An inverse problem for self-adjoint positive Hankel operators

Patrick Gerard, Alexander Pushnitski

AN INVERSE PROBLEM FOR SELF-ADJOINT POSITIVE HANKEL OPERATORS

PATRICK GERARD AND ALEXANDER PUSHNITSKI

Abstract. For a sequence $\{\alpha_n\}_{n=0}^\infty$, we consider the Hankel operator Γ_α, realised as the infinite matrix in ℓ^2 with the entries α_{n+m}. We consider the subclass of such Hankel operators defined by the "double positivity" condition $\Gamma_\alpha \geq 0$, $\Gamma_{S^*\alpha} \geq 0$; here $S^*\alpha$ is the shifted sequence $\{\alpha_{n+1}\}_{n=0}^\infty$. We prove that in this class, the sequence α is uniquely determined by the spectral shift function ξ_α for the pair Γ_α^2, $\Gamma_{S^*\alpha}^2$. We also describe the class of all functions ξ_α arising in this way and prove that the map $\alpha \mapsto \xi_\alpha$ is a homeomorphism in appropriate topologies.

1. Introduction

1.1. **Hankel operators.** Let $\alpha = \{\alpha_n\}_{n=0}^\infty$ be a bounded sequence of complex numbers. The Hankel operator Γ_α in $\ell^2 \equiv \ell^2(\mathbb{Z}_+)$ is formally defined by

$$(\Gamma_\alpha x)_n = \sum_{m=0}^\infty \alpha_{n+m} x_m, \quad x = \{x_n\}_{n=0}^\infty \in \ell^2.$$

In other words, Γ_α is the "infinite matrix"

$$\Gamma_\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.1}$$

Background information on Hankel operators can be found, e.g. in [11]. By Nehari's theorem, Γ_α is bounded on ℓ^2 if and only if there exists a function $f \in L^\infty(\mathbb{T})$ such that $\hat{f}(n) = \alpha_n$ for all $n \geq 0$; here $\hat{f}(n)$ is the n'th Fourier coefficient of f. We denote by e_n, $n \in \mathbb{Z}_+$, the standard basis in ℓ^2. Clearly, $\alpha = \Gamma_\alpha e_0$; thus, if Γ_α is bounded, then the sequence α is automatically in ℓ^2. In what follows we will always assume that Γ_α is bounded and that α is real-valued; in this case Γ_α is self-adjoint.

Let S be the (right) shift operator in ℓ^2:

$$S : (x_0, x_1, x_2, \dots) \mapsto (0, x_0, x_1, x_2, \dots);$$

we will also need its adjoint,

$$S^* : (x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots).$$

Date: December 11, 2013.

2010 *Mathematics Subject Classification.* 47B35.

Key words and phrases. Hankel operators, inverse spectral problem, spectral shift function.

Along with Γ_α, we will consider the Hankel operator $\Gamma_{S^*\alpha}$, corresponding to the shifted sequence $S^*\alpha$. In other words, the matrix of $\Gamma_{S^*\alpha}$ is obtained from the matrix of Γ_α by deleting the first row:

$$\Gamma_{S^*\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots \\ \alpha_3 & \alpha_4 & \alpha_5 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.2}$$

More formally, Γ_α and $\Gamma_{S^*\alpha}$ are related by

$$\Gamma_{S^*\alpha} = S^*\Gamma_\alpha = \Gamma_\alpha S. \tag{1.3}$$

Another important relation between Γ_α and $\Gamma_{S^*\alpha}$ is the formula

$$\Gamma^2_{S^*\alpha} = \Gamma^2_\alpha - (\cdot, \alpha)\alpha, \tag{1.4}$$

where $(\cdot,\cdot)$ is the standard inner product in ℓ^2 (we follow the convention that the inner product is linear in the first component and anti-linear in the second component). Formula (1.4) is an elementary consequence of the matrix representations (1.1), (1.2), or of identities (1.3).

1.2. **The class $\ell^{1,\infty}_{++}$.** We will consider positive bounded Hankel operators:

$$\Gamma_\alpha \geq 0.$$

Here the positivity is understood, as usual, in the quadratic form sense: $(\Gamma_\alpha x, x) \geq 0$ for all $x \in \ell^2$. As a consequence of general results of [10], a positive bounded Hankel operator can have any continuous spectrum of multiplicity ≤ 2 and any set of non-zero eigenvalues of multiplicity one (also zero must be in the spectrum and if zero is an eigenvalue, then it must have infinite multiplicity). In this paper, we make some progress towards the description of *isospectral sets*, i.e. of the sets of all positive Hankel operators with a given spectrum. To simplify the problem, we will consider a special sub-class of positive Hankel operators, those that satisfy the *double positivity condition:*

$$\Gamma_\alpha \geq 0 \quad \text{and} \quad \Gamma_{S^*\alpha} \geq 0. \tag{1.5}$$

In Section 2 we will prove that under this assumption the non-zero spectrum of Γ_α is simple, i.e. has multiplicity one. (In fact, the same applies to $\Gamma_{S^*\alpha}$ and to $\Gamma_{(S^*)^n\alpha}$ for all n.) This property makes the spectral theory of this class of Hankel operators particularly simple.

In order to set up some notation, we first recall that for positive Hankel operators the operator norm $\|\Gamma_\alpha\|$ admits a very simple description in terms of the sequence α. We denote by $\ell^{1,\infty}$ the set of all sequences $\{x_n\}_{n=0}^\infty$ of complex numbers such that

$$\|x\|_{\ell^{1,\infty}} = \sup_{n\geq 0}(n+1)|x_n| < \infty.$$

By [15, Theorem 3.1], under the positivity assumption $\Gamma_\alpha \geq 0$ one has

$$\tfrac{1}{4}\|\alpha\|_{\ell^{1,\infty}} \leq \|\Gamma_\alpha\| \leq \pi\|\alpha\|_{\ell^{1,\infty}} \tag{1.6}$$

(the constants are not written explicitly in [15] but are easy to work out from the argument given there; for completeness we give the proof in Section 2). This motivates the following definition.

Definition 1.1. Let $\ell^{1,\infty}_{++}$ be the class of all sequences $\alpha \in \ell^{1,\infty}$ such that the double positivity conditions (1.5) are fulfilled.

The double positivity condition implies, in particular, that the diagonal elements of the "matrices" Γ_α and $\Gamma_{S^*\alpha}$ are non-negative; hence $\alpha_n \geq 0$ for all $n \in \mathbb{Z}_+$. Notice that, as will be observed in the proof in Section 2, the first inequality (1.6) can then be improved as

$$\tfrac{1}{2}\|\alpha\|_{\ell^{1,\infty}} \leq \|\Gamma_\alpha\| \ . \tag{1.7}$$

1.3. **Main results.** In [6] for $\alpha \in \ell^{1,\infty}_{++}$ and under the additional assumption that Γ_α and $\Gamma_{S^*\alpha}$ are compact, it was proven that the spectrum of Γ_α *and* the spectrum of $\Gamma_{S^*\alpha}$ *together* uniquely determine the sequence α. (In fact, the results of [6] are not limited to the class $\ell^{1,\infty}_{++}$.) In this paper, we consider this inverse spectral problem without the assumption of compactness of Γ_α. In this case it turns out that the spectra of Γ_α and $\Gamma_{S^*\alpha}$ in general *do not* determine the sequence α (see Section 7). We show that the correct way to parametrise the spectral data is to consider the *spectral shift function* (SSF)

$$\xi_\alpha(\lambda) = \xi(\lambda; \Gamma_\alpha^2, \Gamma_{S^*\alpha}^2).$$

We refer to the Appendix for the background information on the SSF theory. In our case, since (by (1.4)) the difference $\Gamma_\alpha^2 - \Gamma_{S^*\alpha}^2$ is a positive operator of rank one, the SSF satisfies

$$0 \leq \xi_\alpha(\lambda) \leq 1, \qquad \text{a.e. } \lambda \in \mathbb{R}.$$

Further, since both $\Gamma_\alpha^2 \geq \Gamma_{S^*\alpha}^2$ are non-negative bounded operators, the SSF is supported on the compact interval $[0, \|\Gamma_\alpha\|^2]$.

We consider the map

$$\alpha \mapsto \xi_\alpha;$$

this gives rise to a *direct problem* (study the properties of ξ_α for a given α) and to an *inverse problem* (recover α from ξ_α). For convenience, let us introduce a piece of notation for the space where ξ_α is going to live:

Definition 1.2. Let Ξ_+ be the set of all functions $\xi \in L^\infty(\mathbb{R})$ with compact support in $[0, \infty)$ and with values in the interval $[0, 1]$.

Our main result is

Theorem 1.3. *The map*

$$\ell^{1,\infty}_{++} \ni \alpha \mapsto \xi_\alpha \in \Xi_+ \tag{1.8}$$

is a bijection between the sets $\ell^{1,\infty}_{++}$ *and* Ξ_+.

Note that this theorem includes two distinct non-trivial statements: the injectivity and the surjectivity of the map (1.8). Next, although the inverse map $\xi_\alpha \mapsto \alpha$ is

rather complicated, some information about α can be obtained directly from ξ_α. Indeed, we have two "trace formulas"

$$\int_0^\infty \xi_\alpha(\lambda)d\lambda = \sum_{n=0}^\infty \alpha_n^2, \tag{1.9}$$
$$\frac{2}{\pi}\int_0^\infty \left\{1 - \exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda+t^2}d\lambda\right)\right\} dt = \alpha_0,$$

see Theorem 3.1. We also have an explicit criterion that allows to decide whether $\operatorname{Ker}\Gamma_\alpha$ is trivial; this happens if and only if both

$$\int_0^1 \frac{\xi_\alpha(\lambda)}{\lambda}d\lambda = \infty \text{ and } \int_0^1 \frac{1-\xi_\alpha(\lambda)}{\lambda}d\lambda = \infty,$$

see Theorem 3.2.

Finally, we prove that the map (1.8) is a homeomorphism with respect to appropriate weak topologies, which we now introduce.

Given $R > 0$, we define the convex subset

$$\ell^{1,\infty}_{++}(R) := \{\alpha \in \ell^{1,\infty}_{++} : \|\alpha\|_{\ell^{1,\infty}} \le R\},$$

which we endow with the weak topology relative to the evaluation linear forms $\alpha \mapsto \alpha_n$. It is well known that $\ell^{1,\infty}_{++}(R)$ is a metrizable space, and that convergence of a sequence $\{\alpha^{(p)}\}_{p=1}^\infty$ to α in this space is equivalent to $\alpha_n^{(p)} \to \alpha_n$ as $p \to \infty$ for all n. Notice that, if $R_1 < R_2$, $\ell^{1,\infty}_{++}(R_1) \subset \ell^{1,\infty}_{++}(R_2)$, and that the corresponding inclusion is a homeomorphism. Since $\ell^{1,\infty}_{++}$ is the union of the increasing family $\{\ell^{1,\infty}_{++}(R)\}_{R>0}$, we endow it with the inductive limit topology, which is the strongest topology such that, for every $R > 0$, the inclusion of $\ell^{1,\infty}_{++}(R)$ into $\ell^{1,\infty}_{++}$ is continuous. A sequence is convergent for this topology if, for some R, it is contained in $\ell^{1,\infty}_{++}(R)$, and if it is convergent for the weak topology of $\ell^{1,\infty}_{++}(R)$. We note that, as it is straightforward to see, the weak convergence $\alpha^{(p)} \to \alpha$ in $\ell^{1,\infty}_{++}$ implies norm convergence of $\alpha^{(p)}$ to α in ℓ^r for any $r > 1$.

We do the same construction with the convex sets $\Xi_+(C)$, $C > 0$, corresponding to those elements of Ξ_+ which are supported on $[0, C]$. In this case, $\Xi_+(C)$ is endowed with the L^∞ weak* topology, which is known to be metrizable, the convergence of a sequence $\{\xi^{(p)}\}_{p=1}^\infty$ to ξ being equivalent to

$$\int_0^\infty \xi^{(p)}(\lambda)\varphi(\lambda)d\lambda \to \int_0^\infty \xi(\lambda)\varphi(\lambda)d\lambda$$

as $p \to \infty$ for all continuous functions φ. Since Ξ_+ is the union of the increasing family $\{\Xi_+(C)\}_{C>0}$, we endow it with the inductive limit topology, for which a sequence is convergent if and only if it is contained in some $\Xi_+(C)$ and if it is convergent for the above weak topology of $\Xi_+(C)$.

Theorem 1.4. *The map* (1.8) *is a homeomorphism with respect to the above weak topologies.*

Finally, we would like to mention a number of open questions related to our results:

- How to extend these results to the case of unbounded Hankel operators? What are the appropriate topologies on the set of sequences α and on the set of functions ξ_α in this case?
- How to extend these results to Hankel operators without the double positivity assumption? The construction of [6] suggests that one needs to introduce some additional spectral variables. An additional problem in this case is that the multiplicity of the spectrum of Γ_α may be non-trivial.

1.4. **Some ideas of the proof of Theorem 1.3.** We introduce the spectral measure ρ_α of Γ_α^2, corresponding to the element e_0:

$$\rho_\alpha(\delta) = (\chi_\delta(\Gamma_\alpha^2)e_0, e_0), \quad \delta \subset \mathbb{R}; \tag{1.10}$$

here and in what follows χ_δ stands for the characteristic function of the set $\delta \subset \mathbb{R}$. The measure ρ_α is related to ξ_α by

$$z\int_0^\infty \frac{d\rho_\alpha(\lambda)}{\lambda - z} = -\exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda - z}d\lambda\right) \tag{1.11}$$

for all z not in the spectrum of Γ_α; this is an easy calculation given in Section 3. The proof of injectivity proceeds as follows. Let ξ_α be given; the relation (1.11) determines the measure ρ_α. Next, we derive a simple recurrence relation which relates $\rho_{S^*\alpha}$ and ρ_α:

$$\int_0^\infty \frac{d\rho_{S^*\alpha}(\lambda)}{\lambda - z} = \int_0^\infty \frac{d\rho_\alpha(\lambda)}{\lambda - z} - \frac{1}{z}\left(\int_0^\infty \frac{d\rho_\alpha(\lambda)}{\lambda - z}\right)^{-1}\left(\int_0^\infty \frac{\sqrt{\lambda}d\rho_\alpha(\lambda)}{\lambda - z}\right)^2. \tag{1.12}$$

This relation allows one to inductively determine the measures $\rho_{(S^*)^n\alpha}$ for all n. This determines the whole sequence α because

$$\alpha_n = (\Gamma_{(S^*)^n\alpha}e_0, e_0) = \int_0^\infty \sqrt{\lambda}d\rho_{(S^*)^n\alpha}(\lambda), \quad n \in \mathbb{N}. \tag{1.13}$$

We also give a second proof of injectivity, which follows the ideas of [5] and [6], and is based on the use of the compressed shift operator.

Surjectivity is the hardest statement to prove in Theorem 1.3. We give two proofs of surjectivity. The first one is based on approximating a general element $\xi \in \Xi_+$ by elements corresponding to finite rank Hankel operators. This approach uses the finite rank surjectivity result, which was proven in [5] — see also [6]. Since we use approximation, this approach also relies on Theorem 1.4. The second proof of surjectivity is based on the identity (1.11) and on constructing the measure ρ_α, following the path of [10]. Finally, Theorem 1.4 (the continuity of the map (1.8)) has a surprisingly easy proof based on (1.11), (1.12), (1.13), on the equivalence of norms (1.6), and on trace formula (1.9).

1.5. **Some notation.** Given a sequence α, let

$$P_\alpha : \ell^2 \to \overline{\operatorname{Ran}\Gamma_\alpha} \tag{1.14}$$

be the orthogonal projection. It is clear from (1.1) and (1.2) that $\operatorname{Ker}\Gamma_\alpha \subset \operatorname{Ker}\Gamma_{S^*\alpha}$. Thus, $\overline{\operatorname{Ran}\Gamma_\alpha}$ is an invariant subspace for both operators Γ_α, $\Gamma_{S^*\alpha}$, and on the orthogonal complement to this subspace both operators are equal to zero. We set

$$\widetilde{\Gamma}_\alpha = \Gamma_\alpha|_{\overline{\operatorname{Ran}\Gamma_\alpha}}, \quad \widetilde{\Gamma}_{S^*\alpha} = \Gamma_{S^*\alpha}|_{\overline{\operatorname{Ran}\Gamma_\alpha}}. \tag{1.15}$$

It follows that

$$\xi_\alpha(\lambda) = \xi(\lambda; \widetilde{\Gamma}_\alpha^2, \widetilde{\Gamma}_{S^*\alpha}^2).$$

2. Operators Γ_α with double positivity condition

2.1. **Description in terms of moment sequences.** The second part of the following proposition is borrowed entirely from [15].

Proposition 2.1. *Let α be a sequence of real numbers such that the corresponding Hankel operator Γ_α is bounded. Then:*

(i) *The double positivity condition (1.5) holds true if and only if there exists a finite positive measure ω supported on $[0,1]$ with $\omega(\{1\}) = 0$ such that α_n can be represented as*

$$\alpha_n = \int_0^1 t^n d\omega(t), \quad n \geq 0. \tag{2.1}$$

(ii) *If $\Gamma_\alpha \geq 0$, then the estimates (1.6) hold true. If moreover $\Gamma_{S^*\alpha} \geq 0$, then (1.7) holds true.*

Proof. (i) Assume that the representation (2.1) holds with some ω. Then it is evident that

$$(\Gamma_\alpha x, x) = \sum_{n,m=0}^\infty \int_0^1 t^{n+m} x_n \overline{x_m}\, d\omega(t) = \int_0^1 \left| \sum_{n=0}^\infty t^n x_n \right|^2 d\omega(t) \geq 0,$$

thus $\Gamma_\alpha \geq 0$. Further, we have

$$\alpha_{n+1} = \int_0^1 t^{n+1} d\omega(t) = \int_0^1 t^n d\omega_1(t), \quad \text{where} \quad d\omega_1(t) = t d\omega(t). \tag{2.2}$$

Thus, by the same reasoning we also have $\Gamma_{S^*\alpha} \geq 0$.

Next, assume that the double positivity condition holds true and Γ_α is bounded. By the solution to the classical Hamburger moment problem (see e.g. [13, Section X.1, Example 3]), condition $\Gamma_\alpha \geq 0$ implies that there exists a measure $\omega \geq 0$ on $\mathbb{R}$ such that

$$\alpha_n = \int_{-\infty}^\infty t^n d\omega(t), \quad n \geq 0.$$

The boundedness of Γ_α implies that $\alpha \in \ell^2$ and then the measure ω is unique (see e.g. [13, Section X.6, Example 4]). By considering even n, it is easy to see that the boundedness of Γ_α implies that $\operatorname{supp}\omega \subset [-1,1]$ and $\omega(\{-1\}) = \omega(\{1\}) = 0$. As the same argument applies to $\Gamma_{S^*\alpha}$, we get that the measure ω_1, given by (2.2), is also positive. Thus, $\operatorname{supp}\omega \subset [0,1]$.

(ii) Assume that $\Gamma_\alpha \geq 0$. The second estimate in (1.6) follows directly from Hilbert's inequality:

$$\left| \sum_{n,m=0}^{N} \frac{x_n \overline{x_m}}{n+m+1} \right| \leq \pi \sum_{n=0}^{N} |x_n|^2.$$

Let us prove the first estimate in (1.6). By the proof of (i), we have

$$\alpha_n = \int_{-1}^{1} t^n d\omega(t), \quad n \geq 0$$

with some finite positive measure ω such that $\omega(\{-1\}) = \omega(\{1\}) = 0$. Fix $\tau \in (0,1)$ and let $x \in \ell^2$ be the element given by $x_n = \tau^n$, $n \geq 0$. We have

$$(\Gamma_\alpha x, x) = \sum_{n,m=0}^{\infty} \tau^n \tau^m \int_{-1}^{1} t^{n+m} d\omega(t) = \int_{-1}^{1} \frac{d\omega(t)}{(1-t\tau)^2}$$
$$\geq \int_{\tau}^{1} \frac{d\omega(t)}{(1-t\tau)^2} \geq \frac{1}{(1-\tau^2)^2} \int_{\tau}^{1} d\omega(t),$$

and therefore

$$\omega([\tau,1)) \leq (1-\tau^2)^2 (\Gamma_\alpha x, x) \leq (1-\tau^2)^2 \|\Gamma_\alpha\| \|x\|^2$$
$$\leq (1-\tau^2)^2 \|\Gamma_\alpha\| \frac{1}{1-\tau^2} = (1-\tau^2)\|\Gamma_\alpha\| \leq 2(1-\tau)\|\Gamma_\alpha\|.$$

Then

$$\int_0^1 t^n d\omega(t) = \omega((0,1)) - n \int_0^1 t^{n-1} \omega((0,t)) dt = n \int_0^1 t^{n-1} \omega([t,1)) dt$$
$$\leq 2\|\Gamma_\alpha\| n \int_0^1 t^{n-1}(1-t) dt = \frac{2}{n+1} \|\Gamma_\alpha\|.$$

In the same way, one proves that

$$\int_{-1}^{0} t^n d\omega(t) \leq \frac{2}{n+1} \|\Gamma_\alpha\|,$$

and so

$$\alpha_n = \int_{-1}^{1} t^n d\omega(t) \leq \frac{4}{n+1} \|\Gamma_\alpha\|.$$

This proves the first estimate in (1.6). The same proof yields (1.7) in the case of the double positivity condition. □

Remark 2.2. By inspection of the "matrix" (1.1) of Γ_α we see that $\Gamma_{(S^*)^2\alpha}$ is a submatrix obtained by deleting the first row and the first column. It follows that the condition $\Gamma_\alpha \geq 0$ implies $\Gamma_{(S^*)^2\alpha} \geq 0$ and then, by iteration, $\Gamma_{(S^*)^{2n}\alpha} \geq 0$. Similarly, $\Gamma_{S^*\alpha} \geq 0$ implies $\Gamma_{(S^*)^{2n+1}\alpha} \geq 0$ for all $n \in \mathbb{Z}_+$. Thus, the double positivity condition implies that $\Gamma_{(S^*)^n\alpha} \geq 0$ for all $n \in \mathbb{Z}_+$. This can be rephrased as

$$\alpha \in \ell^{1,\infty}_{++} \quad \Rightarrow \quad (S^*)^n \alpha \in \ell^{1,\infty}_{++} \text{ for all } n \in \mathbb{N}.$$

This property also follows directly from Proposition 2.1.

Remark 2.3. The sequences (α_n) given by (2.1), for a positive measure ω on $[0,1]$, are clearly completely monotonic, namely

$$\forall k \geq 0,\ \forall n \geq 0,\ ((I - S^*)^k \alpha)_n \geq 0\ .$$

In the classical paper [7], Hausdorff proved that this property is in fact equivalent to the representation (2.1). Hence elements of $\ell^{1,\infty}_{++}$ are special solutions of the classical Hausdorff moment problem, precisely those which belong to $\ell^{1,\infty}$.

2.2. **The simplicity of the spectrum.** We recall that an element ψ of a Hilbert space $\mathcal{H}$ is called a *generating element* of a bounded self-adjoint operator A in $\mathcal{H}$, if

$$\mathcal{H} = \operatorname{clos}\operatorname{span}\{A^n\psi \mid n = 0, 1, 2, \dots\}.$$

If A has a generating element, then it has a *simple spectrum*, i.e. it is unitarily equivalent to the operator of multiplication by an independent variable in some L^2 space of scalar valued functions. More precisely: let ρ_ψ be the measure on $\mathbb{R}$ defined by

$$\rho_\psi(\delta) = (\chi_\delta(A)\psi, \psi), \quad \delta \subset \mathbb{R}, \tag{2.3}$$

and let U_ψ be the operator

$$U_\psi : L^2(\mathbb{R}, d\rho_\psi) \to \mathcal{H}, \quad f \mapsto f(A)\psi. \tag{2.4}$$

Then U_ψ is unitary and

$$U_\psi^* A U_\psi = M_x, \quad U_\psi 1 = \psi, \tag{2.5}$$

where M_x is the operator of multiplication by the independent variable x,

$$(M_x f)(x) = x f(x), \quad x \in \mathbb{R}, \quad f \in L^2(\mathbb{R}, d\rho_\psi), \tag{2.6}$$

and 1 is the function identically equal to 1.

Theorem 2.4. *Let $\alpha \in \ell^{1,\infty}_{++}$. Denote*

$$\mathcal{M}_\alpha = \operatorname{clos}\operatorname{span}\{\Gamma_\alpha^n \alpha \mid n = 0, 1, 2, \dots\},$$
$$\mathcal{M}_{S^*\alpha} = \operatorname{clos}\operatorname{span}\{\Gamma_{S^*\alpha}^n \alpha \mid n = 0, 1, 2, \dots\}.$$

Then:

(i) *The subspaces $\mathcal{M}_\alpha$ and $\mathcal{M}_{S^*\alpha}$ coincide and will henceforth be denoted by $\mathcal{M}$.*
(ii) *$\overline{\operatorname{Ran}\Gamma_\alpha} = \mathcal{M}$; the operator $\widetilde{\Gamma}_\alpha$ (see (1.15)) has a simple spectrum and a generating element $P_\alpha e_0$.*
(iii) *$\overline{\operatorname{Ran}\Gamma_{S^*\alpha}} \subset \mathcal{M}$; the operator $\widetilde{\Gamma}_{S^*\alpha}$ has a simple spectrum and a generating element α.*

Remark. (1) The inclusion $\overline{\operatorname{Ran}\Gamma_{S^*\alpha}} \subset \mathcal{M}$ may be strict. For example, for $\alpha = e_0$ it is easy to see that $\operatorname{Ker}\Gamma_\alpha \neq \operatorname{Ker}\Gamma_{S^*\alpha}$, and so $\overline{\operatorname{Ran}\Gamma_{S^*\alpha}} \neq \overline{\operatorname{Ran}\Gamma_\alpha}$. For a description of when this situation occurs, see [6].
(2) By Remark 2.2, the spectra of all operators $\Gamma_{(S^*)^n\alpha}$ are simple.

In order to prove Theorem 2.4, first we need a general operator theoretic lemma which in some form goes back at least to Kato [8] but is probably much older:

Proposition 2.5. *Let A_1 and A_0 be bounded self-adjoint operators in a Hilbert space such that the difference $A_1 - A_0$ is a rank one operator:*

$$A_1 = A_0 + (\cdot, \psi)\psi. \tag{2.7}$$

Denote

$$\mathcal{M}_j = \operatorname{clos}\operatorname{span}\{A_j^n \psi \mid n = 0, 1, \dots\}, \quad j = 0, 1.$$

Then:

(i) *The subspaces $\mathcal{M}_0$ and $\mathcal{M}_1$ coincide and will henceforth be denoted by $\mathcal{M}$.*
(ii) *$\mathcal{M}$ is an invariant subspace both for A_0 and for A_1.*
(iii) *$A_0|_{\mathcal{M}^\perp} = A_1|_{\mathcal{M}^\perp}$.*

Proof. (i) Let $f = A_1^n \psi \in \mathcal{M}_1$. Using (2.7) and expanding, we see that $f \in \mathcal{M}_0$. Thus, $\mathcal{M}_1 \subset \mathcal{M}_0$; similarly one obtains $\mathcal{M}_0 \subset \mathcal{M}_1$. (ii) It is immediate that $A_0(\mathcal{M}_0) \subset \mathcal{M}_0$ and $A_1(\mathcal{M}_1) \subset \mathcal{M}_1$. (iii) If $f \perp \mathcal{M}$, then in particular $f \perp \psi$. Now apply (2.7). □

Proof of Theorem 2.4. Since $\Gamma_\alpha \geq 0$, one can approximate (in the operator norm) odd powers Γ_α^{2n+1} by polynomials involving only even powers of Γ_α. The same consideration of course applies to $\Gamma_{S^*\alpha}$. It follows that $\mathcal{M}_\alpha$, $\mathcal{M}_{S^*\alpha}$ can be rewritten as

$$\mathcal{M}_\alpha = \operatorname{clos}\operatorname{span}\{\Gamma_\alpha^{2n}\alpha \mid n = 0, 1, 2, \dots\},$$
$$\mathcal{M}_{S^*\alpha} = \operatorname{clos}\operatorname{span}\{\Gamma_{S^*\alpha}^{2n}\alpha \mid n = 0, 1, 2, \dots\}.$$

Now let us apply Proposition 2.5 with $A_0 = \Gamma_{S^*\alpha}^2$, $A_1 = \Gamma_\alpha^2$, $\psi = \alpha$. Part (i) of the Theorem immediately follows from Proposition 2.5(i).

Next, let $f \perp \mathcal{M}$; by Proposition 2.5(iii), we have $\Gamma_\alpha^2 f = \Gamma_{S^*\alpha}^2 f$ and therefore, by the double positivity condition, we get $\Gamma_\alpha f = \Gamma_{S^*\alpha} f$. By (1.3), this can be rewritten as

$$\Gamma_{S^*\alpha} f = S^* \Gamma_\alpha f = \Gamma_\alpha f.$$

Since $\operatorname{Ker}(S^* - I) = \{0\}$, we obtain $\Gamma_\alpha f = 0$ and $\Gamma_{S^*\alpha} f = 0$. Thus,

$$\mathcal{M}^\perp \subset \operatorname{Ker}\Gamma_\alpha \quad \text{and} \quad \mathcal{M}^\perp \subset \operatorname{Ker}\Gamma_{S^*\alpha},$$

and therefore

$$\overline{\operatorname{Ran}\Gamma_\alpha} \subset \mathcal{M} \quad \text{and} \quad \overline{\operatorname{Ran}\Gamma_{S^*\alpha}} \subset \mathcal{M}.$$

Since $\alpha = \Gamma_\alpha P_\alpha e_0$, we also have $\mathcal{M} \subset \overline{\operatorname{Ran}\Gamma_\alpha}$. Thus, we get parts (i) and (ii) of the Theorem. □

3. Direct spectral problem

3.1. **The perturbation determinant and the trace formulas.** Let $\alpha \in \ell^{1,\infty}_{++}$. For $z \notin [0, \infty)$, consider the perturbation determinant (see Appendix) for the pair of operators Γ_α^2, $\Gamma_{S^*\alpha}^2$:

$$\Delta(z) = \Delta_{\Gamma_\alpha^2/\Gamma_{S^*\alpha}^2}(z) = \det((\Gamma_\alpha^2 - z)(\Gamma_{S^*\alpha}^2 - z)^{-1}).$$

By (1.4), it can be explicitly computed as follows:

$$\Delta(z)^{-1} = \det((\Gamma_{S^*\alpha}^2 - z)(\Gamma_\alpha^2 - z)^{-1}) = \det(I + (\Gamma_{S^*\alpha}^2 - \Gamma_\alpha^2)(\Gamma_\alpha^2 - z)^{-1})$$
$$= \det(I - (\cdot, (\Gamma_\alpha^2 - \overline{z})^{-1}\alpha)\alpha) = 1 - ((\Gamma_\alpha^2 - z)^{-1}\alpha, \alpha). \quad (3.1)$$

Recalling that $\alpha = \Gamma_\alpha e_0$ and using (A.5), we obtain

$$(\Gamma_\alpha^2(\Gamma_\alpha^2 - z)^{-1}e_0, e_0) = 1 - \exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda - z}d\lambda\right), \quad z \notin [0, \infty). \quad (3.2)$$

This is one of the key formulas in our construction. It can be alternatively written as

$$z((\Gamma_\alpha^2 - z)^{-1}e_0, e_0) = -\exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda - z}d\lambda\right), \quad z \notin [0, \infty). \quad (3.3)$$

Let ρ_α be the measure on $\mathbb{R}$ defined by (1.10), i.e.

$$\rho_\alpha(\delta) = (\chi_\delta(\Gamma_\alpha^2)e_0, e_0), \quad \delta \subset \mathbb{R}.$$

Using the measure ρ_α, we may rewrite (3.2) as

$$\int_0^\infty \frac{\lambda d\rho_\alpha(\lambda)}{\lambda - z} = 1 - \exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda - z}d\lambda\right). \quad (3.4)$$

This gives a one-to-one correspondence between ξ_α and ρ_α. (In fact, ρ_α is in some respects a more convenient functional parameter than ξ_α.)

It will be also convenient to use the following modification of the measure ρ_α:

$$\widetilde{\rho}_\alpha(\delta) = (\chi_\delta(\widetilde{\Gamma}_\alpha^2)P_\alpha e_0, P_\alpha e_0) \quad (3.5)$$

(see (1.14), (1.15)). Of course, the difference between the measures ρ_α and $\widetilde{\rho}_\alpha$ is only in the weight at zero:

$$\widetilde{\rho}_\alpha(\delta) = \rho_\alpha(\delta) - \rho_\alpha(\delta \cap \{0\}).$$

By Theorem 2.4(ii), the operator $\widetilde{\Gamma}_\alpha^2$ is unitarily equivalent to the operator M_x of multiplication by x in $L^2(\mathbb{R}, d\widetilde{\rho}_\alpha)$; thus, $\widetilde{\rho}_\alpha$ contains all information about the spectrum of $\widetilde{\Gamma}_\alpha$.

Theorem 3.1. *Let $\alpha \in \ell_{++}^{1,\infty}$. Then the identities*

$$\int_0^\infty \xi_\alpha(\lambda)d\lambda = \sum_{n=0}^\infty \alpha_n^2, \quad (3.6)$$

$$\frac{2}{\pi}\int_0^\infty \left\{1 - \exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda + t^2}d\lambda\right)\right\}dt = \alpha_0. \quad (3.7)$$

hold true.

Proof. Formula (3.6) is a direct consequence of (A.2) and of the identity

$$\mathrm{Tr}((\cdot, \alpha)\alpha) = (\alpha, \alpha) = \sum_{n=0}^\infty \alpha_n^2.$$

In order to prove (3.7), we first notice that $\alpha_0 = (\Gamma_\alpha e_0, e_0)$. We also use the integral representation for the square root:

$$\Gamma_\alpha = \sqrt{\Gamma_\alpha^2} = \frac{2}{\pi}\int_0^\infty \Gamma_\alpha^2(\Gamma_\alpha^2 + t^2)^{-1}dt.$$

Putting this together and combining with (3.2), we obtain (3.7). □

3.2. **The kernel of Γ_α.** We recall that (due to the Beurling theorem, see [2]) the kernel of a Hankel operator is either trivial or infinite dimensional. Further, by (1.4), the kernel of Γ_α is infinite dimensional if and only if the kernel of $\Gamma_{S^*\alpha}$ is infinite dimensional. Below we give a concrete criterion for this to happen.

Theorem 3.2. *Let $\alpha \in \ell^{1,\infty}_{++}$; then the kernels of Γ_α, $\Gamma_{S^*\alpha}$ are trivial if and only if both of the following conditions hold true:*

$$\int_0^1 \frac{\xi_\alpha(\lambda)}{\lambda}d\lambda = \infty, \qquad \int_0^1 \frac{1-\xi_\alpha(\lambda)}{\lambda}d\lambda = \infty. \tag{3.8}$$

Of course, the integral $\int_0^1$ in (3.8) can be replaced by $\int_0^a$ for any $a > 0$.

Proof. Exactly as in Theorem 4 of [6], we have

$$\operatorname{Ker}\Gamma_\alpha = \{0\} \quad \Longleftrightarrow \quad e_0 \in \overline{\operatorname{Ran}\Gamma_\alpha} \setminus \operatorname{Ran}\Gamma_\alpha.$$

Let us express the latter condition in terms of the function in the l.h.s. of (3.3). Using the spectral theorem for self-adjoint operators, it is easy to see that the strong limit

$$\operatorname*{s-lim}_{\varepsilon\to 0+} \varepsilon(\Gamma_\alpha^2 + \varepsilon)^{-1}$$

exists and is equal to the orthogonal projection onto $\operatorname{Ker}\Gamma_\alpha$. Thus, using (3.3),

$$e_0 \in \overline{\operatorname{Ran}\Gamma_\alpha} \quad \Longleftrightarrow \quad \lim_{\varepsilon\to 0+} \varepsilon((\Gamma_\alpha^2+\varepsilon)^{-1}e_0, e_0) = 0$$
$$\Longleftrightarrow \quad \lim_{\varepsilon\to 0+}\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda+\varepsilon}d\lambda = \infty \quad \Longleftrightarrow \quad \int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda}d\lambda = \infty,$$

so we obtain the first of the conditions (3.8). Next,

$$e_0 \in \operatorname{Ran}\Gamma_\alpha \quad \Longleftrightarrow \quad \|\Gamma_\alpha^{-1}e_0\|^2 < \infty \quad \Longleftrightarrow \quad \lim_{\varepsilon\to 0+}((\Gamma_\alpha^2+\varepsilon)^{-1}e_0, e_0) < \infty$$

and therefore, using (3.3),

$$e_0 \in \operatorname{Ran}\Gamma_\alpha \quad \Longleftrightarrow \quad \lim_{\varepsilon\to 0+}\frac{1}{\varepsilon}\exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda+\varepsilon}d\lambda\right) < \infty.$$

Finally,

$$\frac{1}{\varepsilon}\exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda+\varepsilon}d\lambda\right) = \exp\left(\int_0^1 \frac{1-\xi_\alpha(\lambda)}{\lambda+\varepsilon}d\lambda\right)\frac{1}{1+\varepsilon}\exp\left(-\int_1^\infty \frac{\xi_\alpha(\lambda)}{\lambda+\varepsilon}d\lambda\right),$$

and therefore

$$e_0 \in \operatorname{Ran}\Gamma_\alpha \quad \Longleftrightarrow \quad \int_0^1 \frac{1-\xi_\alpha(\lambda)}{\lambda}d\lambda < \infty.$$

This yields the second condition (3.8). □

Remark 3.3. In [6, Theorem 2] it was proven that in the case of the compact operators Γ_α, $\Gamma_{S^*\alpha}$ with the eigenvalues $\{\lambda_j\}_{j=1}^\infty$ and $\{\mu_j\}_{j=1}^\infty$, the kernel of Γ_α is trivial if and only if both of the following conditions hold:

$$\sum_{j=1}^\infty \left(1-\frac{\mu_j^2}{\lambda_j^2}\right)=\infty, \qquad \sup_N \frac{1}{\lambda_{N+1}^2}\prod_{j=1}^N \frac{\mu_j^2}{\lambda_j^2}=\infty. \tag{3.9}$$

In this case we have

$$\xi_\alpha(\lambda)=\begin{cases}1, & \mu_j^2\le\lambda\le\lambda_j^2 \text{ for some } j,\\ 0, & \text{otherwise.}\end{cases}$$

Using this formula and some elementary manipulations, it is not difficult to check that (3.9) is in fact equivalent to (3.8).

4. Inverse spectral problem: uniqueness

Theorem 4.1. *The map*

$$\ell^{1,\infty}_{++}\ni\alpha\longmapsto\xi_\alpha\in\Xi_+$$

is injective, i.e. the sequence α is uniquely determined by the function ξ_α.

First proof. Let us derive a recurrence relation for $\rho_{S^*\alpha}$ in terms of ρ_α. ¿From (1.4) by the resolvent identity (A.6), (A.7) we get

$$(\Gamma^2_{S^*\alpha}-z)^{-1}-(\Gamma^2_\alpha-z)^{-1}=\frac{1}{D_\alpha(z)}(\cdot,(\Gamma^2_\alpha-\overline{z})^{-1}\alpha)(\Gamma^2_\alpha-z)^{-1}\alpha, \tag{4.1}$$

where

$$\begin{aligned}D_\alpha(z)=1-((\Gamma^2_\alpha-z)^{-1}\alpha,\alpha)&=1-((\Gamma^2_\alpha-z)^{-1}\Gamma_\alpha e_0,\Gamma_\alpha e_0)\\&=-z((\Gamma^2_\alpha-z)^{-1}e_0,e_0)=-z\int_0^\infty\frac{d\rho_\alpha(\lambda)}{\lambda-z}.\end{aligned}$$

Evaluating the quadratic form of both sides of (4.1) on the element e_0, we obtain

$$\int_0^\infty\frac{d\rho_{S^*\alpha}(\lambda)}{\lambda-z}-\int_0^\infty\frac{d\rho_\alpha(\lambda)}{\lambda-z}=-\frac{1}{z}\left(\int_0^\infty\frac{d\rho_\alpha(\lambda)}{\lambda-z}\right)^{-1}\left(\int_0^\infty\frac{\sqrt{\lambda}d\rho_\alpha(\lambda)}{\lambda-z}\right)^2. \tag{4.2}$$

Now we can complete the proof. It is well known that a finite measure on $\mathbb{R}$ is uniquely determined by its Cauchy transform. Thus, by (3.4), the SSF ξ_α uniquely determines the measure ρ_α. Identity (4.2) allows one to determine $\rho_{(S^*)^n\alpha}$ iteratively for all $n\in\mathbb{Z}_+$. Finally, identity (1.13) uniquely determines the whole sequence α. □

Second proof. Let $\widetilde{\rho}_\alpha$ be the measure given by (3.5). By (3.4), the measure $\widetilde{\rho}_\alpha$ is uniquely determined by ξ_α; thus, it suffices to prove that the sequence α is uniquely determined by the measure $\widetilde{\rho}_\alpha$.

For a given sequence α, let the unitary operator U_α be given by

$$U_\alpha: L^2(\mathbb{R},d\widetilde{\rho}_\alpha)\to\overline{\operatorname{Ran}\Gamma_\alpha},\quad f\mapsto f(\widetilde{\Gamma}^2_\alpha)P_\alpha e_0. \tag{4.3}$$

We have (cf. (2.3)–(2.6))

$$U_\alpha^* \widetilde{\Gamma}_\alpha^2 U_\alpha = M_x, \tag{4.4}$$

where M_x is the operator of multiplication by the independent variable in $L^2(\mathbb{R}, d\widetilde{\rho}_\alpha)$. Applying the unitary transformation U_α to (1.4), we get

$$U_\alpha^* \widetilde{\Gamma}_{S^*\alpha}^2 U_\alpha = M_x - (\cdot, M_x^{1/2}1) M_x^{1/2}1. \tag{4.5}$$

The r.h.s. is an operator in $L^2(\mathbb{R}, d\widetilde{\rho}_\alpha)$ given by an explicit formula independent of α. Thus, the operator $U_\alpha^* \widetilde{\Gamma}_{S^*\alpha}^2 U_\alpha$ (and therefore its square root $U_\alpha^* \widetilde{\Gamma}_{S^*\alpha} U_\alpha$) is uniquely determined by the measure $\widetilde{\rho}_\alpha$.

We will use the the compressed shift operator $P_\alpha S P_\alpha^*$. Denote

$$\Sigma = U_\alpha^* P_\alpha S P_\alpha^* U_\alpha.$$

By (1.3), we have

$$\Sigma^* M_x^{1/2} = U_\alpha^* \widetilde{\Gamma}_{S^*\alpha} U_\alpha,$$

and therefore the operator Σ^* is uniquely determined by $\widetilde{\rho}_\alpha$.

By inspection of (1.1), we find

$$\alpha_n = ((S^*)^n \Gamma_\alpha e_0, e_0) = ((P_\alpha S^* P_\alpha^*)^n \widetilde{\Gamma}_\alpha P_\alpha e_0, P_\alpha e_0).$$

Applying U_α, we get

$$\alpha_n = ((\Sigma^*)^n M_x^{1/2}1, 1);$$

the r.h.s. is uniquely determined by $\widetilde{\rho}_\alpha$, and therefore by ξ_α. □

5. Continuity of the map $\alpha \mapsto \xi_\alpha$ and its inverse

Here we prove Theorem 1.4. It will be useful for us to rephrase it in a slightly different way; the statement below also includes that the range of $\alpha \mapsto \xi_\alpha$ is closed. We refer to the introduction for the definition of weak convergence in $\ell^{1,\infty}_{++}$ and Ξ_+. Notice that property (1.6) implies that $\ell^{1,\infty}_{++}(R)$ is mapped into $\Xi_+(\pi^2 R^2)$. Conversely, if α belongs to the inverse image of $\Xi_+(C)$, using (3.4), it is easy to see that the support of ρ_α is contained in $[0, C]$. Since e_0 is a generating element of Γ_α, we infer that $\|\Gamma_\alpha\| \le \sqrt{C}$, and, by (1.7), that $\alpha \in \ell^{1,\infty}_{++}(2\sqrt{C})$. In order to prove continuity, it is therefore enough to deal with sequences in $\ell^{1,\infty}_{++}(R)$ and in $\Xi_+(C)$.

Theorem 5.1. (i) *Let $\alpha^{(p)}$ be a sequence of elements in $\ell^{1,\infty}_{++}(R)$ and let α be a sequence of real numbers. Assume that $\alpha_n^{(p)} \to \alpha_n$ as $p \to \infty$ for all n. Then $\alpha \in \ell^{1,\infty}_{++}$ and $\xi_{\alpha^{(p)}} \to \xi_\alpha$ weakly in Ξ_+.*

(ii) *Let $\alpha^{(p)}$ be a sequence of elements of $\ell^{1,\infty}_{++}$. If $\xi_{\alpha^{(p)}} \to \xi$ weakly in Ξ_+, then there exists $\alpha \in \ell^{1,\infty}_{++}$ such that $\alpha^{(p)} \to \alpha$ weakly in $\ell^{1,\infty}_{++}$ as $p \to \infty$, and $\xi = \xi_\alpha$.*

Proof. (i) Let $\alpha^{(p)}$, α be as in the hypothesis. As we already mentioned, the supports of $\xi_{\alpha^{(p)}}$ all lie in the compact set $[0, \pi^2 R^2]$.

Next, it is straightforward to see that the weak convergence implies that $\alpha \in \ell^{1,\infty}$ and therefore Γ_α is a bounded operator. Further, since $\ell^{1,\infty} \subset \ell^r$ for any $r > 1$, it is easy to see that $\alpha \in \ell^2$ and we have the norm convergence $\|\alpha^{(p)} - \alpha\|_{\ell^2} \to 0$ as

$p \to \infty$. We conclude that if f is a finite linear combination of the elements $\{e_n\}_{n=0}^{\infty}$, then

$$\|\Gamma_{\alpha^{(p)}} f - \Gamma_\alpha f\|_{\ell^2} \to 0 \quad \text{as } p \to \infty.$$

Using the uniform boundedness of the norms $\|\Gamma_{\alpha^{(p)}}\|$, we obtain that $\Gamma_{\alpha^{(p)}} \to \Gamma_\alpha$ in strong operator topology. Similarly, $\Gamma_{S^*\alpha^{(p)}} \to \Gamma_{S^*\alpha}$ strongly. It follows that $\Gamma_\alpha \geq 0$ and $\Gamma_{S^*\alpha} \geq 0$ and so $\alpha \in \ell^{1,\infty}_{++}$.

The strong convergence of operators yields (see e.g. [12, Theorem VIII.24(a)]) the strong convergence of resolvents. Thus,

$$(\Gamma^2_{\alpha^{(p)}}(\Gamma^2_{\alpha^{(p)}} - z)^{-1} e_0, e_0) \to (\Gamma^2_\alpha(\Gamma^2_\alpha - z)^{-1} e_0, e_0), \qquad \text{as } p \to \infty,$$

for all $\operatorname{Im} z \neq 0$. By (3.2), we get

$$\int_0^\infty \frac{\xi_{\alpha^{(p)}}(\lambda)}{\lambda - z} d\lambda \to \int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda - z} d\lambda \quad \text{as } p \to \infty;$$

this yields the weak convergence $\xi_{\alpha^{(p)}} \to \xi_\alpha$.

(ii) Let $\alpha^{(p)}$ be as in the hypothesis. By our definition of weak convergence in Ξ_+, we have $\operatorname{supp} \xi_{\alpha^{(p)}} \subset [0, C]$ for all p and some $C > 0$. As we already observed, $\rho_{\alpha^{(p)}}$ is supported on $[0, C]$ and $\alpha^{(p)}$ belongs to $\ell^{1,\infty}_{++}(2\sqrt{C})$. Consequently, by a diagonal argument, there exists a subsequence $\alpha^{(p')}$ such that $\alpha_n^{(p')} \to \alpha_n$ as $p' \to \infty$ for all n. Using part i) of the theorem, we infer that $\alpha \in \ell^{1,\infty}_{++}$ and that $\xi_{\alpha^{(p')}} \to \xi_\alpha$ weakly in Ξ_+, so that $\xi = \xi_\alpha$. By the uniqueness Theorem 4.1, we conclude that α is unique, thus the whole sequence $\alpha^{(p)}$ weakly converges to α in $\ell^{1,\infty}_{++}$. □

Remark 5.2. Using the same arguments as in the above proof, one could similarly describe the weak continuity of the map $\alpha \mapsto \rho_\alpha$.

6. Inverse spectral problem: surjectivity

Here we prove that the map (1.8) is surjective. We give two proofs.

6.1. First proof.

The first proof is based on the following result of [5] — see also [6] — about the finite rank Hankel operators:

Proposition 6.1 ([5], Corollary 3). *Let $N \in \mathbb{N}$ and let $\{\lambda_n\}_{n=1}^N$ and $\{\mu_n\}_{n=1}^N$ be two finite sequences of real numbers such that*

$$0 < \mu_N < \lambda_N < \mu_{N-1} < \cdots < \mu_1 < \lambda_1.$$

Then there exists $\alpha \in \ell^{1,\infty}_{++}$ such that the non-zero eigenvalues of Γ_α coincide with $\{\lambda_n\}_{n=1}^N$ and the non-zero eigenvalues of $\Gamma_{S^*\alpha}$ coincide with $\{\mu_n\}_{n=1}^N$.

Note that under the hypothesis of the proposition, we have

$$\xi_\alpha = \sum_{n=1}^N \chi_{\Delta_n}, \quad \Delta_n = [\mu_n^2, \lambda_n^2].$$

This follows from (A.3).

First proof of surjectivity. Let $\xi \in \Xi_+$, $\operatorname{supp}\xi \subset [0, C]$.

Step 1: let us show that there exists a sequence $\xi^{(p)} \in \Xi_+$, $p \in \mathbb{N}$, with the following properties:

(i) $\xi^{(p)} \to \xi$ weakly in Ξ_+;

(ii) for each p, the function $\xi^{(p)}$ has the following structure:

$$\xi^{(p)} = \sum_{n=1}^{N} \chi_{\Delta_n},$$

where $\{\Delta_n\}_{n=1}^N$ is a finite collection of disjoint closed intervals in $(0, C]$.

First note that finite linear combinations of characteristic functions of intervals are dense in $L^1(0, C)$. Approximating ξ by such functions, we can obtain a weakly convergent sequence in Ξ_+. Thus, it suffices to show that if $\xi = A\xi_\Delta$, where $0 < A \le 1$ and $\Delta \subset (0, \infty)$ is a compact interval, then a sequence $\xi^{(p)}$ as above can be constructed. The latter statement is easy to check directly, and is known in signal processing as the pulse width modulation method. Indeed, let $\Delta = [a, b]$; set $L = b - a$ and

$$\Delta_n = \left[a + \tfrac{n-1}{N}L, a + \tfrac{n-1+A}{N}L\right], \quad 1 \le n \le N,$$

and let $\xi^{(N)} = \sum_{n=1}^N \chi_{\Delta_n}$. Then it is easy to see that $\xi^{(N)} \to A\chi_\Delta$ in Ξ_+.

Step 2: Let $\xi^{(p)}$ be as constructed at the previous step. For any fixed p, let us write the intervals Δ_n as $\Delta_n = [\mu_n^2, \lambda_n^2]$ and let us use Proposition 6.1. We obtain an element $\alpha^{(p)} \in \ell^{1,\infty}_{++}$ such that $\xi_{\alpha^{(p)}} = \xi^{(p)}$. Thus, we obtain a sequence of elements $\alpha^{(p)} \in \ell^{1,\infty}_{++}$ such that $\xi_{\alpha^{(p)}} \to \xi$ weakly in Ξ_+.

By Theorem 5.1(ii), there exists $\alpha \in \ell^{1,\infty}_{++}$ such that $\xi = \xi_\alpha$, as required. □

6.2. **Second proof.** The second proof of surjectivity is heavily based on the construction of [10].

Step 1: Let $\xi \in \Xi_+$, $\operatorname{supp}\xi \subset [0, C]$. Consider the analytic function

$$\Delta(z) = \exp\left(\int_0^\infty \frac{\xi(\lambda)}{\lambda - z} d\lambda\right), \quad z \notin [0, \infty). \tag{6.1}$$

For z in the upper half-plane set $z = x + iy$, $y > 0$; then, using the assumption $0 \le \xi \le 1$, we have

$$0 \le y \int_0^\infty \frac{\xi(\lambda)}{(\lambda - x)^2 + y^2} d\lambda \le \pi,$$

and therefore

$$\operatorname{Im}\Delta(x + iy) = \exp\left(\int_0^\infty \frac{(\lambda - x)\xi(\lambda)}{(\lambda - x)^2 + y^2} d\lambda\right) \sin\left(y \int_0^\infty \frac{\xi(\lambda)}{(\lambda - x)^2 + y^2} d\lambda\right) \ge 0.$$

Thus, $\Delta(z)$ is a Herglotz function (analytic function with a positive imaginary part in the upper half-plane). Further, one has

$$\Delta(z) = 1 - \frac{1}{z}\int_0^\infty \xi(\lambda) d\lambda + O(|z|^{-2}), \quad |z| \to \infty.$$

It follows that $1-\Delta(z)^{-1}$ is also a Herglotz function which satisfies

$$1-\Delta(z)^{-1}=-\frac{1}{z}\int_0^\infty \xi(\lambda)d\lambda+O(|z|^{-2}),\quad |z|\to\infty.$$

By a Herglotz representation theorem (see e.g. [9, Section 14]), we obtain

$$1-\Delta(z)^{-1}=\int_{-\infty}^\infty \frac{d\nu(\lambda)}{\lambda-z}, \tag{6.2}$$

where $\nu\geq 0$ is a finite measure on $\mathbb{R}$. Since $\Delta(z)$ is analytic in $\mathbb{C}\setminus[0,C]$, we get that $\operatorname{supp}\nu\subset[0,C]$. Let $z=-\varepsilon$:

$$\int_0^\infty \frac{d\nu(\lambda)}{\lambda+\varepsilon}=1-\Delta(-\varepsilon)^{-1}=1-\exp\left(-\int_0^\infty\frac{\xi(\lambda)}{\lambda+\varepsilon}d\lambda\right)<1.$$

It follows that

$$\int_0^\infty \frac{d\nu(\lambda)}{\lambda}\leq 1.$$

In particular, this means that $\nu(\{0\})=0$. Set $d\widetilde{\rho}(\lambda)=\lambda^{-1}d\nu(\lambda)$. Then $\widetilde{\rho}$ is a non-negative finite measure with a compact support in $[0,\infty)$ and $\widetilde{\rho}(\mathbb{R})\leq 1$. Rewriting the integral representation (6.2) for Δ in terms of $\widetilde{\rho}$, we obtain (cf. (3.4))

$$1-\Delta(z)^{-1}=\int_0^\infty\frac{\lambda d\widetilde{\rho}(\lambda)}{\lambda-z}. \tag{6.3}$$

Step 2: Consider the Hilbert space $L^2(\mathbb{R},d\widetilde{\rho})$ and the operator M_x of multiplication by the independent variable in this space. It what follows, it is important that $\widetilde{\rho}(\{0\})=0$ and therefore $\operatorname{Ker}M_x$ is trivial. Set

$$H_0=M_x^{1/2},\quad H=(H_0^2-(\cdot,H_0 1)H_0 1)^{1/2}.$$

Below we prove that there exists a bounded Hankel operator Γ_α in ℓ^2 with $\Gamma_\alpha\geq 0$, $\Gamma_{S^*\alpha}\geq 0$ and an isometry

$$U: L^2(\mathbb{R},d\widetilde{\rho})\to\ell^2\quad\text{with}\quad \operatorname{Ran}U=\overline{\operatorname{Ran}\Gamma_\alpha} \tag{6.4}$$

such that

$$H_0=U^*\widetilde{\Gamma}_\alpha U,\quad H=U^*\widetilde{\Gamma}_{S^*\alpha}U \tag{6.5}$$

(compare with (4.3), (4.4), (4.5)).

Assume that such operators Γ_α and U have already been found. By (6.3), we have

$$\Delta(z)^{-1}=1-((H_0^2-z)^{-1}H_0 1,H_0 1),$$

and therefore, reversing the steps in calculation (3.1),

$$\Delta(z)=\det((H_0^2-z)(H^2-z)^{-1})=\Delta_{H_0^2/H^2}(z).$$

Thus, from (6.1) and (A.5), we find

$$\xi(\lambda)=\xi(\lambda;H_0^2,H^2).$$

By (6.5), this yields $\xi=\xi_\alpha$, as required.

Step 3: Now we need to construct Γ_α and U satisfying (6.4) and (6.5). The rest of the proof repeats almost verbatim the arguments of [10, Section III.3]. ¿From the definition of H, we have

$$\|Hf\|^2 = (H^2 f, f) \le (H_0^2 f, f) = \|H_0 f\|^2$$

for any f. Thus, there exists a contraction Σ_0 such that $\Sigma_0 H_0 = H$. Let $\Sigma = \Sigma_0^*$; then

$$H_0 \Sigma = H.$$

¿From here we get

$$H_0^2 - (\cdot, H_0 1) H_0 1 = H_0 \Sigma \Sigma^* H_0,$$

and therefore we obtain

$$\Sigma\Sigma^* = I - (\cdot, 1)1\,.$$

For any $f \in L^2(\mathbb{R}, d\widetilde{\rho})$ we have

$$\|f\|^2 = |(f,1)|^2 + \|\Sigma^* f\|^2.$$

Iterating this, we obtain

$$\|f\|^2 = \sum_{n=0}^{\infty} |(f, \Sigma^n 1)|^2 + \lim_{n\to\infty} \|(\Sigma^*)^n f\|^2, \tag{6.6}$$

where the limit necessarily exists and the series necessarily converges. In order to complete the proof, we need

Lemma 6.2. *For any $f \in L^2(\mathbb{R}, d\widetilde{\rho})$, we have*

$$\lim_{n\to\infty} \|(\Sigma^*)^n f\| = 0.$$

This lemma will be proven at the end of the section.

Step 4: Assuming Lemma 6.2, let us complete the proof of the theorem. Consider the map $U : L^2(\mathbb{R}, d\widetilde{\rho}) \to \ell^2$ given by

$$f \mapsto \{(f, \Sigma^n 1)\}_{n=0}^{\infty}.$$

By (6.6) and Lemma 6.2, this map is an isometry. Set

$$\alpha_n = ((\Sigma^*)^n H_0 1, 1).$$

If $\{e_n\}_{n=0}^{\infty}$ is the standard basis in ℓ^2, we have (by the definition of U)

$$U^* e_n = \Sigma^n 1.$$

Thus,

$$(U H_0 U^* e_n, e_m) = (H_0 \Sigma^n 1, \Sigma^m 1).$$

Using $H_0\Sigma = \Sigma^* H_0$, we get

$$(H_0 \Sigma^n 1, \Sigma^m 1) = ((\Sigma^*)^n H_0 1, \Sigma^m 1) = \alpha_{n+m}.$$

Similarly, we get

$$(UHU^* e_n, e_m) = (H\Sigma^n 1, \Sigma^m 1) = (H_0 \Sigma^{n+1} 1, \Sigma^m 1) = \alpha_{n+m+1} = (S^*\alpha)_{n+m}.$$

Thus, we obtain

$$U H_0 U^* = \Gamma_\alpha, \quad UHU^* = \Gamma_{S^*\alpha}. \tag{6.7}$$

Since $H_0 \geq 0$ and $H \geq 0$, the double positivity condition for Γ_α holds true. Since U is an isometry, multiplying (6.7) by U^* on the left and by U on the right gives (6.5). Finally, since $\operatorname{Ker} H_0 = \{0\}$, from (6.7) we obtain $\operatorname{Ker} \Gamma_\alpha = \operatorname{Ker} U^*$. This gives the condition $\operatorname{Ran} U = \overline{\operatorname{Ran} \Gamma_\alpha}$. □

Proof of Lemma 6.2. This is borrowed almost verbatim from [10].

By definition, we have $H^2 \leq H_0^2$. By the Heinz inequality (see e.g. [3, Section 10.4]) $H^{1/2} \leq H_0^{1/2}$, and therefore there exists a (unique) contraction Q in $L^2(\mathbb{R}, d\widetilde{\rho})$ with

$$H^{1/2} = Q H_0^{1/2}.$$

Let us prove that

$$\operatorname{Ker}(Q^*Q - I) = \{0\}. \tag{6.8}$$

We have

$$H^2 = (H_0^{1/2} Q^* Q H_0^{1/2})^2 = H_0^{1/2} Q^* Q H_0 Q^* Q H_0^{1/2},$$
$$H^2 = H_0^2 - (\cdot, H_0 1) H_0 1 = H_0^{1/2} (H_0 - (\cdot, H_0^{1/2} 1) H_0^{1/2} 1) H_0^{1/2}.$$

It follows that

$$Q^* Q H_0 Q^* Q = H_0 - (\cdot, H_0^{1/2} 1) H_0^{1/2} 1, \tag{6.9}$$

and so

$$(H_0 Q^* Q f, Q^* Q f) = (H_0 f, f) - |(f, H_0^{1/2} 1)|^2 \tag{6.10}$$

for any $f \in L^2(\mathbb{R}, d\widetilde{\rho})$. We claim that $\operatorname{Ker}(Q^*Q - I)$ is an invariant subspace of H_0. Indeed, if $Q^*Qf = f$, then by (6.10) we have $f \perp H_0^{1/2} 1$ and so by (6.9)

$$Q^* Q H_0 f = H_0 f,$$

i.e. $H_0 f \in \operatorname{Ker}(Q^*Q - I)$. Thus, $\operatorname{Ker}(Q^*Q - I)$ is an invariant subspace of H_0 which is orthogonal to $H_0^{1/2} 1$. Then it is orthogonal to the minimal invariant subspace of H_0 that contains $H_0^{1/2} 1$. Recalling that $H_0 = M_x^{1/2}$ we see (by an approximation argument based on the Weierstrass theorem) that such minimal subspace coincides with the whole space $L^2(\mathbb{R}, d\widetilde{\rho})$. Thus we get (6.8).

Since $0 \leq Q^*Q \leq I$ and $\operatorname{Ker}(Q^*Q - I) = \{0\}$, by the spectral theorem for self-adjoint operators we get

$$\lim_{n \to \infty} \|(Q^*Q)^n f\| = 0 \quad \forall f \in L^2(\mathbb{R}, d\widetilde{\rho}). \tag{6.11}$$

Next, we have

$$H = H_0^{1/2} Q^* Q H_0^{1/2} = \Sigma^* H_0^{1/2} H_0^{1/2},$$

and therefore

$$H_0^{1/2} Q^* Q = \Sigma^* H_0^{1/2}.$$

Iterating the last identity, we obtain

$$H_0^{1/2} (Q^*Q)^n = (\Sigma^*)^n H_0^{1/2},$$

and so (6.11) implies that

$$\lim_{n \to \infty} \|(\Sigma^*)^n f\| = 0 \qquad \forall f \in \operatorname{Ran} H_0^{1/2}.$$

Since $\|\Sigma^*\| \le 1$, by the density argument we obtain that the last relation in fact holds true for all $f \in L^2(\mathbb{R}, d\widetilde{\rho})$. □

7. Example

Fix a parameter $\gamma > -1/2$. Let

$$\alpha_n = \frac{1}{n+1+\gamma}, \quad n = 0, 1, 2, \dots \tag{7.1}$$

This sequence corresponds to the choice $d\omega(t) = t^\gamma dt$ in (2.1). Consider the Hankel operator Γ_α. An explicit diagonalisation of Γ_α was given by M. Rosenblum in [14]. This diagonalisation shows, in particular, that the spectrum of Γ_α (for any $\gamma > -1/2$) coincides with the interval $[0, \pi]$, is purely absolutely continuous and has multiplicity one. Since $S^*\alpha$ also has the form (7.1) with γ incremented by 1, this yields a whole class of Hankel operators Γ_α with identical spectra of Γ_α and $\Gamma_{S^*\alpha}$. This shows that, unlike in the case of compact Hankel operators, in general the spectra of Γ_α and $\Gamma_{S^*\alpha}$ together do not determine α.

Below we give an explicit formula for the measure ρ_α corresponding to the sequence (7.1). In order to do this, let us recall Rosenblum's diagonalisation of Γ_α. For $k < \frac{1}{2} + \operatorname{Re} m$, let $W_{k,m}$ be the Whittaker function (see e.g. [1, Chapter 6]):

$$\Gamma(m-k+\tfrac{1}{2}) W_{k,m}(x) = x^{m+\frac{1}{2}} e^{-x/2} \int_0^\infty e^{-xt} (t+1)^{m+k-\frac{1}{2}} t^{m-k-\frac{1}{2}} dt. \tag{7.2}$$

For $s > 0$ and $f \in L^2(\mathbb{R}_+)$, set

$$(\mathcal{U}f)(s) = \lim_{\varepsilon \to +0} \int_\varepsilon^\infty \frac{1}{x} W_{-\gamma, i\sqrt{s}}(x) f(x) dx.$$

Then $\mathcal{U}$ is a unitary operator from $L^2(\mathbb{R}_+, dx)$ onto $L^2(\mathbb{R}_+, d\mu(s))$, where

$$d\mu(s) = \frac{1}{2\pi^2} \sinh(2\pi\sqrt{s}) |\Gamma(\tfrac{1}{2} + \gamma - i\sqrt{s})|^2 ds.$$

Further, for $n = 0, 1, 2, \dots$, let L_n be the Laguerre polynomial normalised such that the functions $\phi_n(x) = e^{-x/2} L_n(x)$ form an orthonormal basis in $L^2(\mathbb{R}_+)$. Define a map $V : \ell^2 \to L^2(\mathbb{R}_+, d\mu(s))$ by setting

$$(Va)(s) = \mathcal{U}(\textstyle\sum_{n=0}^\infty a_n \phi_n), \quad \forall a = \{a_n\}_{n=0}^\infty \in \ell^2.$$

Then (see [14] for the proof) V is unitary and V transforms Γ_α into a multiplication operator:

$$(V \Gamma_\alpha V^{-1} g)(s) = \frac{\pi}{\cosh(\pi\sqrt{s})} g(s).$$

Thus, we obtain

$$\rho_\alpha(\delta) = (\chi_\delta(\Gamma_\alpha^2) e_0, e_0) = \int_0^\infty \chi_\delta \left(\frac{\pi^2}{(\cosh(\pi\sqrt{s}))^2} \right) |(Ve_0)(s)|^2 d\mu(s), \tag{7.3}$$

where

$$(Ve_0)(s) = (U\phi_0)(s) = \lim_{\varepsilon \to +0} \int_\varepsilon^\infty \frac{1}{x} W_{-\gamma, i\sqrt{s}}(x) e^{-x/2} dx. \tag{7.4}$$

The measure ρ_α is purely absolutely continuous and is supported on the interval $[0,\pi^2]$. Formula (7.3) can be rewritten as

$$d\rho_\alpha(\lambda)=\frac{1}{\pi}|(Ve_0)(s)|^2|\Gamma(\tfrac12+\gamma-i\sqrt{s})|^2\frac{\sqrt{s}}{\lambda^2}d\lambda,\quad 0<\lambda<\pi^2,$$

where the variables s and λ are related by

$$\lambda=\left(\frac{\pi}{\cosh(\pi\sqrt{s})}\right)^2.$$

In the case $\gamma=0$, the measure ρ_α and even the corresponding function ξ_α can be computed explicitly.

Lemma 7.1. *Let α be defined by (7.1) with $\gamma=0$. Then* $\operatorname{supp}\rho_\alpha=\operatorname{supp}\xi_\alpha=[0,\pi^2]$. *For $\lambda\in(0,\pi^2)$ we have*

$$d\rho_\alpha(\lambda)=\frac{1}{\pi^2\sqrt{\lambda}}\cosh^{-1}\left(\tfrac{\pi}{\sqrt{\lambda}}\right)=\frac{1}{\pi^2\sqrt{\lambda}}\log\left(\tfrac{\pi}{\sqrt{\lambda}}+\sqrt{\tfrac{\pi^2}{\lambda}-1}\right)d\lambda \tag{7.5}$$

and

$$\xi_\alpha(\lambda)=\tfrac{1}{\pi}\tan^{-1}\left(\tfrac{2}{\pi}\cosh^{-1}\left(\tfrac{\pi}{\sqrt{\lambda}}\right)\right)=\tfrac{1}{\pi}\tan^{-1}\left(\tfrac{2}{\pi}\log\left(\tfrac{\pi}{\sqrt{\lambda}}+\sqrt{\tfrac{\pi^2}{\lambda}-1}\right)\right). \tag{7.6}$$

Proof. Substituting the integral representation (7.2) into (7.4), we find

$$\begin{aligned}(Ve_0)(s)&=\frac{1}{\Gamma(\frac12+i\sqrt{s})}\int_0^\infty x^{i\sqrt{s}-\frac12}e^{-x}\int_0^\infty e^{-xt}(t+1)^{i\sqrt{s}-\frac12}t^{-i\sqrt{s}-\frac12}dt\,dx\\&=\frac{1}{\Gamma(\frac12+i\sqrt{s})}\int_0^\infty(t+1)^{i\sqrt{s}-\frac12}t^{-i\sqrt{s}-\frac12}\int_0^\infty x^{i\sqrt{s}-\frac12}e^{-x(t+1)}dx\,dt\\&=\int_0^\infty(t+1)^{-1}t^{-i\sqrt{s}-\frac12}dt=\frac{\pi}{\cosh(\pi\sqrt{s})}.\end{aligned}$$

Further, by the reflection formula for Gamma function, we have

$$|\Gamma(\tfrac12-i\sqrt{s})|^2=\frac{\pi}{\cosh(\pi\sqrt{s})}.$$

Thus, we obtain (7.5).

Next, for $\arg z\in(0,2\pi)$, denote

$$\zeta=\frac{\pi}{\sqrt{-z}},$$

where the branch of the square root is defined so that $\sqrt{-z}>0$ for $z<0$. Let us prove the formula

$$\exp\left(-\int_0^\infty\frac{\xi_\alpha(\lambda)}{\lambda-z}d\lambda\right)=\frac{\sinh^{-1}(\zeta)}{\zeta}=\frac{\log(\zeta+\sqrt{\zeta^2+1})}{\zeta}. \tag{7.7}$$

By (3.3), we have

$$\Delta(z)^{-1} = \exp\left(-\int_0^\infty \frac{\xi_\alpha(\lambda)}{\lambda - z} d\lambda\right) = -z((\Gamma_\alpha^2 - z)^{-1} e_0, e_0)$$
$$= -z\int_0^\infty (\lambda - z)^{-1} d\rho_\alpha(\lambda) = -\frac{z}{\pi^2}\int_0^{\pi^2} \frac{1}{(\lambda - z)\sqrt{\lambda}} \cosh^{-1}(\pi/\sqrt{\lambda}) d\lambda.$$

By a change of variable $x = \cosh^{-1}(\pi/\sqrt{\lambda})$, this transforms into

$$\Delta(z)^{-1} = \frac{2}{\pi}\int_0^\infty \frac{x\sinh(x)}{(\cosh(x))^2 - \frac{\pi^2}{z}} dx = \frac{1}{\pi}\int_{-\infty}^\infty \frac{x\sinh(x)}{(\cosh(x))^2 + \zeta^2} dx.$$

In order to compute the last integral, we write

$$\pi\Delta(z)^{-1} = \int_{-\infty}^\infty f(x) dx, \quad f(x) = \frac{x\sinh(x)}{(\cosh(x))^2 + \zeta^2},$$

and regard it as an integral of the complex variable. The function $f(x)$, $x \in \mathbb{C}$, has poles at $x = i(\pi/2) \pm \sinh^{-1}(\zeta) + 2\pi i n$, $n \in \mathbb{Z}$. Let us move the contour of integration from $\mathbb{R}$ to $i\pi + \mathbb{R}$. We notice that

$$\int_{-\infty}^\infty f(x + \pi i) dx = -\int_{-\infty}^\infty f(x) dx.$$

Moving the contour, we pick up the residues of f at $x_\pm = i(\pi/2) \pm \sinh^{-1}(\zeta)$, and so by a direct calculation we obtain

$$2\int_{-\infty}^\infty f(x) dx = 2\pi i(\operatorname*{Res}_{x_-} f(x) + \operatorname*{Res}_{x_+} f(x)) = 2\pi \frac{\sinh^{-1}(\zeta)}{\zeta},$$

which yields (7.7).

Now let us prove formula (7.6) for ξ_α. Since $\operatorname{supp}\rho_\alpha \subset [0, \pi^2]$, we also have $\operatorname{supp}\xi_\alpha \subset [0, \pi^2]$. Denote

$$F(z) = \int_0^{\pi^2} \frac{\xi_\alpha(\lambda)}{\lambda - z} dz.$$

By (7.5), we have

$$F(z) = -\log\frac{\sinh^{-1}(\zeta)}{\zeta}, \quad \zeta = \frac{\pi}{\sqrt{-z}}.$$

Fix $\lambda_0 \in (0, \pi^2)$ and let $z = \lambda_0 + i0$. Then

$$\zeta = i\zeta_0, \text{ where } \zeta_0 = \frac{\pi}{\sqrt{\lambda_0}} \text{ and } \sqrt{\lambda_0} > 0.$$

We have

$$\xi_\alpha(\lambda_0) = \frac{1}{\pi}\operatorname{Im} F(\lambda_0 + i0) = -\frac{1}{\pi}\arg\frac{\sinh^{-1}(i\zeta_0)}{i\zeta_0}.$$

Now it remains to compute the r.h.s.:

$$\xi_\alpha(\lambda_0) = -\frac{1}{\pi}\arg\frac{i(\pi/2) + \cosh^{-1}(\zeta_0)}{i\zeta_0} = \frac{1}{\pi}\arg((\pi/2) + i\cosh^{-1}(\zeta_0)),$$

which yields (7.6). □

Appendix A. Background information on the SSF theory

Here, for the reader's convenience, we collect key formulas of the SSF theory without proofs or references or much discussion. For the details and history we refer to the survey [4] or the book [16].

Let A, B be bounded self-adjiont operators in a Hilbert space. Assume that $B-A$ is a trace class operator. Then there exists a real valued function $\xi \in L^1(\mathbb{R})$ such that the *Lifshits-Krein* trace formula holds true:

$$\operatorname{Tr}(\varphi(B) - \varphi(A)) = \int_{-\infty}^{\infty} \xi(\lambda)\varphi'(\lambda)d\lambda, \quad \forall \varphi \in C_0^\infty(\mathbb{R}). \tag{A.1}$$

(It is easy to prove that for any $\varphi \in C_0^\infty(\mathbb{R})$, the difference $\varphi(B) - \varphi(A)$ is a trace class operator.) This function is called the *spectral shift function* (SSF) for the pair A, B; notation: $\xi(\lambda) = \xi(\lambda; B, A)$. The SSF has the following properties:

(i) If $\pm(B - A) \geq 0$, then $\pm\xi(\lambda; B, A) \geq 0$ for a.e. $\lambda \in \mathbb{R}$.
(ii) If $\operatorname{rank}(B - A) \leq n$, then $|\xi(\lambda; B, A)| \leq n$ for a.e. $\lambda \in \mathbb{R}$.
(iii) In particular, if $B - A = (\cdot, x)x$ for some element x, then $\xi(\lambda; B, A) \in [0, 1]$ for a.e. $\lambda \in \mathbb{R}$.
(iv) One has the estimate

$$\int_{-\infty}^{\infty} |\xi(\lambda; B, A)|d\lambda \leq \|B - A\|_1,$$

where $\|\cdot\|_1$ is the trace norm.
(v) One has the identity

$$\int_{-\infty}^{\infty} \xi(\lambda; B, A)d\lambda = \operatorname{Tr}(B - A). \tag{A.2}$$

Finally, the SSF is compactly supported; this is a consequence of the boundedness of B and A.

If A, B are compact operators, then the SSF can be expressed in terms of the eigenvalue counting functions of A, B. Denote

$$N_+(\lambda; A) = \operatorname{Tr}(\chi_{(\lambda,\infty)}(A)), \quad \lambda > 0;$$

then

$$\xi(\lambda; B, A) = N_+(\lambda; B) - N_+(\lambda; A), \quad \lambda > 0, \tag{A.3}$$

with a similar formula for $\lambda < 0$. This is a direct consequence of the trace formula (A.1).

The SSF for the pair A, B is closely related to the perturbation determinant $\Delta_{B/A}$ for this pair. The perturbation determinant is defined by

$$\Delta_{B/A}(z) = \det(I + (B - A)(A - z)^{-1}), \tag{A.4}$$

where z is a complex number outside the spectrum of A. The perturbation determinant is an analytic function of z, with poles at isolated eigenvalues of A and zeros

at isolated eigenvalues of B (and may have more complicated singularities at the essential spectra of A, B). One has

$$\Delta_{B/A}(z) = \exp\left\{\int_{-\infty}^{\infty} \frac{\xi(\lambda)}{\lambda - z} d\lambda\right\}. \tag{A.5}$$

Finally, we would like to display a resolvent formula for rank one perturbations. Let $B = A + \gamma(\cdot, x)x$, where x is an element of the Hilbert space and $\gamma \in \mathbb{R}$. Then for all z with $\operatorname{Im} z \neq 0$ we have

$$(B - z)^{-1} = (A - z)^{-1} - \frac{\gamma}{D(z)}(\cdot, (A - \overline{z})^{-1}x)(A - z)^{-1}x, \tag{A.6}$$

$$D(z) = 1 + \gamma((A - z)^{-1}x, x). \tag{A.7}$$

Acknowledgements

Much of the work on the paper was done during P.G.'s visit to King's College London and A.P.'s visit to University of Paris-Sud in 2013. The authors are grateful to both Universities for financial support.

References

[1] H. Bateman, A. Erdelyi, *Higher transcendental functions, Vol. 1,* McGraw-Hill, New York, 1953.
[2] A. Beurling, *On two problems concerning linear transformations in Hilbert space,* Acta Mathematica **81**, no. 1 (1949), 239–255.
[3] M. Sh. Birman, M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space.* D. Reidel, Dordrecht, 1987.
[4] M. Sh. Birman and D. R. Yafaev, *The spectral shift function. The work of M. G. Krein and its further development*, St. Petersburg Math. J. **4** (1993), 833–870.
[5] P. Gérard, S. Grellier, *Invariant rori for the cubic Szegő equation*, Invent. Math. **187** (2012), 707–754.
[6] P. Gérard, S. Grellier, *Inverse spectral problems for compact Hankel operators* Journal of the Institute of Mathematics of Jussieu, available on CJO2013. doi:10.1017/S1474748013000121, published online on 18 April 2013.
[7] F. Hausdorff, *Momentprobleme für ein endlisches Intervall*, Math. Zeit.**16** (1923), 220–248.
[8] T. Kato, *On finite-dimensional perturbations of self-adjoint operators,* J. Math. Soc. Japan **9** (1957), 239–249. 46.2X
[9] B. Ya. Levin, *Lectures on entire functions,* Amer. Math. Soc., Providence, Rhode Island, 1996.
[10] A. V. Megretskii, V. V. Peller, S. R. Treil, *The inverse spectral problem for self-adjoint Hankel operators,* Acta Math. **174** (1995), 241–309.
[11] V. Peller, *Hankel operators and their applications,* Springer, 2003.
[12] M. Reed, B. Simon, *Methods of modern mathematical physics. I: Functional analysis,* Academic Press, New York, 1972.
[13] M. Reed, B. Simon, *Methods of modern mathematical physics. II: Fourier analysis, self-adjointness,* Academic Press, New York, 1975.
[14] M. Rosenblum, *On the Hilbert matrix. I,II,* Proc. Amer. Math. Soc, **9** (1958), 581–585 and 137–140.
[15] H. Widom, *Hankel matrices,* Transactions of the American Mathematical Society **121**, no. 1 (1966), 1–35.

[16] D. R. Yafaev, *Mathematical scattering theory. General theory.* American Mathematical Society, Providence, RI, 1992.

Université Paris-Sud XI, Laboratoire de Mathématiques d'Orsay, CNRS, UMR 8628, et Institut Universitaire de France
E-mail address: Patrick.Gerard@math.u-psud.fr

Department of Mathematics, King's College London, Strand, London, WC2R 2LS, U.K.
E-mail address: alexander.pushnitski@kcl.ac.uk