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# SBV-LIKE REGULARITY FOR HAMILTON-JACOBI EQUATIONS WITH A CONVEX HAMILTONIAN

STEFANO BIANCHINI AND DANIELA TONON

ABSTRACT. In this paper we consider a viscosity solution  $u$  of the Hamilton-Jacobi equation

$$\partial_t u + H(D_x u) = 0 \quad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n,$$

where  $H$  is smooth and convex. We prove that when  $d(t, \cdot) := H_p(D_x u(t, \cdot))$ ,  $H_p := \nabla H$ , is BV for all  $t \in [0, T]$  and suitable hypotheses on the Lagrangian  $L$  hold, the Radon measure  $\text{div}d(t, \cdot)$  can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ . This result extends a result of Robyr for genuinely nonlinear scalar balance laws and a result of Bianchini, De Lellis and Robyr for uniformly convex Hamiltonians.

## 1. INTRODUCTION

We consider the Hamilton-Jacobi equation

$$\partial_t u + H(D_x u) = 0 \quad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n,$$

where  $H$  is a smooth convex Hamiltonian and  $\Omega$  an open set of  $[0, T] \times \mathbb{R}^n$ . A viscosity solution of such an equation is locally Lipschitz but in general it doesn't have any additional regularity. The structure of the non-differentiability set of viscosity solutions has been studied by several authors, see for example Fleming [14], Cannarsa and Soner [10]. The majority of the existing results are in the case of a strictly convex Hamiltonian. Under this assumption the viscosity solution  $u$  is semiconcave, this implies in particular that  $Du$  belongs to BV and  $D^2u$  is a matrix of Radon measures. It is therefore of interest to see when  $Du$  belongs to SBV. The first result in this direction was proven by Cannarsa, Mennucci and Sinestrari in [8]. There, the authors were able to prove the SBV regularity of  $Du$  as a corollary to a more general result on the rectifiability of the closure of the discontinuity set of  $Du$ . Therefore they needed a strongly regular initial datum  $u(0, x) = u_0(x)$  in  $W^{1,\infty}(\mathbb{R}^n) \cap C^{R+1}(\mathbb{R}^n)$ ,  $R \geq 1$ . Less regularity can be asked to the initial datum when attempting directly to the Cantor part of  $D^2u$ . In [5], Bianchini, De Lellis and Robyr proved that, when the Hamiltonian is uniformly convex and the initial datum is bounded Lipschitz,  $D_x u(t, \cdot)$  belongs to  $[SBV(\Omega_t)]^n$ ,  $\Omega_t := \{x \in \mathbb{R}^n \mid (t, x) \in \Omega\}$ , out of a countable number of  $t$ 's in  $[0, T]$ . This means that  $D_x^2 u(t, \cdot)$  can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ . In particular  $Du$  belongs to  $[SBV(\Omega)]^{n+1}$ . This result was first obtained in the one-dimensional case by Ambrosio and De Lellis in [2].

When  $H$  is just convex,  $D_x u(t, \cdot)$  loses in general its BV regularity, an example can be found in Remark 3.7 in Bianchini [4]. However, in this paper, we show that an SBV-like regularity result can be proven for the vector field

$$d(t, x) := H_p(D_x u(t, x)),$$

defined on the set  $U$  of points  $(t, x)$  where  $u(t, x)$  is differentiable in  $x$ . Here  $H_p$  is the gradient of the Hamiltonian  $H(p)$ . Indeed, the divergence  $\text{div}d(t, \cdot)$  is in general a locally finite Radon measure. Moreover when the vector field  $d(t, \cdot)$  is BV and suitable hypotheses are made on the Lagrangian  $L$ , the Legendre transform of  $H$ , the measure  $\text{div}d(t, \cdot)$  has Cantor part only for a countable number of  $t$ 's in  $[0, T]$ .

More precisely let  $H$  be  $C^2(\mathbb{R}^n)$ , convex and superlinear, i.e. such that  $\lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = +\infty$ . (HYP(0)) Suppose the vector field  $d(t, \cdot)$  belongs to  $[BV(\Omega_t)]^n$  for every  $t \in [0, T]$ .

Define  $V_{\pi_n}$  as

$$V_{\pi_n} := \{v \in \mathbb{R}^n \mid L(\cdot) \text{ is not twice differentiable in } v\},$$

and

$$\Sigma_{\pi_n} := \{(t, x) \in U \mid d(t, x) \in V_{\pi_n}\} \quad \text{and} \quad \Sigma_{\pi_n}^c := U \setminus \Sigma_{\pi_n}.$$

(HYP(n)) We suppose  $V_{\pi_n}$  to be contained in a finite union of hyperplanes  $\Pi_{\pi_n}$ .

For  $j = n, \dots, 3$  for every  $(j-1)$ -dimensional plane  $\pi_{j-1}$  in  $\Pi_{\pi_j}$ , let  $L_{\pi_{j-1}} : \pi_{j-1} \rightarrow \mathbb{R}$  be the  $(j-1)$ -dimensional restriction of  $L$  to  $\pi_{j-1}$  and

$$V_{\pi_{j-1}} := \{v \in \pi_{j-1} \mid L_{\pi_{j-1}}(\cdot) \text{ is not twice differentiable in } v\}.$$

Define

$$\Sigma_{\pi_{j-1}} := \{(t, x) \in \Sigma_{\pi_j} \mid d(t, x) \in V_{\pi_j}\} \quad \text{and} \quad \Sigma_{\pi_{j-1}}^c := \Sigma_{\pi_j} \setminus \Sigma_{\pi_{j-1}}.$$

(HYP(j-1)) We suppose  $V_{\pi_{j-1}}$  to be contained in a finite union of  $(j-2)$ -dimensional planes  $\Pi_{\pi_{j-1}}$ , for every  $\pi_{j-1} \in \Pi_{\pi_j}$ .

Let us note that the BV regularity of the vector field  $d(t, \cdot)$  is automatically satisfied by a viscosity solution whose initial datum is semiconcave, as a consequence of Proposition 2.16. However, Remark 3.7 in [4] shows an example of a Hamilton-Jacobi equation with a convex Hamiltonian in which the related vector field  $d(t, \cdot)$  does not belong to BV. Therefore the BV regularity is a property which is not always satisfied by the vector field  $d(t, \cdot)$ .

In Example 5.6 we show a Hamilton-Jacobi equation for which the hypotheses (HYP(n)), ..., (HYP(2)) are satisfied.

**Theorem 1.1.** *Under the assumptions (HYP(0)), (HYP(n)), ..., (HYP(2)), the Radon measure  $\text{div}d(t, \cdot)$  has Cantor part on  $\Omega_t$  only for a countable number of  $t$ 's in  $[0, T]$ .*

The theorem above can be seen as the multi-dimensional version of a result proven by Robyr in [19]. In that paper Robyr studied entropy solutions of the genuinely nonlinear scalar balance laws

$$\partial_t v(t, x) + D_x(f(t, x, v(t, x))) + g(t, x, v(t, x)) = 0 \quad \text{in an open set } \Omega \subset \mathbb{R}^+ \times \mathbb{R},$$

where the source term  $g$  belongs to  $C^1(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$ ,  $f$  belongs to  $C^2(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$  and  $f$  is such that the set  $\{p_i \in \mathbb{R} \mid f_{pp}(t, x, p_i) = 0\}$  is at most countable for every fixed  $(t, x) \in \Omega$ . His main result states that BV entropy solutions of such equations belong to  $SBV_{loc}(\Omega)$ . In the one-dimensional case, genuinely nonlinear scalar balance laws and Hamilton-Jacobi equations are equivalent when  $f(t, x, p) = H(p)$  and  $g(t, x, p) = 0$ . The entropy solution of the first can be seen as the gradient of the viscosity solution of the second,  $v(t, x) = D_x u(t, x)$ . With this consideration our result is a kind of generalization of Robyr's result to the multi-dimensional case. Furthermore, we prove that, in the one-dimensional case, when the Hamiltonian is convex and smooth, the BV regularity of  $d(t, x)$  follows automatically and there is no need to add further hypotheses to prove its SBV regularity out of a countable number of  $t$ 's. Since the BV regularity of  $d(t, x)$  does not imply in general the same regularity for  $\frac{\partial}{\partial x} u(t, \cdot)$ , the BV regularity of  $\frac{\partial}{\partial x} u(t, \cdot)$  has yet to be required to prove the SBV regularity, see Remark 4.3.

*Remark 1.2.* Theorem 1.1 is sharp. Indeed at the end of Remark 3.3 in [2] we can find an example of a viscosity solution to the Hamilton-Jacobi equation

$$\partial_t u + \frac{(D_x u)^2}{2} = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}$$

whose related vector field  $d(t, x) = D_x u(t, x)$  is not SBV in an infinite set of times in  $[0, 1]$ .

In the multi-dimensional case the question on the SBV regularity of  $d(t, \cdot)$  without any additional hypothesis is still open.

The paper is organized as follows. In Section 2 we recall preliminary results on Hamilton-Jacobi equations and viscosity solutions. In Section 3 we extend the definition of the vector field  $d$  to the all  $\Omega$ , we prove that  $\text{div}d(t, \cdot)$  is a locally finite Radon measure on  $\Omega_t$ , for all  $t \in [0, T]$ , and present the general strategy used to prove that  $\text{div}d(t, \cdot)$  has a Cantor part only for a countable number of  $t$ 's in  $[0, T]$ . In Section 4 we study the one-dimensional case and we prove that  $\text{div}d(t, \cdot)$  belongs to  $SBV(\Omega_t)$ , out of a countable number of  $t$ 's in  $[0, T]$ , without any additional hypothesis. In Section 5 we study the multi-dimensional case and prove Theorem 1.1. We also state some easy corollaries.

## 2. PRELIMINARIES

**2.1. Generalized differentials.** We begin with the definition of generalized differential, see Cannarsa and Sinestrari [9] and Cannarsa and Soner [10].

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ .

**Definition 2.1.** Let  $u : \Omega \rightarrow \mathbb{R}$ , for any  $x \in \Omega$  the sets

$$D^-u(x) = \left\{ p \in \mathbb{R}^n \mid \liminf_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\},$$

$$D^+u(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{y \rightarrow x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\},$$

are called, respectively, the *subdifferential* and *superdifferential* of  $u$  at  $x$ .

**2.2. Decomposition of a Radon measure.** Given an  $[L^\infty(\mathbb{R}^n)]^n$  vector field  $d(x)$  such that  $\operatorname{div}d(x) =: \mu(x)$  is a Radon measure on  $\mathbb{R}^n$ , we can decompose  $\mu$  into three mutually singular measures:

$$\mu = \mu_a + \mu_c + \mu_j.$$

$\mu_a$  is the absolutely continuous part with respect to the Lebesgue measure.  $\mu_j$  is the singular part of the measure which is concentrated on an  $\mathcal{H}^{n-1}$ -rectifiable set.  $\mu_c$ , the Cantor part, is the remaining part.

**2.3. BV and SBV functions.** A detailed description of the spaces BV and SBV can be found in Ambrosio, Fusco and Pallara [3], Chapters 3 and 4. For the reader convenience, we briefly recall that, given  $u \in BV(\mathbb{R}^n)$ , the distributional derivative of  $u$ , which by definition must be a measure with bounded total variation, is decomposable into three mutually singular measures:

$$Du = D_a u + D_c u + D_j u.$$

$D_a u$  is the absolutely continuous part with respect to the Lebesgue measure.  $D_j u$  is the part of the measure which is concentrated on the rectifiable  $(n-1)$ -dimensional set  $J$ , where the function  $u$  has jump discontinuities, thus for this reason it is called jump part.  $D_c u$ , the Cantor part, is the singular part which satisfies  $D_c u(E) = 0$  for every Borel set  $E$  with  $\mathcal{H}^{n-1}(E) < \infty$ . If this part vanishes, i.e.  $D_c u = 0$ , we say that  $u \in SBV(\mathbb{R}^n)$ . When  $u \in [BV(\mathbb{R}^n)]^k$  the distributional derivative  $Du$  is a matrix of Radon measures and the decomposition can be applied to every component of the matrix.

We recall here some properties of BV functions which will be useful later on.

**Definition 2.2.** Let  $u \in [L^1_{loc}(\mathbb{R}^n)]^k$ , we say that  $u$  has an *approximate limit* at  $x \in \mathbb{R}^n$  if there exists  $z \in \mathbb{R}^k$  such that

$$\lim_{\rho \rightarrow 0} \int_{B_\rho(x)} |u(y) - z| dy = 0.$$

The set  $S_u$  of points where this property does not hold is called the *approximate discontinuity set*. For any  $x \in \mathbb{R}^n \setminus S_u$  the vector  $z$  is called *approximate limit of  $u$  at  $x$*  and is denoted by  $\tilde{u}(x)$ .

**Proposition 2.3.** Let  $u$  and  $v$  belong to  $[BV(\mathbb{R}^n)]^k$ . Let

$$L := \{x \in \mathbb{R}^n \setminus (S_u \cup S_v) \mid \tilde{u}(x) = \tilde{v}(x)\},$$

where  $\tilde{u}(x), \tilde{v}(x)$  are the approximate limit of  $u$  and  $v$  respectively. Then  $Du$  and  $Dv$  are equal when restricted to  $L$ .

*Proof.* See Remark 3.93 in [3]. □

**Proposition 2.4.** Let  $u$  belong to  $[BV(\mathbb{R}^n)]^k$ . Then  $D_c u$  vanishes on sets which are  $\sigma$ -finite with respect to  $\mathcal{H}^{n-1}$  and on sets of the form  $\tilde{u}^{-1}(E)$  with  $E \subset \mathbb{R}^k$  and  $\mathcal{H}^1(E) = 0$ .

*Proof.* See Proposition 3.92 in [3]. □

**Proposition 2.5.** Let  $u$  belong to  $[BV(\mathbb{R}^n)]^k$ . For  $j = 1, \dots, n-1$  define the  $(n-j)$ -dimensional restriction  $u_{x_1, \dots, x_j}(\cdot) : \mathbb{R}^{n-j} \rightarrow \mathbb{R}^k$  as  $u_{x_1, \dots, x_j}(\hat{x}) = u(x_1, \dots, x_j, \hat{x})$  for fixed  $(x_1, \dots, x_j) \in \mathbb{R}^j$ . Then  $u_{x_1, \dots, x_j}(\cdot)$  is  $[BV(\mathbb{R}^{n-j})]^k$  for  $\mathcal{H}^j$ -a.e.  $(x_1, \dots, x_j)$  in  $\mathbb{R}^j$ .

*Proof.* This is a well-known result. The proof in the case  $j = n-1$  can be found in [3] Section 3.11, in the other cases it is similar. □

**2.4. Semiconcave functions.** For a complete introduction to the theory of semiconcave functions we refer to Cannarsa and Sinestrari [9], Chapter 2 and 3 and Lions [18]. For our purpose we define semiconcave functions with a linear modulus of semiconcavity. In general this class is considered only as a particular subspace of the class of semiconcave functions with general semiconcavity modulus. The proofs of the following statements can be found in the mentioned references.

**Definition 2.6.** We say that a function  $u : \Omega \rightarrow \mathbb{R}$  is *semiconcave* and we denote by  $SC(\Omega)$  the space of functions with such a property, if  $\exists C > 0$  such that for any  $x, z \in \Omega$  such that the segment  $[x - z, x + z]$  is contained in  $\Omega$

$$u(x + z) + u(x - z) - 2u(x) \leq C|z|^2.$$

**Proposition 2.7.** Let  $u : \Omega \rightarrow \mathbb{R}$  belong to  $SC(\Omega)$  with semiconcavity constant  $C \geq 0$ . Then the function

$$w : x \mapsto u(x) - \frac{C}{2}|x|^2$$

is concave, i.e. for any  $x, y$  in  $\Omega$  such that the whole segment  $[x, y]$  is contained in  $\Omega$ ,  $\lambda \in [0, 1]$

$$w(\lambda x + (1 - \lambda)y) \geq \lambda w(x) + (1 - \lambda)w(y).$$

Within all the properties of a semiconcave function let us recall that when  $u$  is semiconcave  $Du$  is a BV map, hence its distributional Hessian  $D^2u$  is a symmetric matrix of Radon measures and can be split into the three mutually singular parts  $D_a^2u, D_j^2u, D_c^2u$ . Moreover the following proposition holds.

**Proposition 2.8.** Let  $u$  be a semiconcave function. If  $D$  denotes the set of points where  $D^+u$  is not single-valued, then  $|D_c^2u|(D) = 0$ .

*Proof.* Indeed, the set of points where  $D^+u$  is not single-valued, i.e. the set of singular points, is an  $\mathcal{H}^{n-1}$ -rectifiable set.  $\square$

**Definition 2.9.** We say that a function  $v : \Omega \rightarrow \mathbb{R}$  is *semiconvex* if  $u := -v$  is semiconcave.

**2.5. Viscosity solutions.** A concept of generalized solution to the equation

$$(2.1) \quad \partial_t u + H(D_x u) = 0 \quad \text{in } \Omega \subset [0, T] \times \mathbb{R}^n,$$

was found to be necessary since classical solutions break down and solutions which satisfy (2.1) almost everywhere are not unique. Crandall and Lions introduced in [12] the notion of viscosity solution to solve both these problems, see also Crandall, Evans and Lions [11].

**Definition 2.10.** A bounded uniformly continuous function  $u : \Omega \rightarrow \mathbb{R}$  is called a *viscosity solution* of (2.1) provided that

i)  $u$  is a viscosity subsolution of (2.1): for each  $v \in C^\infty(\Omega)$  such that  $u - v$  has a maximum at  $(t_0, x_0) \in \Omega$ ,

$$\partial_t v(t_0, x_0) + H(D_x v(t_0, x_0)) \leq 0;$$

ii)  $u$  is a viscosity supersolution of (2.1): for each  $v \in C^\infty(\Omega)$  such that  $u - v$  has a minimum at  $(t_0, x_0) \in \Omega$ ,

$$\partial_t v(t_0, x_0) + H(D_x v(t_0, x_0)) \geq 0.$$

**2.6. Properties of the viscosity solution of Hamilton-Jacobi equations.** We introduce a locality property, whose proof can be found in [5].

**Proposition 2.11.** Let  $u$  be a viscosity solution of (2.1) in  $\Omega$ . Then  $u$  is locally Lipschitz. Moreover for any  $(t_0, x_0) \in \Omega$ , there exist a neighborhood  $\mathcal{U}$  of  $(t_0, x_0)$ , a positive number  $\delta$  and a Lipschitz function  $v_0$  on  $\mathbb{R}^n$  such that  $u$  coincides on  $\mathcal{U}$  with the viscosity solution of

$$\begin{cases} \partial_t v + H(D_x v) = 0 & \text{in } [t_0 - \delta, \infty) \times \mathbb{R}^n \\ v(t_0 - \delta, x) = v_0(x). \end{cases}$$

Motivated by the above proposition, let us consider the Cauchy problem

$$(2.2) \quad \begin{cases} \partial_t u + H(D_x u) = 0 & \text{in } \Omega \subset [0, T] \times \mathbb{R}^n, \\ u(0, x) = u_0(x) & \text{for all } x \in \Omega_0, \end{cases}$$

where  $u_0(x)$  is a bounded Lipschitz function on  $\Omega_0$ .

The proofs of the following statements can be found in Evans [13], Section 3.3 and Chapter 10. See also Cannarsa and Sinestrari [9], Fleming [14], Fleming and Rishel [15], Fleming and Soner [16] and Lions [18].

The convexity of the Hamiltonian in the  $p$ -variable relates Hamilton-Jacobi equations to variational problems.

Let  $L$  be the Lagrangian of our system, i.e. the Legendre transform of the Hamiltonian  $H$

$$L(v) = \sup_{p \in \mathbb{R}^n} \{ \langle v, p \rangle - H(p) \}.$$

When we consider a smooth convex Hamiltonian the corresponding Lagrangian is strictly convex but nonsmooth in general. In the case of a smooth uniformly convex Hamiltonian instead, the Lagrangian inherits the same properties of  $H$ , i.e.  $L$  is itself smooth and uniformly convex.

**Theorem 2.12.** *There exist a sufficiently big  $M > 0$  and a sufficiently small  $\varepsilon > 0$  such that on the set  $\Omega^{\varepsilon, M} := \{(t, x) \in \Omega \mid t \in [0, \varepsilon], \text{dist}((t, x), \partial\Omega) > M\}$  the unique viscosity solution of the Cauchy problem (2.2) is the Lipschitz continuous function  $u(t, x)$  defined as*

$$(2.3) \quad u(t, x) = \min_{y \in \Omega_0^{\varepsilon, M}} \left\{ u(0, y) + tL\left(\frac{x-y}{t}\right) \right\}.$$

From now on we restrict to the set  $\Omega^{\varepsilon, M}$ . To simplify the notations we will call it  $\Omega$  and we will study the equation on the time interval  $[0, T]$  implying that  $T < \varepsilon$ .

**Theorem 2.13.** *Let  $u(t, x)$  be a viscosity solution of the Cauchy problem (2.2) defined on  $\Omega$  as in the previous theorem.*

- i) *The minimum point  $y$  for  $(t, x) \in \Omega$  in (2.3) is unique if and only if  $u(t, x)$  is differentiable in  $x$ . Moreover in this case  $y = x - tH_p(D_x u(t, x))$ .*
- ii) *(Dynamic Programming Principle) Fix  $(t, x) \in \Omega$ , then for all  $t' \in [0, t]$*

$$u(t, x) = \min_{z \in \Omega_{t'}} \left\{ u(t', z) + (t - t')L\left(\frac{x-z}{t-t'}\right) \right\}.$$

- iii) *Let  $0 < s < t$ , let  $(t, x) \in \Omega$  and  $y$  be a minimum point in (2.3). Let  $z = \frac{s}{t}x + (1 - \frac{s}{t})y$ . Then  $y$  is the unique minimum point for*

$$u(s, z) = \min_{w \in \Omega_0} \left\{ u(0, w) + sL\left(\frac{z-w}{s}\right) \right\}.$$

**Definition 2.14.** Let  $y \in \Omega_0$  be a minimizer for  $u(t, x)$ . We call *optimal ray* the segment  $[(t, x), (0, y)]$  defined in  $[0, t]$ .

**Proposition 2.15.** *Let  $[(t, x), (0, y)]$  and  $[(t, x'), (0, y')]$  be two optimal rays in  $[0, t]$ , for  $x, x' \in \Omega_t$   $y, y' \in \Omega_0$  then they cannot intersect except at time 0 or  $t$ .*

*Proof.* It follows from Theorem 2.13(iii). □

**Proposition 2.16.** *Let  $u_0$  be a semiconcave function. Then the unique viscosity solution  $u(t, x)$  of (2.2) is semiconcave in  $x$ , for all  $t \in [0, T]$ .*

**Theorem 2.17** (Semiconcavity Theorem). *Suppose  $H$  is locally uniformly convex. Then for any  $t$  in  $(0, T]$ ,  $u(t, \cdot)$  is locally semiconcave with semiconcavity constant  $C(t) = \frac{C}{t}$ . Thus for any fixed  $\tau > 0$  there exists a constant  $C = C(\tau)$  such that  $u(t, \cdot)$  is semiconcave with constant less than  $C$  for any  $t \geq \tau$ .*

*Moreover  $u$  is also locally semiconcave in both the variables  $(t, x)$  in  $(0, T] \times \mathbb{R}^n$ .*

**2.7. Duality solutions.** We consider a fixed interval of time  $[0, 1]$ , and we define duality solutions in this time interval.

**Definition 2.18.** Setting  $u^+(1, z) := u(1, z)$ , we define *duality solutions* for  $s \in [0, 1]$  and  $z \in \Omega_s$ , the *backward solution*

$$u^-(s, z) := \max_{x \in \Omega_1} \left\{ u^+(1, x) - (1-s)L\left(\frac{x-z}{1-s}\right) \right\},$$

and the forward solution

$$u^+(s, z) := \min_{y \in \Omega_0} \left\{ u^-(0, y) + sL\left(\frac{z-y}{s}\right) \right\}.$$

*Remark 2.19.* Note that the function  $v(\tau, y) := u^-(1-\tau, y)$  is a viscosity solution of

$$\begin{cases} \partial_\tau v - H(D_y v) = 0 & \text{in } \Omega \subset [0, T] \times \mathbb{R}^n \\ v(0, y) = u(1, y) & \text{for all } y \in \Omega_1. \end{cases}$$

Moreover the forward solution is the viscosity solution of

$$\begin{cases} \partial_t u + H(D_x u) = 0 & \text{in } \Omega \subset [0, T] \times \mathbb{R}^n, \\ u(0, x) = u^-(0, x) & \text{for all } x \in \Omega_0. \end{cases}$$

Thanks to the previous remark Theorems 2.13, 2.17 and Propositions 2.15, 2.16 hold for  $v$  and the forward solution  $u^+$ .

**Proposition 2.20.** *From the definitions above,  $u^+$  and  $u^-$  satisfy the following properties for  $x \in \Omega_1$ ,  $y \in \Omega_0$  and  $z \in \Omega_s$  for  $s \in (0, 1)$*

$$u^-(1, x) = u^+(1, x) = u(1, x), \quad u^+(0, y) = u^-(0, y) \leq u(0, y), \quad u^-(s, z) \leq u^+(s, z) \leq u(s, z).$$

*Proof.* The first two equalities are a consequence of the fact that  $u^0$  and  $u^1$ , defined as follows, are  $L(x-y)$  conjugate functions. First, for  $x \in \Omega_1$ , set

$$u^1(x) := \min_{y \in \Omega_0} \{u(0, y) + L(x-y)\},$$

i.e.  $u^1(x) = u(1, x)$ .

Then, for  $y \in \Omega_0$ , set

$$u^0(y) := \max_{x \in \Omega_1} \{u^1(x) - L(x-y)\},$$

i.e.  $u^0(y) = u^-(0, y)$ .

From these definitions it follows  $u^0(y) \leq u(0, y)$  and

$$u^1(x) = \min_{y \in \Omega_0} \{u^0(y) + L(x-y)\}.$$

Indeed, let  $\tilde{x} \in \Omega_1$  be a maximizer for  $u^0(y)$  then

$$u^0(y) = u^1(\tilde{x}) - L(\tilde{x} - y) \leq u(0, y) + L(\tilde{x} - y) - L(\tilde{x} - y) = u(0, y).$$

Nevertheless, from  $u^0(y) \leq u(0, y)$ , it follows

$$\min_{y \in \Omega_0} \{u^0(y) + L(x-y)\} \leq \min_{y \in \Omega_0} \{u(0, y) + L(x-y)\} = u^1(x).$$

On the other hand, let  $\tilde{y}$  be a minimizer for  $\min_{y \in \Omega_0} \{u^0(y) + L(x-y)\}$ , then we have

$$\begin{aligned} \min_{y \in \Omega_0} \{u^0(y) + L(x-y)\} &= u^0(\tilde{y}) + L(x-\tilde{y}) \\ &\geq u^1(x) - L(x-\tilde{y}) + L(x-\tilde{y}) \\ &= u^1(x). \end{aligned}$$

Note that the definition of  $u^-(s, z)$  and  $u^+(s, z)$  implies that  $u^-(1, x) = u^1(x)$  and  $u^+(0, y) = u^0(y)$ .

For  $s$  in  $(0, 1)$ , the last inequality follows by

$$\begin{aligned} u^-(s, z) &= \max_{x \in \Omega_1} \left\{ u^1(x) - (1-s)L\left(\frac{x-z}{1-s}\right) \right\} \\ &= \max_{x \in \Omega_1} \left\{ \min_{y \in \Omega_0} \left\{ u^0(y) + L(x-y) - (1-s)L\left(\frac{x-z}{1-s}\right) \right\} \right\} \\ &\leq \min_{y \in \Omega_0} \left\{ u^0(y) + sL\left(\frac{z-y}{s}\right) \right\} = u^+(s, z), \end{aligned}$$

where the inequality is given by the convexity of  $L$

$$L(x-y) \leq sL\left(\frac{z-y}{s}\right) + (1-s)L\left(\frac{x-z}{1-s}\right).$$

Note that, from the strict convexity of  $L$ , the equality holds if and only if  $\frac{x-z}{1-s} = \frac{z-y}{s}$ , i.e.  $z = sx + (1-s)y$ . That is  $z$  belongs to the segment joining the maximizer  $x$  with the minimizer  $y$ .

Furthermore, due to the fact that  $u^-(0, y) \leq u(0, y)$ , we have  $u^+(s, z) \leq u(s, z)$ .  $\square$

**Proposition 2.21.** *Suppose  $H$  is a smooth uniformly convex Hamiltonian. Then on a smaller set  $\tilde{\Omega} \subset \Omega$  there exists a  $C^{1,1}(\tilde{\Omega})$  function  $w$  such that  $u^-(s, z) \leq w(s, x) \leq u^+(s, z)$ .*

*Proof.* Thanks to the fact that  $H$  is uniformly convex  $u^-$  is semiconvex and  $u^+$  is semiconcave, thus the proposition is a consequence of Ilmanen's Lemma, see Lemma 4G in [17].  $\square$

The definition of backward and forward solutions can be easily generalized for every time interval  $[\tau, t] \subset [0, T]$ . Propositions 2.20 and 2.21 hold even in this case.

**Definition 2.22.** Setting  $u_{t,\tau}^+(t, z) := u(t, z)$ , we define *duality solutions* for  $s \in [\tau, t]$  and  $z \in \Omega_s$ , the *backward solution*

$$u_{t,\tau}^-(s, z) := \max_{x \in \Omega_t} \left\{ u_{t,\tau}^+(t, x) - (t-s)L\left(\frac{x-z}{t-s}\right) \right\},$$

and the *forward solution*

$$u_{t,\tau}^+(s, z) := \min_{y \in \Omega_\tau} \left\{ u_{t,\tau}^-(\tau, y) + (s-\tau)L\left(\frac{z-y}{s-\tau}\right) \right\}.$$

### 3. EXTENSION AND PRELIMINARY PROPERTIES OF THE VECTOR FIELD $d$

We consider a viscosity solution  $u$  of the Hamilton-Jacobi equation

$$\partial_t u + H(D_x u) = 0 \quad \text{in an open set } \Omega \subset \mathbb{R}^+ \times \mathbb{R}^n,$$

where  $H$  is  $C^2(\mathbb{R}^n)$  convex and

$$\lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = +\infty.$$

As already noticed, thanks to the time invariance of the equation and to Proposition 2.11, it is enough to consider the unique viscosity solution of the following Cauchy problem

$$\begin{cases} \partial_t u + H(D_x u) = 0 & \text{in } \Omega \subset [0, T] \times \mathbb{R}^n, \\ u(0, x) = u_0(x) & \text{for all } x \in \Omega_0, \end{cases}$$

where  $u_0(x)$  is a bounded Lipschitz function on  $\Omega_0$ . Here we restrict to the set  $\Omega = \Omega^{\varepsilon, M}$  as in Theorem 2.12.

The vector field  $d(t, x) := H_p(D_x u(t, x))$  is well defined where  $u(t, x)$  is differentiable in  $x$ , i.e.  $\mathcal{H}^n$ -a.e. on  $\Omega_t$ , for every  $t \in [0, T]$ .

Thanks to the Lipschitz regularity of  $u(t, \cdot)$  and the fact that  $H$  is smooth, the vector field  $d(t, \cdot)$  belongs to  $[L^\infty(\Omega_t)]^n$ .

Moreover  $d$  is constant along optimal rays. Indeed, thanks to Theorem 2.13(iii), we have

$$d(t, x) = d(s, x - (t-s)d(t, x))$$

for all  $0 \leq s \leq t$ .

A natural extension of  $d$  to  $\Omega$  is  $\mathcal{D}(\cdot) : \Omega \rightarrow \mathbb{R}^n$

$$\mathcal{D}(t, x) := \left\{ \frac{x-y}{t} \mid y \text{ is a minimum for } u_{t,0}^+(t, z) \right\}.$$

$\mathcal{D}(t, x)$  is a multi-valued function which coincides with  $d(t, x)$  in the points  $(t, x)$  where  $u(t, x)$  is differentiable in  $x$ . Indeed, where  $u(t, \cdot)$  is differentiable,  $u(t, x) = u_{t,0}^+(t, x)$  and they both admit as unique minimizer  $y = x - tH_p(D_x u(t, x))$  in  $\Omega_0$ .

Following the results of Bianchini and Gloyer in [6], we can prove that  $\mathcal{D}(t, x)$  has closed graph and thanks to the fact that  $\mathcal{D}(t, x)$  is closed. For all  $x' \in \Omega_t$ , for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\mathcal{D}(t, x) \subset \mathcal{D}(t, x') + B(0, \varepsilon)$$

for  $x \in B(x', \delta) \subset \Omega_t$ . Moreover  $\mathcal{D}(t, x)$  is a Borel measurable function and  $\text{div}d(t, \cdot)$  is a locally finite Radon measure. We repeat the proof for the reader's convenience.



**Theorem 3.1.** *For every  $t \in (0, T]$ , the divergence  $\operatorname{div}d(t, \cdot)$  is a locally finite Radon measure with negative singular part.*

*Proof.* Consider an approximation of our vector field done by taking a dense sequence of points  $\{y_i\}_{i=1}^{\infty}$  in  $\Omega_0$ . Fix an integer  $I > 0$ , call  $\Omega_0^I := \{y_i \mid i = 1, \dots, I\}$  and define for any  $x \in \Omega_t$

$$u_I^+(t, x) := \min_{i \in I} \left\{ u_{t,0}^-(0, y_i) + tL \left( \frac{x - y_i}{t} \right) \right\}.$$

Through this approximation the set  $\Omega_t$  is split into at most  $I$  open regions  $\Omega_t^i$ ,  $i = 1, \dots, I$ , defined by

$$\Omega_t^i := \text{interior of } \{x \in \Omega_t \mid \exists y_i \text{ minimizer for } u_I^+(t, x)\},$$

together with the set

$$J_t^I := \bigcup_{i \neq j} (\bar{\Omega}_t^i \cap \bar{\Omega}_t^j)$$

of negligible  $\mathcal{H}^n$ -measure. Indeed, even for  $u_I^+(t, \cdot)$  the set of points with more than one minimum is the set of points of nondifferentiability of  $u_I^+(t, \cdot)$  and this set has  $\mathcal{H}^n$ -measure zero. We define the vector field  $d^I$  on  $\Omega$  so that on each open set  $\Omega_t^i$

$$d^I(t, x) := \frac{x - y_i}{t}.$$

Using explicitly the definition of  $d^I$  and the fact that  $\mathcal{H}^n(J_t^I) = 0$ ,

$$\operatorname{div}d^I(t, x) \leq \frac{n}{t}.$$

Thanks to the pointwise convergence of  $d^I$  to  $d$

$$\operatorname{div}d(t, \cdot) - \frac{n}{t}\mathcal{H}^n \leq 0,$$

i.e.  $\operatorname{div}d(t, \cdot) - \frac{n}{t}\mathcal{H}^n$  is a negative definite distribution, hence it is a locally finite Radon measure. Thus  $\operatorname{div}d(t, \cdot)$  is itself a locally finite Radon measure.

Moreover

$$\operatorname{div}d(t, \cdot) \leq \frac{n}{t}\mathcal{H}^n,$$

implies that the singular part of this measure can be only negative.  $\square$

From now on we will denote  $\mu(t, \cdot) := \operatorname{div}d(t, \cdot)$ .

Since we have proven that  $\mu(t, \cdot)$  is a locally finite Radon measure, it makes sense to ask whether is possible or not that  $\mu(t, \cdot)$  has Cantor part for all  $t$  in  $[0, T]$ . Note that if a Cantor part is different from zero then it must be negative for Theorem 3.1.

**3.1. General strategy.** In order to prove that  $\mu(t, \cdot)$  has Cantor part only for a countable number of  $t$ 's, the general idea is now standard, see [2], [5].

We reduce to a smaller interval  $[\tau, T]$ , for a fixed  $\tau > 0$ , and we construct, on this interval, a monotone bounded functional  $F(t)$ . Then, we relate the presence of a Cantor part for the measure  $\mu(t, \cdot)$ , for a certain  $t$  in  $[\tau, T]$ , with a jump of the functional  $F$  in  $t$ . Since this functional is bounded monotone it can have only a countable number of jumps. Thus, the Cantor part of  $\mu(t, \cdot)$  can be different from zero only for a countable number of  $t$ 's.

To define  $F$  we consider the following maps:  $X_{t,\tau}(x) : \Omega_t \rightarrow \Omega_\tau$

$$X_{t,\tau}(x) := x - (t - \tau)\mathcal{D}(t, x),$$

and its restriction to the set  $U_t$  of points where  $\mathcal{D}(t, x)$  is single-valued,  $\chi_{t,\tau}(x) : U_t \rightarrow U_\tau$

$$\chi_{t,\tau}(x) := x - (t - \tau)d(t, x).$$

We will sometimes write  $\chi_{t,\tau}(\Omega_t)$  for  $\chi_{t,\tau}(U_t)$ .

We define the functional  $F : (\tau, T] \rightarrow \mathbb{R}$

$$F(t) := \mathcal{H}^n(\chi_{t,\tau}(U_t)).$$

The functional  $F$  is bounded, and, due to the fact that optimal rays do not intersect except at time  $t$  or 0,  $F$  is a monotone decreasing functional.

In order to apply the strategy above we need two estimates of the following type:

i) For any Borel set  $A \subset U_t$  for  $t$  in  $(\tau, T]$

$$(3.1) \quad \mathcal{H}^n(X_{t,\tau}(A)) \geq C_1 \mathcal{H}^n(A) - (t - \tau)C_2 \mu(t, A),$$

where  $C_1, C_2$  are fixed positive constants.

ii) For any Borel set  $A \subset \Omega_t$ , for  $t$  in  $(\tau, T]$  and for every  $0 \leq \delta \leq t - \tau$

$$(3.2) \quad \mathcal{H}^n(X_{t,\tau+\delta}(A)) \geq \left( \frac{t - (\tau + \delta)}{t - \tau} \right)^m \mathcal{H}^n(X_{t,\tau}(A)),$$

where  $m \in \mathbb{N}, m > 0$  is fixed.

Indeed with the estimates above we can prove the following lemma.

**Lemma 3.2.** *For any  $t$  in  $(\tau, T]$  such that  $\mu_c(t, \Omega_t) < 0$  there exists a Borel set  $A \subset U_t$  such that*

- i)  $\mathcal{H}^n(A) = 0$ ,  $\mu_c(t, A) < 0$  and  $\mu_c(t, \Omega_t \setminus A) = 0$ ;
- ii)  $X_{t,\tau}$  is single-valued on  $A$ ;
- iii) and for any  $\delta$  in  $(0, T - t]$

$$\chi_{t,\tau}(A) \cap \chi_{t+\delta,\tau}(\Omega_{t+\delta}) = \emptyset.$$

*Proof.* The set of points where  $\mathcal{D}(t, \cdot)$  is not single-valued, which coincides with the set of points where  $u(t, \cdot)$  is not differentiable, is an  $\mathcal{H}^{n-1}$ -rectifiable set, due to the Lipschitz regularity of  $u(t, \cdot)$ . Hence, the Radon measure  $\mu(t, \cdot)$  has null Cantor part on it. This and the definition of Cantor part of a measure imply the existence of a Borel set  $A$  such that

- $d(t, x)$  is single-valued for every  $x \in A$ ,
- $\mathcal{H}^n(A) = 0$ ,
- $\mu_c(\Omega_t \setminus A) = 0$  and  $\mu_c(A) < 0$ .

By contradiction suppose there exists a compact set  $K \subset A$  such that

$$\mu_c(t, K) < 0$$

and there exists a  $\delta > 0$  such that

$$X_{t,\tau}(K) = \chi_{t,\tau}(K) \subset \chi_{t+\delta,\tau}(\Omega_{t+\delta}).$$

Then there exists a Borel set  $\tilde{K} \subset \Omega_{t+\delta}$  such that  $\chi_{t,\tau}(K) = \chi_{t+\delta,\tau}(\tilde{K})$ . Moreover, thanks to the fact that we are considering optimal rays starting from  $\tilde{K}$ , we have

$$\chi_{t+\delta,t}(\tilde{K}) = K.$$

Using the estimate (3.2),

$$\mathcal{H}^n(K) = \mathcal{H}^n(X_{t+\delta,t}(\tilde{K})) \geq \left( \frac{\delta}{t + \delta - \tau} \right)^m \mathcal{H}^n(X_{t+\delta,\tau}(\tilde{K})) = \left( \frac{\delta}{t + \delta - \tau} \right)^m \mathcal{H}^n(X_{t,\tau}(K)).$$

Hence

$$\mathcal{H}^n(K) \geq \left( \frac{\delta}{t + \delta - \tau} \right)^m \mathcal{H}^n(X_{t,\tau}(K)).$$

Moreover applying estimate (3.1)

$$\mathcal{H}^n(K) \geq \left( \frac{\delta}{t + \delta - \tau} \right)^m (C_1 \mathcal{H}^n(K) - (t - \tau)C_2 \mu(t, A)).$$

Since  $\mathcal{H}^n(K) = 0$  we obtain  $\mu(t, A) \geq 0$  in contrast with the fact that  $\mu_c(t, A) < 0$ . □

The estimate (3.1) and Lemma 3.2 lead us to the expected conclusion.

Suppose there exists a  $t$  in  $(\tau, T)$  such that

$$\mu_c(t, \Omega_t) < 0,$$

then let  $A$  be the set of Lemma 3.2. According to Lemma 3.2(iii), for any  $\delta > 0$ , we have

$$F(t + \delta) \leq F(t) - \mathcal{H}^n(X_{t,\tau}(A)).$$

Moreover, the estimate (3.1) gives

$$F(t + \delta) \leq F(t) + (t - \tau)C_2 \mu_c(t, A).$$

Hence, letting  $\delta \rightarrow 0$ , we obtain

$$\limsup_{\delta \rightarrow 0} F(t + \delta) < F(t).$$

Therefore  $t$  is a point of discontinuity for  $F$ , as we wanted to prove.

#### 4. ONE-DIMENSIONAL CASE

We first consider the one-dimensional case. In this case we don't need any further assumption on  $d$  or  $L$  to prove the following theorem.

**Theorem 4.1.** *The vector field  $d(t, \cdot)$  belongs to  $SBV(\Omega_t)$ , out of a countable number of  $t \in [0, T]$ .*

In the uniformly convex case, Theorem 4.1 is a corollary of Ambrosio and De Lellis's result in [2].

*Proof.* Since we are in the one-dimensional case,  $\operatorname{div}d(t, x) = \frac{\partial}{\partial x}d(t, x)$ . Hence, Theorem 3.1 implies that  $d(t, x)$  belongs to  $BV(\Omega_t)$ , for every  $t \in (0, T]$ .

Moreover,  $\mathcal{D}(t, \cdot)$  is semimonotone. Indeed, since we are following optimal rays for  $u_{t,0}^+$ , they do not intersect except at time 0 or  $t$ . Thus for  $x_1, x_2 \in \Omega_t$ ,  $x_1 < x_2$  and  $d_1 \in \mathcal{D}(t, x_1), d_2 \in \mathcal{D}(t, x_2)$ , it must hold

$$x_1 - td_1 \leq x_2 - td_2,$$

otherwise the rays cross each other at a time  $s \in (0, t)$ . Hence the function  $\frac{1}{t}x - \mathcal{D}(t, x)$  is monotone increasing and  $\mathcal{D}(t, x)$  is semimonotone with constant  $C = \frac{1}{t}$ .

Let us consider the map  $X_{t,\tau}$  for any  $t \in (\tau, T]$ ,  $\tau > 0$  fixed. The fact that we are in the one-dimensional case implies that for  $t, x$ , such that  $\mathcal{D}(t, x)$  is multi-valued,

$$\mathcal{D}(t, x) = [d_1, d_2],$$

where  $d_1, d_2 \in \mathbb{R}$  are the speeds of the optimal rays for  $u(t, x)$ . Indeed, for every  $\bar{d}, \tilde{d} \in [d_1, d_2]$ , the ray  $[(t, x), (\tau, x - (t - \tau)\bar{d})]$  cannot cross  $[(t, x), (\tau, x - (t - \tau)\tilde{d})]$ , since they are straight lines starting in the same point. So optimal rays fill the triangle delimited by  $[(t, x), (\tau, x - (t - \tau)d_1)]$ ,  $[(t, x), (\tau, x - (t - \tau)d_2)]$ . Moreover, optimal rays starting in other points cannot cross  $[(t, x), (\tau, x - (t - \tau)d_1)]$  and  $[(t, x), (\tau, x - (t - \tau)d_2)]$ , at intermediate time, since they are optimal. Thus they cannot cross any other ray  $[(t, x), (\tau, x - (t - \tau)d)]$ , where  $d \in [d_1, d_2]$ . For this reason these rays are optimal for  $u_{t,0}^+(t, x)$ . Thus optimal rays for the forward solution completely fill the set  $\{\Omega_s \mid s \in [0, t]\}$ .

*Remark 4.2.* This argument holds also in the multi-dimensional case but only for a set of points of non-differentiability of zero dimension. The argument is not true in general when the points of non-differentiability lie on a surface of dimension greater than zero, since rays starting in two different points of this surface can intersect even at intermediate times.

The above consideration ensures that the map  $X_{t,\tau}$  is injective for  $\tau > 0$ , however this map is multi-valued. To recover the Lipschitzianity we use the Hille-Yosida transformation as seen in [1] and [7].

For any Borel set  $A \subset \Omega_t$ , let  $z \in B := A + T(A)$ ,  $T(x) := (Cx - \mathcal{D}(t, x))$  and  $w(z) := (Id_1 + (T)^{-1})^{-1}(z)$ . Then the following 1-Lipschitz transformations

$$\begin{cases} x(z) = z - w(z) \\ p(z) = Cz - (C + 1)w(z), \end{cases}$$

transform our graph

$$\{(x, p) \mid x \in A, p \in \mathcal{D}(t, x)\}$$

into the equivalent graph of a maximal monotone function

$$\{(z - w(z), Cz - (C + 1)w(z)) \mid z \in B\}.$$

Recall that  $C$  is the semimonotonicity constant of  $\mathcal{D}(t, \cdot)$ .

Following optimal rays starting in  $A$  with speed in  $\mathcal{D}(t, A)$ , we can now pass from  $X_{t,\tau}(x)$  to a Lipschitz map defined on  $B$

$$\xi(\tau, z) := z - w(z) - (t - \tau)(Cz - (C + 1)w(z)).$$

Note that

$$\{(Cz - (C + 1)w(z)) \mid z \in x + T(x)\} = \mathcal{D}(t, x)$$

so that  $X_{t,\tau}(x) = \{\xi(\tau, z) \mid z \in x + T(x)\}$  and  $X_{t,\tau}(A) = \xi(\tau, B)$ .

We can now apply the Area Formula to  $\xi(\tau, \cdot)$

$$(4.1) \quad \int_{\xi(\tau, B)} \mathcal{H}^0(\xi(\tau, \cdot)^{-1}(w)) dw = \int_B |\xi_z(\tau, z)| dz.$$

Thanks to the injectivity of the map  $X_{t, \tau}$ , which is preserved when passing to the Lipschitz parametrization, the left term of (4.1) is precisely the measure of the set  $\xi(\tau, B)$ . Hence, we have

$$\int_{\xi(\tau, B)} \mathcal{H}^0(\xi(\tau, \cdot)^{-1}(w)) dw = \mathcal{H}^1(\xi(\tau, B)) = \mathcal{H}^1(X_{t, \tau}(A)).$$

Moreover, differentiating  $\xi$  with respect to  $z$  we denote

$$\xi_z(\tau, z) = \xi_z(t, z) - (t - \tau)\dot{\xi}_z(t, z),$$

where  $\xi_z(t, z) := \frac{\partial}{\partial z}(z - w(z))$  and  $\dot{\xi}_z(t, z) := \frac{\partial}{\partial z}(Cz - (C + 1)w(z))$ .

Thus we have

$$\mathcal{H}^1(X_{t, \tau}(A)) = \int_B |\xi_z(t, z) - (t - \tau)\dot{\xi}_z(t, z)| dz \geq \int_B \xi_z(t, z) dz - (t - \tau) \int_B \dot{\xi}_z(t, z) dz.$$

Observing that

$$\int_B \dot{\xi}_z(t, z) dz = \int_B \frac{\partial}{\partial z}(Cz - (C + 1)w(z)) dz = \mu(t, A),$$

we have proven the following estimate: given a Borel set  $A \subset \Omega_t$  for  $t$  in  $(\tau, T]$ , we have

$$(4.2) \quad \mathcal{H}^1(X_{t, \tau}(A)) \geq \mathcal{H}^1(A) - (t - \tau)\mu(t, A).$$

Moreover, since for every  $0 \leq \delta \leq t - \tau$

$$\xi_z(t, z) - (t - (\tau + \delta))\dot{\xi}_z(t, z) = \frac{\delta}{t - \tau}\xi_z(t, z) + \frac{t - (\tau + \delta)}{t - \tau}(\xi_z(t, z) - (t - \tau)\dot{\xi}_z(t, z)),$$

and  $\xi_z(t, z) > 0$ , we have

$$\xi_z(t, z) - (t - (\tau + \delta))\dot{\xi}_z(t, z) \geq \frac{t - (\tau + \delta)}{t - \tau}(\xi_z(t, z) - (t - \tau)\dot{\xi}_z(t, z)).$$

Thus, integrating the last equation over  $B$ , we obtain the following estimate: given a Borel set  $A \subset \Omega_t$  for  $t$  in  $(\tau, T]$ , then for every  $0 \leq \delta \leq t - \tau$  we have

$$(4.3) \quad \mathcal{H}^1(X_{t, \tau + \delta}(A)) \geq \frac{t - (\tau + \delta)}{t - \tau} \mathcal{H}^1(X_{t, \tau}(A)).$$

The estimates (4.2) and (4.3) are of type (3.1) and (3.2) respectively, thus they are enough to prove the SBV regularity of  $d$ , as seen in Section 3.1.  $\square$

*Remark 4.3.* The SBV regularity of  $d(t, \cdot)$  does not necessarily imply the one of  $\frac{\partial}{\partial x}u(t, \cdot)$  as well as the BV regularity of  $d(t, \cdot)$  does not necessarily imply the one of  $\frac{\partial}{\partial x}u(t, \cdot)$ . However, in the one-dimensional case when  $H$  is strictly convex, the divergence of  $d(t, \cdot)$  controls  $\frac{\partial^2}{\partial x^2}u(t, \cdot)$  when  $\frac{\partial}{\partial x}u(t, \cdot)$  is BV. Therefore, this result can be seen as an extension of the one in [19].

## 5. THE MULTI-DIMENSIONAL CASE

In [5] Bianchini, De Lellis and Robyr proved that the estimates (3.1) and (3.2) hold for the uniformly convex Hamiltonian  $H_\epsilon(p) := H(p) + \frac{\epsilon}{2}|p|^2$  for every  $\epsilon > 0$  in a small interval of time and with constants strictly depending on  $\epsilon$ . Thus, the two estimates cannot pass to the limit.

Nevertheless, we can prove that the divergence  $\operatorname{div}d(t, \cdot)$  has Cantor part only for a countable number of  $t$ 's, adding some hypotheses on the regularity of  $d$  and on the structure of the set of points where  $L$  is not twice differentiable.

As already noticed, the Lagrangian corresponding to a smooth convex Hamiltonian is strictly convex but nonsmooth in general. Particular conditions on the set of points where  $L$  is not twice differentiable will allow us to reduce iteratively our problem to a problem of lower dimension, down to the one-dimensional case, where, as we have seen, SBV regularity can be proven without additional assumptions.

Before going on with the proof we set some notations. We will denote by  $(x_1, x_2, \dots, x_n)$  the components of the vector  $x \in \mathbb{R}^n$  and, to contract the notation, for a fixed  $j = 1, \dots, n-1$  we call  $\hat{x} \in \mathbb{R}^{n-j}$  the vector defined so that

$$(x_1, \dots, x_j, \hat{x}) = (x_1, x_2, \dots, x_n).$$

Given a set  $E \subset [0, T] \times \mathbb{R}^n$  we will denote

$$E_t := \{x \in \mathbb{R}^n \mid (t, x) \in E\}$$

and for  $j = 1, \dots, n-1$

$$E_{x_1, \dots, x_j} := \left\{ (t, x_{j+1}, \dots, x_n) \mid (t, x_1, \dots, x_j, x_{j+1}, \dots, x_n) \in E \right\}.$$

As before we will sometimes denote with  $\mu(t, \cdot)$  the Radon measure  $\text{div}d(t, \cdot)$  defined on  $\Omega_t$ .

(HYP(0)) Suppose that the vector field  $d(t, \cdot)$  belongs to  $[BV(\Omega_t)]^n$  for any  $t \in [0, T]$ .

*Remark 5.1.* This hypothesis is certainly satisfied when the initial datum is semiconcave, as a consequence of Proposition 2.16. However it is not true in general, see Remark 3.7. in [4] for an example of a Hamilton-Jacobi equation with convex Hamiltonian whose vector field  $d$  is not BV.

The measure  $\text{div}d$  can have Cantor part only on a subset of the points of differentiability in  $x$  of  $u(t, x)$ , i.e. the points where  $\mathcal{D}(t, x)$  is single-valued. Thus we can reduce to the study of our measure on the set

$$U := \Omega \setminus \{(t, x) \mid \mathcal{D}(t, x) \text{ is multi-valued}\}.$$

Call  $V$  the set of points where  $L$  is not twice differentiable:

$$V := \{v \in \mathbb{R}^n \mid L(\cdot) \text{ is not twice differentiable in } v\}.$$

Then the set  $U$  can be split into two subsets:

$$\Sigma := \{(t, x) \in U \mid d(t, x) \in V\} \quad \text{and} \quad \Sigma^c := U \setminus \Sigma.$$

(HYP(n)) Suppose  $V$  is contained in a finite union of hyperplanes.

*Remark 5.2.* The set  $V^c := \mathbb{R}^n \setminus V$ , of points where  $L$  is twice differentiable, is clearly open because of (HYP(n)). Moreover, since  $d$  is continuous,  $\Sigma^c$ , the pre-image of  $V^c$  through  $d$ , is relatively open in  $U$ .

**Claim 1.(n)** The vector field  $d(t, \cdot)$  belongs to  $[SBV(\Sigma_t^c)]^n$  out of a countable number of  $t$ 's in  $[0, T]$ .

**Claim 2.(n)** The Radon measure  $\text{div}d(t, \cdot)$ , restricted to  $\Sigma_t$ , can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ .

The regularity of  $\text{div}d$  will follow from the previous claims and the fact that  $U = \Sigma \cup \Sigma^c$ .

*Proof of Claim 1.(n).* For a fixed  $(\bar{t}, \bar{x}) \in \Sigma^c$ , the Hessian of  $L$  exists and is continuous in  $\bar{v} := d(\bar{t}, \bar{x})$ . Thus there exist  $r > 0$  and an  $(n+1)$ -dimensional ball  $B_r^{n+1}(\bar{t}, \bar{x}) \subset \Omega \setminus \Sigma$  where  $L$  and  $H$  are uniformly convex.

We can also find an open cone  $C_{n+1}(\bar{t}, \bar{x}) \subset B_r^{n+1}(\bar{t}, \bar{x})$ , properly containing  $(\bar{t}, \bar{x})$ , over which an Hamilton-Jacobi equation can be solved. Indeed, we take an  $n$ -dimensional ball as base,

$$B^n \subset (B_r^{n+1}(\bar{t}, \bar{x}))_{\bar{t}-\sigma} \subset (\Omega \setminus \Sigma)_{\bar{t}-\sigma},$$

for a certain  $0 < \sigma < r$ , and we fix the height of length  $l \in \mathbb{R}$ ,  $0 < l < 2r$ . The height must be chosen according to the speed of propagation of the solution and such that  $\bar{t} < \bar{t} - \sigma + l$ .

Consider now the viscosity solution  $\bar{u}$  of the Cauchy problem

$$\begin{cases} \partial_t \bar{u} + H(D_x \bar{u}) = 0 & \text{in } C_{n+1}(\bar{t}, \bar{x}), \\ \bar{u}(t - \sigma, x) = u(t - \sigma, x) \mathbb{1}_{B^n}(x), \end{cases}$$

where  $\mathbb{1}_E(x)$  is the indicator function of the set  $E$ . Note that  $u(t, x) = \bar{u}(t, x)$  on  $C_{n+1}(\bar{t}, \bar{x})$ .

Thanks to the uniform convexity of  $H$  over  $C_{n+1}(\bar{t}, \bar{x})$ , the main theorem of [5] ensures that the vector field

$$\bar{d}(t, \cdot) := H_p(D_x \bar{u}(t, \cdot))$$

is SBV out of a countable number of  $t$ 's in  $[\bar{t} - \sigma, \bar{t} - \sigma + l]$ .

The vector fields  $d(t, \cdot)$  and  $\bar{d}(t, \cdot)$  are both BV and coincide on  $(C_{n+1}(\bar{t}, \bar{x}))_t$ , thus, by Proposition 2.3,

$$D_x d(t, \cdot) = D_x \bar{d}(t, \cdot).$$

Therefore  $d(t, \cdot)$  belongs to  $SBV((C_{n+1}(\bar{t}, \bar{x}))_t)$  out of a countable number of  $t$ 's in  $[\bar{t} - \sigma, \bar{t} - \sigma + l]$ .

Finally, using the fact that  $\mathbb{R}^n$  is a countable union of bounded sets, we can apply Besicovitch covering theorem, see [3], to prove that the set  $\Sigma^c$  can be fully covered by a countable number of cones  $C_{n+1}^i$ , for  $i \in \mathbb{N}$ , with the property stated above. Thus  $d(t, \cdot)$  belongs to  $[SBV(\Sigma_t^c)]^n$  out of a countable number of  $t$ 's in  $[0, T]$ .  $\square$

We consider now the behavior of  $\text{div}d$  on the set  $\Sigma$ . In order to prove Claim 2.(n), in the  $n$ -dimensional case,  $n > 2$ , we need some other hypotheses on  $L$  and its restriction to the set of points where  $L$  is not twice differentiable. No additional hypotheses are needed in the case  $n = 2$ .

*Proof of Claim 2.(n). 2-Dimensional case.* First, suppose  $V$  is a single straight line. Without loss of generality we can fix  $V = \{v \in \mathbb{R}^2 \mid v_1 = 0\}$ .

Call  $L_V : \mathbb{R} \rightarrow \mathbb{R}$  the restriction of the Lagrangian  $L$  to  $V$ ,

$$L_V(v_2) := L(0, v_2)$$

for any  $v_2 \in \mathbb{R}$ . Call  $I \subset \mathbb{R}$  the set of every  $x_1$  in  $\mathbb{R}$  such that  $\Sigma_{x_1}$  is nonempty. Note that if  $(t, x_2) \in \Sigma_{x_1}$  then  $(0, x_2 - td_2(t, (x_1, x_2)))$  belongs to  $\Sigma_{x_1}$  because  $d(t, (x_1, x_2)) = (0, d_2(t, (x_1, x_2)))$ .

For every  $x_1 \in I$ , we consider the one-dimensional Hamilton-Jacobi equation for the function  $u_{x_1}(t, x_2)$ .

$$\begin{cases} \partial_t u_{x_1} + H_V(D_{x_2} u_{x_1}) = 0 & \text{in } \Sigma_{x_1}, \\ u_{x_1}(0, x_2) = u(0, (x_1, x_2)) & \forall x_2 \in (\Sigma_{x_1})_0, \end{cases}$$

where  $H_V(p)$  is the Hamiltonian associated to  $L_V(v)$ .

The viscosity solution  $u_{x_1}(t, x_2)$  is equal to  $u(t, (x_1, x_2))$  for every  $(t, (x_1, x_2)) \in \Sigma$ . Indeed

$$u_{x_1}(t, x_2) = \min_{y_2 \in \mathbb{R}} \left\{ u(0, x_1, y_2) + tL_V \left( \frac{x_2 - y_2}{t} \right) \right\} = \min_{y_2 \in \mathbb{R}} \left\{ u(0, x_1, y_2) + tL \left( 0, \frac{x_2 - y_2}{t} \right) \right\} = u(t, (x_1, x_2)),$$

where the last equality follows from the fact that, for  $(t, x)$  in  $\Sigma$ , the unique minimizer in the representation formula (2.3) is  $y = (x_1 - td_1(t, x), x_2 - td_2(t, x))$  and  $d(t, (x_1, x_2)) = (0, d_2(t, (x_1, x_2))) = d(0, (x_1, x_2 - td_2(t, (x_1, x_2))))$ . Let us define as usual

$$d_{x_1}(t, x_2) := (H_V)_{p_2}(D_{x_2} u_{x_1}(t, x_2))$$

and

$$\mu_{x_1}(t, \cdot) := \frac{\partial}{\partial x_2} d_{x_1}(t, \cdot).$$

The vector field  $d_{x_1}(t, \cdot)$  is one-dimensional. Hence, by Theorem 3.1,  $d_{x_1}(t, \cdot)$  belongs to  $BV((\Sigma_{x_1})_t)$  for any  $x_1 \in I$ , for any  $t \in [0, T]$ .

On the set  $\Sigma \subset U$ , the matrix of Radon measures  $D_x d$  has no jump part. Moreover, since  $\Sigma_t$  is contained in the set  $\{x \mid d_1(t, x) = 0\}$  and  $d(t, \cdot)$  is BV, Proposition 2.4 implies

$$\frac{\partial}{\partial x_1} d_1(t, \Sigma_t) = 0 \quad \text{and} \quad \frac{\partial}{\partial x_2} d_1(t, \Sigma_t) = 0.$$

Therefore

$$\text{div}d(t, \cdot) = \frac{\partial}{\partial x_2} d_2(t, \cdot) \quad \text{on } \Sigma_t.$$

For every  $(t, x) \in \Sigma$ ,  $u_{x_1}(t, x_2) = u(t, (x_1, x_2))$  implies

$$d_2(t, x) = d_{x_1}(t, x_2).$$

The vector field  $d_2(t, (x_1, \cdot))$  is a one-dimensional restriction of  $d_2(t, \cdot)$  thus, by Proposition 2.5, belongs to  $BV((\Sigma_{x_1})_t)$  for  $\mathcal{H}^1$ -a.e.  $x_1 \in I$ . Since even  $d_{x_1}(t, \cdot)$  is BV on  $(\Sigma_{x_1})_t$ , Proposition 2.3 implies

$$\frac{\partial}{\partial x_2} d_2(t, (x_1, \cdot)) = \frac{\partial}{\partial x_2} d_{x_1}(t, \cdot)$$

for  $\mathcal{H}^1$ -a.e.  $x_1 \in I$ . Therefore taken a Borel set  $A \subset \Sigma_t$  and any  $\phi \in C_c^\infty(\Sigma_t)$ ,

$$\int_A \phi(x) d\mu(t, x) = \int_I \int_{A_{x_1}} \phi(x) d\mu_{x_1}(t, x_2) dx_1.$$

Thanks to the convexity of  $L_V$ , we can apply Theorem 4.1 to  $\mu_{x_1}(t, \cdot)$  and obtain the following estimates.

For any  $\tau > 0$ , let  $A$  be a Borel set in  $\Sigma_t$ , for  $t \in (\tau, T]$ . Then for any  $0 \leq \delta \leq t - \tau$ , and every section  $A_{x_1}$ , for  $x_1 \in I$ , we have

$$\begin{aligned}\mathcal{H}^1(X_{t,\tau}^{x_1}(A_{x_1})) &\geq \mathcal{H}^1(A_{x_1}) - (t - \tau)\mu_{x_1}(t, A_{x_1}), \\ \mathcal{H}^1(X_{t,\tau+\delta}^{x_1}(A_{x_1})) &\geq \frac{t - (\tau + \delta)}{t - \tau} \mathcal{H}^1(X_{t,\tau}^{x_1}(A_{x_1})).\end{aligned}$$

Here we denote by  $X_{t,\tau}^{x_1}(x_2)$  the one-dimensional map defined on  $(\Sigma_{x_1})_t$

$$X_{t,\tau}^{x_1}(x_2) := x_2 - (t - \tau)d_{x_1}(t, x_2).$$

The corresponding 2-dimensional map

$$X_{t,\tau}(x) := x - (t - \tau)d(t, x),$$

reduces to

$$X_{t,\tau}(x) = (x_1, X_{t,\tau}^{x_1}(x_2))$$

for every  $x \in \Sigma_t$ .

We can integrate the previous estimates with respect to  $\mathcal{H}^1$  on  $I \subset \mathbb{R}$  to recover estimates of type (3.1) and (3.2).

For any  $\tau > 0$ , given a Borel set  $A \subset \Sigma_t$ , for  $t$  in  $(\tau, T]$ , we have

$$\mathcal{H}^2(X_{t,\tau}(A)) \geq \mathcal{H}^2(A) - (t - \tau)\mu(t, A).$$

For any  $\tau > 0$ , given a Borel set  $A \subset \Sigma_t$ , for  $t$  in  $[\tau, T]$  and  $0 \leq \delta \leq t - \tau$  we have

$$\mathcal{H}^2(X_{t,\tau+\delta}(A)) \geq \frac{t - (\tau + \delta)}{t - \tau} \mathcal{H}^2(X_{t,\tau}(A)).$$

Thus the strategy seen in the Section 3.1 can be easily applied to prove that  $\mu(t, \cdot)$ , restricted to  $\Sigma_t$ , can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ .

*Remark 5.3.* Note that in this case nothing can be said about the Cantor part of  $\frac{\partial}{\partial x_1} d_2(t, \cdot)$ . Thus we cannot say that  $d(t, \cdot)$  belongs to  $[SBV(\Omega_t)]^2$ .

Consider now the case in which  $V$  consists of a finite number of straight lines. When we consider  $\mu(\cdot, \cdot)$  restricted to the points of  $\Sigma$  such that  $d(t, x)$  belongs only to a part of one of the straight lines, we can apply the considerations done in the case where  $V$  consists only of a single straight line. On the other hand, when we consider  $\mu(\cdot, \cdot)$  restricted to the points of  $\Sigma$  such that  $d(t, x)$  belongs to an intersection point  $(v_1, v_2)$  of two, or more, straight lines, the divergence  $\text{div}d(t, \cdot)$  must be zero on every Borel subset of  $\{x \mid d_1(t, x) = v_1, d_2(t, x) = v_2\}$ , by Proposition 2.4. Thus the measure  $\mu(t, \cdot)$ , restricted to  $\Sigma_t$ , can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$  even when  $V$  consists of a finite number of straight lines. The case in which  $V$  is contained in a finite number of straight lines is analogous.

***n*-Dimensional case.** We prove the claim iterating a subdivision of  $\Sigma$  down to the dimension one.

Call  $V_n := V$ . At the step  $n - j$ , for  $j = n, \dots, 3$ , we first suppose that  $V_j$  consists of a single  $(j - 1)$ -dimensional plane, without loss of generality we can fix

$$V_j = \{v \in \mathbb{R}^n \mid v_1 = 0, \dots, v_{n+1-j} = 0\}.$$

Call  $L_{V_j} : \mathbb{R}^{j-1} \rightarrow \mathbb{R}$  the restriction of  $L_{V_{j+1}}$  to  $V_j$ ,

$$L_{V_j}(\hat{v}) := L_{V_{j+1}}(0, \hat{v}) = L(0, \dots, 0, \hat{v})$$

for any  $\hat{v} \in \mathbb{R}^{j-1}$ .

(HYP(j-1)) We require that the restriction  $L_{V_j}$  is twice  $(j - 1)$ -differentiable out of the set  $V_{j-1}$ ,

$$V_{j-1} := \{(0, \dots, 0, \hat{v}) \in \mathbb{R}^n \mid \hat{v} \in \mathbb{R}^{j-1}, L_{V_j}(\cdot) \text{ is not twice differentiable in } \hat{v}\},$$

and  $V_{j-1}$  is contained in a finite number of  $(j - 2)$ -dimensional planes.

Then we can subdivide  $\Sigma_j$  into two sets:

$$\Sigma_{j-1} := \{(t, x) \in \Sigma_j \mid d(t, x) \in V_{j-1}\} \quad \text{and} \quad \Sigma_{j-1}^c := \Sigma_j \setminus \Sigma_{j-1}.$$

Thus, at every step, we have to prove the following claims.

**Claim 1.(j-1)** The Radon measure  $\text{div}d(t, \cdot)$ , restricted to  $(\Sigma_{j-1}^c)_t$ , can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ .

**Claim 2.(j-1)** The Radon measure  $\text{div}d(t, \cdot)$ , restricted to  $(\Sigma_{j-1})_t$ , can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ .

*Proof of Claim 1.(j-1)* . We will prove it for  $j = n$ , in the other cases the proof is similar.

For a fixed  $(\bar{t}, \bar{x}) \in \Sigma_{n-1}^c$ , the Hessian of  $L_V$  exists and is continuous in  $\hat{v} := (d_2(\bar{t}, \bar{x}), \dots, d_n(\bar{t}, \bar{x})) \in \mathbb{R}^{n-1}$ . Thus there exist  $r > 0$  and an  $(n+1)$ -dimensional ball  $B_r^{n+1}(\bar{t}, \bar{x}) \subset \Omega \setminus \Sigma_{n-1}$  where  $L_V$  and  $H_V$  are uniformly convex.

We can also find, as we did in the proof of Claim 1.(n), an open cone  $C_{n+1}(\bar{t}, \bar{x}) \subset B_r^{n+1}(\bar{t}, \bar{x})$  of height  $[\bar{t} - \sigma, \bar{t} - \sigma + l]$ , for a certain  $0 < \sigma < r$ ,  $\bar{t} < \bar{t} - \sigma + l$  and base  $B^n$ , which contains properly  $(\bar{t}, \bar{x})$ . On every section  $(C_{n+1}(\bar{t}, \bar{x}))_{x_1}$ , for every  $x_1 \in I := \{z \in \mathbb{R} \mid (C_{n+1}(\bar{t}, \bar{x}))_z \neq \emptyset\}$ , we can consider the viscosity solution  $\bar{u}_{x_1}$  of the  $(n-1)$ -dimensional Hamilton-Jacobi equation

$$\begin{cases} \partial_t \bar{u}_{x_1} + H_V(D_{\hat{x}} \bar{u}_{x_1}) = 0 & \text{in } (C_{n+1}(\bar{t}, \bar{x}))_{x_1}, \\ \bar{u}_{x_1}(\bar{t} - \sigma, \hat{x}) = u(\bar{t} - \sigma, x) \mathbb{1}_{B^n}(x). \end{cases}$$

As usual we define

$$\bar{d}_{x_1}(t, \hat{x}) := (H_V)_{\hat{p}}(D_{\hat{x}} \bar{u}_{x_1}(t, \hat{x})).$$

and

$$\bar{\mu}_{x_1}(t, \cdot) := \text{div}_{n-1} \bar{d}_{x_1}(t, \cdot).$$

The vector field  $\bar{d}_{x_1}(t, \cdot)$  belongs to  $[BV(((C_{n+1}(\bar{t}, \bar{x}))_{x_1})_t)]^{n-1}$  for any  $x_1 \in I$ , for any  $t \in [\bar{t} - \sigma, \bar{t} - \sigma + l]$ . Indeed in every  $(C_{n+1}(\bar{t}, \bar{x}))_{x_1}$   $H_V$  is uniformly convex.

Since we have a uniform convexity constant for  $H_V$ , which holds on every  $(C_{n+1}(\bar{t}, \bar{x}))_{x_1}$ , for  $x_1 \in I$ , we can arrange  $l$  small enough, eventually subdividing the cone, so that the following two estimates hold with uniform constants  $C_1, C_2 > 0$ , which do not depend on  $x_1$ .

Let  $\bar{t} - \sigma < \tau < \bar{t} - \sigma + l$ , let  $A$  be a Borel set in  $(C_{n+1}(\bar{t}, \bar{x}))_t$ , for  $t$  in  $[\tau, \bar{t} - \sigma + l]$ . Then, for any  $0 \leq \delta \leq t - \tau$  and every set  $A_{x_1}$ , for  $x_1 \in I$ , we have

$$\begin{aligned} \mathcal{H}^{n-1}(\bar{X}_{t, \tau}^{x_1}(A_{x_1})) &\geq C_1 \mathcal{H}^{n-1}(A_{x_1}) - (t - \tau) C_2 \bar{\mu}_{x_1}(t, A_{x_1}), \\ \mathcal{H}^{n-1}(\bar{X}_{t, \tau + \delta}^{x_1}(A_{x_1})) &\geq \left( \frac{t - (\tau + \delta)}{t - \tau} \right)^{n-1} \mathcal{H}^{n-1}(\bar{X}_{t, \tau}^{x_1}(A_{x_1})). \end{aligned}$$

Here the  $(n-1)$ -dimensional map  $\bar{X}_{t, \tau}^{x_1}(\hat{x})$  is defined as

$$\bar{X}_{t, \tau}^{x_1}(\hat{x}) := \hat{x} - (t - \tau) \bar{d}_{x_1}(t, \hat{x}).$$

Consider now the vector field  $d$ .

On the set  $C_{n+1}(\bar{t}, \bar{x}) \subset U$ , the matrix of Radon measures  $D_x d$  has no jump part. Moreover, since  $(C_{n+1}(\bar{t}, \bar{x}))_t$  is contained in the set  $\{x \mid d_1(t, x) = 0\}$  and  $d(t, \cdot)$  is BV, Proposition 2.4 implies

$$\frac{\partial}{\partial x_j} d_1(t, (C_{n+1}(\bar{t}, \bar{x}))_t) = 0 \quad \text{for } j = 1, \dots, n.$$

Therefore

$$\text{div}d(t, \cdot) = \text{div}_{n-1} \hat{d}(t, \cdot) \quad \text{on } (C_{n+1}(\bar{t}, \bar{x}))_t,$$

$$\hat{d}(t, x) := (d_2(t, x), \dots, d_n(t, x)).$$

For every  $(t, x) \in C_{n+1}(\bar{t}, \bar{x})$ ,  $u_{x_1}(t, x_2) = u(t, (x_1, x_2))$  implies

$$\hat{d}(t, x) = \bar{d}_{x_1}(t, \hat{x}).$$

The vector field  $\hat{d}(t, x_1, \cdot)$ , being an  $(n-1)$ -dimensional section of the BV vector field  $d(t, \cdot)$ , belongs, by Proposition 2.5, to  $[BV((\Sigma_{n-1})_{x_1})]^{n-1}$  for  $\mathcal{H}^1$ -a.e.  $x_1$  such that  $(\Sigma_{n-1})_{x_1}$  is nonempty.

Since even  $\bar{d}_{x_1}(t, \cdot)$  is BV on  $(C_{n+1}(\bar{t}, \bar{x}))_t$ , Proposition 2.3 implies

$$\text{div}_{n-1} \hat{d}(t, (x_1, \cdot)) = \text{div}_{n-1} \bar{d}_{x_1}(t, \cdot)$$

for almost every  $x_1$  such that  $(\Sigma_{n-1})_{x_1}$  is non empty. Therefore taken a Borel set  $A \subset (C_{n+1}(\bar{t}, \bar{x}))_t$  and any  $\phi \in C_c^\infty((C_{n+1}(\bar{t}, \bar{x}))_t)$ ,

$$\int_A \phi(x) d\mu(t, x) = \int_I \int_{A_{x_1}} \phi(x) d\bar{\mu}_{x_1}(t, \hat{x}) dx_1.$$



Moreover, for every  $x \in (C_{n+1}(\bar{t}, \bar{x}))_t$

$$X_{t,\tau}(x) = x - (t - \tau)d(t, x) = (x_1, \bar{X}_{t,\tau}^{x_1}(\hat{x})).$$

The uniformity on every  $A_{x_1}$  allows us to integrate with respect to  $\mathcal{H}^1$ , over the set  $I$ , to obtain the following estimates.

Let  $\bar{t} - \sigma < \tau < t$ , let  $A$  be a Borel set in  $(C_{n+1}(\bar{t}, \bar{x}))_t$ , for  $t$  in  $[\bar{t} - \sigma, \bar{t} - \sigma + l]$ . Then for any  $0 \leq \delta \leq t - \tau$ , it holds

$$\begin{aligned} \mathcal{H}^n(X_{t,\tau}(A)) &\geq C_1 \mathcal{H}^n(A) - (t - \tau)C_2 \mu(t, A), \\ \mathcal{H}^n(X_{t,\tau+\delta}(A)) &\geq \left( \frac{t - (\tau + \delta)}{t - \tau} \right)^{n-1} \mathcal{H}^n(X_{t,\tau}(A)). \end{aligned}$$

Therefore, repeating the standard procedure seen in Section 3.1, we can prove that  $\mu(t, \cdot) := \text{divd}(t, \cdot)$  has Cantor part only for a countable number of  $t$ 's in  $[\bar{t} - \sigma, \bar{t} - \sigma + l]$ .

Finally, using again Besicovitch theorem, the set  $\Sigma_{n-1}^c$  can be fully covered by a countable number of cones  $C_{n+1}^i$  for  $i \in \mathbb{N}$  with the property stated above. Thus the Radon measure  $\text{divd}(t, \cdot)$  can have Cantor part on  $(\Sigma_{n-1}^c)_t$  only for a countable number of  $t$ 's in  $[0, T]$ .  $\square$

We iterate the procedure subdividing  $\Sigma_{j-1}$  in  $\Sigma_{j-2}$  and  $\Sigma_{j-2}^c$ . Hence to prove Claim 2.(j-1) is enough to prove Claim 2.(2), i.e. for  $j = 3$ .

**Claim 2.(2)** The Radon measure  $\text{divd}(t, \cdot)$ , restricted to  $(\Sigma_2)_t$ , can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ .

*Proof.* The proof is equal to the one done in the 2-dimensional case. We rewrite it with the notation which applies in this case.

First, suppose  $V_2$  is a single straight line. Without loss of generality we can fix

$$V_2 = \{v \in \mathbb{R}^n \mid v_1 = 0, \dots, v_{n-1} = 0\}.$$

Recall that  $V_2$  is a straight line in  $V_3 = \{v \in \mathbb{R}^n \mid v_1 = 0, \dots, v_{n-2} = 0\}$ .

Call  $L_{V_2} : \mathbb{R} \rightarrow \mathbb{R}$  the restriction of the Lagrangian  $L_{V_3}$  to  $V_2$ ,

$$L_{V_2}(v_n) := L_{V_3}(0, v_n) = L(0, \dots, 0, v_n)$$

for any  $v_n \in \mathbb{R}$ . For  $i = 1, \dots, n-1$ , call  $I_i \subset \mathbb{R}$  the set of every  $x_i$  in  $\mathbb{R}$  such that  $(\Sigma_2)_{x_i}$  is nonempty and  $I := I_1 \times \dots \times I_{n-1} \subset \mathbb{R}^{n-1}$ .

For every  $(x_1, \dots, x_{n-1}) \in I$ , we consider the one-dimensional Hamilton-Jacobi equation for the function  $u_{x_1, \dots, x_{n-1}}(t, x_n)$ .

$$\begin{cases} \partial_t u_{x_1, \dots, x_{n-1}} + H_{V_2}(D_{x_n} u_{x_1, \dots, x_{n-1}}) = 0 & \text{in } (\Sigma_2)_{x_1, \dots, x_{n-1}}, \\ u_{x_1, \dots, x_{n-1}}(0, x_n) = u(0, (x_1, \dots, x_n)) & \forall x_n \in ((\Sigma_2)_{x_1, \dots, x_{n-1}})_0, \end{cases}$$

where  $H_{V_2}(p_n)$  is the Hamiltonian associated to  $L_{V_2}(v_n)$ .

The viscosity solution  $u_{x_1, \dots, x_{n-1}}(t, x_n)$  is equal to  $u(t, (x_1, \dots, x_n))$  for  $(t, (x_1, \dots, x_n)) \in \Sigma_2$ . Indeed

$$u_{x_1, \dots, x_{n-1}}(t, x_n) = \min_{y_n \in \mathbb{R}} \left\{ u(0, (x_1, \dots, x_{n-1}, y_n)) + t L_{V_2} \left( \frac{x_n - y_n}{t} \right) \right\} = u(t, (x_1, \dots, x_n)),$$

where the last equality follows from the fact that, for  $(t, x)$  in  $\Sigma_2$ , the unique minimizer in (2.3) is  $y = (x_1 - t d_1(t, x), \dots, x_n - t d_n(t, x))$  and  $d(t, x) = (0, \dots, 0, d_n(t, x))$  on  $\Sigma_2$ .

Let us define as usual

$$d_{x_1, \dots, x_{n-1}}(t, x_n) := (H_{V_2})_{p_n}(D_{x_n} u_{x_1, \dots, x_{n-1}}(t, x_n)),$$

and

$$\mu_{x_1, \dots, x_{n-1}}(t, \cdot) := \frac{\partial}{\partial x_n} d_{x_1, \dots, x_{n-1}}(t, \cdot).$$

The vector field  $d_{x_1, \dots, x_{n-1}}(t, \cdot)$  is one-dimensional. Hence, by Theorem 3.1,  $d_{x_1, \dots, x_{n-1}}(t, \cdot)$  belongs to  $BV(((\Sigma_2)_{x_1, \dots, x_{n-1}})_t)$  for any  $(x_1, \dots, x_{n-1}) \in I$ , for any  $t \in [0, T]$ .

On the set  $\Sigma_2 \subset U$ , the matrix of Radon measures  $D_x d$  has no jump part. Moreover, since  $(\Sigma_2)_t$  is contained on the set  $\{x \mid d_1(t, x) = 0, \dots, d_{n-1}(t, x) = 0\}$  and  $d(t, \cdot)$  is BV, Proposition 2.4 implies

$$\frac{\partial}{\partial x_l} d_i(t, (\Sigma_2)_t) = 0 \quad \text{for } i = 1, \dots, n-1 \text{ and } l = 1, \dots, n.$$

Therefore

$$\operatorname{div}d(t, \cdot) = \frac{\partial}{\partial x_n} d_n(t, \cdot) \quad \text{on } (\Sigma_2)_t.$$

For every  $(t, x) \in \Sigma_2$ ,  $u_{x_1, \dots, x_{n-1}}(t, x_n) = u(t, (x_1, \dots, x_n))$  implies

$$d_n(t, x) = d_{x_1, \dots, x_{n-1}}(t, x_n).$$

The vector field  $d_n(t, (x_1, \dots, x_{n-1}, \cdot))$  is a one-dimensional restriction of  $d_n(t, \cdot)$  thus, by Proposition 2.5, belongs to  $BV(((\Sigma_2)_{x_1, \dots, x_{n-1}})_t)$  for almost every  $(x_1, \dots, x_{n-1}) \in I$ . Since even  $d_{x_1, \dots, x_{n-1}}(t, \cdot)$  is BV on  $((\Sigma_2)_{x_1, \dots, x_{n-1}})_t$ , Proposition 2.3 implies

$$\frac{\partial}{\partial x_n} d_n(t, (x_1, \dots, x_{n-1}, \cdot)) = \frac{\partial}{\partial x_n} d_{x_1, \dots, x_{n-1}}(t, \cdot)$$

for  $\mathcal{H}^{n-1}$ -a.e.  $(x_1, \dots, x_{n-1}) \in I$ . Therefore taken a Borel set  $A \subset (\Sigma_2)_t$  and any  $\phi \in C_c^\infty((\Sigma_2)_t)$ ,

$$\int_A \phi(x) d\mu(t, x) = \int_I \int_{A_{x_1, \dots, x_{n-1}}} \phi(x) d\mu_{x_1, \dots, x_{n-1}}(t, x_n) d(x_1, \dots, x_{n-1}).$$

Thanks to the convexity of  $L_{V_2}$ , we can apply Theorem 4.1 to  $\mu_{x_1, \dots, x_{n-1}}(t, \cdot)$  and obtain the following estimates.

For any  $\tau > 0$ , let  $A$  be a Borel set in  $(\Sigma_2)_t$ ,  $t \in (\tau, T]$ . Then for any  $0 \leq \delta \leq t - \tau$  and every section  $A_{x_1, \dots, x_{n-1}}$ , for  $(x_1, \dots, x_{n-1}) \in I$ , we have

$$\mathcal{H}^1(X_{t, \tau}^{x_1, \dots, x_{n-1}}(A_{x_1, \dots, x_{n-1}})) \geq \mathcal{H}^1(A_{x_1, \dots, x_{n-1}}) - (t - \tau) \mu_{x_1, \dots, x_{n-1}}(t, A_{x_1, \dots, x_{n-1}}),$$

$$\mathcal{H}^1(X_{t, \tau + \delta}^{x_1, \dots, x_{n-1}}(A_{x_1, \dots, x_{n-1}})) \geq \frac{t - (\tau + \delta)}{t - \tau} \mathcal{H}^1(X_{t, \tau}^{x_1, \dots, x_{n-1}}(A_{x_1, \dots, x_{n-1}})).$$

Here we denote by  $X_{t, \tau}^{x_1, \dots, x_{n-1}}(x_n)$  the one-dimensional map defined on  $((\Sigma_2)_{x_1, \dots, x_{n-1}})_t$

$$X_{t, \tau}^{x_1, \dots, x_{n-1}}(x_n) := x_n - (t - \tau) d_{x_1, \dots, x_{n-1}}(t, x_n).$$

The corresponding  $n$ -dimensional map defined on  $(\Sigma_2)_t$

$$X_{t, \tau}(x) := x - (t - \tau) d(t, x),$$

reduces to

$$X_{t, \tau}(x) = (x_1, \dots, x_{n-1}, X_{t, \tau}^{x_1, \dots, x_{n-1}}(x_n))$$

for every  $x \in (\Sigma_2)_t$ .

We can integrate the previous estimates with respect to  $\mathcal{H}^{n-1}$  over  $I$  to recover estimates of type (3.1) and (3.2). For any  $\tau > 0$ , given a Borel set  $A \subset (\Sigma_2)_t$ , for  $t$  in  $[\tau, T]$ , we have

$$(5.1) \quad \mathcal{H}^n(X_{t, \tau}(A)) \geq \mathcal{H}^n(A) - (t - \tau) \mu(t, A).$$

For any  $\tau > 0$ , given a Borel set  $A \subset \Sigma_t$ , for  $t$  in  $[\tau, T]$  and  $0 \leq \delta \leq t - \tau$  we have

$$(5.2) \quad \mathcal{H}^n(X_{t, \tau + \delta}(A)) \geq \frac{t - (\tau + \delta)}{t - \tau} \mathcal{H}^n(X_{t, \tau}(A)).$$

Thus the strategy seen in the Section 3.1 can be easily applied to prove that  $\mu(t, \cdot)$ , restricted to  $(\Sigma_2)_t$ , can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$ .

Consider now the case in which  $V_2$  consists of a finite number of straight lines. When we consider  $\mu(\cdot, \cdot)$  restricted to the points of  $\Sigma_2$  such that  $d(t, x)$  belongs only to a part of one of the straight lines, we can apply the considerations done in the case where  $V_2$  consists only of a single straight line. On the other hand, when we consider  $\mu(\cdot, \cdot)$  restricted to the points of  $\Sigma_2$  such that  $d(t, x)$  belongs to an intersection point of two, or more, straight lines, the divergence  $\operatorname{div}d(t, \cdot) = \mu(t, \cdot)$  must be zero on every Borel set, as seen in the 2-dimensional case. The case in which  $V_2$  is contained in a finite number of straight lines is analogous.

Thus the measure  $\mu(t, \cdot)$  can have Cantor part only for a countable number of  $t$ 's in  $[0, T]$  even when  $V_2$  is contained in a finite number of straight lines.  $\square$

Once Claim 2.(2) is proved, we can iteratively prove all the others Claims 2.(j-1) for  $j = 4, \dots, n$  just by repeating the same considerations for the general case in which  $V_j$  consists of a finite union of  $(j-1)$ -dimensional planes. This case can be treated as usual distinguishing the two cases. When we consider  $\mu(\cdot, \cdot)$  restricted to the points of  $\Sigma_j$  such that  $d(t, x)$  belongs only to a part of one of the  $(j-1)$ -dimensional planes, we can apply the considerations done in the case where  $V_j$  consists only of a single  $(j-1)$ -dimensional plane. On the other hand, when we consider  $\mu(\cdot, \cdot)$  restricted to the points of  $\Sigma_j$  such that  $d(t, x)$  belongs to a  $(j-2)$ -dimensional plane intersection of two, or more,  $(j-1)$ -dimensional planes, we can reduce the problem to the  $(j-2)$ -dimensional case. Indeed in this case we can apply again the iterative proof. The case in which  $V_j$  is contained in a finite number of  $(j-1)$ -dimensional planes is analogous.

The considerations above done for  $j = n+1$  conclude even the proof of Claim 2.(n).  $\square$

Let us recall all the necessary assumptions.

Suppose  $H$  is  $C^2(\mathbb{R}^n)$  convex and

$$\lim_{|p| \rightarrow \infty} \frac{H(p)}{|p|} = +\infty.$$

(HYP(0)) The vector field  $d(t, \cdot)$  belongs to  $[BV(\Omega_t)]^n$  for every  $t \in [0, T]$ .

Define  $V_{\pi_n}$  as

$$V_{\pi_n} := \{v \in \mathbb{R}^n \mid L(\cdot) \text{ is not twice differentiable in } v\},$$

and

$$\Sigma_{\pi_n} := \{(t, x) \in U \mid d(t, x) \in V_{\pi_n}\} \quad \text{and} \quad \Sigma_{\pi_n}^c := U \setminus \Sigma_{\pi_n}.$$

(HYP(n)) We suppose  $V_{\pi_n}$  to be contained in a finite union of hyperplanes  $\Pi_{\pi_n}$ .

For  $j = n, \dots, 3$  for any  $(j-1)$ -dimensional plane  $\pi_{j-1}$  in  $\Pi_{\pi_j}$ , let  $L_{\pi_{j-1}} : \pi_{j-1} \rightarrow \mathbb{R}$  be the  $(j-1)$ -dimensional restriction of  $L$  to  $\pi_{j-1}$  and

$$V_{\pi_{j-1}} := \{v \in \pi_{j-1} \mid L_{\pi_{j-1}}(\cdot) \text{ is not twice differentiable in } v\}.$$

Define

$$\Sigma_{\pi_{j-1}} := \{(t, x) \in \Sigma_{\pi_j} \mid d(t, x) \in V_{\pi_{j-1}}\} \quad \text{and} \quad \Sigma_{\pi_{j-1}}^c := \Sigma_{\pi_j} \setminus \Sigma_{\pi_{j-1}}.$$

(HYP(j-1)) We suppose  $V_{\pi_{j-1}}$  to be contained in a finite union of  $(j-2)$ -dimensional planes  $\Pi_{\pi_{j-1}}$ , for every  $\pi_{j-1} \in \Pi_{\pi_j}$ .

*Remark 5.4.* There is no need to ask any assumption on the one-dimensional restriction of  $L$  to a straight line in any of the  $V_{\pi_2}$  for a plane  $\pi_2$ , since in the one-dimensional case the SBV regularity is proven without any other assumptions on  $L$ .

**Theorem 5.5.** *With the above assumptions (HYP(0)), (HYP(n)), ..., (HYP(2)), the Radon measure  $\text{div}d(t, \cdot)$  has Cantor part on  $\Omega_t$  only for a countable number of  $t$ 's in  $[0, T]$ .*

*Example 5.6.* The Hamiltonian

$$H(p) := \sum_{i=1}^{n-1} \frac{(p_i)^4}{4} + \frac{(p_n)^2}{2}$$

is such that the hypotheses (HYP(n)), ..., (HYP(2)) are satisfied. Indeed the corresponding Lagrangian

$$L(v) = \sum_{i=1}^{n-1} \frac{3}{4} v_i (v_i)^{\frac{1}{3}} + \frac{(v_n)^2}{2}$$

is not twice differentiable on the set  $V = \{v \in \mathbb{R}^n \mid v_1 = 0\} \cup \dots \cup \{v \in \mathbb{R}^n \mid v_{n-1} = 0\}$  which is a finite union of hyperplanes. Every restriction on one of these hyperplanes is not twice differentiable on a finite union of  $(n-2)$ -planes and so on.

**Corollary 5.7.** *Let  $D_x u(t, \cdot)$  belong to  $[BV(\Omega_t)]^n$  for every  $t \in [0, T]$  and let  $L$  satisfy (HYP(n)), ..., (HYP(2)), then the Radon measure  $\text{div}d(t, \cdot)$  has Cantor part on  $\Omega_t$  only for a countable number of  $t$ 's in  $[0, T]$ .*

*Proof.* If  $D_x u(t, \cdot)$  belongs to  $[BV(\Omega_t)]^n$  for every  $t \in [0, T]$ , then  $d(t, \cdot) = H_p(D_x u(t, \cdot))$  belongs to  $[BV(\Omega_t)]^n$  for every  $t \in [0, T]$ .  $\square$

**Corollary 5.8.** *Let  $u(0, \cdot)$  be semiconcave and let  $L$  satisfy (HYP( $n$ )), ..., (HYP(2)), then the Radon measure  $\text{divd}(t, \cdot)$  has Cantor part on  $\Omega_t$  only for a countable number of  $t$ 's in  $[0, T]$ .*

*Proof.* It follows from Proposition 2.16. □

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