac{1}{2} + \frac{t - \mathfrak{f}_{T_{i \rightarrow o}}^\varepsilon(x_i)}{2\varepsilon}. \quad (20)$$

Comparing (19) and (20), we see that  $\theta(o, i) = [x_i]_0^1$ , i.e.  $\theta(i, o) = [1 - x_i]_0^1$ . Re-injecting this into (18), we arrive exactly at the desired  $x = g(t)$ .  $\square$

In the remainder of this section, we fix a vanishing sequence  $\{\varepsilon_n\}_{n \geq 1}$  and study the pointwise limit  $\mathfrak{f} = \lim_{n \rightarrow \infty} \mathfrak{f}_{(T,o)}^{\varepsilon_n}$ , when it exists. Note that  $\mathfrak{f}$  needs not be invertible. However, (15) and (16) guarantee that  $\mathfrak{f}$  is non-decreasing with  $\mathfrak{f}(\pm\infty) = \pm\infty$ , so that it admits a well-defined right-continuous inverse

$$\mathfrak{f}^{-1}(t) := \sup\{x \in \mathbb{R} : \mathfrak{f}(x) \leq t\} \quad (t \in \mathbb{R}).$$

**Proposition 9.** Assume that  $\ell_o := \lim_{n \rightarrow \infty} \partial\Theta_{\varepsilon_n}(T, o)$  exists for each  $o \in V$ . Then  $\mathfrak{f}_{T_{i \rightarrow j}} := \lim_{n \rightarrow \infty} \mathfrak{f}_{T_{i \rightarrow j}}^{\varepsilon_n}$  exists pointwise for each  $(i, j) \in \vec{E}$ , and

$$\mathfrak{f}_{T_{i \rightarrow j}}^{-1}(t) = t - \sum_{k \sim i, k \neq j} \left[ 1 - \mathfrak{f}_{T_{k \rightarrow i}}^{-1}(t) \right]_0^1, \quad (21)$$

for every  $t \in \mathbb{R}$ . Moreover, for every  $o \in V$ ,

$$\ell_o > t \iff \sum_{i \sim o} [1 - \mathfrak{f}_{T_{i \rightarrow o}}^{-1}(t)]_0^1 > t. \quad (22)$$

*Proof.* Fix  $(i, j) \in \vec{E}$ ,  $x \in \mathbb{R}$  and let us show that  $\mathfrak{f}_{T_{i \rightarrow j}}(x) := \lim_{n \rightarrow \infty} \mathfrak{f}_{T_{i \rightarrow j}}^{\varepsilon_n}(x)$  exists. By definition,  $\mathfrak{f}_{T_{i \rightarrow j}}^\varepsilon(x) = x + \partial\theta_\varepsilon(i)$ , where  $\theta_\varepsilon$  is the  $\varepsilon$ -balanced allocation on  $T_{i \rightarrow j}$  with baseload  $x$  at  $i$  and 0 elsewhere. Since the set of allocations on  $T_{i \rightarrow j}$  is compact, it is enough to consider two subsequential limits  $\theta, \theta'$  of  $\{\theta_{\varepsilon_n}\}_{n \geq 1}$  and prove that  $\partial\theta = \partial\theta'$ . Passing to the limit in (13), we know that  $\theta, \theta'$  are balanced with respect to the above baseload. Writing  $V_{i \rightarrow j}$  for the vertex set of  $T_{i \rightarrow j}$ , Lemma 7 reduces our task to proving

$$\|\partial\theta - \partial\theta'\|_{\ell^1(V_{i \rightarrow j})} < \infty. \quad (23)$$

Let  $\theta_\varepsilon^*$  be the restriction of  $\Theta_\varepsilon$  to  $T_{i \rightarrow j}$ . Thus,  $\theta_\varepsilon^*$  is an allocation on  $T_{i \rightarrow j}$  and it is  $\varepsilon$ -balanced with baseload  $\theta_\varepsilon^*(j, i)$  at  $i$  and 0 elsewhere. Consequently, Proposition 6 guarantees that for any finite  $K \subseteq V_{i \rightarrow j} \setminus \{i\}$ ,

$$\|\partial\theta_\varepsilon - \partial\theta_\varepsilon^*\|_{\ell^1(K)} \leq |x| + 1.$$

Applying this to  $\varepsilon, \varepsilon' > 0$  and using the triangle inequality, we obtain

$$\|\partial\theta_\varepsilon - \partial\theta_{\varepsilon'}\|_{\ell^1(K)} \leq 2|x| + 2 + \|\partial\theta_\varepsilon^* - \partial\theta_{\varepsilon'}^*\|_{\ell^1(K)}.$$

Since  $\{\partial\theta_{\varepsilon_n}^*\}_{n \geq 1}$  converges by assumption, we may pass to the limit to obtain  $\|\partial\theta - \partial\theta'\|_{\ell^1(K)} \leq 2|x| + 2$ . But  $K$  is arbitrary, so (23) follows. This shows that  $\mathfrak{f}_{T_{i \rightarrow j}} := \lim_{n \rightarrow \infty} \mathfrak{f}_{T_{i \rightarrow j}}^{\varepsilon_n}$  exists pointwise. We now recall two classical facts about non-decreasing functions  $\mathfrak{f}: \mathbb{R} \rightarrow \mathbb{R}$  with  $\mathfrak{f}(\pm\infty) = \pm\infty$ . First,  $\mathfrak{f}^{-1}$  is non-decreasing, so that its discontinuity set  $\mathcal{D}(\mathfrak{f}^{-1})$  is countable. Second, the pointwise convergence  $\mathfrak{f}_n \rightarrow \mathfrak{f}$  implies  $\mathfrak{f}_n^{-1}(t) \rightarrow \mathfrak{f}^{-1}(t)$  for every  $t \in \mathbb{R} \setminus \mathcal{D}(\mathfrak{f}^{-1})$ . Consequently, letting  $\varepsilon \rightarrow 0$  in (17) proves (21) for  $t \notin \mathcal{D} := \mathcal{D}(\mathfrak{f}_{T_{i \rightarrow j}}^{-1}) \cup \bigcup_{k \sim i} \mathcal{D}(\mathfrak{f}_{T_{k \rightarrow i}}^{-1})$ . The equality then extends to  $\mathbb{R}$  since  $\mathcal{D}$  is countable and both sides of (21) are right-continuous in  $t$ . Replacing  $T_{i \rightarrow j}$  with  $(T, o)$  in the above argument shows that  $\mathfrak{f}_{(T, o)} := \lim_{n \rightarrow \infty} \mathfrak{f}_{(T, o)}^{\varepsilon_n}$  exists and satisfies

$$\mathfrak{f}_{(T, o)}^{-1}(t) = t - \sum_{i \sim o} [1 - \mathfrak{f}_{T_{i \rightarrow o}}^{-1}(t)]_0^1 \quad (t \in \mathbb{R}).$$

Finally, recall that  $\mathfrak{f}_{(T, o)}^{\varepsilon_n}(0) = \partial\Theta_{\varepsilon_n}(T, o)$  for all  $n \geq 1$ , so that  $\mathfrak{f}_{(T, o)}(0) = \ell_o$ . But  $\mathfrak{f}_{(T, o)}(0) > t \iff \mathfrak{f}_{(T, o)}^{-1}(t) < 0$  by definition of  $\mathfrak{f}_{(T, o)}^{-1}$ , so (22) follows.  $\square$

## 10 Proof of Theorem 2

In all this section, we consider networks rather than graphs, where each oriented edge  $(i, j)$  is equipped with a mark  $\xi(i, j) \in [0, 1]$ . Given  $t \in \mathbb{R}$ , we shall be interested in marks that satisfy the local recursion

$$\xi(i, j) = \left[ 1 - t + \sum_{k \sim i, k \neq j} \xi(k, i) \right]_0^1, \quad (i, j) \in \vec{E}. \quad (24)$$

We start with a simple Lemma.

**Lemma 1.** *Under (24),  $\partial\xi(i) \wedge \partial\xi(j) > t \iff \xi(i, j) + \xi(j, i) > 1$ .*

*Proof.* We prove the equivalence case by case. Note that by assumption,

$$\xi(i, j) = [1 - t + \partial\xi(i) - \xi(j, i)]_0^1 \quad (25)$$

$$\xi(j, i) = [1 - t + \partial\xi(j) - \xi(i, j)]_0^1. \quad (26)$$

- If  $0 < \xi(i, j), \xi(j, i) < 1$ , then the equivalence trivially holds since we may safely remove the truncation  $[\cdot]_0^1$  from (25)-(26) to obtain

$$\partial\xi(i) - t = \xi(i, j) + \xi(j, i) - 1 = \partial\xi(j) - t.$$

- If  $\xi(j, i) = 0$ , then we have  $1 - t + \partial\xi(j) - \xi(i, j) \leq 0$  thanks to (26), and hence  $\partial\xi(j) \leq t$ . Thus, both sides of the equivalence are false.
- If  $\xi(i, j) = 1, \xi(j, i) > 0$ , then using  $\xi(i, j) = 1$  in (25) gives  $\partial\xi(i) - t \geq \xi(j, i)$  and since  $\xi(j, i) > 0$  we obtain  $\partial\xi(i) > t$ . On the other hand, using  $\xi(j, i) > 0$  in (26) gives  $\partial\xi(j) > t + \xi(i, j) - 1$  and since  $\xi(i, j) = 1$  we obtain  $\partial\xi(j) > t$ . Thus, both sides of the equivalence are true.

The other possible cases follow by exchanging  $\xi(i, j)$  and  $\xi(j, i)$ .  $\square$

We are ready for the proof of Theorem 2, which we divide into two parts.

**Proposition 10.** *Let  $Q \in \mathcal{P}([0, 1])$  be any solution to  $Q = F_{\pi, t}(Q)$ . Then,*

$$\Phi_\mu(t) \geq \frac{\mathbb{E}[D]}{2} \mathbb{P}(\xi_1 + \xi_2 > 1) - t \mathbb{P}(\xi_1 + \dots + \xi_D > t),$$

where  $D \sim \pi$  and where  $\{\xi_n\}_{n \geq 1}$  are i.i.d. with law  $Q$ , independent of  $D$ .

*Proof.* Kolmogorov's extension Theorem allows us to convert the consistency equation  $Q = F_{\pi,t}(Q)$  into a random rooted tree  $\mathbb{T} \sim \text{UGWT}(\pi)$  equipped with marks satisfying (24) a-s, such that conditionally on the structure of  $[\mathbb{T}, o]_h$ , the marks from generation  $h$  to  $h-1$  are IID with law  $Q$ . This random rooted network is easily checked to be unimodular. Thus, we may apply Proposition 4 with  $f = \mathbf{1}_{\partial\xi > t}$ . By Lemma 1, we have  $\widehat{f} = \mathbf{1}_{\xi + \xi^* > 1}$  and hence

$$\Phi_\mu(t) \geq \frac{1}{2}\vec{\mu}(\xi + \xi^* > 1) - t\mu(\partial\xi > t).$$

This is precisely the desired result, since we have by construction

$$\mu(\partial\xi > t) = \mathbb{P}(\xi_1 + \dots + \xi_D > t), \quad \vec{\mu}(\xi + \xi^* > 1) = \mathbb{E}[D]\mathbb{P}(\xi_1 + \xi_2 > 1)$$

where  $D \sim \pi$  and  $\xi_1, \xi_2, \dots$  are IID with law  $Q$ , independent of  $D$ .  $\square$

**Proposition 11.** *There is a solution  $Q \in \mathcal{P}([0, 1])$  to  $Q = F_{\pi,t}(Q)$  such that*

$$\Phi_\mu(t) = \frac{\mathbb{E}[D]}{2}\mathbb{P}(\xi_1 + \xi_2 > 1) - t\mathbb{P}(\xi_1 + \dots + \xi_D > t).$$

where  $D \sim \pi$  and where  $\{\xi_n\}_{n \geq 1}$  are IID with law  $Q$ , independent of  $D$ .

*Proof.* Let  $\mathbb{T} \sim \text{UGWT}(\pi)$ . Thanks to Proposition 2, we have

$$\partial\Theta_\varepsilon(\mathbb{T}, o) \xrightarrow[\varepsilon \rightarrow 0]{L^2} \partial\Theta_0(\mathbb{T}, o).$$

In particular, there is a deterministic vanishing sequence  $\varepsilon_1, \varepsilon_2, \dots$  along which the convergence holds almost surely. This almost-sure convergence automatically extends from the root to all vertices, since *everything shows up at the root* of a unimodular random network [2, Lemma 2.3]. Therefore,  $\mathbb{T}$  satisfies almost-surely the assumption of Proposition 9. Consequently, the marks  $\xi(i, j) := [1 - \mathfrak{f}_{\mathbb{T}_{i \rightarrow j}}^{-1}(t)]_0^1$  satisfy (24) almost-surely, and

$$\partial\Theta_0(\mathbb{T}, o) > t \iff \partial\xi(o) > t.$$

This ensures that  $f = \mathbf{1}_{\partial\xi > t}$  satisfies the requirement for equality in Proposition 4, and we may then use Lemma 1 to rewrite the conclusion as

$$\Phi_\mu(t) = \frac{1}{2}\vec{\mu}(\xi + \xi^* > 1) - t\mu(\partial\xi > t).$$

Now,  $D = \deg(\mathbb{T}, o)$  has law  $\pi$ , and conditionally on  $D$ , the subtrees  $\{\mathbb{T}_{i \rightarrow o}\}_{i \sim o}$  are IID copies of a homogenous Galton-Watson tree  $\widehat{\mathbb{T}}$  with offspring distribution  $\widehat{\pi}$ . Since  $\xi(i, o)$  depends only on the subtree  $\mathbb{T}_{i \rightarrow o}$ , we obtain

$$\mu(\partial\xi > t) = \mathbb{P}(\xi_1 + \dots + \xi_D > t), \quad \vec{\mu}(\xi + \xi^* > 1) = \mathbb{E}[D]\mathbb{P}(\xi_1 + \xi_2 > 1)$$

where  $\xi_1, \xi_2, \dots$  are IID copies of  $[1 - \mathfrak{f}_{\widehat{\mathbb{T}}}^{-1}(t)]_0^1$ , independent of  $D$ . In turn, removing the root of  $\widehat{\mathbb{T}}$  splits it into a  $\widehat{\pi}$ -distributed number of IID copies of  $\widehat{\mathbb{T}}$ , so that the law  $Q$  of  $[1 - \mathfrak{f}_{\widehat{\mathbb{T}}}^{-1}(t)]_0^1$  satisfies  $Q = F_{\pi,t}(Q)$ .  $\square$

## 11 Proof of Theorem 3

Fix a degree sequence  $\mathbf{d} = \{d(i)\}_{1 \leq i \leq n}$  and set  $2m = \sum_{i=1}^n d(i)$ .

**Lemma 2.** *The number of edges of  $\mathbb{G}[\mathbf{d}]$  with both end-points in  $S \subseteq \{1, \dots, n\}$  is stochastically dominated by a Binomial with mean  $\frac{1}{m} \left(\sum_{i \in S} d_i\right)^2$ .*

*Proof.* We assume that  $s := \sum_{i \in S} d_i < m$ , otherwise the claim is trivial. It is classical that  $\mathbb{G}[\mathbf{d}]$  can be generated sequentially: at each step  $1 \leq t \leq m$ , a half-edge is selected and paired with a uniformly chosen other half-edge. The selection rule is arbitrary, and we choose to give priority to half-edges whose end-point lies in  $S$ . Let  $X_t$  be the number of edges with both end-points in  $S$  after  $t$  steps. Then  $\{X_t\}_{0 \leq t \leq m}$  is a Markov chain with  $X_0 = 0$  and transitions

$$X_{t+1} := \begin{cases} X_t + 1 & \text{with conditional probability } \frac{(s - X_t - t - 1)^+}{2m - 2t - 1} \\ X_t & \text{otherwise.} \end{cases}$$

For every  $0 \leq t < m$ , the fact that  $X_t \geq 0$  ensures that

$$\frac{(s - X_t - t - 1)^+}{2m - 2t - 1} \leq \frac{s - t - 1}{2m - 2t - 1} \mathbf{1}_{(t < s)} \leq \frac{s}{2m} \mathbf{1}_{(t < s)},$$

where the second inequality uses the condition  $s < m$ . This shows that  $X_m$  is in fact stochastically dominated by a Binomial  $(s, \frac{s}{2m})$ , which is enough.  $\square$

**Lemma 3.** *Let  $X_{k,r}$  be the number of induced subgraphs with  $k$  vertices and at least  $r$  edges in  $\mathbb{G}[\mathbf{d}]$ . Then, for any  $\theta > 0$ ,*

$$\mathbb{E}[X_{k,r}] \leq \left(\frac{2r}{\theta^2 m}\right)^r \left(\frac{e}{k} \sum_{i=1}^n e^{\theta d_i}\right)^k.$$

*Proof.* First observe that if  $Z \sim \text{Bin}(n, p)$  then by a simple union-bound,

$$\mathbb{P}(Z \geq r) \leq \binom{n}{r} p^r \leq \frac{n^r p^r}{r!} = \frac{\mathbb{E}[Z]^r}{r!}.$$

Thus, by Lemma 2, the number  $Z_S$  of edges with both end-points in  $S$  satisfies

$$\mathbb{P}(Z_S \geq r) \leq \frac{1}{r! m^r} \left(\sum_{i \in S} d_i\right)^{2r} \leq \left(\frac{2r}{\theta^2 m}\right)^r \prod_{i \in S} e^{\theta d_i},$$

where we have used the crude bounds  $x^{2r} \leq (2r)! e^x$  and  $(2r)!/r! \leq (2r)^r$ . The result follows by summing over all  $S$  with  $|S| = k$  and observing that

$$\sum_{|S|=k} \prod_{i \in S} e^{\theta d_i} \leq \frac{1}{k!} \left(\sum_{i=1}^n e^{\theta d_i}\right)^k \leq \left(\frac{k}{e} \sum_{i=1}^n e^{\theta d_i}\right)^k.$$

The second inequality follows from the classical lower-bound  $k! \geq \left(\frac{k}{e}\right)^k$ .  $\square$

We now fix  $\{\mathbf{d}_n\}_{n \geq 1}$  as in Theorem 3. Let  $Z_{\delta,t}^{(n)}$  be the number of subsets  $\emptyset \subsetneq S \subseteq \{1, \dots, n\}$  such that  $|S| \leq \delta n$  and  $|E(S)| \geq t|S|$  in  $\mathbb{G}_n := \mathbb{G}[\mathbf{d}_n]$ .

**Proposition 12.** *For each  $t > 1$ , there is  $\delta > 0$  and  $\kappa < \infty$  such that*

$$\mathbb{E} \left[ Z_{\delta,t}^{(n)} \right] \leq \kappa \left( \frac{\ln n}{n} \right)^{t-1},$$

uniformly in  $n \geq 1$ . In particular,  $Z_{\delta,t}^{(n)} = 0$  w.h.p. as  $n \rightarrow \infty$ .

*Proof.* The assumptions of Theorem 3 guarantee that for some  $\theta > 0$ ,

$$\alpha := \inf_{n \geq 1} \left\{ \frac{1}{n} \sum_{i=1}^n d_n(i) \right\} > 0 \quad \text{and} \quad \lambda := \sup_{n \geq 1} \left\{ \frac{1}{n} \sum_{i \in V} e^{\theta d_n(i)} \right\} < \infty.$$

Now, fix  $t > 1$  and choose  $\delta > 0$  small enough so that  $f(\delta) < 1$ , where

$$f(\delta) := \left( 1 \vee \frac{2(1+t)}{\alpha \theta^2} \right)^{t+1} e \lambda \delta^{t-1}.$$

Using Lemma 3 and the trivial inequality  $kt \leq [kt] \leq k(t+1)$ , we have

$$\mathbb{E} \left[ X_{k, [kt]}^{(n)} \right] \leq \left( \frac{2[kt]}{\theta^2 k \alpha} \right)^{[kt]} (e \lambda)^k \left( \frac{k}{n} \right)^{[kt]-k} \leq f^k \left( \frac{k}{n} \right).$$

Since  $f$  is increasing, we see that for any  $1 \leq m \leq \delta n$ ,

$$\begin{aligned} \mathbb{E} \left[ Z_{\delta,t}^{(n)} \right] &= \sum_{k=1}^{\lfloor \delta n \rfloor} \mathbb{E} \left[ X_{k, [kt]}^{(n)} \right] \leq \sum_{k=1}^{m-1} f^k \left( \frac{m}{n} \right) + \sum_{k=m}^{\lfloor \delta n \rfloor} f^k(\delta) \\ &\leq \frac{f\left(\frac{m}{n}\right)}{1 - f\left(\frac{m}{n}\right)} + \frac{f(\delta)^m}{1 - f(\delta)}. \end{aligned}$$

Choose  $m \sim c \ln n$  with  $c$  fixed. As  $n \rightarrow \infty$ , the first term is of order  $\left(\frac{\ln n}{n}\right)^{t-1}$  while the second is of order  $f(\delta)^{c \ln n} \ll \left(\frac{\ln n}{n}\right)^{t-1}$ , if  $c$  is large enough.  $\square$

*Proof of Theorem 3.* The assumptions on  $\{\mathbf{d}_n\}_{n \geq 1}$  are more than sufficient to guarantee that a-s, the local weak limit of  $\{\mathbb{G}_n\}_{n \geq 1}$  is  $\mu := \text{UGWT}(\pi)$  (see e.g [9]). Thus, the weak convergence  $\mathcal{L}_{\mathbb{G}_n} \rightarrow \mathcal{L}$  holds a-s, where  $\mathcal{L}$  is the law of  $\partial\Theta_0$  under  $\mu$ . Now, if  $t < \varrho(\mu)$  then  $\mathcal{L}((t, \infty)) > 0$ , so the Portmanteau Theorem ensures that  $\liminf_n \mathcal{L}_{\mathbb{G}_n}((t, \infty)) > 0$  a-s. Consequently,

$$\mathbb{P}(\varrho(\mathbb{G}_n) \leq t) = \mathbb{P}(\mathcal{L}_{\mathbb{G}_n}((t, \infty)) = 0) \xrightarrow[n \rightarrow \infty]{} 0.$$

On the other-hand, if  $t > \varrho(\mu)$  then  $\mathcal{L}([t, \infty)) = 0$ , so the Portmanteau Theorem gives  $\mathcal{L}_{\mathbb{G}_n}([t, \infty)) \rightarrow 0$  a-s. Thus, with  $\delta$  as in Proposition 12,

$$\mathbb{P}(\varrho(\mathbb{G}_n) > t) \leq \mathbb{P}(\mathcal{L}_{\mathbb{G}_n}([t, \infty)) > \delta) + \mathbb{P}\left(Z_{\delta,t}^{(n)} > 0\right) \xrightarrow{n \rightarrow \infty} 0.$$

Note that the requirement  $t > 1$  is fulfilled, since  $\varrho(\mu) \geq 1$ . Indeed, every node in a tree of size  $n$  has load  $1 - \frac{1}{n}$ , and the assumption  $\pi_0 + \pi_1 < 1$  guarantees that the size of the random tree  $\mathbb{T} \sim \text{UGWT}(\pi)$  is unbounded.  $\square$

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