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Abstract

In this paper, we study general linear programs in which right handsides are interval numbers. This model is relevant when uncertain and inaccurate factors make difficult the assignment of a single value to each right handside. When objective function coefficients are interval numbers in a linear program, it is used to determine optimal solutions according to classical criteria coming from decision theory (like the worst case criterion). When the feasible solutions set is uncertain, another approach consists in determining the worst and best optimum solutions. We study the complexity of these two optimization problems when each right handside is an interval number. Moreover, we analysis the relationship between these two problems and the classical approach coming from decision theory. We exhibit some duality relation between the worst optimum solution problem and the best optimum solution problem in the dual. This study highlights some duality property in robustness analysis.

Keywords: linear programming, interval right handside, robustness analysis, worst case criteria, maximum regret criteria, complexity theory.

1 Problem statement

In optimization, it is used to deal with uncertain and inaccurate factors which make difficult the assignment of a single plausible value to each model parameters. Two approaches are possible: in the first one, a single nominal value is assigned to each parameter, the corresponding optimal solution is computed, then the interval in which each parameter can vary in order to preserve optimality solution is determined; the second approach consists in taking into account in the model to optimize, the possible variations of each parameter. In mathematical programming, the first approach is known as sensibility analysis (see e.g. [7]). For the second approach, stochastic optimization may be applied for some problems in which parameters value can be described by probability laws (see for example [5]). When it is not possible nor relevant to associate probability laws to parameters, another way amounts to assign a set of possible values to each parameter. Two models may be considered: in the first one, a finite set of values is assigned to each uncertain model coefficient; in the second one, each uncertain model coefficient is associated with an interval number. In this paper, we only consider this second model called interval linear programming.

The choice of one value in each interval corresponds to a scenario. The induced robust optimization problem is to determine a single solution which is optimal for all scenarios. In general, such a solution does not exist and the problem is to determine a "relatively good" solution for all scenarios (see for example [2, 13, 15]). In this context, classical criteria coming from decision theory may be used. In linear programming, when objective function coefficients are interval numbers, the worst case and the maximum regret criteria have been extensively studied ([1, 8, 9, 10, 11]). When uncertainty concerns feasible solution set, robustness problems have been less studied (see for example [14, 12]). Nevertheless, a lot of real optimization problems includes uncertainty and inaccuracy factors on feasible solutions set. For example, when a linear program represents a production problem in which right handsides equal to some forecast demands on several periods, it may be much more relevant to replace each right handside coefficient by a suitable interval number.

In this paper, we consider general linear program in which each right handside b_i is an interval number $[\underline{b_i}, \overline{b_i}]$. It is assumed that each b_i can take on any value from the corresponding interval regardless of the values taken by other coefficients.

In the first part, we present some classical decision problems dealing with uncertainty on objective function coefficients. We introduce three criteria classically considered: the worst case criterion, the best case criterion and the maximum regret criterion. Then, we recall few results obtained in case of uncertainty on right handsides. We define the best and worst optimum problems (firstly introduced by Chinneck and Ramadan in [6]). In the second part, we extensively study the theoretical complexity of the best and worst optimum problems when each right handside is an interval number. We consider separately linear programs with inequality constraints and those with equality constraints. In each case, we analysis the relationship between best (worst) optimum and optimal solutions according to the best (worst respectively) case criterion. In case of linear programs with equality constraints, we propose a model, called the penalty model, which allows to apply the classical worst case criterion. We show that this model is tractable in polynomial time. Finally, in the last part of the paper, we highlight the duality relation between these problems and we discuss about another approach of robustness suggested by Bertsimas and Sim in [4].

2 Decision problems dealing with uncertainty

2.1 Uncertainty on objective function coefficients: main results

Linear programs with interval coefficients in the objective function have been the subject of numerous studies. Some of them deal with the following problem

$$(P^c)\begin{cases} \min & cx \\ s.t & x \in X \end{cases}$$

with $c \in [\underline{c}, \overline{c}]$ and X being a nonempty bounded polyhedron. As usual, we denote in this paper $v(P)$ the optimal solution value of the optimization problem P.

Three criteria, coming from decision theory, are classically considered: the worst case criterion, the best case criterion and the maximum regret criterion.

2.1.1 The worst case criterion

Given a solution $x \in X$, the scenario to be considered is the one that gives the worst value for this solution. In this context, the value of x, noted $f_{\text{WOR}}(x)$, is defined by:

$$f_{\text{WOR}}(x) = \max_{\underline{c} \leq c \leq \overline{c}} cx$$

The problem is to determine the solution $x_{\text{WOR}} \in X$ which minimizes f_{WOR} as follows:

$$f_{\text{WOR}}(x_{\text{WOR}}) = \min_{x \in X} f_{\text{WOR}}(x)$$

Under any scenario c, x_{WOR} has a value lower or equal to $f_{\text{WOR}}(x_{\text{WOR}})$. Thus, $f_{\text{WOR}}(x_{\text{WOR}})$ can be considered as an upper bound which offers an absolute guarantee. That is why in the work of Kouvelis and Yu [9], solutions which optimize this criterion are called absolute robust solutions.

In [1], Averbakh and Lebedev observe that the problem of finding x_{WOR} is polynomial. Indeed, by strong duality theorem, we have

$$f_{\text{WOR}}(x) = \min\{\overline{c}u - \underline{c}l : u - l = x, u \geq 0, l \geq 0\}$$

with u being the dual variables associated with constraints $c \leq \overline{c}$ and l being the dual variables associated with $c \geq \underline{c}$. Thus, the problem of finding x_{WOR} can be written as the following linear program:

$$\left\{\begin{array}{rl} \min & \overline{c}u - \underline{c}l \\ s.t & u - l = x \\ & x \in X \\ & u \geq 0, l \geq 0 \end{array}\right.$$

Let us remark that, when all x variables are subject to positivity constraint, the problem of finding x_{WOR} is obviously polynomial since, for all x, $f_{\text{WOR}}(x) = \overline{c}x$. Thus, one have to solve:

$$\left\{\begin{array}{rl} \min & \overline{c}x \\ s.t & x \in X \\ & x \geq 0 \end{array}\right.$$

2.1.2 The best case criterion

Given a solution $x \in X$, the scenario to be considered is the one that gives the best value for this solution. In this context, the value of x, noted $f_{\text{BES}}(x)$, is defined by:

$$f_{\text{BES}}(x) = \min_{\underline{c} \leq c \leq \overline{c}} cx$$

The problem is to determine the solution $x_{\text{BES}} \in X$ which minimizes f_{BES} as follows:

$$f_{\text{BES}}(x_{\text{BES}}) = \min_{x \in X} f_{\text{BES}}(x)$$

This problem has received much less attention. Indeed, the best case criterion is not relevant to guarantee some kind of robustness as the worst case criterion do.

When all x variables are subject to positivity constraints, the problem of determining x_{BES} is obviously polynomial. Indeed, the problem is equivalent to:

$$\begin{cases} \min & \underline{c}x \\ s.t & x \in X \\ & x \geq 0 \end{cases}$$

In general case, when some variables are not restricted in sign, the complexity of this problem has not been studied. In this paper, we prove that this problem is NP-hard and we establish its equivalence with some linear programs with interval right handsides.

2.1.3 The maximum regret criterion

Given a vector c, the choice of a solution x which is not necessarily an optimal solution for P^c, generates a regret denoted $r(x, c) = cx - v(P^c)$. A solution x is then evaluated by $f_{\text{REG}}(x)$ on the basis of the maximum regret value:

$$f_{\text{REG}}(x) = \max_{\underline{c} \leq c \leq \overline{c}} r(x, c)$$

The optimal solution according to the maximum regret criterion will be denoted x_{REG} and checks:

$$f_{\text{REG}}(x_{\text{REG}}) = \min_{x \in X} f_{\text{REG}}(x)$$

This criterion has received much more attention than the two previous ones. In the work of Kouvelis and Yu [9], a solution which optimizes the maximum regret criterion is called a robust deviation solution.

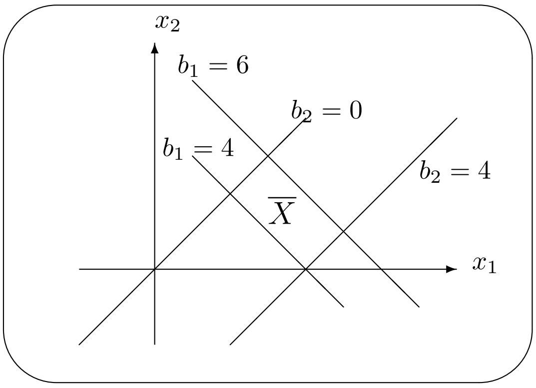
In [1], Averbakh and Lebedev recently prove that the problem of determining x_{REG} is strongly NP-hard and, moreover, that the problem of computing $f_{\text{REG}}(x)$ for a specific x is also strongly NP-hard. Several formulations and algorithms have been proposed to solve approximately this problem (see e.g. [8, 10, 11]).

2.2 Uncertainty on right handsides: main results

When uncertainty (representing by interval numbers) concerns right handside constraints, only few results have already been obtained. The difficulty comes from the fact that the set of feasible solutions is not exactly known. Thus, any solution may not be feasible for all interval right handside. Consequently, classical criteria cannot be directly applied as we illustrate in the following example.

Let us consider the linear program

$$\begin{cases} \min & x_1 + 2x_2 \\ s.t & x_1 + x_2 = b_1 \in [4,6] \\ & x_1 - x_2 = b_2 \in [0,4] \\ & x_1, x_2 \geq 0 \end{cases}$$



For a given $x \in \overline{X}$, it always exists a scenario (b_1, b_2) for which x is not feasible. Thus, any solution x is not feasible in the worst case and the worst case criterion is no more relevant.

Another approach is suggested by Chinneck and Ramadan in [6]. In this paper, general linear programs with interval coefficients (simultaneously in objective function, matrix constraints and right handsides) are considered. The goal is to compute the best possible optimum and the worst one over all possible scenarios in order to provide a kind of robustness information: "The range of the objective function between the best and the worst optimum values gives a sense of the risk involved... For example, the specific values of the uncertain coefficients can be chosen to reflect a conservative or a risk-taking strategy."

Remark 1 *When only objective function coefficients are defined by intervals, the worst optimal solution is equal to the solution which minimizes the worst case criterion. Indeed*

$$f_{\mathrm{WOR}}(x_{\mathrm{WOR}}) = \min_{x \in X} \max_{\underline{c} \leq c \leq \overline{c}} cx$$

is equal to

$$\max_{\underline{c} \leq c \leq \overline{c}} \min_{x \in X} cx$$

when we inverse the min *and* max *operators. And, this optimization problem is exactly the problem of determining the worst optimum. Equivalently, the determination of the best optimum and the optimization of the best case criterion are exactly the same optimization problems when each objective function coefficient are interval numbers.*

In [6], algorithms are proposed to determine best and worst optimum but none complexity result is given. They consider separately linear problems with variables restricted in sign and equality or inequality constraints. They propose polynomial time algorithms for determining the best optimum of linear program with variables restricted in sign, and the worst optimum of linear program with inequality constraints and variables restricted in sign. They define an exponential time algorithm for computing the worst optimum of linear program with equality constraints and variables restricted in sign. Moreover, the authors remark that algorithms complexity grows when variables are not restricted in sign.

When only right handsides are interval numbers in a linear program, we show in this paper that only two cases have to be distinguished for complexity analysis. Firstly, we consider the easier case of linear programs with general inequality constraints (whatever the sign of each variable is), and secondly, we study the much more difficult case of linear programs with equality constraints. In each case, we exhibit the relationship between the best (worst) optimum and the optimal solution of the best (worst) case criterion.

3 Linear programs with interval right handsides: the case of inequality constraints

We consider the following linear program with n variables and m constraints

$$(P_{\geq}^{b}) \begin{cases} \min & cx \\ s.t & Ax \geq b \end{cases}$$

We suppose that each b_i varies in the interval $[\underline{b_i}, \overline{b_i}]$. For all $b \in [\underline{b}, \overline{b}]$, we denote $X_{\geq}^{b}$ the polyhedron defined by $\{x \in \mathbb{R}^n : Ax \geq b\}$ and we suppose that $X_{\geq}^{b}$ is a nonempty bounded polyhedron.

3.1 Best optimal solution

Our objective is to determine the minimum value of the optimal solution of $(P_{\geq}^{b})$ when b varies in the interval $[\underline{b}, \overline{b}]$. The best optimal solution problem can be written as follows

$$(B_{\geq}) \begin{cases} \min & v(P_{\geq}^{b}) \\ s.t & \underline{b} \leq b \leq \overline{b} \end{cases}$$

Theorem 1 $(B_{\geq})$ *can be solved in polynomial time.*

Proof 1 *It is sufficient to remark that* $(B_{\geq})$ *is equivalent to the following linear program*

$$\begin{cases} \min & cx \\ s.t & Ax \geq b \\ & \underline{b} \leq b \leq \overline{b} \end{cases}$$

Moreover, it is possible to characterize the scenario which gives the best optimal solution. Since $X^{b}_{\geq} \subseteq X^{\underline{b}}_{\geq}$ for all b in $[\underline{b}, \overline{b}]$, we have $v(P^{b}_{\geq}) \geq v(P^{\underline{b}}_{\geq})$ and consequently $v(B_{\geq}) = v(P^{\underline{b}}_{\geq})$.

3.2 Worst optimal solution

Our objective is to determine the maximum value of the optimal solution of $(P^{b}_{\geq})$ when b varies in the interval $[\underline{b}, \overline{b}]$. The worst optimal solution problem can be written as follows

$$(W_{\geq}) \left\{ \begin{array}{rl} \max & v(P^{b}_{\geq}) \\ s.t & \underline{b} \leq b \leq \overline{b} \end{array} \right.$$

As seen in the previous section, $X^{\overline{b}}_{\geq} \subseteq X^{b}_{\geq}$ for all b in $[\underline{b}, \overline{b}]$. Thus we have $v(P^{b}_{\geq}) \leq v(P^{\overline{b}}_{\geq})$ and the theorem 2 follows.

Theorem 2 *$(W_{\geq})$ can be solved in polynomial time since $v(W_{\geq}) = v(P^{\overline{b}}_{\geq})$.*

3.3 Best case and worst case criteria

The largest (respectively smallest) feasible solution set of $P^{b}_{\geq}$ is obtained when $b = \underline{b}$ (respectively $b = \overline{b}$). For a given b, the value of any solution $x \in X^{\underline{b}}_{\geq}$, denoted $f^{b}(x)$, is cx when $x \in X^{b}_{\geq}$ and, by convention, $+\infty$ otherwise. Thus,

$$f_{\text{BES}}(x) = \min_{\underline{b} \leq b \leq \overline{b}} f^{b}(x) = \min\{cx, +\infty\} = cx$$

Consequently, for any solution $x \in X^{\underline{b}}_{\geq}$, the best case value is always cx. Thus, the optimal solution according to the best case criterion is obtained by solving $(P^{\underline{b}}_{\geq})$. As seen in section 3.1, $P^{\underline{b}}_{\geq}$ is also the problem to solve for determining the best optimal solution.

Moreover, for any solution $x \in X^{\overline{b}}_{\geq}$, the worst value is equal to cx because $x \in X^{b}_{\geq}$ for all b. Otherwise, any solution x which does not belong to $X^{\overline{b}}_{\geq}$ has, by convention, a worst value equals to $+\infty$. Consequently, the optimal solution according to the worst case criterion necessarily belongs to $X^{\overline{b}}_{\geq}$ and one have to solve $P^{\overline{b}}_{\geq}$. It is then equivalent to the worst optimal solution problem as shown in section 3.2

Consequently, if $X^b_{\geq} \neq \emptyset$ for all $b \in [\underline{b}, \overline{b}]$, the best and worst optimum problem are equivalent to the problem of determining the optimal solution according to the best and worst case criteria respectively.

In conclusion, the problem of determining an optimal solution of a linear program, with general inequality constraints, according to the worst case criterion (or the best case criterion) can be solved in polynomial time even if variables are not restricted in sign.

4 Linear programs with interval right hand-sides: the case of equality constraints

In this section, we consider the following linear program with n variables and m equality constraints

$$(P^b_=) \begin{cases} \min & cx \\ s.t & Ax = b \end{cases}$$

We suppose that each b_i varies in the interval $[\underline{b_i}, \overline{b_i}]$. For all $b \in [\underline{b}, \overline{b}]$, we denote $X^b_=$ the polyhedron defined by $\{x \in \mathbb{R}^n : Ax = b\}$ and we suppose that $X^b_=$ is a nonempty bounded polyhedron. We introduce two sets $\overline{X} = \bigcup_{b \in [\underline{b}, \overline{b}]} X^b_=$ and $\underline{X} = \bigcap_{b \in [\underline{b}, \overline{b}]} X^b_=$. Given a solution $x \in \overline{X}$ and a scenario b, two cases have to be considered:

- x belongs to $X^b_=$ and its value is equal to cx,
- x does not belong to $X^b_=$ and, by convention, we set its value to $+\infty$.

4.1 Best optimal solution

In this case, the best optimal solution problem is

$$(B_=) \begin{cases} \min & v(P^b_=) \\ s.t & \underline{b} \leq b \leq \overline{b} \end{cases}$$

Theorem 3 $(B_=)$ *can be solved in polynomial time.*

The proof is equivalent to the proof 1 given in case of linear program with inequality constraints.

Another formulation of $(B_=)$ can be obtained by introducing additional variables denoted by $z \in \mathbb{R}^m$. For $i = 1, \ldots, m$, each z_i variable, defined in $[0,1]$, represents the deviation from the lower bound $\underline{b_i}$ in the interval $[\underline{b_i}, \overline{b_i}]$ and we have

$$\forall b_i \in [\underline{b_i}, \overline{b_i}], b_i = \underline{b_i} + z_i(\overline{b_i} - \underline{b_i}) \text{ with } z_i \in [0,1]$$

So, $(B_=)$ can be written

$$\begin{cases} \min & cx \\ s.t & Ax = \underline{b} + z(\overline{b} - \underline{b}) \\ & 0 \le z \le 1 \end{cases}$$

Let us remark that this reformulation, with z_i variables, is inspired by the robustness approach proposed by Bertsimas and Sim in [4] presented in section 5.

With this formulation, one may characterize the scenario which leads to the best optimal solution.

Theorem 4 *The best optimal solution can be obtained with an extreme scenario, that is to say, $\forall i = 1, \ldots, m$, z_i equals to 1 or 0.*

Proof 2 *Let us consider the formulation of $(B_=)$ with z_i variable:*

$$(B_=)\begin{cases} \min & \displaystyle\sum_{j=1}^{n} c_j x_j & \\ s.t & \displaystyle\sum_{j=1}^{n} a_{ij} x_j - z_i(\overline{b}_i - \underline{b}_i) = \underline{b}_i & i = 1, \ldots, m \\ & -z_i \ge -1 & i = 1, \ldots, m \\ & z_i \ge 0 & i = 1, \ldots, m \end{cases}$$

The dual program of $(B_=)$ denoted $(D_=)$ is:

$$(D_=)\begin{cases} \max & \displaystyle\sum_{i=1}^{m} \underline{b}_i \lambda_i - \sum_{i=1}^{m} \mu_i & \\ s.t & \displaystyle\sum_{i=1}^{m} a_{ij} \lambda_i = c_j & j = 1, \ldots, n \\ & -(\overline{b}_i - \underline{b}_i)\lambda_i - \mu_i \le 0 & i = 1, \ldots, m \\ & \mu_i \ge 0 & i = 1, \ldots, m \end{cases}$$

Let us remark that each μ_i value must be separately minimized and consequently

$$\mu_i^* = \max\{0; -(\overline{b}_i - \underline{b}_i)\lambda_i^*\} \tag{1}$$

By strong duality, a feasible solution (x, z) of $(B_=)$ and a feasible solution (λ, μ) of $(D_=)$ are optimal if and only if the following exclusion relations are satisfied:

$$\mu_i(-z_i + 1) = 0 \; \forall i = 1, \ldots, m \tag{2}$$
$$z_i(-(\overline{b}_i - \underline{b}_i)\lambda_i - \mu_i) = 0 \; \forall i = 1, \ldots, m \tag{3}$$

Considering a dual optimal solution (λ^*, μ^*), three cases must be considered:

- *If $\lambda_i^* < 0$, from equation 1, we have: $\mu_i^* = -(\overline{b}_i - \underline{b}_i)\lambda_i^*$ which is strictly greater than 0. So, the relation 2 implies $z_i^* = 1$.*
- *If $\lambda_i^* > 0$, from equation 1, we have: $\mu_i^* = 0$. So, the relation 3 implies $z_i^* = 0$.*
- *If $\lambda_i^* = 0$, from equation 1, we have: $\mu_i^* = 0$. The relation 2 and 3 are verified for all z_i belonging to $[0, 1]$. It always exists a feasible primal solution (x, z) since, by hypothesis, $X_=^b \neq \emptyset$ for all b or equivalently for all $z_i \in [0, 1]$. In particular, one can consider the optimal solution induced by an extreme scenario with z_i^* equals 1 or 0.*

4.2 Worst optimal solution

The problem of determining the worst optimal solution can be formulated as follows

$$(W_=) \begin{cases} \max & v(P_=^b) \\ s.t & \underline{b} \leq b \leq \overline{b} \end{cases}$$

Theorem 5 *$(W_=)$ is NP-hard even if any x variable is restricted in sign.*

Proof 3 *For a given b, the dual program of $(P_=^b)$, with additional positivity constraints on x variables, is*

$$(D_=^b) \begin{cases} \max & b^t\lambda \\ s.t & A^t\lambda \leq c^t \end{cases}$$

where $\lambda = (\lambda_i)_{i=1,\ldots,m}$ *and* λ_i *is the dual variable of the* i^{th} *constraint* $\sum_{j=1}^{n} a_{ij}x_j = b_i$.

According to the strong duality theorem, one can replace $v(P^b_=)$ *by* $v(D^b_=)$ *in* $(W_=)$ *as follows*

$$v(W_=) = \max_{\underline{b}\leq b\leq \overline{b}} \max_{A^t\lambda\leq c^t} b^t\lambda$$

which is equivalent to the following quadratic linear program

$$(Q)\begin{cases} \max & b^t\lambda \\ s.t & A^t\lambda \leq c^t \\ & \underline{b} \leq b \leq \overline{b} \end{cases}$$

To prove the NP-completness, we establish a reduction from the problem of computing the maximum regret value of a given $u \in X^{\beta}_{\leq}$ *for the following problem*

$$\begin{cases} \max & \gamma u \\ s.t & \Delta u \leq \beta \end{cases}$$

with $\gamma \in [\underline{\gamma}, \overline{\gamma}]$, $u, \gamma \in \mathbb{R}^m$, $\beta \in \mathbb{R}^n$ *and* $\Delta \in \mathbb{R}^{n\times m}$. *Averbakh and Lebedev prove in [1] that the problem of computing the maximum regret value of a given u is NP-hard. This problem can be written as follows*

$$f_{\text{REG}}(u) = \max_{\substack{\underline{\gamma} \leq \gamma \leq \overline{\gamma} \\ \Delta v \leq \beta}} \{\gamma(v-u)\}$$

By setting $\lambda = v - u$, *we obtain*

$$f_{\text{REG}}(u) = \begin{cases} \max & \gamma\lambda \\ s.t & \Delta\lambda \leq \beta - \Delta u \\ & \underline{\gamma} \leq \gamma \leq \overline{\gamma} \end{cases}$$

For a given u, $\beta' = \beta - \Delta u$ *is fixed and we have* $v(Q) = f_{\text{REG}}(u)$ *with* $\gamma = b^t$, $\beta' = c^t$ *and* $\Delta = A^t$. *So, if* $(W_=)$ *can be solved in polynomial time, we can also compute* $f_{\text{REG}}(u)$ *in polynomial time which contradicts the Averbakh result.*

In conclusion, the worst optimum problem is NP-hard for linear program including equality constraints with right handside coefficients equal to interval numbers.

Moreover, the scenario which leads to a worst optimal solution is an extreme scenario. Considering Q, one can remark that for a given feasible λ, the b_i variables can be separately optimized since $\max_{\underline{b}\leq b\leq \bar{b}} \sum_{i=1}^m b_i\lambda_i = \sum_{i=1}^m \max_{\underline{b}_i\leq b_i\leq \bar{b}_i} b_i\lambda_i$. Thus, for all $i = 1,\ldots,m$, if $\lambda_i \geq 0$ then $b_i^* = \bar{b}_i$ otherwise, if $\lambda_i < 0$ then $b_i^* = \underline{b}_i$. Chinneck and Ramadan in [6] observe also that extreme scenarios are those of interest to determine a worst optimum and they give an exact algorithm which enumerates the 2^m extreme scenarios.

4.3 Best case criterion

As equivalently mentioned in the section 3.3, the best case value of any solution $x \in \overline{X}$ is cx. Thus, the optimal solution according to the best case criterion is obtained by solving

$$\begin{cases} \min & cx \\ s.t & Ax = b \\ & \underline{b} \leq b \leq \bar{b} \end{cases}$$

which is exactly $(B_=)$.

In conclusion, the best optimal solution is also the optimal solution according to the best case criterion that can be obtained in polynomial time.

4.4 Worst case criterion: the penalty model

Due to infeasibility, the worst case value of any solution $x \in \overline{X}$ is $+\infty$ and the worst case criterion is no more equivalent to the worst optimal solution problem.

However, the problem of determining a solution for which infeasibility is limited, may be of interest for some decision problem. This way has been extensively explored in stochastic programming in the framework of recourse models ([5]).

We suggest to measure infeasibility on each constraint i with an additional variable, denoted e_i, for $i = 1,\ldots,m$. And we set:

$$e_i = |b_i - \sum_{j=1}^n a_{ij}x_j|$$

To evaluate any solution $x \in \overline{X}$ on any scenario b, we introduce e_i variables with a high penalty coefficient, denoted p, as follows:

$$f_b(x) = \sum_{j=1}^{n} c_j x_j + p \sum_{i=1}^{m} e_i$$

And we have:

$$\begin{aligned} f_{\text{WOR}}(x) &= cx + p \max_{\underline{b} \le b \le \overline{b}} \{\sum_{i=1}^{m} e_i\} \\ &= cx + p \sum_{i=1}^{m} \max_{\underline{b_i} \le b_i \le \overline{b_i}} \{e_i\} \end{aligned}$$

Let us now remark that for a given constraint i:

$$\begin{aligned} \max_{\underline{b_i} \le b_i \le \overline{b_i}} \{|b_i - \sum_{j=1}^{n} a_{ij} x_j|\} &= \max\{|\bar{b}_i - \sum_{j=1}^{n} a_{ij} x_j|; |\underline{b}_i - \sum_{j=1}^{n} a_{ij} x_j|\} \\ &= \max\{\bar{b}_i - \sum_{j=1}^{n} a_{ij} x_j; \sum_{j=1}^{n} a_{ij} x_j - \bar{b}_i; \underline{b}_i - \sum_{j=1}^{n} a_{ij} x_j; \sum_{j=1}^{n} a_{ij} x_j - \underline{b}_i\} \\ &= \max\{\bar{b}_i - \sum_{j=1}^{n} a_{ij} x_j; \sum_{j=1}^{n} a_{ij} x_j - \underline{b}_i\}. \end{aligned}$$

So,

$$f_{\text{WOR}}(x) = \left\{ \begin{array}{rll} \min & cx + p \sum_{i=1}^{m} z_i & \\ s.t & z_i \ge \bar{b}_i - \sum_{j=1}^{n} a_{ij} x_j & \forall i = 1, \ldots, m \\ & z_i \ge \sum_{j=1}^{n} a_{ij} x_j - \underline{b}_i & \forall i = 1, \ldots, m \\ & z_i \ge 0 & \forall i = 1, \ldots, m \end{array} \right.$$

which can be computed in polynomial time.

If we want to determine the solution which minimizes the worst case criterion, we have to solve:

$$\left\{ \begin{array}{rl} \min & f_{\text{WOR}}(x) \\ s.t & Ax = b \\ & \underline{b} \le b \le \bar{b} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{ll} \min & cx + p\sum_{i=1}^{m}(\max\{\overline{b}_i - \sum_{j=1}^{n} a_{ij}x_j; \sum_{j=1}^{n} a_{ij}x_j - \underline{b}_i\}) \\ s.t & Ax = b \\ & \underline{b} \leq b \leq \overline{b} \end{array} \right.$$

$$\Leftrightarrow \left\{ \begin{array}{lll} \min & cx + \sum_{i=1}^{m} pz_i & \\ s.t & Ax \leq \overline{b} & \\ & Ax \geq \underline{b} & \\ & z_i \geq \overline{b}_i - \sum_{j=1}^{n} a_{ij}x_j & \forall i = 1, \ldots, m \\ & z_i \geq \sum_{j=1}^{n} a_{ij}x_j - \underline{b}_i & \forall i = 1, \ldots, m \\ & z_i \geq 0 & \forall i = 1, \ldots, m \end{array} \right.$$

It is a linear program and, consequently, the optimal solution according the worst case criterion for the penalty model can be computed in polynomial time.

5 Duality and robustness

As proved in section 4.2, the problem $(W_=)$ of determining the worst optimum is equivalent to the quadratic linear program (Q). This quadratic linear program can be seen as the problem of determining λ_{BES} which optimizes the best case criterion when b varying between $\underline{b}$ and $\overline{b}$ in $(D^b_=)$. Indeed, $f_{\text{BES}}(\lambda) = \max\limits_{\underline{b} \leq b \leq \overline{b}} b^t\lambda$ and

$$f_{\text{BES}}(\lambda_{\text{BES}}) = \max_{A^t\lambda \leq c^t} \max_{\underline{b} \leq b \leq \overline{b}} b^t\lambda = \max_{\substack{A^t\lambda \leq c^t \\ \underline{b} \leq b \leq \overline{b}}} b^t\lambda = v(Q)$$

Thus, $(W_=)$ is equivalent to the problem of determining the optimal solution according to the best case criterion applied to a maximization linear program, with inequality constraints and variables unrestricted in sign, where each objective function coefficient is an interval number. So, the theorem follows:

Theorem 6 *Considering the following linear program*

$$(P^c_{\leq}) \left\{ \begin{array}{ll} \max & cx \\ s.t & Ax \leq b \end{array} \right.$$

with c varying between $\underline{c}$ and $\overline{c}$, the problem of determining the solution which optimizes the best case criterion is NP-hard.

Moreover as shown in section 4.3, λ_{BES} is also an optimal solution of $(B_=)$ and a kind of duality relationship exists between $(W_=)$ and $(B_=)$. In other words, the worst optimum problem for a minimization linear program (P) with interval right handsides is equivalent to the best optimum problem for the dual linear program (D) with interval objective function coefficients. On one hand, when constraints in (P) are inequalities, variables in (D) are restricted in sign and worst optimum problem of (P) and best optimum problem of (D) are both polynomial. On the other hand, when constraints in (P) are equalities, variables in (D) are unrestricted in sign and worst optimum problem of (P) and best optimum problem of (D) become NP-hard. Furthermore, if a particular robust criterion is applied on a problem P, the "dual" criterion must be applied to the dual problem D in order to obtain the same optimal value.

More recently, Bertsimas and Sim in [4] suggest another polynomial approach of robustness in linear programming. Their approach includes the case under consideration in this paper of interval right handside. For each constraint i, they consider the following interval model for any coefficient: $\delta \in [\widetilde{\delta} - \widehat{\delta}, \widetilde{\delta} + \widehat{\delta}]$ where $\widetilde{\delta}$ is the nominal value and $\widehat{\delta} \geq 0$ represents the deviation from the nominal value $\widetilde{\delta}$. They suppose the quite natural idea that the worst case will not simultaneously happen for all coefficients in a same constraints. In [3], authors "stipulate that nature will be restricted in its behavior, in that only a subset of coefficients will change in order to adversely affect the solution". They introduce a parameter Γ_i which represents the maximum number of coefficients that can deviate from their nominal value in the constraint i: $\Gamma_i = 0$ means that none coefficient will vary, while $\Gamma_i = n+1$ means that all coefficients will vary in the worst case sense. So, Γ_i is interpreted as a level of robustness. When uncertainty concerns only right handsides, $0 \leq \Gamma_i \leq 1$ and we have to consider separately linear programs with inequality constraints and those with equality constraints. In the first case, the worst case sense is identified and fixing Γ_i remains to choose a particular value for the constraint i right handside: for a $\geq$ constraint, the value is $\widetilde{b}_i + \Gamma_i \widehat{b}_i$, otherwise for a $\leq$ constraint, the value is $\widetilde{b}_i - \Gamma_i \widehat{b}_i$. In particular $\Gamma_i = 1$ for all $i = 1, \ldots, m$ corresponds to the worst case criterion. In the second case, the worst case sense cannot be identified. Thus, each equality constraint i is replaced by two opposite inequalities and, fixing Γ_i remains to choose two differents values for b_i: $\widetilde{b}_i + \Gamma_i \widehat{b}_i$ for one constraint and $\widetilde{b}_i - \Gamma_i \widehat{b}_i$ for the other one. It does not correspond to any scenario for the initial equality constraint. Thus, Bertsimas and Sim approach is not very suitable when the uncertainty only concerns right handsides.

6 Conclusion

In this article, we study theoretical complexity of the best (worst) optimum problem for linear program with interval right handsides and we highlight the relationship with the best (worst) case criterion. In the following array, the main results are summarized:

	best opt and best case criterion	*worst opt*	*worst case criterion*
$\begin{cases} \min & cx \\ s.t & Ax \geq b \end{cases}$ with $\underline{b} \leq b \leq \overline{b}$	polynomial	polynomial	Equivalent to the worst optimum
$\begin{cases} \min & cx \\ s.t & Ax = b \end{cases}$ with $\underline{b} \leq b \leq \overline{b}$	polynomial	NP-hard	Non relevant

In case of linear program with equality constraints, we propose a penalty model useful to apply the worst case criterion and tractable in polynomial time.

Even if the best (worst) optimum and the best (worst) case criterion often lead to the same solutions, they correspond to different decision context. For the best (worst) optimum, the underlying decision context is the following: in a first step, a scenario is given and, in a second step, one choose the best solution under this scenario. So, in the real decision step, they is no more uncertainty and the computation of the best and the worst optimum gets ahead of the decision step to provide additional information. For the best (worst) case criterion, the underlying decision context is different: in a first step, one choose a solution, and in a second step, one record the scenario carried out. Consequently, one can choose an infeasible solution for the scenario carried out which, in some decision context, leads to pay some penalties. In this decision context, the worst case criterion may be efficiently compared with other criteria (like the maximum regret criterion), this will be the subject of future research.

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