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# Comment on “Spin in an arbitrary gravitational field”

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## Abstract

The authors of that work [Phys. Rev. D **88**, 084014 (2013)] derive quantum-mechanical equations valid for the covariant Dirac equation by restricting the choice of the tetrad field through the use of the “Schwinger gauge”. Yet it has been shown previously that this gauge leaves space for a physical ambiguity of the Hamiltonian operator. It is shown here precisely how this ambiguity occurs with their settings. There is another ambiguity in the Foldy-Wouthuysen Hamiltonian, for the time-dependent case which is relevant here. However, their equations of motion for classical spinning particles are unambiguous.

Obukhov, Silenko & Teryaev [1] consider the (standard form of the) covariant Dirac equation in an arbitrary coordinate system in a general spacetime. They derive quantum-mechanical equations and compare them with classical equations. Their first step (i) consists in restricting the choice of the tetrad field by using the “Schwinger gauge”. In a second step (ii), by writing the corresponding expression of the covariant Dirac equation in the Schrödinger form while reexpressing the wave function with a specific non-unitary transformation, they get a Hermitian Hamiltonian  $\mathcal{H}$ . In a third step (iii), they transform it into a new Hamiltonian  $\mathcal{H}_{\text{FW}}$  by using a Foldy-Wouthuysen transformation; based on  $\mathcal{H}_{\text{FW}}$ , they compute the time derivative of a polarization operator. Then (iv) they take the semi-classical limit, which (v) they compare with the Hamiltonian and the equations of motion got for a classical spinning particle.

However, in Refs. [2, 3], it has been shown that a whole functional space of different tetrad fields still remains available in the Schwinger gauge, and that the Hermitian Hamiltonians deduced from any two of them by precisely the same non-unitary transformation of the wave function as the one used in Ref. [1] are in general physically inequivalent. This means that step (i) is not unique and leads to a physical non-uniqueness at step (ii). It will be shown how exactly this applies. Hence, the subsequent steps of the work [1] are *a priori* non-unique, too. This will be discussed also.

**1. Non-uniqueness of a tetrad field in the Schwinger gauge.** — With the parameterization of the spacetime metric used by the authors of Ref. [1] (hereafter OST for short), Eq. (2.1), it appears in the following way.<sup>1</sup> As noted by OST, “the line element (2.1) is invariant under redefinitions  $W^{\hat{a}}_b \longrightarrow L^{\hat{a}}_{\hat{c}} W^{\hat{c}}_b$  using arbitrary local rotations  $L^{\hat{a}}_{\hat{c}}(t, x) \in SO(3)$ .” [Clearly, here  $x$  denotes the triplet  $x \equiv (x^a)$  ( $a = 1, 2, 3$ ). Note that OST consider an arbitrary spacetime coordinate system, but do not envisage its change.] Under such a redefinition, the cotetrad field defined by Eq. (2.2):  $\theta^\alpha \equiv e^\alpha_i dx^i$  ( $\alpha, i = 0, \dots, 3$ ) changes to  $\theta'^\alpha \equiv e'^\alpha_i dx^i$  [( $dx^i$ ) being the basis of one-forms dual to the natural basis ( $\partial_i$ ) of the spacetime coordinate system ( $x^i$ ), where  $x^0 \equiv t$  according to OST’s convention], with

$$e^{\hat{0}}_i \equiv e^{\hat{0}}_i \quad (\theta^{\hat{0}} \equiv \theta^{\hat{0}}), \quad e^{\hat{a}}_i \equiv W^{\hat{a}}_b (\delta_i^b - cK^b \delta_i^0), \quad (1)$$

and where

$$W^{\hat{a}}_b \equiv L^{\hat{a}}_{\hat{c}} W^{\hat{c}}_b. \quad (2)$$

The tetrad field  $u_\alpha \equiv e^i_\alpha \partial_i$  defined by Eq. (2.3) changes to  $u'_\alpha \equiv e'^i_\alpha \partial_i$ , with

$$e'^i_{\hat{0}} \equiv e^i_{\hat{0}} \quad (u'_{\hat{0}} \equiv u_{\hat{0}}), \quad e'^i_{\hat{a}} \equiv \delta_b^i W^b_{\hat{a}}, \quad (3)$$

where

$$W^b_{\hat{a}} \equiv W^b_{\hat{c}} P^{\hat{c}}_{\hat{a}}, \quad (4)$$

with  $P = (P^{\hat{c}}_{\hat{a}})$  ( $c, a = 1, 2, 3$ ) the inverse matrix of the matrix ( $L^{\hat{a}}_{\hat{c}}$ ):  $P = P(t, x) \in SO(3)$ , hence  $P^{\hat{c}}_{\hat{a}} = L^{\hat{a}}_{\hat{c}}$ . Thus, the new tetrad field in the Schwinger gauge, Eq. (3), is deduced from the first one by a local Lorentz transformation  $\Lambda = \Lambda(t, x) \in SO(1, 3)$ :

$$u'_\beta = \Lambda^\alpha_\beta u_\alpha, \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & P & & \\ 0 & & & \end{pmatrix}. \quad (5)$$

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<sup>1</sup> All equation numbers of the form ( $m.n$ ) refer to the corresponding equations in Ref. [1].

Such a redefinition, envisaged by OST themselves, affects many relevant quantities. E.g. in (2.6),  $\mathcal{F}^a_b$  becomes

$$\mathcal{F}'^a_b = \mathcal{F}^a_c P^{\widehat{c}}_{\widehat{b}}. \quad (6)$$

It affects also  $\mathcal{Q}_{\widehat{a}\widehat{b}}$  in (2.11),  $\mathcal{C}_{\widehat{a}\widehat{b}}^{\widehat{c}}$  in (2.12), hence also  $\Gamma_{i\alpha\beta}$  in (2.9), (2.10) and (2.8), etc. Hence, *a priori*, every result in the “quantum” part of the paper (sections II–III), as well as every comparison between quantum and classical results (section IV), may depend on the admissible choice of the field of the  $3 \times 3$  real matrix  $W \equiv (W^{\widehat{c}}_{\widehat{b}})$  — that field being determined by the data of the spacetime metric only up to a spacetime dependent rotation matrix  $(L^{\widehat{a}}_{\widehat{c}}(t, x)) \in SO(3)$ , Eq. (2).

**2. Non-uniqueness of the Hermitian Hamiltonian operator (2.15).** — As OST note, the non-unitary transformation (2.14) “also appears in the framework of the pseudo-Hermitian quantum mechanics” as it is used in Ref. [4]. More precisely, since the usual relation  $g_{\alpha\beta} e^{\alpha}_i e^{\beta}_j = g_{ij}$  mentioned (with a misprint) by OST is equivalent to  $g^{\alpha\beta} e^i_{\alpha} e^j_{\beta} = g^{ij}$ , the “Schwinger gauge” condition  $e^0_{\widehat{a}} = 0$  gives

$$g^{\widehat{0}\widehat{0}} (e^0_{\widehat{0}})^2 = g^{00}, \quad (7)$$

thus with OST’s convention  $x^0 = t$ :  $e^0_{\widehat{0}} = c\sqrt{g^{00}} = 1/V$  [Eqs. (2.3) and (4.33)]. Instead, with  $x^0 = ct$ , (7) implies  $|e^0_{\widehat{0}}| = \sqrt{|g^{00}|}$  independently of the signature [3, 4]. Therefore, the transformation (2.14):

$$\psi = (\sqrt{-g} e^0_{\widehat{0}})^{\frac{1}{2}} \Psi \quad (8)$$

is exactly the one used by Gorbatenko & Neznamov [4] to transform the Hamiltonian which they note  $\widetilde{H}$ , got with some Schwinger tetrad, into the Hermitian Hamiltonian noted  $H_{\eta}$  by them, Eqs. (67) and (72) in Ref. [4]. It is also the particular case of the local similarity transformation  $T$  in Eq. (18) of Ref. [3], corresponding with  $S = \mathbf{1}_4$  (one starts from a Schwinger tetrad). A crucial property of the transformation (8), that it brings the Hilbert-space scalar product to the “flat” form [3, 4]: <sup>2</sup>

$$(\Psi | \Phi) \equiv \int \Psi^{\dagger} \sqrt{-g} \gamma^{\widehat{0}} \gamma^0 \Phi \, d^3\mathbf{x} \longrightarrow (\psi | \phi) = \int \psi^{\dagger} \phi \, d^3\mathbf{x}, \quad (10)$$

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<sup>2</sup> In Eq. (10)<sub>1</sub>,  $\gamma^{\widehat{0}}$  is the “ $\alpha = 0$ ” constant Dirac matrix (assumed to be “hermitizing” as is standard), which is noted  $\gamma^{\natural 0}$  in Ref. [3]. Whereas,  $\gamma^0$  is the “ $i = 0$ ” matrix of the *field of Dirac matrices in the curved spacetime*,  $\gamma^i \equiv e^i_{\alpha} \gamma^{\alpha}$  ( $\gamma^{\mu} \equiv a^{\mu}_{\alpha} \gamma^{\natural\alpha}$  in the notation of Refs. [2, 3]). Whether one mentions it or not, the field  $\gamma^i$  can be defined as soon as one has a tetrad field and a set of constant Dirac matrices  $\gamma^{\alpha}$  valid for the Minkowski metric, and it depends on both. In fact, the field  $\gamma^i$ , or at least the field of the  $\gamma^0$  matrix, is needed to define the scalar product — as shown precisely by Eq. (10)<sub>1</sub>. The field  $\gamma^i$  is also there in the Dirac equation, though not explicitly with

is ensured by Eq. (7) above, due to Eq. (19) in Ref. [3]. Recall that the scalar product has to be specified before one can state that some operator is Hermitian. In this Comment, *Hermitian* operators are stated to be so w.r.t. the “flat” product (10)<sub>2</sub>.

Thus, starting from one tetrad field ( $u_\alpha$ ) in the Schwinger gauge (2.2)–(2.3) and getting then, in general, a non-Hermitian Hamiltonian [w.r.t. the product (10)<sub>1</sub>], the Hermitian Hamiltonian  $\mathcal{H}$  obtained by OST using the transformation (2.14) [Eq. (8) here] is just the one denoted  $H_\eta$  in Refs. [3, 4], with here  $\eta = (\sqrt{-g} e^0_{\hat{0}})^{\frac{1}{2}} \mathbf{1}_4$ . Now, consider another tetrad field ( $u'_\alpha$ ) in the Schwinger gauge (2.2)–(2.3). It is thus related with the first one ( $u_\alpha$ ) by the local Lorentz transformation  $\Lambda$ , Eq. (5): essentially,  $\Lambda$  is the arbitrary rotation field  $P(t, x) = (L^{\hat{a}_{\hat{c}}})^{-1} \in SO(3)$ . Define  $S'$ , the local similarity transformation got by “lifting” the local Lorentz transformation  $\Lambda$  to the spin group. At this point, we could repeat *verbatim* what is written in Ref. [3] after the first sentence following Eq. (19). Thus, let  $'\mathcal{H} \equiv H_{\eta'}$  be the Hermitian Hamiltonian obtained by OST using the transformation (8), but starting from the Schwinger tetrad field ( $u'_\alpha$ ) instead of the other one ( $u_\alpha$ ). (The notation  $\mathcal{H}'$  designates something else in Ref. [1].) The change from the Hamiltonian  $\mathcal{H} = H_\eta$  to the Hamiltonian  $'\mathcal{H} = H_{\eta'}$  is through the local similarity transformation  $U = S'S^{-1}$ , Eq. (20) of Ref. [3]. (Here,  $U = S'$ , because we have  $S = \mathbf{1}_4$  as noted after Eq. (8) above.) This implies that the Schrödinger equation  $i\hbar \frac{\partial \psi}{\partial t} = \mathcal{H}\psi$  is equivalent to  $i\hbar \frac{\partial \psi'}{\partial t} = '\mathcal{H}\psi'$ , with  $\psi' = U^{-1}\psi$ . Moreover, the similarity matrix  $U$  is a unitary matrix. {This property of the gauge transformations internal to the Schwinger gauge, derived in [3] from the invariance of the scalar product (10)<sub>2</sub> under  $U$ , can be seen also from the fact that the Lorentz transformation (5) is a rotation.} Hence the transformation

$$\psi \mapsto \psi' = U^{-1}\psi \tag{11}$$

is a unitary transformation internal to the Hilbert space  $\mathbf{H}$  made, in view of (10)<sub>2</sub>, of the usual square-integrable functions of the spatial coordinates,  $x \mapsto \psi(x) \in \mathbb{C}^4$  such that  $(\psi | \psi) = \int \psi^\dagger \psi \, d^3\mathbf{x} < \infty$ . And  $'\mathcal{H}$  is physically inequivalent to  $\mathcal{H}$ , unless  $\partial_t U = 0$ , i.e., unless the arbitrary rotation field  $P$  is chosen independent of  $t$ . To be

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the form (2.7) used by OST: (2.7) rewrites using (2.8)<sub>1</sub> in the slightly more standard form

$$(i\hbar \gamma^i D_i - mc)\Psi = 0. \tag{9}$$

more precise, we have in fact [3, 4]:

$${}'\mathcal{H} = U^{-1}\mathcal{H}U - i\hbar U^{-1}\partial_t U \quad (12)$$

(with here  $U^{-1} = U^\dagger$ ,  $U$  being a unitary matrix).<sup>3</sup> This is actually the relation between two Hamiltonians exchanging by the most general “operator gauge transformation” [5, 6] (in the present case a local similarity transformation, i.e.  $U = U(X)$  is a regular complex matrix depending smoothly on the spacetime position  $X$ ).

**3. Physical inequivalence of  $\mathcal{H}$  and  ${}'\mathcal{H}$ .** — Indeed, contrary to what is stated by Gorbatenko & Neznamov [9], this inequivalence is what is expressed by Eq. (12), unless  $\partial_t U = 0$ . This had been already demonstrated in detail in Ref. [7]. It has been redemonstrated in Ref. [3]; that time with emphasis on the notions of a unitary transformation and of the mean value of a quantum-mechanical operator, invoked in Ref. [9]. Recall that the mean value  $\langle \mathcal{H} \rangle$  of an operator such as the Hamiltonian operator  $\mathcal{H}$  depends on the *state*  $\psi$ , belonging to the “domain” (of definition)  $\mathcal{D}$  of the operator  $\mathcal{H}$ . Here, in view of (10)<sub>2</sub>:

$$\langle \mathcal{H} \rangle = \langle \mathcal{H} \rangle_\psi \equiv (\psi | \mathcal{H} \psi) = \int \psi^\dagger(\mathcal{H}\psi) \, d^3\mathbf{x} \quad \text{when } \psi \in \mathcal{D}. \quad (13)$$

(The domain  $\mathcal{D}$  is a linear subspace of the whole Hilbert space  $\mathbf{H}$ , and  $\mathcal{D}$  should be dense in  $\mathbf{H}$ . The precise definition of  $\mathcal{D}$  should ensure that the integral above makes sense, for any  $\psi \in \mathcal{D}$ .) The mean value  $\langle {}'\mathcal{H} \rangle$  for the corresponding state  $\psi'$  after the transformation (11) is given by the same Eq. (13), with primes. It has been proved in Ref. [3] that, if the similarity matrix  $U(t, x)$  in Eq. (12) depends indeed on  $t$ , then not only the mean values  $\langle \mathcal{H} \rangle$  and  $\langle {}'\mathcal{H} \rangle$  are in general different, but in addition the difference  $\langle {}'\mathcal{H} \rangle - \langle \mathcal{H} \rangle$  depends on the state  $\psi \in \mathcal{D}$ , so that the two Hamiltonians  $\mathcal{H}$  and  ${}'\mathcal{H}$  are physically inequivalent. {This is true also [3] for the case of general “non-Schwinger” gauge transformations, for which the regular complex matrix  $U$  in Eq. (11) is not unitary: then, the transformation (11) is a unitary transformation between *two* Hilbert spaces [3, 7].} Moreover, the difference  $\langle {}'\mathcal{H} \rangle - \langle \mathcal{H} \rangle$  can be calculated explicitly when the tetrad  $(u_\alpha(t, x))$  is deduced from

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<sup>3</sup> It is noted in Refs. [3, 8] that  $\mathcal{H}$  [respectively  ${}'\mathcal{H}$ ] is also equal to the energy operator (the Hermitian part of the Hamiltonian) corresponding to the Schwinger tetrad  $(u_\alpha)$  [respectively  $(u'_\alpha)$ ]. Accounting for Eq. (7) and for the fact that here  $U$  is unitary, the same relation (12) is got by using the general relationship [7] between two energy operators related by an admissible local similarity transformation.

$(u'_\alpha(t, x))$  by the rotation of angle  $\omega t$  around the vector  $u_3(t, x) = u'_3(t, x)$ . In that case, one has the explicit expression

$$U(t) = e^{\omega t N}, \quad N \equiv (\alpha^1 \alpha^2)/2 \quad (N^\dagger = -N). \quad (14)$$

Here,  $\alpha^a \equiv \gamma^{\hat{0}} \gamma^a$  ( $a = 1, 2, 3$ ), in the notation of [1], that is  $\alpha'^j \equiv \gamma'^0 \gamma'^j$  in the notation of Ref. [9]. Equation (14) [9] applies to this more general situation as well. Indeed, the spin transformation  $U$  (noted  $R$  in Ref. [9]) that lifts the local Lorentz transformation (5) with  $P^T$  the rotation of angle  $\omega t$  around  $u_3$  is independent of whether or not  $u_3$  and  $\partial_3$  coincide (as was the case in Ref. [9]). We get from (14)<sub>2</sub>:  $N = \frac{i}{2} \Sigma^3 \equiv \frac{i}{2} \text{diag}(1, -1, 1, -1)$  [3], whence by (12), (13) and (14)<sub>1</sub> [3]:

$$\langle \mathcal{H}' \rangle - \langle \mathcal{H} \rangle = \frac{\omega}{2} \langle \Sigma^3 \rangle = \frac{\omega}{2} \int (|\psi^0|^2 + |\psi^2|^2 - |\psi^1|^2 - |\psi^3|^2) d^3 \mathbf{x}. \quad (15)$$

This even holds true in the presence of an electromagnetic field [10]. This equation applies to *any* possible state  $\psi \in \mathcal{D}$ . It implies that, for the states  $\psi \in \mathcal{D}$  such that  $\langle \Sigma^3 \rangle \neq 0$ , i.e. the integral in (15) does not vanish, then definitely  $\langle \mathcal{H}' \rangle \neq \langle \mathcal{H} \rangle$ . The difference,  $\delta = \frac{\omega}{2} \langle \Sigma^3 \rangle$ , depends on the state  $\psi$ . For a normed state:  $(\psi | \psi) = 1$ ,  $\delta$  can take any value between  $\frac{\omega}{2}$  and  $-\frac{\omega}{2}$ . Note that  $\omega$  is the rotation rate of the tetrad  $(u_\alpha)$  w.r.t.  $(u'_\alpha)$  and can be made arbitrarily large. Of course there are states for which  $\langle \Sigma^3 \rangle = 0$ ; e.g., when the metric (2.1) is the Minkowski metric of a flat spacetime, the coordinates  $(x^i)$  being Cartesian and  $u_3$  being parallel to  $Ox^3$ : an average state  $\psi_{\text{av}} \equiv (\psi_1 + \psi_2)/\sqrt{2}$ , with  $\psi_j \in \mathcal{D}$  of the form  $\varphi(x) A_j$  ( $j = 1, 2$ ) where  $\varphi(x)$  is a square-integrable scalar function and  $A_j \in \mathbb{R}^4$  are the amplitude vectors of two plane wave solutions of the free Dirac equation, *with momentum parallel to  $Ox^3$  i.e. to the rotation axis of the tetrad  $(u_\alpha)$  w.r.t.  $(u'_\alpha)$* , and with opposite helicities  $\pm \frac{1}{2}$  (see Ref. [12]). (Such an average state has to be defined before the mean value is calculated, for the mean value is not linear.) In that particular metric as well as in a general one, the states for which  $\langle \Sigma^3 \rangle = 0$  are a very small subset of all physical states  $\psi \in \mathcal{D}$ . Moreover, by choosing another rotation field  $P$ , one may easily show that, also for the states having  $\langle \Sigma^3 \rangle = 0$ , the energy mean values are indeterminate.

**4. Inequivalence of  $\mathcal{H}$  and  $\mathcal{H}_{\text{FW}}$  in the time-dependent case.** — It has just been proved that, when the rotation field  $P$  in Eq. (5) depends on  $t$ , the two Hermitian Hamiltonians (2.15)  $\mathcal{H}$  and  $\mathcal{H}'$  are inequivalent. This inequivalence would transmit automatically to that of the corresponding Foldy-Wouthuysen (FW) Hamiltonians (3.11), say  $\mathcal{H}_{\text{FW}}$  and  $\mathcal{H}'_{\text{FW}}$  — if the FW transformation (3.2) would lead to an equivalent Hamiltonian to the starting one, i.e., if  $\mathcal{H}$  were equivalent to

$\mathcal{H}_{\text{FW}}$ , and  $'\mathcal{H}$  to  $'\mathcal{H}_{\text{FW}}$ . However, the issue of inequivalence enters the scene an *additional time* here, because the unitary transformation (3.2) has just the form (11)–(12) (albeit with  $U \rightarrow U^{-1}$ ), which, for the case  $\partial_t U \neq 0$ , is responsible for an inequivalence of the Hamiltonians before and after transformation. This issue may have been noted by Eriksen [11], perhaps even by Foldy & Wouthuysen themselves [5, 6]. Eriksen [11] limited the application of the FW transformation to “non-explicitly time-dependent” transformations, i.e. indeed  $\partial_t U = 0$  in Eqs. (11)–(12). That issue has been discussed in detail by Goldman [6], for the FW transformation applied to the Dirac equation in a flat spacetime in Cartesian coordinates in the presence of an electromagnetic field. He noted explicitly that, in the time-dependent case, the transformed Hamiltonian [here  $'\mathcal{H}$  in (12)] “is unphysical (in the sense of EEV’s)” (energy expectation values), given that (in his case) “the EEV’s of the original  $\mathcal{H}$  [are supposed to] have physical meaning”.

The FW transformation  $U$  (3.3) used in effect by OST has the form  $U = \mathcal{N} \cdot (\mathcal{N}^2)^{-1/2} \beta$  with  $\mathcal{N} = \mathcal{N}(t) \equiv \beta\epsilon + \beta\mathcal{M} - \mathcal{O}$  a time-dependent operator (implicitly assumed to be Hermitian and have an inverse), where  $\mathcal{H} = \beta\mathcal{M} + \mathcal{E} + \mathcal{O}$  is a decomposition, stated by OST, of the Hermitian Hamiltonian  $\mathcal{H}$ , Eq. (3.1);<sup>4</sup> and  $\epsilon \equiv \sqrt{\mathcal{M}^2 + \mathcal{O}^2}$ . Assume for simplicity that there is a subdomain  $\mathcal{D}_e(t) \subset \mathcal{D}$  to which the restriction of  $\mathcal{N}(t)$  has a decomposition in eigenspaces:  $\mathcal{D}_e = \oplus_j \mathbb{E}_j$  and  $\mathcal{N}|_{\mathcal{D}_e} = \sum_j \lambda_j \text{Pr}_{\mathbb{E}_j}$  with  $\lambda_j(t)$  nonzero reals and  $\text{Pr}_{\mathbb{E}_j}$  the orthogonal projection on the eigenspace  $\mathbb{E}_j(t)$ . Then, if  $\psi \in \mathbb{E}_j$ , we have  $\mathcal{N} \cdot (\mathcal{N}^2)^{-1/2} \psi = \varepsilon_j \psi$  with  $\varepsilon_j \equiv \text{sgn}(\lambda_j) = \pm 1$ . Hence, defining  $\mathcal{D}_e^+ = \oplus_{\varepsilon_j=+1} \mathbb{E}_j$  and the like for  $\mathcal{D}_e^-$ , we have  $\mathcal{D}_e = \mathcal{D}_e^+ \oplus \mathcal{D}_e^-$  and  $[\mathcal{N} \cdot (\mathcal{N}^2)^{-1/2}]|_{\mathcal{D}_e} = \text{Pr}_{\mathcal{D}_e^+} - \text{Pr}_{\mathcal{D}_e^-}$ , whence  $U|_{\beta^{-1}(\mathcal{D}_e^\pm)} = \pm\beta$  and

$$U|_{\beta^{-1}(\mathcal{D}_e)} = (\text{Pr}_{\mathcal{D}_e^+} - \text{Pr}_{\mathcal{D}_e^-}) \beta, \quad (16)$$

with in fact  $\beta^{-1} = \beta$ . Now, clearly the eigenspaces  $\mathbb{E}_j$  of the operator  $\mathcal{N}(t)$  evolve with time in a general metric, due to the time-dependence of the metric and that of the tetrad, hence  $\mathcal{D}_e^+$  and  $\mathcal{D}_e^-$  also evolve. So Eq. (16) confirms that  $U$  is time-dependent. Thus in the general case the FW transformation (3.3) depends on time, so that the FW Hamiltonian (3.11) is physically inequivalent to the starting Hermitian Hamiltonian  $\mathcal{H}$  (2.15). This is in addition to the physical inequivalence of two Hermitian Hamiltonians  $\mathcal{H}$  and  $'\mathcal{H}$  (2.15) got from two choices of the

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<sup>4</sup> Any operator acting on four-component functions decomposes uniquely into “even” and “odd” parts, i.e. commuting and anticommuting with  $\beta \equiv \gamma^0 \equiv \text{diag}(1, 1, -1, -1)$  [5, 11]. The decomposition (3.1) of the given operator  $\mathcal{H}$  is hence unique for *any* even operator  $\mathcal{M}$  which is also *given*. In the present case,  $\mathcal{M}$  appears to be given by Eq. (3.8). Then one can easily express  $\mathcal{E}$  and  $\mathcal{O}$  in terms of the Hermitian operators  $\mathcal{H}$ ,  $\beta$  and  $\mathcal{M}$ , and check that  $\mathcal{N}$  is indeed Hermitian.



Schwinger tetrad, proved at points **(2)** and **(3)**, and which applies even in the case of a Minkowski spacetime in Cartesian coordinates.

### 5. Semi-classical limit and comparison with classical spinning particles.

— The quantum equation of motion of spin (3.15):  $\frac{d\mathbf{\Pi}}{dt} = \frac{i}{\hbar}[\mathcal{H}_{\text{FW}}, \mathbf{\Pi}]$ , should be ambiguous as is  $\mathcal{H}_{\text{FW}}$ . In fact, the explicit dependence of  $\mathcal{F}^a_b$  on the choice of the Schwinger tetrad field has been noted in Eq. (6). Hence, the three-vector operators of the angular velocity of spin precessing  $\mathbf{\Omega}_{(1)}$  and  $\mathbf{\Omega}_{(2)}$ , Eqs. (3.16) and (3.17), depend *a priori* also on that choice. This dependence may survive in the semi-classical limit (3.19)–(3.20). To check these two points would be somewhat cumbersome. However, we note that the operator  $p_a \equiv -i\hbar\frac{\partial}{\partial x^a}$ , as well as the c-number  $p_a$  given by (4.30), are independent of the choice. Therefore, using  $PP^T = \mathbf{1}_3$ , one finds that  $\epsilon'$  in Eq. (3.21) is invariant under the change of the Schwinger tetrad involving the substitution (6), for

$$\delta^{cd}\mathcal{F}^a_c\mathcal{F}^b_dp_ap_b = \delta^{cd}\mathcal{F}^a_eP^e_c\mathcal{F}^b_fP^f_d p_ap_b = \delta^{ef}\mathcal{F}^a_e\mathcal{F}^b_f p_ap_b. \quad (17)$$

This applies whether in (3.21) one considers  $p_a$  and  $\epsilon'$  as operators or as c-numbers. It follows, using again (17), that the semi-classical velocity operator  $\frac{dx^a}{dt}$  as given by the last member of Eq. (3.23) is invariant under the change of the Schwinger tetrad. The semi-classical velocity operator in the Schwinger frame (3.24) is covariant under the change of the Schwinger tetrad:  $v'_a = P^b_a v_b$ , if  $\epsilon'$  is regarded as a c-number in (3.24).

The degree to which “the classical equation of the spin motion (4.22) agrees with the quantum equation (3.15) and with the semiclassical one (3.18)” [1] does not appear very clearly: these are three rather complex expressions which do not seem to coincide, and as noted above the quantum equation (3.15) looks ambiguous, to the very least *a priori*. We note that the classical spin rate  $\mathbf{\Omega}$  (4.36), as well as [using (17) with  $p_a \rightarrow \pi_a$ ] the classical Hamiltonian (4.38), are independent of the choice of the Schwinger tetrad. In their conclusion, the authors of Ref. [1] state that a “complete consistency of the quantum-mechanical and classical descriptions of spinning particles is also established using the Hamiltonian approach in Sec. IV B”. However, the quantum-mechanical description — in particular, the Hermitian Hamiltonian operator, be it  $\mathcal{H}$  (2.15) or  $\mathcal{H}_{\text{FW}}$  (3.11) — is seriously non-unique as demonstrated at points **(2)**, **(3)** and **(4)** above. Whereas, the classical description is unique as we just saw. The ambiguity of the covariant Dirac theory regards the energy mean values and eigenvalues [3, 7], but the probability current and its

motion are unambiguous. In the wave packet approximation, the latter motion can be rewritten as the exact equations of motion of classical (non-spinning) particles in the electromagnetic field, without any use of the semi-classical limit  $\hbar \rightarrow 0$  [13].

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