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Quasimodes and instability for linear Schrödinger equation on manifolds

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Abstract

We consider the evolution operator $\exp(-it(-\Delta+V))$ associated with a Schrödinger operator on a Riemannian manifold (M, g) . We are interested in the dependence of this operator on V running in $L^p(M)$. Under a geometrical hypothesis, we show the instability for $p < \infty$ and give examples for which the hypothesis is satisfied. Then we show in the general case the instability for $p < \dim M/2$.

1 Introduction

These last years several papers were published about the stability of the nonlinear Schrödinger equation, that is the uniform continuity of the map $u_0 \mapsto u$ where u is the solution of the equation

$$i\partial_t u + \Delta u = \varepsilon |u|^2 u, \quad u(0, x) = u_0 \quad (1.1)$$

where $\varepsilon = \pm 1$, $u(t, \cdot)$ is defined on a Riemannian manifold M and Δ is the Laplace-Beltrami operator on this manifold. More precisely :

Definition 1.1. *Let $\sigma \in \mathbb{R}$, and denote by $B_{R,\sigma}$ the ball of radius R in $H^\sigma(M)$. We say that the problem 1.1 is uniformly well-posed in H^σ if for any $R > 0$, there exists $T > 0$ such that the map :*

$$B_{R,\sigma} \cap H^1(M) \ni u_0 \mapsto u \in L^\infty([-T, T]; H^\sigma(M))$$

is uniformly continuous ($B_{R,\sigma} \cap H^1(M)$ is endowed with the H^σ norm). Otherwise, we say that the Cauchy problem (1.1) is unstable.

Let us recall some results :

In 1993, J.Bourgain proved in [2] that the Cauchy problem is uniformly well-posed on the rational torus \mathbb{T}^2 when $\sigma > 0$,

in 2002, N.Burq, P.Gérard and N.Tzevtkov showed in [4] the instability on \mathbb{S}^2 when $0 \leq \sigma < 1/4$,

in 2004, the same authors proved in [5] that the Cauchy problem is uniformly well-posed when M is a compact manifold of dimension $d \geq 2$ and $\sigma > (d-1)/2$,

in 2005, they showed in [6] that the Cauchy problem is well-posed in \mathbb{S}^2 when $\sigma > 1/4$,

in 2008, L.Thomann [13] proved the instability in the case when M is a surface with a stable and not degenerate periodic geodesic and $0 < \sigma < 1/4$.

For results in one dimension, we refer to the work of M.Christ, J.Colliander and T.Tao [7].

In the linear case, the propagators are unitary operators so the dependence on the initial data has no interest. However, the stability property is the continuity of the evolution operator $\exp(-it(-\Delta + V))$ with respect to V running in a proper functions space. Recently in [3] J.Bourgain, N. Burq and M.Zworski proved the following stability result on the torus $\mathbb{T}^2 = \mathbb{R}^2/(a\mathbb{Z} \times b\mathbb{Z})$, $(a, b) \in \mathbb{R}^2$

Theorem 1.2. 1. Let K be a compact subset of $L^2(\mathbb{T}^2)$. Then for any $T > 0$ the map

$$K \ni V \mapsto e^{-it(-\Delta+V)} \in L^\infty((0, T), \mathcal{L}(L^2(\mathbb{T})))$$

is lipschitz continuous.

2. Let $p > 2$ and A be a bounded subset of $L^p(\mathbb{T}^2)$. Then for any $T > 0$ the map

$$A \ni V \mapsto e^{-it(-\Delta+V)} \in L^\infty((0, T), \mathcal{L}(L^2(\mathbb{T})))$$

is lipschitz continuous.

The authors notice that “it would be interesting to investigate such properties on other manifolds, as they seem to depend strongly on the geometry”. The aim of this article is to answer partially to this remark.

This article is organized as following

- In section 2 we state our main theorem : assuming a geometrical condition on a sequence of quasimodes for a Schrödinger operator $-\Delta + V$ on a manifold M , we show the unstability near V of the maps $W \mapsto e^{-it(-\Delta+W)} \in L^\infty((0, T), \mathcal{L}(L^2(\mathbb{T})))$ with respect to the L^p norm, $1 \leq p < +\infty$.
- In section 3 we show examples for which this theorem applies.
- In section 4 we prove that the geometrical hypothesis is not necessary for $p < \dim M/2$.

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2 Main theorem

Theorem 2.1. Let (M, g) be a Riemannian manifold of finite dimension, $\Delta_g = \Delta$ the Laplace-Beltrami operator, $d\mu$ the riemannian volume form and V a continuous nonnegative potentiel. Assume

1. There exists a sequence (λ_n, u_n) of quasi-eigenvalues and associated quasimodes for the Schrödinger operator $-\Delta + V$

$$\lim_{n \rightarrow +\infty} \|(-\Delta + V - \lambda_n)u_n\|_{L^2(M)} = 0, \quad \|u_n\|_{L^2(M)} = 1.$$

2. The sequence of measures $|u_n(x)|^2 d\mu$ tends to a measure ν for the weak* topology, that is for any continuous f vanishing at infinity

$$\lim_{n \rightarrow +\infty} \int_M f(x) |u_n(x)|^2 d\mu = \int_M f(x) d\nu$$

and ν is not absolutely continuous with respect to μ .

Then there exists a sequence of smooth bounded potentiels W_k such that

$$\begin{aligned} \|W_k\|_{L^\infty(M)} &\leq 1 \\ \forall p \in [1, +\infty[, \quad \lim_{k \rightarrow +\infty} \|W_k\|_{L^p(M)} &= 0 \\ \forall t > 0, \quad \liminf_{k \rightarrow +\infty} \|e^{-it(-\Delta+V+W_k)} - e^{-it(-\Delta+V)}\|_{\mathcal{L}(L^2(M))} &> 0 \end{aligned}$$

Proof : First notice that the operators $-\Delta + V$ and $-\Delta + V + W_n$ are self-adjointed with the same domain $\{u \in L^2(M); \Delta u \in L^2(M), V u \in L^2(M)\}$.

The second hypothetis ensures the existence of a compact subset Γ of M such that $\mu(\Gamma) = 0$ and $\nu(\Gamma) > 0$. Let $\kappa \in]0, 1]$ be a parameter to be fixed later and $(\varphi_k)_{k \in \mathbb{N}^*}$ be a sequence of smooth fonctions on M with values in $[0, 1]$ such that

$$\begin{cases} \varphi_k = 1 \text{ on } \Gamma \\ \mu(\text{supp}(\varphi_k)) \leq \frac{1}{k} \end{cases}$$

We observe

$$\begin{aligned} (-i\partial_t + (-\Delta + V(x) + \kappa\varphi_k(x))) \left(e^{-i(\lambda_n + \kappa)t} u_n(x) \right) \\ = e^{-i(\lambda_n + \kappa)t} (-\Delta + V(x) - \lambda_n) u_n(x) + \kappa(\varphi_k(x) - 1) e^{-i(\lambda_n + \kappa)t} u_n(x) \end{aligned}$$

By hypothetis 1

$$\lim_{n \rightarrow +\infty} \|e^{-i(\lambda_n + \kappa)t} (-\Delta + V - \lambda_n) u_n\|_{L^2(M)} = 0$$

On the other hand

$$\|(\varphi_k(x) - 1) e^{-i(\lambda_n + \kappa)t} u_n(x)\|_{L^2(M)}^2 = 1 - \int_M (2\varphi_k(x) - \varphi_k(x)^2) |u_n(x)|^2 d\mu$$

tends to $1 - \int_M (2\varphi_k - \varphi_k^2) d\nu$ when n tends to $+\infty$, and this quantity tends to $1 - \nu(\Gamma)$ when k tends to $+\infty$.

So there exists a sequence of integers n_k tending to $+\infty$ and a sequence ε_k tending to 0 such that :

$$\left\| (-i\partial_t + (-\Delta + V(x) + \kappa\varphi_k(x))) \left(e^{-i(\lambda_{n_k} + \kappa)t} u_{n_k}(x) \right) \right\|_{L^2(M)} \leq \varepsilon_k + \kappa(1 - \nu(\Gamma) + \varepsilon_k)^{\frac{1}{2}}$$

Put

$$v_k(t, x) = (-i\partial_t + (-\Delta + V(x) + \kappa\varphi_k(x))) \left(e^{-i(\lambda_{n_k} + \kappa)t} u_{n_k}(x) \right)$$

Duhamel's formula gives

$$\left(e^{-it(-\Delta+V+\kappa\varphi_{n_k})} u_{n_k} \right) (t, x) = \left(e^{-i(\lambda_{n_k} + \kappa)t} u_{n_k}(x) \right) - i \int_0^t e^{i(s-t)(-\Delta+V+\kappa\varphi_{n_k})} v_k(s, x) ds$$

So

$$\left\| e^{-it(-\Delta+V+\kappa\varphi_{n_k})} u_{n_k} - \left(e^{-i(\lambda_{n_k} + \kappa)t} u_{n_k} \right) \right\|_{L^2(M)} \leq t \left(\varepsilon_k + \kappa(1 - \nu(\Gamma) + \varepsilon_k)^{\frac{1}{2}} \right)$$

Similarly

$$\left\| e^{-it(-\Delta+V)} u_{n_k} - e^{-i\lambda_{n_k}t} u_{n_k} \right\|_{L^2(M)} \leq \varepsilon_k t$$

Obviously

$$\left\| e^{-i(\lambda_{n_k} + \kappa)t} u_{n_k} - e^{-i\lambda_{n_k}t} u_{n_k} \right\|_{L^2(M)} = |e^{i\kappa t} - 1|$$

Hence

$$\left\| e^{-it(-\Delta + V + \kappa\varphi_{n_k})} u_{n_k} - e^{-it(-\Delta + V)} u_{n_k} \right\|_{L^2(M)} \geq |e^{i\kappa t} - 1| - \left(2\varepsilon_k + \kappa(1 - \nu(\Gamma) + \varepsilon_k)^{\frac{1}{2}} \right) t$$

So

$$\liminf_{k \rightarrow +\infty} \left\| e^{-it(-\Delta + V + \kappa\varphi_{n_k})} - e^{-it(-\Delta + V)} \right\|_{\mathcal{L}(L^2(M))} \geq 2 \sin \frac{\kappa t}{2} - (1 - \nu(\Gamma))^{\frac{1}{2}} \kappa t$$

For κ small enough the right hand side is positive.

Remark 2.2. For $p = \infty$ there is always stability. Indeed, Duhamel's formula gives for u_0 in the domain of $-\Delta + V$

$$e^{-it(-\Delta + V + W_n)} u_0 - e^{-it(-\Delta + V)} u_0 = -i \int_0^t e^{-i(t-s)(-\Delta + V + W_n)} W_n e^{-is(-\Delta + V)} u_0 ds$$

so

$$\left\| e^{-it(-\Delta + V + W_n)} - e^{-it(-\Delta + V)} \right\|_{\mathcal{L}(L^2(M))} \leq t \|W_n\|_{L^\infty(M)}$$

Remark 2.3. D. Jakobson (referring to a communication of J. Bourgain) proves in [11] that the second hypothesis of Theorem 2.1 is not satisfied in the case of the Laplacian on tori. The stability is proved in dimension 2 in [3], the problem is open for $d \geq 3$.

3 Examples of applications

3.1 The sphere \mathbb{S}^d

We consider the sphere \mathbb{S}^d , $d \geq 2$ endowed with the usual metric induced by that one of \mathbb{R}^{d+1} . We will show that Theorem 2.1 gives the unstability near $-\Delta_{\mathbb{S}^d}$.

Let e_n be the restriction to \mathbb{S}^d of the harmonic polynomial $(x_1 + i x_2)^n$ (called equatorial spherical harmonic) and $u_n = e_n / \|e_n\|_{L^2(\mathbb{S}^d)}$ which satisfies

$$-\Delta_{\mathbb{S}^d} u_n = n(n + d - 1) u_n, \quad \|u_n\|_{L^2(\mathbb{S}^d)} = 1$$

and we have to study the weak* limit of $|u_n|^2 d\mu$ where $d\mu$ is the Riemannian volume form on \mathbb{S}^d .

Proposition 3.1. The sequence of measures $|u_n(x)|^2 d\mu$ on \mathbb{S}^d tends to the measure $\frac{d\theta}{2\pi}$ on the circle $\{(\cos \theta, \sin \theta, 0, \dots, 0); \theta \in [0, \pi]\}$ for the weak* topology.

Proof. We parameter \mathbb{S}^d by

$$x = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, (\sin \varphi) t), \quad (\theta, \varphi, t) \in [0, 2\pi] \times [0, \frac{\pi}{2}] \times \mathcal{S}^{d-2}$$

so $d\mu = \cos \varphi (\sin \varphi)^{d-2} d\theta d\varphi dt$.

Let f be a continuous function on \mathbb{S}^d .

$$\int_{\mathbb{S}^d} f |e_n|^2 d\mu = \int_0^{\frac{\pi}{2}} g(\varphi) (\cos \varphi)^{2n+1} (\sin \varphi)^{d-2} d\varphi$$

where

$$g(\varphi) = \int_{[0,2\pi] \times \mathcal{S}^{d-2}} f(\cos \theta \cos \varphi, \sin \theta \cos \varphi, (\sin \varphi) t) d\theta dt$$

Laplace's methods gives

$$\int_{\mathbb{S}^d} f |e_n|^2 d\mu = \left(\int_0^{+\infty} e^{-(n+\frac{1}{2})\varphi^2} \varphi^{d-2} d\varphi \right) (g(0) + o(1))$$

Notice that $\|e_n\|_{L^2(\mathbb{S}^d)}^2$ is given by $f = 1$ so

$$\int_{\mathbb{S}^d} f(x) |u_n(x)|^2 d\mu = \frac{1}{2\pi} \int_0^{2\pi} f(\cos \theta, \sin \theta, 0, \dots, 0) d\theta + o(1)$$

which achieves the proof.

Thus Theorem 2.1 applies on the sphere near $-\Delta_{\mathbb{S}^d}$.

Remark 3.2. *The result applies whenever the manifold (M, g) only coincide with the sphere near a closed geodesic.*

3.2 Periodic stable geodesic

We refer here to the work of J.V.Ralston [12] who developed the ideas of [1] and [8] to obtain quasimodes using WKB constructions. The author considers a Riemannian manifold (M, g) of dimension $d \geq 2$ and assumes the existence of a periodic closed non degenerate geodesic γ . The eigenvalues λ_j , $1 \leq j \leq 2(d-1)$ of the Poincaré application associated to γ are supposed to have modulus 1 (so γ is stable) and to satisfy the diophantian condition

$$\forall n \in \mathbb{N}^{2d-2}, \quad \prod_{j=1}^{2d-2} \lambda_j^{n_j} \neq 1$$

V is a smooth potential on M . Under these hypotheses, J.V. Ralston proves

Theorem 3.3. *For any nonnegative integer N and real number ε , there exist sequences of quasimodes E_j tending to $+\infty$ and associated normalized quasi-eigenfunctions u_n , and a constant $C_{N,\varepsilon}$ such that*

1. $\|(-\Delta_g + V - E_n) u_n\|_{L^2(M)} = \mathcal{O}(E_n^{-N})$
2. $\|u_n\|_{L^2(\gamma_{N,\varepsilon})} < \varepsilon$, where $\gamma_{N,\varepsilon} = \{x \in M; d(x, \gamma) > C_{N,\varepsilon} E_n^{-1/4}\}$.

As a consequence, Theorem 2.1 applies on M for any smooth nonnegative potential V .

3.3 Hyperbolic surfaces

We refer here to the work of Y. Colin de Verdière and B. Parisse [9]. They state their theorem in a particular case, but it is easy to check that they proved the following

Theorem 3.4. *Consider $a < b$ two real numbers, f a smooth positive function on $[a, b]$, H the cylinder $[a, b] \times \mathbb{Z}/2\pi\mathbb{Z}$ endowed with the metric $dt^2 + f^2(t) d^2\theta$, and Δ the Laplace-Beltrami operator on H associated with this metric with Dirichlet boundary condition. Assume that f reaches its minimum on $[a, b]$ in a unique point $t_0 \in]a, b[$, and that this minimum is not degenerate.*

Then Δ admits a sequence of normalized eigenfunctions u_n such that the sequence of measures $|u_n|^2 d\mu$ tends to $\delta_{t_0} \otimes \frac{d\theta}{2\pi}$ for the weak topology.*

In particular, we can consider a rotation invariant surface. Let $a < b$ two real numbers, f and g two smooth functions on $[a, b]$ such that $f > 0$ and the curve $\mathcal{C} : [a, b] \ni t \mapsto (g(t), f(t), 0)$ in \mathbb{R}^3 have no multiple point and is parametrized by its curvi-linear abscissa, i.e. $f'^2 + g'^2 = 1$. The surface

$$H = \{(g(t), f(t) \cos \theta, f(t) \sin \theta); (t, \theta) \in [a, b] \times \mathbb{Z}/2\pi\mathbb{Z}\}$$

generated by the rotation of \mathcal{C} on the X-axis is endowed with the metric induced by that one of $\mathbb{R}^3 dt^2 + f^2(t) d\theta^2$. Theorem 2.1 applies to the Laplacian on H with Dirichlet boundary condition.

4 Unstability for small p

In the case when $p < \dim M/2$, the unstability can be proved by local constructions and does not request the geometrical hypothesis.

Theorem 4.1. *Let (M, g) be a Riemannian manifold of dimension $d \geq 3$, Δ the associated Laplace-Betrami operator, V a smooth nonnegative potential.*

Then there exists a sequence of smooth bounded potentials W_n such that

$$\forall p \in [1, \frac{d}{2}[, \quad \lim_{n \rightarrow +\infty} \|W_n\|_{L^p(M)} = 0$$

$$\forall T > 0, \quad \lim_{n \rightarrow +\infty} \|e^{-it(-\Delta+V+W_n)} - e^{-it(-\Delta+V)}\|_{L^\infty([0,T]; \mathcal{L}(L^2(M)))} = 2$$

Remark 4.2. *Here the smooth potential V is estimated in L^p -norm. Notice that the threshold $d/2$ is natural, as it is the same one above which the operator $-\Delta + V$ is well defined (in the quadratic form sense).*

Proof of Theorem 4.1: In local coordinates we have

$$\Delta = \frac{1}{\sqrt{\det G}} \nabla \sqrt{\det G} G^{-1} \nabla \quad (4.1)$$

where G is the matrix of the metric g .

Let $u^0 \in C_0^\infty(\mathbb{R}^d)$ normalized by $\|u^0\|_{L^2(\mathbb{R}^d)} = 1$ and $W \in C_0^\infty(\mathbb{R}^d)$ real valued such that $W = 1$ on the support of u^0 .

We define for $n \in \mathbb{N}^*$

$$u_n^0(\cdot) = n^{d/2} u^0(n \cdot) \quad (4.2)$$

$$W_n(\cdot) = n^2 \ln(n+1) W_0(n \cdot) \quad (4.3)$$

so $\|u_n^0\|_{L^2(\mathbb{R}^d)} = 1$ and $\|W_n\|_{L^2(\mathbb{R}^d)} = n^{2-\frac{d}{p}} \ln(n+1)$ tends to 0 for any $p < d/2$.

By Duhamel's formula we have for any positive t

$$e^{-it(-\Delta+V)} u_n^0 - e^{-itV} u_n^0 = i \int_0^t e^{-i(t-s)(-\Delta+V)} \Delta (e^{-isV} u_n^0) ds \quad (4.4)$$

$$e^{-it(-\Delta+V+W_n)} u_n^0 - e^{-it(V+W_n)} u_n^0 = i \int_0^t e^{-i(t-s)(-\Delta+V+W_n)} \Delta (e^{-is(V+W_n)} u_n^0) ds \quad (4.5)$$

Definitions (4.2) and (4.3) of u_n^0 and W_n , together with (4.1) yield for a suitable $C > 0$

$$\|\Delta (e^{-isV} u_n^0)\|_{L^2(\mathbb{R}^d)} \leq C n^2 \quad (4.6)$$

$$\|\Delta (e^{-is(V+W_n)} u_n^0)\|_{L^2(\mathbb{R}^d)} \leq C(s^2 n^6 \ln^2(n+1) + n^2) \quad (4.7)$$

Then (4.4) and (4.5) give

$$\|e^{-it(-\Delta+V)}u_n^0 - e^{-itV}u_n^0\|_{L^2(\mathbb{R}^d)} \leq Cn^2t \quad (4.8)$$

$$\|e^{-it(-\Delta+V+W_n)}u_n^0 - e^{-it(V+W_n)}u_n^0\|_{L^2(\mathbb{R}^d)} \leq C(t^3n^6\ln^2(n+1) + n^2t) \quad (4.9)$$

We consider the time

$$t_n = \frac{\pi}{n^2\ln(n+1)} \quad (4.10)$$

From (4.8) and (4.9) we get

$$\|e^{-it_n(-\Delta+V)}u_n^0 - e^{-it_nV}u_n^0\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{\ln(n+1)} \quad (4.11)$$

$$\|e^{-it_n(-\Delta+V+W_n)}u_n^0 - e^{-it_n(V+W_n)}u_n^0\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{\ln(n+1)} \quad (4.12)$$

Since $W_n = 1$ on the support of u_n^0 and $\|u_n^0\|_{L^2(\mathbb{R}^d)} = 1$ we have

$$\|e^{-it_n(V+W_n)}u_n^0 - e^{-it_nV}u_n^0\|_{L^2(\mathbb{R}^d)} = 2 \quad (4.13)$$

So (4.8), (4.9) and (4.13) give

$$\|e^{-it_n(-\Delta+V+W_n)}u_n^0 - e^{-it_n(-\Delta+V)}u_n^0\|_{L^2(\mathbb{R}^d)} \geq 2 - \frac{C}{\ln(n+1)} \quad (4.14)$$

which achieves the proof.

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