Infinite Games Specified by 2-Tape Automata

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Abstract

We prove that the determinacy of Gale-Stewart games whose winning sets are infinitary rational relations accepted by 2-tape Büchi automata is equivalent to the determinacy of (effective) analytic Gale-Stewart games which is known to be a large cardinal assumption. Then we prove that winning strategies, when they exist, can be very complex, i.e. highly non-effective, in these games. We prove the same results for Gale-Stewart games with winning sets accepted by real-time 1-counter Büchi automata, then extending previous results obtained about these games.

1. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that: (a) there is a model of ZFC in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$ but σ cannot be recursive and not even in the class $(\Sigma_2^1 \cup \Pi_2^1)$; (b) there is a model of ZFC in which the game $G(L(\mathcal{A}))$ is not determined.
2. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that $L(\mathcal{A})$ is an arithmetical Δ_3^0-set and Player 2 has a winning strategy in the game $G(L(\mathcal{A}))$ but has no hyperarithmetical winning strategies in this game.
3. There exists a recursive sequence of 2-tape Büchi automata (respectively, of real-time 1-counter Büchi automata) $\mathcal{A}_n$, $n \geq 1$, such that all games $G(L(\mathcal{A}_n))$ are determined, but for which it is Π_2^1-complete hence highly undecidable to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{A}_n))$.

Then we consider the strenghs of determinacy for these games, and we prove the following results.

1. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $A_\sharp$ such that the game $G(A_\sharp)$ is determined iff the effective analytic determinacy holds.
2. There is a transfinite sequence of 2-tape Büchi automata (respectively, of real-time 1-counter Büchi automata) $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, indexed by recursive ordinals, such that the games $G(L(\mathcal{A}_\alpha))$ have strictly increasing strenghs of determinacy.

We show also that the determinacy of Wadge games between two players in charge of infinitary rational relations accepted by 2-tape Büchi automata is equivalent to the (effective) analytic Wadge determinacy and thus also equivalent to the (effective) analytic determinacy.

Keywords: Automata and formal languages; logic in computer science; Gale-Stewart games; 2-tape Büchi automaton; 1-counter automaton; determinacy; effective analytic determinacy; models of set theory; independence from the axiomatic system ZFC; complexity of winning strategies; Wadge games.

1 Introduction

In Computer Science, non terminating systems in relation with an environment may be specified with some particular infinite games of perfect information, called Gale Stewart games since they have been firstly studied by Gale and Stewart in 1953 in [GS53]. The two players in such a game are respectively a non terminating reactive program and the "environment". A Gale-Stewart game is defined as follows. If X is a (countable) alphabet having at least two letters and $A \subseteq X^\omega$, then the Gale-Stewart game $G(A)$ is an infinite game with perfect information between two players. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in X$, then Player 1 writes $a_2 \in X$, and so on ... After ω steps, the two players have composed an infinite word $x = a_1 b_1 a_2 b_2 \ldots$ of X^ω. Player 1 wins the play iff $x \in A$, otherwise Player 2 wins the play. The game $G(A)$ is said to be determined iff one of the two players has a winning strategy.

Then the problem of the synthesis of winning strategies is of great practical interest for the problem of program synthesis in reactive systems. In particular, if $A \subseteq X^\omega$, where X is here a finite alphabet, and A is effectively presented, i.e. accepted by a given finite machine or defined by a given logical formula, the following questions naturally arise, see [Tho95, LT94]: (1) Is the game $G(A)$ determined? (2) If Player 1 has a winning strategy, is it effective, i.e. computable? (3) What are the amounts of space and time necessary to compute such a winning strategy? Büchi and Landweber gave a solution to the famous Church's Problem, posed in 1957, by proving that in a Gale Stewart game $G(A)$, where A is a regular ω-language, one can decide who the winner is and compute a winning strategy given by a finite state transducer, see [Tho08]. Walukiewicz extended Büchi and Landweber's Theorem to the case of a winning set A which is deterministic context-free, i.e. accepted by some deterministic pushdown automaton, answering a question of Thomas and Lescow in [Tho95, LT94]. He first showed in [Wal00] that one can effectively construct winning strategies in parity games played on pushdown graphs and that these strategies can be computed by pushdown transducers. Notice that later some extensions to the case of higher-order pushdown automata have been established [Cac03, CHM$^+$08].

In [Fin12, Fin13] we have studied Gale-Stewart games $G(A)$, where A is a context-free ω-language accepted by a *non-deterministic* pushdown automaton, or even by a 1-counter automaton. We have proved that the determinacy of Gale-Stewart games $G(A)$, whose winning sets A are accepted by real-time 1-counter Büchi automata, is equivalent to the determinacy of (effective) analytic Gale-Stewart games. On the other hand Gale-Stewart games have been much studied in Set Theory and in Descriptive Set Theory, see [Kec95, Jec02]. It has been proved by Martin that every Gale-Stewart game $G(A)$, where A is a Borel set, is determined [Kec95]. Notice that this is proved in ZFC, the commonly accepted axiomatic framework for Set Theory in which all usual mathematics can be developped. But the determinacy of Gale-Stewart games $G(A)$, where A is an (effective) analytic set, is not provable in ZFC; Martin and Harrington have proved that it is a large cardinal assumption equivalent to the existence of a particular real, called the real $0^\sharp$, see [Jec02, page 637]. Thus we proved in [Fin12, Fin13] that the determinacy of Gale-Stewart games $G(A)$, whose winning sets A are accepted by real-time 1-counter Büchi automata, is also equivalent to the existence of the real $0^\sharp$, and thus not provable in ZFC.

In this paper we consider Gale-Stewart games $G(L(\mathcal{A}))$, where $L(\mathcal{A})$ is an infinitary rational relation, i.e. an ω-language over a product alphabet $X = \Sigma \times \Gamma$, which is accepted by a 2-tape (non-deterministic) Büchi automaton $\mathcal{A}$. In such a game, the two players alternatively write letters from the product alphabet $X = \Sigma \times \Gamma$, and after ω steps they have produced an infinite word over X which may be identified with a pair of infinite words $(u, v) \in \Sigma^\omega \times \Gamma^\omega$. Then Player 1 wins the play if $(u, v) \in L(\mathcal{A})$. Notice that if the 2-tape Büchi automaton $\mathcal{A}$ is synchronous then the winning set is actually a regular ω-language over the product alphabet $X = \Sigma \times \Gamma$. Then the infinitary rational relation $L(\mathcal{A})$ is Borel, the game $G(L(\mathcal{A}))$ is determined, and it follows from Büchi and Landweber's Theorem that one can decide who the winner is and compute a winning strategy given by a finite state transducer. We show in this paper that the situation is very different when the 2-tape Büchi automaton may be asynchronous.

We firstly prove that the determinacy of Gale-Stewart games whose winning sets are infinitary rational relations accepted by 2-tape Büchi automata is equivalent to the determinacy of Gale-Stewart games whose winning sets are accepted by 1-counter Büchi automata and thus also equivalent to the existence of the real $0^\sharp$. In particular, it is not provable in ZFC.

Next we prove numerous more results on these games along with similar results about 1-counter games which extend the previous results obtained in [Fin12, Fin13]. In particular, we prove that winning strategies in these games, when they exist, can be very complex, i.e. highly non-effective.

1. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that: (a) there is a model of ZFC in which Player 1 has a winning strategy σ in the game $G(L(\mathcal{A}))$ but σ cannot be recursive and not even in the class $(\Sigma_2^1 \cup \Pi_2^1)$; (b) there is a model of ZFC in which the game $G(L(\mathcal{A}))$ is not determined.

2. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $\mathcal{A}$ such that the infinitary rational relation (respectively, the 1-counter ω-language) $L(\mathcal{A})$ is an arithmetical Δ_3^0-set and Player 2 has a winning strategy in the game $G(L(\mathcal{A}))$ but has no hyperarithmetical winning strategies in this game.

3. There exists a recursive sequence of 2-tape Büchi automata (respectively, of real-time 1-counter Büchi automata) $\mathcal{A}_n$, $n \geq 1$, such that all games $G(L(\mathcal{A}_n))$ are determined, but for which it is Π_2^1-complete, hence highly undecidable, to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{A}_n))$.

Then we consider the possible strenghs of determinacy for these games, and prove the following results, using results of Harrington and Stern on effective analytic games, [Har78, Ste82].

1. There exists a 2-tape Büchi automaton (respectively, a real-time 1-counter Büchi automaton) $A_\sharp$ such that the game $G(L(A_\sharp))$ is determined iff the effective analytic determinacy holds.

2. There is a transfinite sequence of 2-tape Büchi automata (respectively, of real-time 1-counter Büchi automata) $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, indexed by recursive ordinals, such that the games $G(L(\mathcal{A}_\alpha))$ have strictly increasing strenghs of determinacy.

On the other hand, there is another class of infinite games of perfect information which have been much studied in Set Theory and in Descriptive Set Theory: the Wadge games firstly studied by Wadge in [Wad83] where he determined a great refinement of the Borel hierarchy defined via the notion of reduction by continuous functions. The Wadge games are closely related to the notion of reducibility by continuous functions. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' iff there exists a continuous function $f : X^\omega \to Y^\omega$, such that $L = f^{-1}(L')$; this is then denoted by $L \leq_W L'$. On the other hand, the Wadge game $W(L, L')$ is an infinite game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'. And it turned out that Player 2 has a winning strategy in the Wadge game $W(L, L')$ iff $L \leq_W L'$. The Wadge games have also been considered in Computer Science since they are important in the study of the topological complexity of languages of infinite words or trees accepted by various kinds of automata, [PP04, Sta97, Fin06a, Fin08, Sel03, Sel08, ADNM08]. We proved in [Fin12, Fin13] that the determinacy of Wadge games between two players in charge of ω-languages accepted by real-time 1-counter Büchi automata is equivalent to the (effective) analytic Wadge determinacy, which is known to be equivalent to the (effective) analytic determinacy (see [LSR88]) and thus also equivalent to the existence of the real $0^\sharp$. We consider here Wadge games between two players in charge of infinitary rational relations accepted by 2-tape Büchi automata and we prove that the determinacy of these Wadge games is equivalent to the determinacy of Wadge games between two players in charge of ω-languages accepted by real-time 1-counter Büchi automata and thus also equivalent to the (effective) analytic determinacy. In particular, the determinacy of these games is not provable in ZFC.

Notice that as the results presented in this paper might be of interest to both set theorists and theoretical computer scientists, we shall recall in detail some notions of automata theory which are well known to computer scientists but not to set theorists. In a similar way, we give a presentation of some results of set theory which are well known to set theorists but not to computer scientists.

The paper is organized as follows. We recall some known notions in Section 2. We study Gale-Stewart games with winning sets accepted by 2-tape Büchi automata or by 1-counter Büchi automata in Section 3. In Section 4 we study Wadge games between two players in charge of infinitary rational relations. Some concluding remarks are given in Section 5.

2 Recall of some known notions

We assume the reader to be familiar with the theory of formal (ω-)languages [Sta97, PP04]. We recall the usual notations of formal language theory.

If Σ is a finite or countably infnite alphabet, a *non-empty finite word* over Σ is any sequence $x = a_1 \ldots a_k$, where $a_i \in \Sigma$ for $i = 1, \ldots, k$, and k is an integer ≥ 1. The *length* of x is k, denoted by $|x|$. The *empty word* is denoted by λ; its length is 0. $\Sigma^\star$ is the *set of finite words* (including the empty word) over Σ. A (finitary) *language* V over an alphabet Σ is a subset of $\Sigma^\star$.

The *first infinite ordinal* is ω. An ω*-word* over Σ is an ω -sequence $a_1 \ldots a_n \ldots$, where for all integers $i \geq 1$, $\quad a_i \in \Sigma$. When $\sigma = a_1 \ldots a_n \ldots$ is an ω-word over Σ, we write $\sigma(n) = a_n$, $\sigma[n] = \sigma(1)\sigma(2) \ldots \sigma(n)$ for all $n \geq 1$ and $\sigma[0] = \lambda$.

The usual concatenation product of two finite words u and v is denoted $u.v$ (and sometimes just uv). This product is extended to the product of a finite word u and an ω-word v: the infinite word $u.v$ is then the ω-word such that:

$(u.v)(k) = u(k)$ if $k \leq |u|$, and $(u.v)(k) = v(k - |u|)$ if $k > |u|$.

The *set of* ω*-words* over the alphabet Σ is denoted by Σ^ω. An ω*-language* V over an alphabet Σ is a subset of Σ^ω, and its complement (in Σ^ω) is $\Sigma^\omega - V$, denoted V^-.

The *prefix relation* is denoted $\sqsubseteq$: a finite word u is a *prefix* of a finite word v (respectively, an infinite word v), denoted $u \sqsubseteq v$, if and only if there exists a finite word w (respectively, an infinite word w), such that $v = u.w$.

If L is a finitary language (respectively, an ω-language) over the alphabet Σ then the set $\mathrm{Pref}(L)$ of prefixes of elements of L is defined by $\mathrm{Pref}(L) = \{u \in \Sigma^\star \mid \exists v \in L \;\; u \sqsubseteq v\}$.

We now recall the definition of k-counter Büchi automata, reading ω-words over a *finite* alphabet, which will be useful in the sequel.

Let k be an integer ≥ 1. A k-counter machine has k *counters*, each of which containing a non-negative integer. The machine can test whether the content of a given counter is zero or not. And transitions depend on the letter read by the machine, the current state of the finite control, and the tests about the values of the counters. Notice that in this model some λ-transitions are allowed. During these transitions the reading head of the machine does not move to the right, i.e.

the machine does not read any more letter.

Formally a k-counter machine is a 4-tuple $\mathcal{M}$=(K, Σ, Δ, q_0), where K is a finite set of states, Σ is a finite input alphabet, $q_0 \in K$ is the initial state, and $\Delta \subseteq K \times (\Sigma \cup \{\lambda\}) \times \{0,1\}^k \times K \times \{0,1,-1\}^k$ is the transition relation. The k-counter machine $\mathcal{M}$ is said to be *real time* iff: $\Delta \subseteq K \times \Sigma \times \{0,1\}^k \times K \times \{0,1,-1\}^k$, i.e. iff there are no λ-transitions.

If the machine $\mathcal{M}$ is in state q and $c_i \in \mathbf{N}$ is the content of the i^{th} counter $\mathcal{C}_i$ then the configuration (or global state) of $\mathcal{M}$ is the $(k+1)$-tuple $(q, c_1, \ldots, c_k)$.

For $a \in \Sigma \cup \{\lambda\}$, $q, q' \in K$ and $(c_1, \ldots, c_k) \in \mathbf{N}^k$ such that $c_j = 0$ for $j \in E \subseteq \{1, \ldots, k\}$ and $c_j > 0$ for $j \notin E$, if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ where $i_j = 0$ for $j \in E$ and $i_j = 1$ for $j \notin E$, then we write:

$$a : (q, c_1, \ldots, c_k) \mapsto_{\mathcal{M}} (q', c_1 + j_1, \ldots, c_k + j_k).$$

Thus the transition relation must obviously satisfy:
if $(q, a, i_1, \ldots, i_k, q', j_1, \ldots, j_k) \in \Delta$ and $i_m = 0$ for some $m \in \{1, \ldots, k\}$ then $j_m = 0$ or $j_m = 1$ (but j_m may not be equal to -1).

Let $\sigma = a_1a_2 \ldots a_n \ldots$ be an ω-word over Σ. An ω-sequence of configurations $r = (q_i, c_1^i, \ldots c_k^i)_{i\geq 1}$ is called a run of $\mathcal{M}$ on σ iff:

(1) $(q_1, c_1^1, \ldots c_k^1) = (q_0, 0, \ldots, 0)$

(2) for each $i \geq 1$, there exists $b_i \in \Sigma \cup \{\lambda\}$ such that $b_i : (q_i, c_1^i, \ldots c_k^i) \mapsto_{\mathcal{M}} (q_{i+1}, c_1^{i+1}, \ldots c_k^{i+1})$ and such that $a_1a_2 \ldots a_n \ldots = b_1b_2 \ldots b_n \ldots$

For every such run r, $\mathrm{In}(r)$ is the set of all states entered infinitely often during r.

Definition 2.1 *A Büchi k-counter automaton is a 5-tuple $\mathcal{M}$=$(K, \Sigma, \Delta, q_0, F)$, where $\mathcal{M}'$=(K, Σ, Δ, q_0) is a k-counter machine and $F \subseteq K$ is the set of accepting states. The ω-language accepted by $\mathcal{M}$ is: $L(\mathcal{M})$= $\{\sigma \in \Sigma^\omega \mid$ there exists a run r of $\mathcal{M}$ on σ such that $\mathrm{In}(r) \cap F \neq \emptyset\}$*

The class of ω-languages accepted by Büchi k-counter automata is denoted $\mathbf{BCL}(k)_\omega$. The class of ω-languages accepted by *real time* Büchi k-counter automata will be denoted **r**-$\mathbf{BCL}(k)_\omega$. The class $\mathbf{BCL}(1)_\omega$ is a strict subclass of the class $\mathbf{CFL}_\omega$ of context free ω-languages accepted by Büchi pushdown automata.

Infinitary rational relations are subsets of $\Sigma^\omega \times \Gamma^\omega$, where Σ and Γ are finite alphabets, which are accepted by 2-tape Büchi automata.

Definition 2.2 *A 2-tape Büchi automaton is a sextuple $\mathcal{A} = (K, \Sigma, \Gamma, \Delta, q_0, F)$, where K is a finite set of states, Σ and Γ are finite alphabets, Δ is a finite subset of $K \times \Sigma^\star \times \Gamma^\star \times K$ called the set of transitions, q_0 is the initial state, and $F \subseteq K$ is the set of accepting states.*
A computation $\mathcal{C}$ of the 2-tape Büchi automaton $\mathcal{A}$ is an infinite sequence of transitions

$$(q_0, u_1, v_1, q_1), (q_1, u_2, v_2, q_2), \ldots, (q_{i-1}, u_i, v_i, q_i), (q_i, u_{i+1}, v_{i+1}, q_{i+1}), \ldots$$

The computation is said to be successful iff there exists a final state $q_f \in F$ and infinitely many integers $i \geq 0$ such that $q_i = q_f$.
The input word of the computation is $u = u_1.u_2.u_3 \ldots$
The output word of the computation is $v = v_1.v_2.v_3 \ldots$
Then the input and the output words may be finite or infinite.
The infinitary rational relation $L(\mathcal{A}) \subseteq \Sigma^\omega \times \Gamma^\omega$ accepted by the 2-tape Büchi automaton $\mathcal{A}$ is the set of pairs $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ such that u and v are the input and the output words of some successful computation $\mathcal{C}$ of $\mathcal{A}$.
The set of infinitary rational relations will be denoted by $\mathbf{RAT}_\omega$.

We assume the reader to be familiar with basic notions of topology which may be found in [Kec95, LT94, Sta97, PP04]. There is a natural metric on the set Σ^ω of infinite words over a finite or countably infinite alphabet Σ containing at least two letters which is called the *prefix metric* and is defined as follows. For $u, v \in \Sigma^\omega$ and $u \neq v$ let $\delta(u, v) = 2^{-l_{\mathrm{pref}(u,v)}}$ where $l_{\mathrm{pref}(u,v)}$ is the first integer n such that the $(n+1)^{st}$ letter of u is different from the $(n+1)^{st}$ letter of v. This metric induces on Σ^ω the usual topology in which the *open subsets* of Σ^ω are of the form $W.\Sigma^\omega$, for $W \subseteq \Sigma^\star$. A set $L \subseteq \Sigma^\omega$ is a *closed set* iff its complement $\Sigma^\omega - L$ is an open set. If the alphabet Σ is finite then the set Σ^ω equipped with this topology is a Cantor space, and if $\Sigma = \omega$ then the set ω^ω equipped with this topology is the classical Baire space. We shall consider only these two cases in the sequel.

For $V \subseteq \Sigma^\star$ we denote $\mathrm{Lim}(V) = \{x \in \Sigma^\omega \mid \exists^\infty n \geq 1 \;\; x[n] \in V\}$ the set of infinite words over Σ having infinitely many prefixes in V. Then the topological closure $\mathrm{Cl}(L)$ of a set $L \subseteq \Sigma^\omega$ is equal to $\mathrm{Lim}(\mathrm{Pref}(L))$. Thus we have also the following characterization of closed subsets of Σ^ω: a set $L \subseteq \Sigma^\omega$ is a closed subset of the space Σ^ω iff $L = \mathrm{Lim}(\mathrm{Pref}(L))$.

We now recall the definition of the *Borel Hierarchy* of subsets of X^ω.

Definition 2.3 *For a non-null countable ordinal α, the classes $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ of the Borel Hierarchy on the topological space X^ω are defined as follows: $\mathbf{\Sigma}^0_1$ is the class of open subsets of X^ω, $\mathbf{\Pi}^0_1$ is the class of closed subsets of X^ω, and for any countable ordinal $\alpha \geq 2$:*
$\mathbf{\Sigma}^0_\alpha$ is the class of countable unions of subsets of X^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Pi}^0_\gamma$.
$\mathbf{\Pi}^0_\alpha$ is the class of countable intersections of subsets of X^ω in $\bigcup_{\gamma<\alpha} \mathbf{\Sigma}^0_\gamma$.

A set $L \subseteq X^\omega$ is Borel iff it is in the union $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$, where ω_1 is the first uncountable ordinal.

There are also some subsets of X^ω which are not Borel. In particular, the class of Borel subsets of X^ω is strictly included into the class $\mathbf{\Sigma}^1_1$ of *analytic sets* which are obtained by projection of Borel sets. The *co-analytic sets* are the complements of analytic sets.

Definition 2.4 *A subset A of X^ω is in the class $\mathbf{\Sigma}^1_1$ of analytic sets iff there exist a finite alphabet Y and a Borel subset B of $(X \times Y)^\omega$ such that $x \in A \leftrightarrow \exists y \in Y^\omega$ such that $(x, y) \in B$, where (x, y) is the infinite word over the alphabet $X \times Y$ such that $(x, y)(i) = (x(i), y(i))$ for each integer $i \geq 1$.*

We now recall the notion of completeness with regard to reduction by continuous functions. For a countable ordinal $\alpha \geq 1$, a set $F \subseteq X^\omega$ is said to be a $\mathbf{\Sigma}^0_\alpha$ (respectively, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_1$)-*complete set* iff for any set $E \subseteq Y^\omega$ (with Y a finite alphabet): $E \in \mathbf{\Sigma}^0_\alpha$ (respectively, $E \in \mathbf{\Pi}^0_\alpha$, $E \in \mathbf{\Sigma}^1_1$) iff there exists a continuous function $f : Y^\omega \to X^\omega$ such that $E = f^{-1}(F)$.

We now recall the definition of classes of the arithmetical hierarchy of ω-languages, see [Sta97]. Let X be a finite alphabet or $X = \omega$. An ω-language $L \subseteq X^\omega$ belongs to the class Σ_n if and only if there exists a recursive relation $R_L \subseteq (\mathbb{N})^{n-1} \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists a_1 \ldots Q_n a_n \quad (a_1, \ldots, a_{n-1}, \sigma[a_n + 1]) \in R_L\},$$

where Q_i is one of the quantifiers $\forall$ or $\exists$ (not necessarily in an alternating order). An ω-language $L \subseteq X^\omega$ belongs to the class Π_n if and only if its complement $X^\omega - L$ belongs to the class Σ_n. The class Σ^1_1 is the class of *effective analytic sets* which are obtained by projection of arithmetical sets. An ω-language $L \subseteq X^\omega$ belongs to the class Σ^1_1 if and only if there exists a recursive relation $R_L \subseteq \mathbb{N} \times \{0, 1\}^\star \times X^\star$ such that:

$$L = \{\sigma \in X^\omega \mid \exists \tau (\tau \in \{0, 1\}^\omega \wedge \forall n \exists m ((n, \tau[m], \sigma[m]) \in R_L))\}.$$

Then an ω-language $L \subseteq X^\omega$ is in the class Σ_1^1 iff it is the projection of an ω-language over the alphabet $X \times \{0, 1\}$ which is in the class Π_2. The class Π_1^1 of *effective co-analytic sets* is simply the class of complements of effective analytic sets.

Recall that the (lightface) class Σ_1^1 of effective analytic sets is strictly included into the (boldface) class $\boldsymbol{\Sigma}_1^1$ of analytic sets.

Recall that a Büchi Turing machine is just a Turing machine working on infinite inputs (over a finite alphabet) with a Büchi-like acceptance condition, and that the class of ω-languages accepted by Büchi Turing machines is the class Σ_1^1 of effective analytic sets [CG78, Sta97]. On the other hand, one can construct, using a classical construction (see for instance [HMU01]), from a Büchi Turing machine $\mathcal{T}$, a 2-counter Büchi automaton $\mathcal{A}$ accepting the same ω-language. Thus one can state the following proposition.

Proposition 2.5 ([Sta97, Sta00]) *Let X be a finite alphabet. An ω-language $L \subseteq X^\omega$ is in the class Σ_1^1 iff it is accepted by a non deterministic Büchi Turing machine, hence iff it is in the class $\mathbf{BCL}(2)_\omega$.*

We assume also the reader to be familiar with the arithmetical and analytical hierarchies on subsets of $\mathbb{N}$, these notions may be found in the textbooks on computability theory [Rog67] [Odi89, Odi99].

3 Gale-Stewart games specified by 2-tape automata

We first recall the definition of Gale-Stewart games.

Definition 3.1 ([Jec02]) *Let $A \subseteq X^\omega$, where X is a finite or countably infinite alphabet. The Gale-Stewart game $G(A)$ is a game with perfect information between two players. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in X$, then Player 1 writes $a_2 \in X$, and so on ... After ω steps, the two players have composed a word $x = a_1 b_1 a_2 b_2 \ldots$ of X^ω. Player 1 wins the play iff $x \in A$, otherwise Player 2 wins the play.*

Let $A \subseteq X^\omega$ and $G(A)$ be the associated Gale-Stewart game. A strategy for Player 1 is a function $F_1 : (X^2)^\star \to X$ and a strategy for Player 2 is a function $F_2 : (X^2)^\star X \to X$. Player 1 follows the strategy F_1 in a play if for each integer $n \geq 1$ $\quad a_n = F_1(a_1 b_1 a_2 b_2 \cdots a_{n-1} b_{n-1})$. If Player 1 wins every play in which she has followed the strategy F_1, then we say that the strategy F_1 is a winning strategy (w.s.) for Player 1. The notion of winning strategy for Player 2 is defined in a similar manner.

The game $G(A)$ is said to be determined if one of the two players has a winning strategy.

We shall denote **Det**$(\mathcal{C})$*, where $\mathcal{C}$ is a class of ω-languages, the sentence : "Every Gale-Stewart game $G(A)$, where $A \subseteq X^\omega$ is an ω-language in the class $\mathcal{C}$, is determined".*

Notice that, in the whole paper, we assume that ZFC is consistent, and all results, lemmas, propositions, theorems, are stated in ZFC unless we explicitely give another axiomatic framework.

Notice that it is known that the determinacy of effective analytic games for $X = \omega$, i.e. for a countably infinite alphabet, is equivalent to the determinacy of effective analytic games for a finite alphabet X. This follows easily from Lemma 3.14 below. In the sequel the determinacy of effective analytic games will be denoted by **Det**(Σ_1^1).

The following results were successively proved in [Fin13].

Proposition 3.2 **Det**(Σ_1^1) $\Longleftrightarrow$ **Det**($\mathbf{r}$-$\mathbf{BCL}(8)_\omega$).

Theorem 3.3 **Det**(Σ_1^1) $\Longleftrightarrow$ **Det**($\mathbf{CFL}_\omega$) $\Longleftrightarrow$ **Det**($\mathbf{BCL}(1)_\omega$).

Theorem 3.4 **Det**(Σ_1^1) $\Longleftrightarrow$ **Det**($\mathbf{CFL}_\omega$) $\Longleftrightarrow$ **Det**($\mathbf{r}$-$\mathbf{BCL}(1)_\omega$).

We now consider Gale-Stewart games of the form $G(A)$ where $A \subseteq X^\omega$, $X = \Sigma \times \Gamma$ is the product of two finite alphabets, and $A = L(\mathcal{A}) \subseteq (\Sigma \times \Gamma)^\omega$ is an infinitary rational relation accepted by a 2-tape Büchi automaton $\mathcal{A}$.

Recall that an infinite word over the alphabet $X = \Sigma \times \Gamma$ may be identified with a pair of infinite words $(u, v) \in \Sigma^\omega \times \Gamma^\omega$ and so we often identify $(\Sigma \times \Gamma)^\omega$ and $\Sigma^\omega \times \Gamma^\omega$.

We are going to prove the following result.

Theorem 3.5 **Det**(Σ_1^1) $\Longleftrightarrow$ **Det**($\mathbf{RAT}_\omega$).

In order to prove this result, we shall use the equivalence **Det**(Σ_1^1) $\Longleftrightarrow$ **Det**($\mathbf{r}$-$\mathbf{BCL}(1)_\omega$) which was proved in [Fin12, Fin13].

We now first define a coding of an ω-word over a finite alphabet Σ, by an ω-word over the alphabet $\Sigma_1 = \Sigma \cup \{0, A\}$, where $0, A$ are additional letters not in Σ.

For $x \in \Sigma^\omega$ the ω-word $h(x)$ is defined by:

$$h(x) = 0.Ax(1).0^2.x(2).0^3.A.x(3).0^4.x(4).\ldots 0^{2n}.x(2n).0^{2n+1}.A.x(2n+1)\ldots$$

Notice that the ω-word $h(x)$ is obtained from the ω-word

$$0.x(1).0^2.x(2).0^3.x(3).0^4.x(4)\ldots$$

by adding a letter A before each letter $x(2n+1)$, where $n \geq 0$ is an integer.

Let also

$$\alpha = 0.AA.0^2.A.0^3.AA.0^4.A.0^5\ldots AA.0^{2n}.A.0^{2n+1}.AA.0^{2n+2}\ldots$$

Notice that this ω-word α is easily obtained from the ω-word

$$\alpha' = 0.A.0^2.A.0^3.A.0^4.A.0^5.A\ldots A.0^n.A.0^{n+1}.A\ldots$$

by adding a letter A before each segment $A.0^{2n}.A$, where $n \geq 1$ is an integer.

Then it is easy to see that the mapping h from Σ^ω into $(\Sigma \cup \{0, A\})^\omega$ is continuous and injective.

We can now state the following Lemma.

Lemma 3.6 *Let Σ be a finite alphabet and $0, A$ be two additional letters not in Σ. Let α be the ω-word over $\Gamma = \{0, A\}$ defined as above, and $L \subseteq \Sigma^\omega$ be in $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$. Then there exists an infinitary rational relation $R_1 \subseteq (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$ such that:*

$$\forall x \in \Sigma^\omega \quad (x \in L) \textit{ iff } ((h(x), \alpha) \in R_1)$$

Proof. Let Σ be a finite alphabet, $0, A$ be two additional letters not in Σ. Let α be the ω-word over $\{0, A\}$ defined as above, and $L = L(\mathcal{A}) \subseteq \Sigma^\omega$, where $\mathcal{A}$=$(K, \Sigma, \Delta, q_0, F)$ is a real time 1-counter Büchi automaton.

We now define the relation R_1.

A pair $y = (y_1, y_2) \in (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$ is in R_1 if and only if it is in the form

$y_1 = u_1.v_1.A.x(1).u_2.v_2.x(2).u_3.v_3.A.x(3)\ldots.u_{2n}.v_{2n}.x(2n).u_{2n+1}.v_{2n+1}.A.x(2n{+}1)\ldots.$

$y_2 = w_1.z_1.AA.w_2.z_2.A.w_3.z_3.AA\ldots AAw_{2n}.z_{2n}.A.w_{2n+1}.z_{2n+1}\ldots$

where $|v_1| = 0$ and for all integers $i \geq 1$,

$$u_i, v_i, w_i, z_i \in 0^\star \text{ and } x(i) \in \Sigma \text{ and}$$

$$|u_{i+1}| = |z_i| + 1$$

and there is a sequence $(q_i)_{i\geq 0}$ of states of K such that for all integers $i \geq 1$:

$$x(i) : (q_{i-1}, |v_i|) \mapsto_{\mathcal{A}} (q_i, |w_i|)$$

Moreover some state $q_f \in F$ occurs infinitely often in the sequence $(q_i)_{i\geq 0}$.
Notice that the state q_0 of the sequence $(q_i)_{i\geq 0}$ is also the initial state of $\mathcal{A}$.

Notice that the main idea is that we try to simulate, using a 2-tape automaton, the reading of the infinite word $x(1).x(2).x(3)\ldots$ by the real time 1-counter Büchi automaton $\mathcal{A}$. The initial value of the counter is $|v_1|$ and the value of the counter after the reading of the letter $x(1)$ by $\mathcal{A}$ is $|w_1|$ which is on the second tape. Now the 2-tape automaton accepting R_1 would need to read again the value $|w_1|$ in order to compare it to the value of the counter after the reading of $x(2)$ by the 1-counter automaton $\mathcal{A}$. This is not directly possible so the simulation does not work on every pair of R_1. However, using the very special shape of pairs in $h(\Sigma^\omega) \times \{\alpha\}$, the simulation will be possible on a pair $(h(x), \alpha)$. Then for such a pair $(h(x), \alpha) \in R_1$ written in the above form (y_1, y_2), we have $|v_2| = |w_1|$ and then the simulation can continue from the value $|v_2|$ of the counter, and so on.

We now give the details of the proof.
Let $x \in \Sigma^\omega$ be such that $(h(x), \alpha) \in R_1$. We are going to prove that $x \in L$.

By hypothesis $(h(x), \alpha) \in R_1$ thus there are finite words $u_i, v_i, w_i, z_i \in 0^\star$ such that $|v_1| = 0$ and for all integers $i \geq 1$, $|u_{i+1}| = |z_i| + 1$, and

$y_1 = u_1.v_1.A.x(1).u_2.v_2.x(2).u_3.v_3.A.x(3)\ldots.u_{2n}.v_{2n}.x(2n).u_{2n+1}.v_{2n+1}.A.x(2n{+}1)\ldots.$

$y_2 = w_1.z_1.AA.w_2.z_2.A.w_3.z_3.AA\ldots AAw_{2n}.z_{2n}.A.w_{2n+1}.z_{2n+1}\ldots$

Moreover there is a sequence $(q_i)_{i\geq 0}$ of states of K such that for all integers $i \geq 1$:

$$x(i) : (q_{i-1}, |v_i|) \mapsto_{\mathcal{A}} (q_i, |w_i|)$$

and some state $q_f \in F$ occurs infinitely often in the sequence $(q_i)_{i \geq 0}$.

On the other side we have:
$h(x) = 0.Ax(1).0^2.x(2).0^3.A.x(3).0^4.x(4).\ldots.0^{2n}.x(2n).0^{2n+1}.A.x(2n+1)\ldots$
$\alpha = 0.AA.0^2.A.0^3.AA.0^4.A.0^5\ldots AA.0^{2n}.A.0^{2n+1}.AA.0^{2n+2}\ldots$

So we have $|u_1.v_1| = 1$ and $|v_1| = 0$ and $x(1) : (q_0, |v_1|) \mapsto_{\mathcal{A}} (q_1, |w_1|)$. But $|w_1.z_1| = 1$, $|u_2.v_2| = 2$, and $|u_2| = |z_1| + 1$ thus $|v_2| = |w_1|$.

We are going to prove in a similar way that for all integers $i \geq 1$ it holds that $|v_{i+1}| = |w_i|$. We know that $|w_i.z_i| = i$, $|u_{i+1}.v_{i+1}| = i+1$, and $|u_{i+1}| = |z_i| + 1$ thus $|w_i| = |v_{i+1}|$.

Then for all $i \geq 1$, $x(i) : (q_{i-1}, |v_i|) \mapsto_{\mathcal{A}} (q_i, |v_{i+1}|)$.
So if we set $c_i = |v_i|$, $(q_{i-1}, c_i)_{i \geq 1}$ is an accepting run of $\mathcal{A}$ on x and this implies that $x \in L$. Conversely it is easy to prove that if $x \in L$ then $(h(x), \alpha)$ may be written in the form of $(y_1, y_2) \in R_1$.

It remains to prove that the above defined relation R_1 is an infinitary rational relation. It is easy to find a 2-tape Büchi automaton $\mathcal{A}$ accepting the relation R_1. □

Lemma 3.7 *The set*
$$R_2 = (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega - (h(\Sigma^\omega) \times \{\alpha\})$$
is an infinitary rational relation.

Proof. By definition of the mapping h, we know that a pair of ω-words (σ_1, σ_2) is in $h(\Sigma^\omega) \times \{\alpha\}$ iff it is of the form:

$$\sigma_1 = h(x) = 0.Ax(1).0^2.x(2).0^3.A.x(3).0^4.x(4).\ldots.0^{2n}.x(2n).0^{2n+1}.A.x(2n+1)\ldots$$

$$\sigma_2 = \alpha = 0.AA.0^2.A.0^3.AA.0^4.A.0^5\ldots AA.0^{2n}.A.0^{2n+1}.AA.0^{2n+2}\ldots$$

where for all integers $i \geq 1$, $x(i) \in \Sigma$.

So it is easy to see that $(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega - (h(\Sigma^\omega) \times \{\alpha\})$ is the union of the sets $\mathcal{C}_j$ where:

- $\mathcal{C}_1$ is formed by pairs (σ_1, σ_2) where
  σ_1 has not any initial segment in $0.A.\Sigma.0^2.\Sigma.0^3A.\Sigma$, or
  σ_2 has not any initial segment in $0.AA.0^2.A.0^3AA$.
- $\mathcal{C}_2$ is formed by pairs (σ_1, σ_2) where
  $\sigma_2 \notin (0^+AA0^+A)^\omega$, or
  $\sigma_1 \notin (0^+.A.\Sigma.0^+.\Sigma)^\omega$.
- $\mathcal{C}_3$ is formed by pairs (σ_1, σ_2) where
  $\sigma_1 = w_1.u.A.z_1$
  $\sigma_2 = w_2.v.A.z_2$
  where n is an integer ≥ 1, $w_1 \in (0^+.A.\Sigma.0^+.\Sigma)^n$, $w_2 \in (0^+AA0^+A)^n$,
  $u, v \in 0^+$, $z_1 \in (\Sigma \cup \{0, A\})^\omega$, $z_2 \in \Gamma^\omega$, and
  $$|u| \neq |v|$$

- $\mathcal{C}_4$ is formed by pairs (σ_1, σ_2) where
  $\sigma_1 = w_1.u.z_1$
  $\sigma_2 = w_2.v.A.z_2$

  where n is an integer ≥ 1,
  $w_1 \in (0^+.A.\Sigma.0^+.\Sigma)^n.0^+.A.\Sigma.$,
  $w_2 \in (0^+AA0^+A)^n.0^+AA$,
  $u, v \in 0^+$, $z_1 \in \Sigma.(\Sigma \cup \{0, A\})^\omega$, $z_2 \in \Gamma^\omega$, and

$$|u| \neq |v|$$

- $\mathcal{C}_5$ is formed by pairs (σ_1, σ_2) where
  $\sigma_1 = w_1.u.A.b.w.c.A.z_1$
  $\sigma_2 = w_2.v.A.z_2$
  where n is an integer ≥ 1,

  where n is an integer ≥ 1, $w_1 \in (0^+.A.\Sigma.0^+.\Sigma)^n$, $w_2 \in (0^+AA0^+A)^n$,
  $u, v, w \in 0^+$, $b, c \in \Sigma$, $z_1 \in (\Sigma \cup \{0, A\})^\omega$, $z_2 \in \Gamma^\omega$, and

$$|w| \neq |v| + 1$$

- $\mathcal{C}_6$ is formed by pairs (σ_1, σ_2) where
  $\sigma_1 = w_1.u.A.b.w.c.w''.A.z_1$
  $\sigma_2 = w_2.v.AA.w'.Az_2$
  where n is an integer ≥ 1,

  where n is an integer ≥ 1, $w_1 \in (0^+.A.\Sigma.0^+.\Sigma)^n$, $w_2 \in (0^+AA0^+A)^n$,
  $u, v, w, w', w'' \in 0^+$, $b, c \in \Sigma$, $z_1 \in (\Sigma \cup \{0, A\})^\omega$, $z_2 \in \Gamma^\omega$, and

$$|w''| \neq |w'| + 1$$

It is easy to see that for each integer $j \in [1, 6]$, the set $\mathcal{C}_j \subseteq (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$ is an infinitary rational relation. The class $\mathbf{RAT}_\omega$ is closed under finite union thus

$$R_2 = (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega - (h(\Sigma^\omega) \times \{\alpha\}) = \bigcup_{1 \leq j \leq 6} \mathcal{C}_j$$

is an infinitary rational relation. $\square$

End of Proof of Theorem 3.5.
The implication **Det**$(\Sigma^1_1) \Longrightarrow$ **Det**$(\mathbf{RAT}_\omega)$ follows directly from the inclusion $\mathbf{RAT}_\omega \subseteq \Sigma^1_1$.

To prove the reverse implication **Det**$(\mathbf{RAT}_\omega) \Longrightarrow$ **Det**(Σ^1_1), we assume that **Det**$(\mathbf{RAT}_\omega)$ holds and we show that every Gale-Stewart game $G(L)$, where $L \subseteq \Sigma^\omega$ is an ω-language in the class **r**-$\mathbf{BCL}(1)_\omega$ is determined. Then Theorem 3.4 will imply that **Det**(Σ^1_1) also holds.

Let then $L = L(\mathcal{A}) \subseteq \Sigma^\omega$ be an ω-language in the class **r**-$\mathbf{BCL}(1)_\omega$ which is accepted by a real-time 1-counter Büchi automaton $\mathcal{A}$=$(K, \Sigma, \Delta, q_0, F)$.

We shall consider a Gale-Stewart game $G(\mathcal{L})$ where $\mathcal{L} \subseteq (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$, the letters $0, A$ are not in Σ and $\Gamma = \{0, A\}$, and we are going to define a suitable winning set $\mathcal{L}$ accepted by a 2-tape Büchi automaton.

Notice first that in such a game, the players alternatively write letters (a_i, b_i), $i \geq 1$, from the product alphabet $X = (\Sigma \cup \{0, A\}) \times \Gamma$. After ω steps they have produced an ω-word $y \in X^\omega$ where y may be identified with a pair $(y_1, y_2) \in (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$.

Consider now the coding defined above with the function $h : \Sigma^\omega \to (\Sigma \cup \{0, A\})^\omega$, and the ω-word $\alpha \in \Gamma^\omega$. This coding is inspired from a previous one we have used to study the topological complexity of infinitary rational relations [Fin06b, Fin08]. We have here modified this previous coding to get some useful properties for the game we are going to define.

Assume that two players alternatively write letters from the alphabet $X = (\Sigma \cup \{0, A\}) \times \Gamma$ and that they finally produce an ω-word in the form $y = (h(x), \alpha)$ for some $x \in \Sigma^\omega$. We now have the two following properties which will be useful in the sequel.

(1) The letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2.

(2) After a sequence of consecutive letters 0, either on the first component $h(x)$ or on the second component α, the first letter which is not a 0 has always been written by Player 2.

This is due in particular to the following fact: the sequences of letters 0 on the first component $h(x)$ or on the second component α are alternatively of odd and even lengths.

On the other hand we can remark that all ω-words in the form $h(x)$ belong to the ω-language $H \subseteq (\Sigma \cup \{0, A\})^\omega$ defined by:

$$H = [(0^2)^\star.0.A.\Sigma.(0^2)^+.\Sigma]^\omega$$

In a similar way the ω-word α belongs to the ω-language $H' \subseteq \Gamma^\omega$ defined by:

$$H' = [(0^2)^\star.0.AA.(0^2)^+.A]^\omega$$

An important fact is the following property of $H \times H'$ which extends the same property of the set $h(\Sigma^\omega) \times \{\alpha\}$. Assume that two players alternatively write letters from the alphabet $X = (\Sigma \cup \{0, A\}) \times \Gamma$ and that they finally produce an ω-word $y = (y_1, y_2)$ in $H \times H'$ in the following form:

$$y_1 = 0^{n_1}.Ax(1).0^{n_2}.x(2).0^{n_3}.A.x(3).0^{n_4}.x(4).\ldots 0^{n_{2k}}.x(2k).0^{n_{2k+1}}.A.x(2k+1)\ldots$$

$$y_2 = \alpha = 0^{n'_1}.AA.0^{n'_2}.A.0^{n'_3}.AA.0^{n'_4}.A.0^{n'_5}\ldots AA.0^{n'_{2k}}.A.0^{n'_{2k+1}}.AA.0^{n'_{2k+2}}\ldots$$

where for all integers $i \geq 1$, $n_i \geq 1$ (respectively, n'_i) is an odd integer iff i is an odd integer and n_i (respectively, n'_i) is an even integer iff i is an even integer.

Then we have the two following facts:

(1) The letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2.

(2) After a sequence of consecutive letters 0 (either on the first component y_1 or on the second component y_2), the first letter which is not a 0 has always been written by Player 2.

Let now

$$V = \mathrm{Pref}(H) \cap (\Sigma \cup \{0, A\})^\star.0$$

So a finite word over the alphabet $\Sigma \cup \{0, A\}$ is in V iff it is a prefix of some word in H and its last letter is a 0. It is easy to see that the topological closure of H is $\mathrm{Cl}(H) = H \cup V.0^\omega$.

In a similar manner let

$$V' = \mathrm{Pref}(H') \cap (\Gamma)^\star.0$$

So a finite word over the alphabet Γ is in V' iff it is a prefix of some word in H' and its last letter is a 0. It is easy to see that the topological closure of H' is $\mathrm{Cl}(H') = H' \cup V'.0^\omega$.

Notice that an ω-word x in $\mathrm{Cl}(H)$ is not in $h(\Sigma^\omega)$ iff a sequence of consecutive letters 0 in x has not the good length. And an ω-word y in $\mathrm{Cl}(H')$ is not equal to α iff a sequence of consecutive letters 0 in y has not the good length.

Thus if two players alternatively write letters from the alphabet $X = (\Sigma \cup \{0, A\}) \times \Gamma$ and that they finally produce an ω-word in the form $y = (y_1, y_2) \in \mathrm{Cl}(H) \times \mathrm{Cl}(H') - h(\Sigma^\omega) \times \{\alpha\}$ then it is Player 2 who "has gone out" of the *closed* set $h(\Sigma^\omega) \times \{\alpha\}$ at some step of the play. This means that there is an integer $n \geq 1$ such that $y[2n-1] \in \mathrm{Pref}(h(\Sigma^\omega) \times \{\alpha\})$ and $y[2n] \notin \mathrm{Pref}(h(\Sigma^\omega) \times \{\alpha\})$. In a similar way we shall say that, during an infinite play, Player 1 "goes out" of the *closed* set $h(\Sigma^\omega) \times \{\alpha\}$ if the final play y composed by the two players has a prefix $y[2n] \in \mathrm{Pref}(h(\Sigma^\omega) \times \{\alpha\})$ such that $y[2n+1] \notin \mathrm{Pref}(h(\Sigma^\omega) \times \{\alpha\})$. This will be important in the sequel.

From Lemmas 3.6 and 3.7 we know that we can effectively construct a 2-tape Büchi automaton $\mathcal{B}$ such that

$$L(\mathcal{B}) = [h(L(\mathcal{A})) \times \{\alpha\}] \cup [(h(\Sigma^\omega) \times \{\alpha\})^-]$$

On the other hand it is very easy to see that the ω-language H (respectively, H') is regular and to construct a Büchi automaton $\mathcal{H}$ (respectively, $\mathcal{H}'$) accepting it. Therefore one can also construct a 2-tape Büchi automaton $\mathcal{B}'$ such that

$$L(\mathcal{B}') = [h(L(\mathcal{A})) \times \{\alpha\}] \cup [(h(\Sigma^\omega) \times \{\alpha\})^- \cap H \times H']$$

Notice also that $\mathrm{Pref}(H)$ (respectively, $\mathrm{Pref}(H')$) is a regular finitary language since H (respectively, H') is a regular ω-language. Thus the ω-languages $V.0^\omega$ and $V'.0^\omega$ are also regular. Moreover the closure of a regular ω-language is a regular ω-language thus $\mathrm{Cl}(H)$ and $\mathrm{Cl}(H')$ are also regular, and we can construct, from the Büchi automata $\mathcal{H}$ and $\mathcal{H}'$, some other Büchi automata $\mathcal{H}_c$ and $\mathcal{H}'_c$ acccepting the regular ω-languages $\mathrm{Cl}(H)$ and $\mathrm{Cl}(H')$, [PP04]. Thus one can construct a 2-tape Büchi automaton $\mathcal{C}$ such that:

$$L(\mathcal{C}) = [V.0^\omega \times \mathrm{Cl}(H')] \cup [\mathrm{Cl}(H) \times V'.0^\omega]$$

We denote also U the set of finite words u over $X = (\Sigma \cup \{0, A\}) \times \Gamma$ such that $|u| = 2n$ for some integer $n \geq 1$ and $u[2n-1] \in \mathrm{Pref}(H) \times \mathrm{Pref}(H')$ and $u = u[2n] \notin \mathrm{Pref}(H) \times \mathrm{Pref}(H')$. Since the regular languages $\mathrm{Pref}(H)$ and $\mathrm{Pref}(H')$ are accepted by finite automata, one can construct a 2-tape Büchi automaton $\mathcal{C}'$ such that:

$$L(\mathcal{C}') = U.[(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega]$$

Now we set:

$$\mathcal{L} \;=\; L(\mathcal{B}') \;\cup\; L(\mathcal{C}) \;\cup\; L(\mathcal{C}')$$

i.e.

$$\mathcal{L} \;=\; [h(L(\mathcal{A})) \times \{\alpha\}] \;\cup\; [(h(\Sigma^\omega) \times \{\alpha\})^- \cap H \times H'] \;\cup\; L(\mathcal{C}) \;\cup\; L(\mathcal{C}')$$

The class of infinitary rational relations is effectively closed under finite union, thus we can construct a 2-tape Büchi automaton $\mathcal{D}$ such that $\mathcal{L} = L(\mathcal{D})$.

By hypothesis we assume that $\mathbf{Det}(\mathbf{RAT}_\omega)$ holds and thus the game $G(\mathcal{L})$ is determined. We are going to show that this implies that the game $G(L(\mathcal{A}))$ itself is determined.

Assume firstly that Player 1 has a winning strategy F_1 in the game $G(\mathcal{L})$.

If during an infinite play, the two players compose an infinite word $z \in X^\omega$, and Player 2 "does not go out of the set $h(\Sigma^\omega) \times \{\alpha\}$" then we claim that also Player 1, following her strategy F_1, "does not go out of the set $h(\Sigma^\omega) \times \{\alpha\}$". Indeed if Player 1 goes out of this set then due to the above remark this would imply that Player 1 also goes out of the set $\mathrm{Cl}(H) \times \mathrm{Cl}(H')$: there is an integer $n \geq 0$ such that $z[2n] \in \mathrm{Pref}(H \times H')$ but $z[2n+1] \notin \mathrm{Pref}(H \times H')$. So $z \notin h(L(\mathcal{A})) \times \{\alpha\} \;\cup\; [(h(\Sigma^\omega) \times \{\alpha\})^- \cap H \times H'] \;\cup\; L(\mathcal{C})$. Moreover it follows from the definition of U that $z \notin L(\mathcal{C}') = U.[(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega]$. Thus If Player 1 goes out of the set $h(\Sigma^\omega) \times \{\alpha\}$ then she looses the game.

Consider now an infinite play in which Player 2 "does not go out of the set $h(\Sigma^\omega) \times \{\alpha\}$". Then Player 1, following her strategy F_1, "does not go out of the set $h(\Sigma^\omega) \times \{\alpha\}$". Thus the two players write an infinite word $z = (h(x), \alpha)$ for some infinite word $x \in \Sigma^\omega$. But the letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2. Player 1 wins the play iff $x \in L(\mathcal{A})$ and Player 1 wins always the play when she uses her strategy F_1. This implies that Player 1 has also a w.s. in the game $G(L(\mathcal{A}))$.

Assume now that Player 2 has a winning strategy F_2 in the game $G(\mathcal{L})$.

If during an infinite play, the two players compose an infinite word z, and Player 1 "does not go out of the set $h(\Sigma^\omega) \times \{\alpha\}$" then we claim that also Player 2, following his strategy F_2, "does not go out of the set $h(\Sigma^\omega) \times \{\alpha\}$". Indeed if Player 2 goes out of the set $h(\Sigma^\omega) \times \{\alpha\}$ and the final play z remains in $\mathrm{Cl}(H \times H') = \mathrm{Cl}(H) \times \mathrm{Cl}(H')$ then $z \in [(h(\Sigma^\omega) \times \{\alpha\})^- \cap H \times H'] \;\cup\; L(\mathcal{C}) \subseteq \mathcal{L}$ and Player 2 looses. If Player 1 does not go out of the set $\mathrm{Cl}(H \times H')$ and at some step of the play, Player 2 goes out of $\mathrm{Cl}(H) \times \mathrm{Cl}(H')$, i.e. there is an integer $n \geq 1$ such that $z[2n-1] \in \mathrm{Pref}(H) \times \mathrm{Pref}(H')$ and $z[2n] \notin \mathrm{Pref}(H) \times \mathrm{Pref}(H')$, then $z \in U.[(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega] \subseteq \mathcal{L}$ and Player 2 looses.

Assume now that Player 1 "does not go out of the set $h(\Sigma^\omega) \times \{\alpha\}$". Then Player 2 follows his w. s. F_2, and then "never goes out of the set $h(\Sigma^\omega) \times \{\alpha\}$". Thus the two players write an infinite word $z = (h(x), \alpha)$ for some infinite word $x \in \Sigma^\omega$. But the letters $x(2n+1)$, for $n \geq 0$, have been written by Player 1, and the letters $x(2n)$, for $n \geq 1$, have been written by Player 2. Player 2 wins the play iff $x \notin L(\mathcal{A})$ and Player 2 wins always the play when he uses his strategy F_2. This implies that Player 2 has also a w.s. in the game $G(L(\mathcal{A}))$. $\square$

Recall the following effective result cited in [Fin13, remark 3.5] which follows from the proofs of Proposition 3.2 and Theorems 3.3 and 3.4.

Proposition 3.8 *Let $L \subseteq X^\omega$ be an ω-language in the class Σ_1^1, or equivalently in the class $\mathbf{BCL}(2)_\omega$, which is accepted by a Büchi 2-counter automaton $\mathcal{A}$. Then one can effectively construct from $\mathcal{A}$ a real time Büchi 1-counter automaton $\mathcal{B}$ such that the game $G(L)$ is determined if and only if the game $G(L(\mathcal{B}))$ is determined. Moreover Player 1 (respectively, Player 2) has a w.s. in the game $G(L)$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{B}))$.*

We can easily see, from the proofs of Proposition 3.2 and Theorems 3.3 and 3.4 in [Fin13], that we have also the following additional property which strengthens the above one.

Proposition 3.9 *With the same notations as in the above Proposition, if σ is a winning strategy for Player 1 (respectively, Player 2) in the game $G(L)$ then one can construct a w.s. σ' for Player 1 (respectively, Player 2) in the game $G(L(\mathcal{B}))$ such that σ' is recursive in σ. And conversely, if σ is a winning strategy for Player 1 (respectively, Player 2) in the game $G(L(\mathcal{B}))$ then one can construct a w.s. σ' for Player 1 (respectively, Player 2) in the game $G(L)$ such that σ' is recursive in σ.*

Moreover we can easily see, from the proof of the above Theorem 3.5, that we have also the following property.

Proposition 3.10 *Let $\mathcal{A}$ be a real time Büchi 1-counter automaton. Then one can effectively construct from $\mathcal{A}$ a 2-tape Büchi automaton $\mathcal{B}$ such that the game $G(L(\mathcal{A}))$ is determined if and only if the game $G(L(\mathcal{B}))$ is determined. Moreover Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{A}))$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{B}))$ and if σ is a winning strategy for Player 1 (respectively, Player 2) in the game $G(L(\mathcal{A}))$ then one can construct a w.s. σ' for Player 1 (respectively, Player 2) in the game $G(L(\mathcal{B}))$ such that σ' is recursive in σ. And similarly if σ is a winning strategy for Player 1 (respectively, Player 2) in the game $G(L(\mathcal{B}))$ then one can construct a w.s. σ' for Player 1 (respectively, Player 2) in the game $G(L(\mathcal{A}))$ such that σ' is recursive in σ.*

Recall that, assuming that ZFC is consistent, there are some models of ZFC in which **Det**(Σ_1^1) does not hold. Therefore there are some models of ZFC in which some Gale-Stewart games $G(L(\mathcal{A}))$, where $\mathcal{A}$ is a one-counter Büchi automaton or a 2-tape Büchi automaton, are not determined.

Some very natural questions now arise.

Question 1. If we live in a model of ZFC in which **Det**(Σ_1^1) holds, then all Gale-Stewart games $G(L(\mathcal{A}))$, where $\mathcal{A}$ is a one-counter Büchi automaton or a 2-tape Büchi automaton, are determined. Is it then possible to construct the winning strategies in an effective way ?

Question 2. We know from Martin's Theorem that in any model of ZFC the Gale-Stewart Borel games are determined. Is it possible to construct effectively the winning strategies in games $G(L(\mathcal{A}))$, when $L(\mathcal{A})$ is a Borel set, or even a Borel set of low Borel rank ?

We are going to give some answers to these questions. We now firstly recall some basic notions of set theory which will be useful in the sequel, and which are exposed in any textbook on set theory, like [Jec02].

The usual axiomatic system ZFC is Zermelo-Fraenkel system ZF plus the axiom of choice AC. The axioms of ZFC express some natural facts that we consider to hold in the universe of sets. For instance a natural fact is that two sets x and y are equal iff they have the same elements. This is expressed by the *Axiom of Extensionality*:

$$\forall x \forall y\, [\, x = y \leftrightarrow \forall z (z \in x \leftrightarrow z \in y)\,].$$

Another natural axiom is the *Pairing Axiom* which states that for all sets x and y there exists a set $z = \{x, y\}$ whose elements are x and y:

$$\forall x \forall y\, [\, \exists z (\forall w (w \in z \leftrightarrow (w = x \vee w = y)))]$$

Similarly the *Powerset Axiom* states the existence of the set of subsets of a set x. Notice that these axioms are first-order sentences in the usual logical language of set theory whose only non logical symbol is the membership binary relation symbol $\in$. We refer the reader to any textbook on set theory for an exposition of the other axioms of ZFC.

A model (**V**, $\in$) of an arbitrary set of axioms $\mathbb{A}$ is a collection **V** of sets, equipped with the membership relation $\in$, where "$x \in y$" means that the set x is an element of the set y, which satisfies the axioms of $\mathbb{A}$. We often say " the model **V**" instead of "the model (**V**, $\in$)".

We say that two sets A and B have same cardinality iff there is a bijection from A onto B and we denote this by $A \approx B$. The relation $\approx$ is an equivalence relation. Using the axiom of choice AC, one can prove that any set A can be well-ordered and thus there is an ordinal γ such that $A \approx \gamma$. In set theory the cardinal of the set A is then formally defined as the smallest such ordinal γ.

The infinite cardinals are usually denoted by $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\alpha, \ldots$ The cardinal $\aleph_\alpha$ is also denoted by ω_α, when it is considered as an ordinal. The first uncountable ordinal is ω_1, and formally $\aleph_1 = \omega_1$. The ordinal ω_2 is the first ordinal of cardinality greater than $\aleph_1$, and so on.

Let $\mathbf{ON}$ be the class of all ordinals. Recall that an ordinal α is said to be a successor ordinal iff there exists an ordinal β such that $\alpha = \beta + 1$; otherwise the ordinal α is said to be a limit ordinal and in this case $\alpha = \sup\{\beta \in \mathbf{ON} \mid \beta < \alpha\}$.

The class **L** of *constructible sets* in a model **V** of ZF is defined by $\mathbf{L} = \bigcup_{\alpha \in \mathbf{ON}} \mathbf{L}(\alpha)$, where the sets $\mathbf{L}(\alpha)$ are constructed by induction as follows:

(1). $\mathbf{L}(0) = \emptyset$

(2). $\mathbf{L}(\alpha) = \bigcup_{\beta < \alpha} \mathbf{L}(\beta)$, for α a limit ordinal, and

(3). $\mathbf{L}(\alpha{+}1)$ is the set of subsets of $\mathbf{L}(\alpha)$ which are definable from a finite number of elements of $\mathbf{L}(\alpha)$ by a first-order formula relativized to $\mathbf{L}(\alpha)$.

If **V** is a model of ZF and **L** is the class of *constructible sets* of **V**, then the class **L** is a model of ZFC. Notice that the axiom (V=L), which means "every set is constructible", is consistent with ZFC because **L** is a model of ZFC + V=L.

Consider now a model **V** of ZFC and the class of its constructible sets $\mathbf{L} \subseteq \mathbf{V}$ which is another model of ZFC. It is known that the ordinals of **L** are also the ordinals of **V**, but the cardinals in **V** may be different from the cardinals in **L**.

In particular, the first uncountable cardinal in **L** is denoted $\aleph_1^{\mathbf{L}}$, and it is in fact an ordinal of **V** which is denoted $\omega_1^{\mathbf{L}}$. It is well-known that in general this ordinal satisfies the inequality $\omega_1^{\mathbf{L}} \leq \omega_1$. In a model **V** of the axiomatic system ZFC + V=L the equality $\omega_1^{\mathbf{L}} = \omega_1$ holds, but in some other models of ZFC the inequality may be strict and then $\omega_1^{\mathbf{L}} < \omega_1$: notice that in this case $\omega_1^{\mathbf{L}} < \omega_1$ holds because there is actually a bijection from ω onto $\omega_1^{\mathbf{L}}$ in **V** (so $\omega_1^{\mathbf{L}}$ is countable in **V**) but no such bijection exists in the inner model **L** (so $\omega_1^{\mathbf{L}}$ is uncountable in **L**). The construction of such a model is presented in [Jec02, page 202]: one can start from a model **V** of ZFC + V=L and construct by forcing a generic extension **V[G]** in which $\omega_1^{\mathbf{V}}$ is collapsed to ω; in this extension the inequality $\omega_1^{\mathbf{L}} < \omega_1$ holds.

We can now state the following result, which gives an answer to Question 1.

Theorem 3.11 *There exists a real-time* 1*-counter Büchi automaton* $\mathcal{A}$ *and a* 2*-tape Büchi automaton* $\mathcal{B}$ *such that:*

1. *There is a model* V_1 *of* ZFC *in which Player 1 has a winning strategy* σ *in the game* $G(L(\mathcal{A}))$ *(respectively,* $G(L(\mathcal{B}))$*). But* σ *cannot be recursive and not even in the class* $(\Sigma_2^1 \cup \Pi_2^1)$.
2. *There is a model* V_2 *of* ZFC *in which the game* $G(L(\mathcal{A}))$ *(respectively,* $G(L(\mathcal{B}))$*) is not determined.*

Moreover these are the only two possibilities: there are no models of ZFC in which Player 2 has a winning strategy in the game $G(L(\mathcal{A}))$ *(respectively,* $G(L(\mathcal{B}))$*).*

To prove this result, we shall use some set theory, a result of Stern in [Ste82] on coanalytic games, and the Shoenfield Absolutenesss Theorem.

We first recall Stern's result.

Theorem 3.12 (Stern [Ste82]) *For every recursive ordinal* ξ *there exists an effective coanalytic set* $L_\xi \subseteq \omega^\omega$ *such that the Gale-Stewart game* $G(L_\xi)$ *is determined if and only if the ordinal* $\aleph_\xi^{\mathbf{L}}$ *is countable. Moreover if the game* $G(L_\xi)$ *is determined then Player 2 has a winning strategy (and thus Player 1 cannot have a w.s. in this game).*

We also state the following lemmas.

Lemma 3.13 *Let* $L \subseteq \omega^\omega$ *be an effective coanalytic subset of the Baire space. Then there is an effective analytic subset* $L' \subseteq \omega^\omega$ *such that Player 1 (respectively, Player 2) has a w.s. in the game* $G(L)$ *iff Player 2 (respectively, Player 1) has a w.s. in the game* $G(L')$*. In particular, the game* $G(L)$ *is determined iff the game* $G(L')$ *is determined.*

Proof. As noticed for instance in [McA79], we can associate to every effective coanalytic set $L \subseteq \omega^\omega$ the effective analytic set $L' \subseteq \omega^\omega$ which is the complement of the set $L + 1$ defined by:

$$L + 1 = \{x \in \omega^\omega \mid \exists y \; [y \in L \text{ and } \forall n \geq 1 \; x(n+1) = y(n)]\}.$$

It is then easy to see that Player 1 (respectively, Player 2) has a w.s. in the game $G(L)$ iff Player 2 (respectively, Player 1) has a w.s. in the game $G(L')$. □

Lemma 3.14 *Let* $L \subseteq \omega^\omega$ *be an effective analytic subset of the Baire space. Then there exists an effective analytic set* $L' \subseteq \{0,1\}^\omega$ *such that Player 1 (respectively, Player 2) has a w.s. in the game* $G(L)$ *iff Player 1 (respectively, Player 2) has a w.s. in the game* $G(L')$*. In particular, the game* $G(L)$ *is determined iff the game* $G(L')$ *is determined. If* L *is an (effective)* Σ_1^0 *subset of* ω^ω *then the set* L' *can be chosen to be an (arithmetical)* Δ_3^0*-subset of the Cantor space* $\{0,1\}^\omega$*. Moreover if* σ *is a winning strategy for Player 1 (respectively, Player 2) in the game* $G(L)$ *then one can construct a w.s.* σ' *for Player 1 (respectively, Player 2) in the game* $G(L')$ *such that* σ' *is recursive in* σ*. And conversely, if* σ *is a winning strategy for Player 1 (respectively, Player 2) in the game* $G(L')$ *then one can construct a w.s.* σ' *for Player 1 (respectively, Player 2) in the game* $G(L)$ *such that* σ' *is recursive in* σ*.*

Proof. Let $L \subseteq \omega^\omega$ be an effective analytic subset of the Baire space, and let φ be the mapping from the Baire space ω^ω into the Cantor space $\{0,1\}^\omega$ defined by:

$$\varphi((n_i)_{i\geq 1}) = (11)^{n'_1}0(11)^{n'_2}0\ldots(11)^{n'_i}0(11)^{n'_{i+1}}0\ldots$$

where for each integer $i \geq 1$ $n_i \in \omega$ and $n'_i = n_i + 1$.

Notice that $\varphi(\omega^\omega) = [(11)^+.0]^\omega$ is a regular ω-language accepted by a deterministic Büchi automaton, hence it is an arithmetical Π^0_2-subset of $\{0,1\}^\omega$.

We now define the set L' as the union of the following sets D_i, for $1 \leq i \leq 4$:

- $D_1 = \varphi(L)$,
- $D_2 = \{y \mid \exists n, k \geq 0 \;\; y \in [(11)^+.0]^{2n}.(1)^{2k+1}.0.\{0,1\}^\omega\}$,
- $D_3 = \{y \mid \exists n \geq 0 \;\; y \in [(11)^+.0]^{2n+1}.1^\omega\}$,
- $D_4 = \{y \mid \exists n \geq 0 \;\; y \in [(11)^+.0]^{2n+1}.0.\{0,1\}^\omega\}$,

We now explain the meaning of these sets. The first set D_1 codes the set $L \subseteq \omega^\omega$. The other sets D_i, for $2 \leq i \leq 4$ are the results of infinite plays where two players alternatively write letters 0 or 1 and the infinite word written by the players in ω steps is out of the set $\varphi(\omega^\omega)$, *due to the letters written by Player 2*.

Notice first that if the two players alternatively write letters 0 or 1 and the infinite word written by the players in ω steps is in the form

$$\varphi((n_i)_{i\geq 1}) = (11)^{n'_1}0(11)^{n'_2}0\ldots(11)^{n'_i}0(11)^{n'_{i+1}}0\ldots$$

then the letters 0 have been written alternatively by Player 1 and by Player 2 and the writing of these letters 0 determines the integers n'_i and therefore also the integers n_i. Thus the integers n_{2i+1}, $i \geq 0$, have been chosen by Player 1 and the integers n_{2i}, $i \geq 1$, have been chosen by Player 2.

We can now see that D_2 is the set of plays where Player 2 write the $(2n+1)$ th letter 0 while it was Player 1's turn to do this. The set D_3 is the set of plays where Player 2 does not write any letter 0 for the rest of the play when it is his turn to do this. And the set D_4 is the set of plays where Player 2 writes a letter 0 immediately after Player 1 writes a letter 0, while Player 2 should then writes a letter 1 to respect the codes of integers given by the function φ.

Moreover it is easy to see that the mapping φ is a recursive isomorphism between the Baire space ω^ω and its image $\varphi(\omega^\omega) \subseteq \{0,1\}^\omega$ which is an arithmetical Π^0_2-subset of $\{0,1\}^\omega$. And it is easy to see that D_2 and D_4 are ω-regular (arithmetical) Σ^0_1-subsets of $\{0,1\}^\omega$, and that D_3 is an ω-regular (arithmetical) Σ^0_2-subset of $\{0,1\}^\omega$. Therefore this implies the following facts:
(1) If L is a Σ^1_1-subset (respectively, a Δ^1_1-subset, a Σ^0_1-subset) of ω^ω then $\varphi(L)$ is a Σ^1_1-subset (respectively, a Δ^1_1-subset, a Δ^0_3-subset) of $\{0,1\}^\omega$.
(2) If L is a Σ^1_1-subset (respectively, a Δ^1_1-subset, a Σ^0_1-subset) of ω^ω then L' is a Σ^1_1-subset (respectively, a Δ^1_1-subset, a Δ^0_3-subset) of $\{0,1\}^\omega$.

We now prove that Player 1 (respectively, Player 2) has a w.s. in the game $G(L)$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L')$.

Assume firstly that Player 1 has a w.s. F_1 in the game $G(L)$. Consider a play in the game $G(L')$. If the two players alternatively write letters 0 or 1 and the infinite word written by the players in ω steps is in the form

$$\varphi((n_i)_{i\geq 1}) = (11)^{n'_1}0(11)^{n'_2}0\ldots(11)^{n'_i}0(11)^{n'_{i+1}}0\ldots$$

then we have already seen that the integers n'_{2i+1}, $i \geq 0$, have been chosen by Player 1 and the integers n'_{2i}, $i \geq 1$, have been chosen by Player 2, and this is also the case for the corresponding integers n_{2i+1}, $i \geq 0$, and n_{2i}, $i \geq 1$. Thus the game is like a game where each player writes some integer at each step of the play, and Player 1 can apply the strategy F_1 to ensure that $(n_i)_{i\geq 1} \in L$ and this implies that $\varphi((n_i)_{i\geq 1}) \in \varphi(L) \subseteq L'$, so Player 1 wins the play. On the other hand we have seen that if the two players alternatively write letters 0 or 1 and the infinite word x written by the players in ω steps is out of the set $\varphi(\omega^\omega)$, *due to the letters written by Player 2*, then the ω-word x is in $D_2 \cup D_3 \cup D_4$, and thus Player 1 wins also the play. Finally this shows that Player 1 has a w. s. in the game $G(L')$.

Assume now that Player 2 has a winning strategy F_2 in the game $G(L)$.
Consider a play in the game $G(L')$. If the two players alternatively write letters 0 or 1 and the infinite word written by the players in ω steps is in the form

$$\varphi((n_i)_{i\geq 1}) = (11)^{n'_1}0(11)^{n'_2}0\ldots(11)^{n'_i}0(11)^{n'_{i+1}}0\ldots$$

then we have already seen that the integers n'_{2i+1}, $i \geq 0$, have been chosen by Player 1 and the integers n'_{2i}, $i \geq 1$, have been chosen by Player 2, and this is also the case for the corresponding integers n_{2i+1}, $i \geq 0$, and n_{2i}, $i \geq 1$. Thus the game is like a game where each player writes some integer at each step of the play, and Player 2 can apply the strategy F_2 to ensure that $(n_i)_{i\geq 1} \notin L$ and this implies that $\varphi((n_i)_{i\geq 1}) \notin \varphi(L)$, and also $\varphi((n_i)_{i\geq 1}) \notin L'$ because $L' \cap \varphi(\omega^\omega) = \varphi(L)$, so Player 2 wins the play. On the other hand we can easily see that if the two players alternatively write letters 0 or 1 and the infinite word y written by the players in ω steps is out of the set $\varphi(\omega^\omega)$, *due to the letters written by Player 1*, then the ω-word y is not in $D_2 \cup D_3 \cup D_4$, and thus y is not in L' and Player 2 wins also the play. Finally this shows that Player 2 has a w. s. in the game $G(L')$.

Conversely assume now that Player 1 has a w.s. F'_1 in the game $G(L')$. Consider a play in the game $G(L')$ in which Player 2 does not make that the final ω-word x written by the two players is in $D_2 \cup D_3 \cup D_4$. Then Player 1, following the strategy F'_1, must write letters so that the final ω-word x belongs to $\varphi(\omega^\omega)$. Then the game is reduced to the game $G(L)$ in which the two players alternatively write some integers n_i, $i \geq 1$. But Player 1 wins the game and this implies that Player 1 has actually a w.s. in the game $G(L)$.

Assume now that Player 2 has a w.s. F'_2 in the game $G(L')$. By a very similar reasoning as in the preceding case we can see that Player 2 has also a w.s. in the game $G(L)$; details are here left to the reader.

From the construction of the strategies given in the previous paragraphs, it is now easy to see that if F is a winning strategy for Player 1 (respectively, Player 2) in the game $G(L)$ then one can construct a w.s. F' for Player 1 (respectively, Player 2) in the game $G(L')$ such that F' is recursive in F. And conversely, if F' is a winning strategy for Player 1 (respectively, Player 2) in the game $G(L')$ then one can construct a w.s. F for Player 1 (respectively, Player 2) in the game $G(L)$ such that F is recursive in F'. □

We can now give the proof of the above Theorem 3.11.

Proof of Theorem 3.11. We know from Stern's Theorem 3.12 that there exists an effective coanalytic set $L_1 \subseteq \omega^\omega$ such that the Gale-Stewart game $G(L_1)$ is determined if and only if the ordinal $\omega_1^{\mathbf{L}}$ is countable. Moreover if the game $G(L_1)$ is determined then Player 2 has a winning strategy. Then Lemmas 3.13 and 3.14 imply that there exists a effective analytic set $L \subseteq \{0,1\}^\omega$ such that $G(L)$ is determined if and only if the ordinal $\omega_1^{\mathbf{L}}$ is countable. And moreover if the game $G(L)$ is determined then Player 1 has a winning strategy. We can now infer from Propositions 3.8 and 3.10 that there there exists a real-time 1-counter Büchi automaton $\mathcal{A}$, reading words over a finite alphabet X, and a 2-tape Büchi automaton $\mathcal{B}$, reading words over a finite alphabet Y, such that the game $G(L(\mathcal{A}))$ (respectively, $G(L(\mathcal{B}))$) is determined if and only if $\omega_1^{\mathbf{L}}$ is countable. Moreover if the game $G(L(\mathcal{A}))$ (respectively, $G(L(\mathcal{B}))$) is determined then Player 1 has a winning strategy.

Assume now that V_1 is a model of ZFC in which $\omega_1^{\mathbf{L}}$ is countable, i.e. is a model of (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$). Then Player 1 has a winning strategy in the game $G(L(\mathcal{A}))$. This strategy is a mapping $F : (X^2)^\star \to X$ hence it can be coded in a recursive manner by an infinite word $X_F \in \{0,1\}^\omega$ which may be identified with a subset of the set $\mathbb{N}$ of natural numbers. We now claim that this strategy is not constructible, or equivalently that the set $X_F \subseteq \mathbb{N}$ does not belong to the class $\mathbf{L}^{V_1}$ of constructible sets in the model V_1. Recall that a real-time 1-counter Büchi automaton $\mathcal{A}$ has a finite description to which can be associated, in an effective way, a unique natural number called its index, so we have a Gödel numbering of real-time 1-counter Büchi automata. We denote $\mathcal{A}_z$ the real time Büchi 1-counter automaton of index z reading words over X. Then there exists an integer z_0 such that $\mathcal{A} = \mathcal{A}_{z_0}$. If $x \in X^\omega$ is the ω-word written by Player 2 during a play of the game $G(L(\mathcal{A}))$, and Player 1 follows a strategy G, the ω-word $(G \star x) \in X^\omega$ is defined by $(G\star x)(2n) = x(n)$ and $(G\star x)(2n+1) = G((G\star x)[2n])$ for all integers $n \geq 1$ so that $(G\star x)$ is the ω-word composed by the two players during the play. We can now easily see that the sentence: "G is a winning strategy for Player 1 in the game $G(L(\mathcal{A}_z))$" can be expressed by the following Π_2^1-formula $P(z, G) : \quad \forall x \in X^\omega \; [\; (G \star x) \in L(\mathcal{A}_z) \;]$

Recall that $x \in L(\mathcal{A}_z)$ can be expressed by a Σ_1^1-formula (see [Fin09b]). And $(G \star x) \in L(\mathcal{A}_z)$ can be expressed by $\exists y \in X^\omega (y = (G \star x)$ and $y \in L(\mathcal{A}_z))$, which is also a Σ_1^1-formula since $(G \star x)$ is recursive in x and G. Finally the formula $P(z, G)$ is a Π_2^1-formula (with parameters z and G).

Towards a contradiction, assume now that the winning strategy F for Player 1 in the game $G(L(\mathcal{A}))$ belongs to the class $\mathbf{L}^{V_1}$ of constructible sets in the model V_1. The relation $P_F \subseteq \mathbb{N}$ defined by $P_F(z)$ iff $P(z, F)$ is a $\Pi_2^1(F)$-relation, i.e. a relation with is Π_2^1 with parameter F. By Shoenfield's Absoluteness Theorem (see [Jec02, page 490]), the relation $P_F \subseteq \mathbb{N}$ would be absolute for the models $\mathbf{L}^{V_1}$ and V_1 of ZFC. This means that the set $\{z \in \mathbb{N} \mid P_F(z)\}$ would be the same set in the two models $\mathbf{L}^{V_1}$ and V_1. In particular, the integer (z_0) belongs to P_F in the model V_1 since F is a w.s. for Player 1 in the game $G(L(\mathcal{A}))$. This would imply that F is also a w.s. for Player 1 in the game $G(L(\mathcal{A}))$ in the model $\mathbf{L}^{V_1}$. But $\mathbf{L}^{V_1}$ is a model of ZFC + V=L so in this model $\omega_1^{\mathbf{L}} = \omega_1$ holds and the game $G(L(\mathcal{A}))$ is not determined. This contradiction shows that the w.s. F is not constructible in V_1. On the other hand every set $A \subseteq \mathbb{N}$ which is Π_2^1 or Σ_2^1 is constructible, see [Jec02, page 491]. Thus X_F is neither a Π_2^1-set nor a Σ_2^1-set; in particular, the strategy F is not recursive and not even hyperarithmetical, i.e. not Δ_1^1.

The case of the game $G(L(\mathcal{B}))$, for the 2-tape Büchi automaton $\mathcal{B}$, is proved in a similar way. $\square$

Remark 3.15 *The* 1*-counter Büchi automaton* $\mathcal{A}$ *and the* 2*-tape Büchi automaton* $\mathcal{B}$*, given by Theorem 3.11, can be effectively constructed, although the automata might have a great number of states. Indeed the effective coanalytic set* $L_1 \subseteq \omega^\omega$ *such that the Gale-Stewart game* $G(L_1)$ *is determined if and only if the ordinal* $\aleph_1^{\mathbf{L}}$ *is countable is explicitly given by a formula* ψ*. Then the effective analytic set* $L \subseteq \{0,1\}^\omega$ *such that* $G(L)$ *is determined if and only if the ordinal* $\aleph_1^{\mathbf{L}}$ *is countable is also given by a* Σ_1^1*-formula from which on can construct a Büchi Turing machine and thus a 2-counter Büchi automaton accepting it. The constructions given in the proofs of Propositions 3.8 and 3.10 lead then to the effective construction of* $\mathcal{A}$ *and* $\mathcal{B}$*.*

Remark 3.16 *In the above proof of Theorem 3.11 we have not used any large cardinal axiom or even the consistency of such an axiom, like the axiom of analytic determinacy.*

We now prove some lemmas which will be useful later to give some answer to Question 2.

Lemma 3.17 *Let* $L \subseteq \Sigma^\omega$ *be a* Δ_3^0*-subset of a Cantor space, accepted by a Büchi* 2*-counter automaton* $\mathcal{A}$ *and let* $\mathcal{B}$ *be the real time Büchi* 1*-counter automaton which can be effectively constructed from* $\mathcal{A}$ *by Proposition 3.8. Then* $L(\mathcal{B})$ *is also a* Δ_3^0*-subset of a Cantor space* Y^ω *for some finite alphabet* Y *containing* Σ*.*

Proof. We refer now to the proofs of Proposition 3.2 and Theorems 3.3 and 3.4 in [Fin13], and we use here the same notations as in [Fin13].

In the proof of Proposition 3.2 it is firstly proved that, from a Büchi 2-counter automaton $\mathcal{A}$ accepting L, one can construct a real time Büchi 8-counter automaton $\mathcal{A}_3$ accepting $\theta_S(L) \cup L'$, where $\theta_S : \Sigma^\omega \to (\Sigma \cup \{E\})^\omega$ is a function defined, for all $x \in \Sigma^\omega$, by:

$$\theta_S(x) = x(1).E^S.x(2).E^{S^2}.x(3).E^{S^3}.x(4)\ldots x(n).E^{S^n}.x(n+1).E^{S^{n+1}}\ldots$$

It is easy to see that θ_S is a recursive homeomorphism from Σ^ω onto the image $\theta_S(\Sigma^\omega)$ which is a closed subset of the Cantor space $(\Sigma \cup \{E\})^\omega$. It is then easy to se that if L is a Δ_3^0-subset of Σ^ω then $\theta_S(L)$ is also a Δ_3^0-subset $(\Sigma \cup \{E\})^\omega$. Moreover the ω-language L' is defined as the set of ω-words $y \in (\Sigma \cup \{E\})^\omega$ for which there is an integer $n \geq 1$ such that $y[2n-1] \in \mathrm{Pref}(\theta_S(\Sigma^\omega))$ and $y[2n] \notin \mathrm{Pref}(\theta_S(\Sigma^\omega))$. Then it is easy to see that L' is an arithmetical Σ_1^0-subset of $(\Sigma \cup \{E\})^\omega$, and thus the union $\theta_S(L) \cup L'$ is a Δ_3^0-set as the union of two Δ_3^0-sets.

Recall also that Player 1 (respectively, Player 2) has a w.s. in the game $G(L)$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(\theta_S(L) \cup L')$.

In a second step, in the proof of Theorem 3.3, it is proved that, from a real time Büchi 8-counter automaton $\mathcal{A}$ accepting an ω-language $L(\mathcal{A}) \subseteq \Gamma^\omega$, where Γ is a finite alphabet, one can construct a Büchi 1-counter automaton $\mathcal{A}_4$ accepting the ω-language

$$\mathcal{L} \;=\; h(L(\mathcal{A})) \;\cup\; [h(\Gamma^\omega)^- \cap H] \;\cup\; V.C^\omega \;\cup\; U.(\Gamma_1)^\omega$$

Moreover it is proved that Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{A}))$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(\mathcal{L})$.

On the other hand the mapping h is a recursive homeomorphism from Γ^ω onto its image $h(\Gamma^\omega) \subseteq (\Gamma_1)^\omega$ where Γ_1 is the finite alphabet $\Gamma \cup \{A, B, C\}$ and A, B, C, are additional letters not in Γ. It is then easy to see that if $L(\mathcal{A}) \subseteq \Gamma^\omega$ is a Δ_3^0-set then $h(L(\mathcal{A}))$ is a Δ_3^0-subset of $(\Gamma_1)^\omega$. On the other hand the ω-language H is accepted by a deterministic Büchi automaton and hence it is an arithmetical Π_2^0-set, see [PP04, LT94]. Thus $[h(\Gamma^\omega)^- \cap H]$ is also a Π_2^0-set since it is the intersection of a Σ_1^0-set and of a Π_2^0-set. Moreover it is easy to see that $V.C^\omega$ is a Σ_2^0-set since it is accepted by a deterministic automaton with co-Büchi acceptance condition, and that $U.(\Gamma_1)^\omega$ is a Σ_1^0-subset of $(\Gamma_1)^\omega$ since U is regular and hence recursive. Finally this shows that if $L(\mathcal{A}) \subseteq \Gamma^\omega$ is a Δ_3^0-set then $\mathcal{L}$ is a Δ_3^0-subset of $(\Gamma_1)^\omega$.

In a third step, in the proof of Theorem 3.4, it is proved that, from the Büchi 1-counter automaton $\mathcal{A}_4$ accepting the ω-language $\mathcal{L}$, one can construct a *real time* Büchi 1-counter automaton $\mathcal{B}''$ accepting the ω-language $\phi_K(L(\mathcal{A}_4)) \cup L''$. It is easy to see, as in the above first step, that if $\mathcal{L} = L(\mathcal{A}_4)$ is a Δ_3^0-subset of $(\Gamma_1)^\omega$, then the ω-language $\phi_K(L(\mathcal{A}_4)) \cup L''$ is also a Δ_3^0-subset of $(\Gamma_1 \cup \{F\})^\omega$. Moreover Player 1 (respectively, Player 2) has a w.s. in the game $G(\mathcal{L})$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(\phi_K(\mathcal{L}) \cup L'')$. □

Lemma 3.18 *Let $\mathcal{A}$ be a real time Büchi 1-counter automaton accepting a Δ_3^0-set $L \subseteq \Sigma^\omega$ and let $\mathcal{B}$ be the 2-tape Büchi automaton which can be effectively constructed from $\mathcal{A}$ by Proposition 3.10. Then $L(\mathcal{B})$ is a Δ_3^0-subset of the Cantor space $(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$, where $0, A$ are additional letters not in Σ and $\Gamma = \{0, A\}$.*

Proof. We refer now to the proof of the above Theorem 3.5 and we use here the same notations. We showed above that, from a a real-time 1-counter Büchi automaton $\mathcal{A}$ accepting an ω-language $L = L(\mathcal{A}) \subseteq \Sigma^\omega$, we can effectively construct a 2-tape Büchi automaton $\mathcal{D}$ accepting the infinitary rational relation $\mathcal{L} \subseteq (\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$, where the letters $0, A$ are not in Σ and $\Gamma = \{0, A\}$, and

$$\mathcal{L} \ = \ L(\mathcal{B}') \ \cup \ L(\mathcal{C}) \ \cup \ L(\mathcal{C}')$$

where

$$L(\mathcal{B}') = [h(L(\mathcal{A})) \times \{\alpha\}] \cup [(h(\Sigma^\omega) \times \{\alpha\})^- \cap H \times H']$$
$$L(\mathcal{C}) = [V.0^\omega \times \text{Cl}(H')] \ \cup \ [\text{Cl}(H) \times V'.0^\omega]$$
$$L(\mathcal{C}') = U.[(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega]$$

We now assume that $L = L(\mathcal{A})$ is a Δ_3^0-subset of Σ^ω.

It is easy to see that the mapping h is a recursive homeomorphism from Σ^ω onto its image $h(\Sigma^\omega) \subseteq (\Sigma \cup \{0, A\})^\omega$. Moreover α is recursive and $\{\alpha\}$ is a Π_1^0-subset of Γ^ω. Therefore $h(L(\mathcal{A})) \times \{\alpha\}$ is a Δ_3^0-subset of $(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$. On the other hand $(h(\Sigma^\omega) \times \{\alpha\})$ is a Π_1^0-set, and so $(h(\Sigma^\omega) \times \{\alpha\})^-$ is a Σ_1^0-subset of $(\Sigma \cup \{0, A\})^\omega \times \Gamma^\omega$. And it is easy to see that H and H' are accepted by deterministic Büchi automata and thus are (arithmetical) Π_2^0-sets. Thus $[(h(\Sigma^\omega) \times \{\alpha\})^- \cap H \times H']$ is also a Π_2^0-set and finally this shows that $L(\mathcal{B}')$ is a Δ_3^0-set.

The ω-languages H and H' being ω-regular, their closures Cl(H) and Cl(H') are closed and ω-regular and thus they are (arithmetical) Π_1^0-sets (see [PP04, LT94]) . On the other hand the finitary languages V and V' are regular thus $V.0^\omega$ and $V'.0^\omega$ are (arithmetical) Σ_2^0-sets. This implies that $L(\mathcal{C}) = [V.0^\omega \times \text{Cl}(H')] \ \cup \ [\text{Cl}(H) \times V'.0^\omega]$ is also a Δ_3^0-set.

The ω-language $L(\mathcal{C}')$ is an open ω-regular set since the finitary language U is regular. Thus $L(\mathcal{C}')$ is also an (arithmetical) Σ^0_1-set.

Finally the ω-language $\mathcal{L}$ is the union of three Δ^0_3-sets and thus it is also a Δ^0_3-set. $\square$

We can now state the following result which gives an answer to Question 2.

Theorem 3.19 *There exist a real-time* 1*-counter Büchi automaton* $\mathcal{A}$ *and a* 2*-tape Büchi automaton* $\mathcal{B}$ *such that the* ω*-language* $L(\mathcal{A})$ *and the infinitary rational relation* $L(\mathcal{B})$ *are arithmetical* Δ^0_3*-sets and such that Player 2 has a winning strategy in the games* $G(L(\mathcal{A}))$ *and* $G(L(\mathcal{B}))$ *but has no hyperarithmetical winning strategies in these games.*

Proof. It is proved in [Bla72, Theorem 3] that there exists an arithmetical Σ^0_1-set $L \subseteq \omega^\omega$ such that Player 2 has a winning strategy in the game $G(L)$ but has no hyperarithmetical winning strategies in this game. Using Lemmas 3.14, 3.17, 3.18, we see that one can construct a real-time 1-counter Büchi automaton $\mathcal{A}$ and a 2-tape Büchi automaton $\mathcal{B}$ such that the ω-language $L(\mathcal{A})$ and the infinitary rational relation $L(\mathcal{B})$ are arithmetical Δ^0_3-sets and such that Player 2 has a winning strategy in the games $G(L(\mathcal{A}))$ and $G(L(\mathcal{B}))$.

Moreover, by Propositions 3.9 and 3.10, if F was an hyperarithmetical winning strategy for Player 2 in the game $G(L(\mathcal{A}))$ or $G(L(\mathcal{B}))$ then there would exist a winning strategy T for Player 2 in the game $G(L)$ which would be recursive in F and thus also hyperarithmetical. This implies that F can not be hyperarithmetical since Player 2 has no hyperarithmetical winning strategies in the game $G(L)$. $\square$

The above negative results given by Theorems 3.11 and 3.19 show that one cannot effectively construct winning strategies in Gale-Stewart games with winning sets accepted by 1-counter Büchi automata or 2-tape Büchi automata. We are going to see that, even when we know that the games are determined, one cannot determine the winner of such games.

Theorem 3.20 *There exists a recursive sequence of real time* 1*-counter Büchi automata* $\mathcal{A}_n$*, (respectively, of* 2*-tape Büchi automata* $\mathcal{B}_n$*),* $n \geq 1$*, such that all games* $G(L(\mathcal{A}_n))$ *(respectively,* $G(L(\mathcal{B}_n))$*) are determined. But it is* Π^1_2*-complete (hence highly undecidable) to determine whether Player 1 has a winning strategy in the game* $G(L(\mathcal{A}_n))$ *(respectively,* $G(L(\mathcal{B}_n))$*).*

Proof. We first define the following operation on ω-languages. For $x, x' \in \Sigma^\omega$ the ω-word $x \otimes x'$ is defined by : for every integer $n \geq 1$ $(x \otimes x')(2n-1) = x(n)$ and $(x \otimes x')(2n) = x'(n)$. For two ω-languages $L, L' \subseteq \Sigma^\omega$, the ω-language $L \otimes L'$ is defined by $L \otimes L' = \{x \otimes x' \mid x \in L \text{ and } x' \in L'\}$. Let now $\Sigma = \{0, 1\}$ and let T_n be the Büchi Turing machine of index n reading ω-words over the alphabet Σ. Let also $\mathcal{T}_n$ be a Büchi Turing machine constructed from T_n such that $L(\mathcal{T}_n) = \Sigma^\omega \otimes L(T_n)$. Notice that $\mathcal{T}_n$ can easily be constructed in a recursive manner from T_n, and that on can also construct some Büchi 2-counter automata $\mathcal{C}_n$ such that $L(\mathcal{T}_n) = L(\mathcal{C}_n)$.

Consider now the game $G(L(\mathcal{C}_n))$. It is easy to see that this game is always determined. Indeed if $L(T_n) = \Sigma^\omega$ then Player 1 always wins the play so Player 1 has an obvious winning strategy. And if $L(T_n) \neq \Sigma^\omega$ then Player 2 can win by playing an ω-word not in $L(T_n)$ so that the final ω-word written by the two players will be outside $L(\mathcal{C}_n) = \Sigma^\omega \otimes L(T_n)$. Recall now that Castro and Cucker proved in [CC89] that it is Π^1_2-complete (hence highly undecidable) to determine whether $L(T_n) = \Sigma^\omega$. Thus it is Π_2^1-complete (hence highly undecidable) to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{C}_n))$.

Using the constructions we made in the proofs of Theorems 3.4 and 3.5 and Propositions 3.8 and 3.10, we can effectively construct from $\mathcal{C}_n$ a real time Büchi 1-counter automaton $\mathcal{A}_n$ and a 2-tape Büchi automaton $\mathcal{B}_n$ such that Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{C}_n))$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{A}_n))$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{B}_n))$. This implies that it is Π_2^1-complete (hence highly undecidable) to determine whether Player 1 has a winning strategy in the game $G(L(\mathcal{A}_n))$ (respectively, $G(L(\mathcal{B}_n))$). □

We now consider the strength of determinacy of a game $G(L(\mathcal{A}))$, where $\mathcal{A}$ is a Büchi 1-counter automaton or a 2-tape Büchi automaton. We first recall that there exists some effective analytic set $L_\sharp \subseteq \{0,1\}^\omega$ such that the determinacy of the game $G(L_\sharp)$ is equivalent to the effective analytic determinacy, i.e. to the determinacy of all effective analytic Gale-Stewart games: a first example was given by Harrington in [Har78], Stern gave another one in [Ste82]. We can now infer from this result a similar one for games specified by automata.

Theorem 3.21 *There exists a real time 1-counter Büchi automaton $A_\sharp$ (respectively, a 2-tape Büchi automaton $B_\sharp$) such that the game $G(A_\sharp)$ (respectively, the game $G(B_\sharp)$) is determined iff the effective analytic determinacy holds iff all 1-counter games are determined iff all games specified by 2-tape Büchi automata are determined.*

Proof. The effective analytic set $L_\sharp \subseteq \{0,1\}^\omega$ is defined by a Σ_1^1-formula from which one can construct a Büchi Turing machine and a 2-counter Büchi automaton $\mathcal{C}_\sharp$ accepting it. Using the constructions we made in the proofs of Theorems 3.4 and 3.5, we can effectively construct from $\mathcal{C}_\sharp$ a real time Büchi 1-counter automaton $\mathcal{A}_\sharp$ and a 2-tape Büchi automaton $\mathcal{B}_\sharp$ such that the game $G(L(\mathcal{C}_\sharp))$ is determined iff the game $G(L(\mathcal{A}_\sharp))$ is determined iff the game $G(L(\mathcal{B}_\sharp))$ is determined. □

This shows that there exists a real time 1-counter Büchi automaton $A_\sharp$ (respectively, a 2-tape Büchi automaton $B_\sharp$) such that the determinacy strength of the game $G(L(\mathcal{A}_\sharp))$ (respectively, $G(L(\mathcal{B}_\sharp)))$ is the strongest possible. Then the following question naturally arises.

Question 3. Are there many different strengths of determinacy for games specified by 1-counter Büchi automata (respectively, by 2-tape Büchi automata) ?

We now give a positive answer to this question, stating the following result. Notice that below $Det(G(L))$ means "the game $G(L)$ is determined". We recall that ω_1^{CK} is the Church-Kleene ordinal, which is the first non-recursive ordinal.

Theorem 3.22 *There is a transfinite sequence of real-time 1-counter Büchi automata $(\mathcal{A}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$, (respectively, of 2-tape Büchi automata $(\mathcal{B}_\alpha)_{\alpha<\omega_1^{\mathrm{CK}}}$), indexed by recursive ordinals, s.t.:*

$$\forall \alpha < \beta < \omega_1^{\mathrm{CK}} \;\; [\, Det(G(L(\mathcal{A}_\beta))) \Longrightarrow Det(G(L(\mathcal{A}_\alpha))) \,]$$

$$\forall \alpha < \beta < \omega_1^{\mathrm{CK}} \;\; [\, Det(G(L(\mathcal{B}_\beta))) \Longrightarrow Det(G(L(\mathcal{B}_\alpha))) \,]$$

but the converse is not true:

For each recursive ordinal α there is a model $\mathbf{V}_\alpha$ of ZFC such that in this model the game $G(L(\mathcal{A}_\beta))$ (respectively, $G(L(\mathcal{B}_\beta))$) is determined iff $\beta < \alpha$.

Proof. It follows from Stern's Theorem 3.12 and from Lemmas 3.13 and 3.14 that for each recursive ordinal ξ there exists an effective analytic set $L_\xi \subseteq \{0,1\}^\omega$ such that the game $G(L_\xi)$ is determined if and only if the ordinal $\aleph_\xi^{\mathbf{L}}$ is countable. Notice that each set L_ξ is accepted by a Büchi Turing machine T_ξ and by a 2-counter Büchi automaton $\mathcal{C}_\xi$.

Using the constructions we made in the proofs of Theorems 3.4 and 3.5 and Propositions 3.8 and 3.10, we can construct from $\mathcal{C}_\xi$ a real time Büchi 1-counter automaton $\mathcal{A}'_\xi$ and a 2-tape Büchi automaton $\mathcal{B}'_\xi$ such that Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{C}_\xi))$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{A}'_\xi))$ iff Player 1 (respectively, Player 2) has a w.s. in the game $G(L(\mathcal{B}'_\xi))$. Thus the game $G(L(\mathcal{A}'_\xi))$ is determined if and only if the game $G(L(\mathcal{B}'_\xi))$ is determined if and only if the ordinal $\aleph_\xi^{\mathbf{L}}$ is countable. We set $\mathcal{A}_\xi = \mathcal{A}'_{\xi+1}$ and $\mathcal{B}_\xi = \mathcal{B}'_{\xi+1}$.

The first part of the theorem follows easily from the obvious implication [$\aleph_\xi^{\mathbf{L}}$ is countable] $\Longrightarrow$ [$\aleph_\alpha^{\mathbf{L}}$ is countable, for all ordinals $\alpha < \xi$].

Let now α be a recursive ordinal and $\mathbf{V}$ be a model of ZFC + V=L. The cardinal $\aleph_{\alpha+1}$ in $\mathbf{V}$ is a successor cardinal hence also a regular cardinal (the reader may find these notions in any textbook of set theory like [Kun80] or [Jec02]). One can then construct from the model $\mathbf{V}$, using a forcing method due to Lévy, a generic extension $\mathbf{V}_\alpha$ of $\mathbf{V}$ which is another model of ZFC in which the cardinal $\aleph_{\alpha+1}$ has been "collapsed" in such a way that in the new model $\aleph_{\alpha+1}$ becomes $\omega_1^{\mathbf{V}_\alpha}$. Notice that the two models have the same ordinals, and the above sentence means that the ordinal of $\mathbf{V}$ which plays the role of $\aleph_{\alpha+1}$ in $\mathbf{V}$ plays the role of the cardinal $\aleph_1$ in $\mathbf{V}_\alpha$ (we refer the reader to [Kun80, page 231] for more details about Lévy's forcing).

Another crucial point here is that the two models $\mathbf{V}$ and $\mathbf{V}_\alpha$ have the same constructible sets (this is always true for generic extensions obtained by the method of forcing), i.e. $\mathbf{L}^{\mathbf{V}} = \mathbf{L}^{\mathbf{V}_\alpha}$. Notice also that $\aleph_{\alpha+1}^{\mathbf{L}} = \aleph_{\alpha+1}$ since $\mathbf{V}$ is a model of ZFC + V=L. For a recursive ordinal β, we have now the following equivalences:

[$\aleph_{\beta+1}^{\mathbf{L}}$ is countable in $\mathbf{V}_\alpha$] $\Longleftrightarrow$ [$\aleph_{\beta+1}^{\mathbf{L}} < \omega_1^{\mathbf{V}_\alpha} = \aleph_{\alpha+1}^{\mathbf{L}}$] $\Longleftrightarrow \beta + 1 < \alpha + 1 \Longleftrightarrow \beta < \alpha$

And thus $G(L(\mathcal{A}_\beta))$ (respectively, $G(L(\mathcal{B}_\beta))$ is determined in the model $\mathbf{V}_\alpha$ if and only if $\beta < \alpha$. □

Remark 3.23 *We can add the real time 1-counter Büchi automaton $A_\sharp$ and the 2-tape Büchi automaton $B_\sharp$ to the sequences given by Theorem 3.22. The determinacy of $G(L(\mathcal{A}_\sharp))$ (respectively, $G(L(\mathcal{B}_\sharp))$) implies the determinacy of all games $G(L(\mathcal{A}_\alpha))$ (respectively, $G(L(\mathcal{B}_\alpha))$, $\alpha < \omega_1^{\mathrm{CK}}$, but the converse is not true. Then we get a transfinite sequence of real time 1-counter Büchi automata (respectively, of 2-tape Büchi automata) of length $\omega_1^{\mathrm{CK}} + 1$.*

Remark 3.24 *One can actually see from [McA79] that the situation is even more complicated. Indeed Mc Aloon proved that there exists some analytic game whose determinacy is equivalent to the fact that the first inaccessible cardinal in the constructible universe $\mathbf{L}$ of a model $\mathbf{V}$ of ZFC is countable in $\mathbf{V}$. And this property implies that $\aleph_\alpha^{\mathbf{L}}$, for a recursive ordinal α, is countable in $\mathbf{V}$, but does not imply the existence of $0^\sharp$. We refer the interested reader to [Jec02] for the notion of inaccessible cardinals and of other large cardinals, and to [McA79] for more results of this kind.*

4 Wadge games between 2-tape automata

The now called Wadge games have been firstly considered by Wadge to study the notion of reduction of Borel sets by continuous functions. We firstly recall the notion of Wadge reducibility; notice that we give the definition in the case of ω-languages over *finite* alphabets since we have only to consider this case in the sequel.

Definition 4.1 (Wadge [Wad83]) *Let X, Y be two finite alphabets. For $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, L is said to be Wadge reducible to L' ($L \leq_W L'$) iff there exists a continuous function $f : X^\omega \rightarrow Y^\omega$, such that $L = f^{-1}(L')$. L and L' are Wadge equivalent iff $L \leq_W L'$ and $L' \leq_W L$. This will be denoted by $L \equiv_W L'$. And we shall say that $L <_W L'$ iff $L \leq_W L'$ but not $L' \leq_W L$.*
The relation $\leq_W$ is reflexive and transitive, and $\equiv_W$ is an equivalence relation.
The equivalence classes of $\equiv_W$ are called Wadge degrees.

We now recall the definition of Wadge games.

Definition 4.2 (Wadge [Wad83]) *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$. The Wadge game $W(L, L')$ is a game with perfect information between two players, Player 1 who is in charge of L and Player 2 who is in charge of L'. Player 1 first writes a letter $a_1 \in X$, then Player 2 writes a letter $b_1 \in Y$, then Player 1 writes a letter $a_2 \in X$, and so on. The two players alternatively write letters a_n of X for Player 1 and b_n of Y for Player 2. After ω steps, Player 1 has written an ω-word $a \in X^\omega$ and Player 2 has written an ω-word $b \in Y^\omega$. Player 2 is allowed to skip, even infinitely often, provided he really writes an ω-word in ω steps. Player 2 wins the play iff $[a \in L \leftrightarrow b \in L']$, i.e. iff: $[(a \in L$ and $b \in L')$ or $(a \notin L$ and $b \notin L'$ and b is infinite$)]$.*

Recall that a strategy for Player 1 is a function $\sigma : (Y \cup \{s\})^\star \rightarrow X$. And a strategy for Player 2 is a function $f : X^+ \rightarrow Y \cup \{s\}$. The strategy σ is a winning strategy for Player 1 iff she always wins a play when she uses the strategy σ, i.e. when the n^{th} letter she writes is given by $a_n = \sigma(b_1 \ldots b_{n-1})$, where b_i is the letter written by Player 2 at step i and $b_i = s$ if Player 2 skips at step i. A winning strategy for Player 2 is defined in a similar manner.

The game $W(L, L')$ is said to be determined if one of the two players has a winning strategy. In the sequel we shall denote **W-Det**($\mathcal{C}$), where $\mathcal{C}$ is a class of ω-languages, the sentence: "All Wadge games $W(L, L')$, where $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ are ω-languages in the class $\mathcal{C}$, are determined".

Recall that the determinacy of Borel Gale-Stewart games implies easily the determinacy of Wadge games $W(L, L')$, where $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ are Borel ω-languages. Thus it follows from Martin's Theorem that these Wadge games are determined. We also recall that the determinacy of effective analytic Gale-Stewart games is equivalent to the determinacy of effective analytic Wadge games, i.e. **Det**(Σ^1_1) $\Longleftrightarrow$ **W-Det**(Σ^1_1), see [LSR88].

The close relationship between Wadge reducibility and Wadge games is given by the following theorem.

Theorem 4.3 (Wadge) *Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$ where X and Y are finite alphabets. Then $L \leq_W L'$ if and only if Player 2 has a winning strategy in the Wadge game $W(L, L')$.*

The Wadge hierarchy WH is the class of Borel subsets of a set X^ω, where X is a finite set, equipped with $\leq_W$ and with $\equiv_W$. Using Wadge games, Wadge proved that, up to the complement and $\equiv_W$, it is a well ordered hierarchy which provides a great refinement of the Borel hierarchy.

Theorem 4.4 (Wadge) *The class of Borel subsets of X^ω, for a finite alphabet X, equipped with $\leq_W$, is a well ordered hierarchy. There is an ordinal $|WH|$, called the length of the hierarchy, and a map d_W^0 from WH onto $|WH| - \{0\}$, such that for all $L, L' \subseteq X^\omega$:*
$d_W^0 L < d_W^0 L' \leftrightarrow L <_W L'$ and
$d_W^0 L = d_W^0 L' \leftrightarrow [L \equiv_W L'$ or $L \equiv_W L'^-]$.

We proved in [Fin13] the following result on the determinacy of Wadge games between two players in charge of ω-languages of one-counter automata.

Theorem 4.5 **Det**(Σ_1^1) $\Longleftrightarrow$ **W-Det**($\mathbf{r}$-$\mathbf{BCL}(1)_\omega$).

Using this result we are now going to prove the following one on determinacy of Wadge games between two players in charge of ω-languages accepted by 2-tape Büchi automata.

Theorem 4.6 **Det**(Σ_1^1) $\Longleftrightarrow$ **W-Det**($\mathbf{RAT}_\omega$).

In order to prove this theorem, we first recall the notion of operation of sum of sets of infinite words which has as counterpart the ordinal addition over Wadge degrees, and which will useful later.

Definition 4.7 (Wadge) *Assume that $X \subseteq Y$ are two finite alphabets, $Y - X$ containing at least two elements, and that $\{X_+, X_-\}$ is a partition of $Y - X$ in two non empty sets. Let $L \subseteq X^\omega$ and $L' \subseteq Y^\omega$, then*

$$L' + L =_{df} L \cup \{u.a.\beta \mid u \in X^\star,\ (a \in X_+ \text{ and } \beta \in L') \text{ or } (a \in X_- \text{ and } \beta \in L'^-)\}$$

Notice that a player in charge of a set $L' + L$ in a Wadge game is like a player in charge of the set L but who can, at any step of the play, erase his previous play and choose to be this time in charge of L' or of L'^-. But he can do this only one time during a play. This property will be used below.

We now recall the following lemma, proved in [Fin13].

Lemma 4.8 *Let $L \subseteq \Sigma^\omega$ be an analytic but non Borel set. Then it holds that $L \equiv_W \emptyset + L$.*

Notice that in this lemma, $\emptyset$ is viewed as the empty set over an alphabet Γ such that $\Sigma \subseteq \Gamma$ and cardinal $(\Gamma - \Sigma) \geq 2$. Recall also that the emptyset and the whole set Γ^ω are located at the first level of the Wadge hierarchy and that their Wadge degree is equal to 1.

proof of Theorem 4.6.

The implication **Det**(Σ_1^1) $\Longrightarrow$**W-Det**($\mathbf{RAT}_\omega$) is obvious since **Det**(Σ_1^1) is known to be equivalent to **W-Det**(Σ_1^1) and $\mathbf{RAT}_\omega \subseteq \Sigma_1^1$.

To prove the reverse implication, we assume that **W-Det**($\mathbf{RAT}_\omega$) holds and we are going to show that every Wadge game $W(L, L')$, where $L \subseteq (\Sigma_1)^\omega$ and $L' \subseteq (\Sigma_2)^\omega$ are ω-languages in the class $\mathbf{r}$-$\mathbf{BCL}(1)_\omega$, is determined. Then this will imply that **Det**(Σ_1^1) holds by Theorem 4.5. Notice that if the two ω-languages are Borel we already know that the game $W(L, L')$ is determined; thus we have only to consider the case where at least one of these languages is non-Borel.

We now assume that the letters 0 and A do not belong to the alphabets Σ_1 and Σ_2, and recall that we have used in the proof of Theorem 3.5 a mapping $h_1 : (\Sigma_1)^\omega \to (\Sigma_1 \cup \{0, A\})^\omega$ to code ω-words over Σ_1 by ω-words over $\Sigma_1 \cup \{0, A\}$; and we can define similarly $h_2 : (\Sigma_2)^\omega \to (\Sigma_2 \cup \{0, A\})^\omega$. Recall also that we have defined an ω-word $\alpha \in \{0, A\}^\omega = \Gamma^\omega$.

It follows from Lemmas 3.6 and 3.7 that one can effectively construct, from real-time Büchi 1-counter automata $\mathcal{A}_1$ and $\mathcal{A}_2$ accepting L and L', some 2-tape Büchi automata $\mathcal{B}_1$ and $\mathcal{B}_2$ accepting the ω-languages

$$\mathcal{L}_1 = [h_1(L) \times \{\alpha\}] \cup [h_1(\Sigma_1^\omega) \times \{\alpha\}]^-$$

and

$$\mathcal{L}_2 = [h_2(L') \times \{\alpha\}] \cup [h_2(\Sigma_2^\omega) \times \{\alpha\}]^-$$

Then the Wadge game $W(\mathcal{L}_1, \mathcal{L}_2)$ is determined. We consider now the two following cases:

First case. Player 2 has a w.s. in the game $W(\mathcal{L}_1, \mathcal{L}_2)$.

If L' is Borel then $h_2(L') \times \{\alpha\}$ is easily seen to be Borel and then $\mathcal{L}_2$ is also Borel since $h_2(\Sigma_2^\omega) \times \{\alpha\}$ is a closed set and hence $[h_2(\Sigma_2^\omega) \times \{\alpha\}]^-$ is an open set. Then $\mathcal{L}_1$ is also Borel because $\mathcal{L}_1 \leq_W \mathcal{L}_2$ and thus L is also Borel and the game $W(L, L')$ is determined.

Assume now that L' is not Borel, and consider the Wadge game $W(L, \emptyset + L')$.

We claim that Player 2 has a w.s. in that game which is easily deduced from a w.s. of Player 2 in the Wadge game $W(\mathcal{L}_1, \mathcal{L}_2)$. Consider a play in this latter game where Player 1 remains in the closed set $h_1(\Sigma_1^\omega) \times \{\alpha\}$: she writes a beginning of a word in the form

$$(0.Ax(1).0^2.x(2).0^3.A.x(3)\ldots 0^{2n}.x(2n).0^{2n+1}\ldots\ ;\ 0.AA.0^2.A.0^3.AA.\ldots AA.0^{2n}.A.0^{2n+1}\ldots)$$

Then player 2 writes a beginning of a word in the form

$$(0.Ax'(1).0^2.x'(2).0^3.A.x'(3)\ldots 0^{2p}.x'(2p).0^{2p+1}\ldots\ ;\ 0.AA.0^2.A.0^3.AA.\ldots AA.0^{2p}.A.0^{2p+1}\ldots)$$

where $p \leq n$. Then the strategy for Player 2 in $W(L, \emptyset + L')$ consists to write $x'(1).x'(2)\ldots x'(p)$. when Player 1 writes $x(1).x(2)\ldots x(n)$.. If the strategy for Player 2 in $W(\mathcal{L}_1, \mathcal{L}_2)$ was at some step to go out of the set $h_2(\Sigma_2^\omega) \times \{\alpha\}$ then this means that his final word is *surely inside* $\mathcal{L}_2$, and that the final word of Player 1 is also surely inside $\mathcal{L}_1$, because Player 2 wins the play. Then Player 2 in the Wadge game $W(L, \emptyset + L')$ can make as he is now in charge of the wholeset and play anything (without skipping anymore) so that his final ω-word is also inside $\emptyset + L'$. So we have proved that Player 2 has a w.s. in the Wadge game $W(L, \emptyset + L')$ or equivalently that $L \leq_W \emptyset + L'$. But by Lemma 4.8 we know that $L' \equiv_W \emptyset + L'$ and thus $L \leq_W L'$ which means that Player 2 has a w.s. in the Wadge game $W(L, L')$.

Second case. Player 1 has a w.s. in the game $W(\mathcal{L}_1, \mathcal{L}_2)$.

Notice that this implies that $\mathcal{L}_2 \leq_W \mathcal{L}_1^-$. Thus if L is Borel then $\mathcal{L}_1$ is Borel, $\mathcal{L}_1^-$ is also Borel, and $\mathcal{L}_2$ is Borel as the inverse image of a Borel set by a continuous function, and thus L' is also Borel, so the Wadge game $W(L, L')$ is determined. We assume now that L is not Borel and we consider the Wadge game $W(L, L')$. Player 1 has a w.s. in this game which is easily constructed from a w.s. of the same player in the game $W(\mathcal{L}_1, \mathcal{L}_2)$ as follows. For this consider a play in this latter game where Player 2 does not go out of the closed set $h_2(\Sigma_2^\omega) \times \{\alpha\}$. Then player 2 writes a beginning of a word in the form

$$(0.Ax'(1).0^2.x'(2).0^3.A.x'(3)\ldots 0^{2p}.x'(2p).0^{2p+1}\ldots\ ;\ 0.AA.0^2.A.0^3.AA.\ldots AA.0^{2p}.A.0^{2p+1}\ldots)$$

Player 1, following her w.s. composes a beginning of a word in the form

$$(0.Ax(1).0^2.x(2).0^3.A.x(3)\ldots 0^{2n}.x(2n).0^{2n+1}\ldots\ ;\ 0.AA.0^2.A.0^3.AA.\ldots AA.0^{2n}.A.0^{2n+1}\ldots)$$

where $p \leq n$. Then the strategy for Player 1 in $W(L, L')$ consists to write $x(1).x(2)\ldots x(n)$ when Player 2 writes $x'(1).x'(2)\ldots x'(p)$.

If the strategy for Player 1 in $W(\mathcal{L}_1, \mathcal{L}_2)$ was at some step to go out of the closed set $h_1(\Sigma_1^\omega) \times \{\alpha\}$ then this means that her final word is surely inside $\mathcal{L}_1$, and that the final word of Player 2 is also surely outside the set $\mathcal{L}_2$ (at least if he produces really an infinite word in ω steps). This case is actually not possible because Player 2 can always go out of the closed set $h_2(\Sigma_2^\omega) \times \{\alpha\}$ and then his final word is surely in the set $\mathcal{L}_2$.

We have then proved that Player 1 has a w.s. in the Wadge game $W(L, L')$. □

In order to prove our next result we recall that the following result was proved in [Fin09a].

Theorem 4.9 *There exists a 2-tape Büchi automaton $\mathcal{A}$, which can be effectively constructed, such that the topological complexity of the infinitary rational relation $L(\mathcal{A})$ is not determined by the axiomatic system* ZFC. *Indeed it holds that :*

(1) (ZFC + V=L*).* *The ω-language $L(\mathcal{A})$ is an analytic but non-Borel set.*

(2) (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$*).* *The ω-language $L(\mathcal{A})$ is a $\mathbf{\Pi}_2^0$-set.*

We now state the following new result.

Theorem 4.10 *Let $\mathcal{B}$ be a Büchi automaton accepting the regular ω-language $(0^\star.1)^\omega \subseteq \{0,1\}^\omega$. Then one can effectively construct a 2-tape Büchi automaton $\mathcal{A}$ such that:*

(1) (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$*). Player 2 has a winning strategy F in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$. But F can not be recursive and not even in the class $(\Sigma_2^1 \cup \Pi_2^1)$.*

(2) (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$*). The Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$ is not determined.*

Proof. It is very similar to the proof of [Fin13, Theorem 4.12], replacing "1-counter automaton" by "2-tape Büchi automaton" and using the above Theorem 4.9 instead of the corresponding result for a real-time 1-counter automaton proved in [Fin09a]. In the proof we use in particular the above Theorem 4.9, the link between Wadge games and Wadge reducibility, the $\mathbf{\Pi}_2^0$-completeness of the regular ω-language $(0^\star.1)^\omega \subseteq \{0,1\}^\omega$, the Shoenfield's Absoluteness Theorem, and the notion of extensions of a model of ZFC. □

Notice that every model of ZFC is either a model of (ZFC + $\omega_1^{\mathbf{L}} < \omega_1$) or a model of (ZFC + $\omega_1^{\mathbf{L}} = \omega_1$). Thus there are no models of ZFC in which Player 1 has a winning strategy in the Wadge game $W(L(\mathcal{A}), L(\mathcal{B}))$.

Notice also that, to prove Theorems 4.9 and 4.10, we do not need to use any large cardinal axiom or even the consistency of such an axiom, like the axiom of analytic determinacy.

5 Concluding remarks

We have proved that the determinacy of Gale-Stewart games whose winning sets are accepted by *non-deterministic* 2-tape Büchi automata is equivalent to the determinacy of (effective) analytic Gale-Stewart games which is known to be a large cardinal assumption equivalent to the existence of the real $0^\sharp$. Then we have proved that the winning strategies in these games, when they exist, may be very complex, i.e. highly non-effective. Moreover we have proved that, even if we know that some of these games are determined, it may be highly undecidable to determine whether Player 1 has a winning strategy.

On the other hand, we know that the infinitary rational relations accepted by *deterministic* 2-tape Büchi automata are always Borel $\mathbf{\Delta}^0_3$-sets. Thus this implies that Gale-Stewart games whose winning sets are accepted by *deterministic* 2-tape Büchi automata are always determined. It would be interesting to study these games for which the following questions naturally arises: can we decide who the winner is in such a game? can we compute a winning strategy given by a transducer?

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