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# Better differential approximation for symmetric TSP

Bruno Escoffier\*      Jérôme Monnot\*

July 3, 2007

## Abstract

In this paper, we study the approximability properties of symmetric TSP under an approximation measure called the *differential ratio*. More precisely, we improve up to  $3/4 - \varepsilon$  (for any  $\varepsilon > 0$ ) the best differential ratio of  $2/3$  known so far, given in Hassin and Khuller, “z-Approximations”, J. of Algorithms, 41(2), 429-442, 2001.

**Keywords:** Approximation algorithm; Differential ratio; Traveling salesman problem.

## 1 Introduction

Due to both its practical and theoretical interests, symmetric TSP is one of the most famous combinatorial optimization problems. Given a complete edge-weighted graph, one seeks a tour (Hamiltonian cycle) either of minimum length (MinTSP) or maximum length (MaxTSP). Shown to be *NP*-hard in the very early development of the complexity theory ([24]), it has been widely studied since then from an approximate point of view. A polynomial algorithm  $A$  is said to be  $\rho$ -approximate if for any instance  $I$ ,  $m(A(I)) \leq \rho \text{opt}(I)$  for a minimization problem (resp.  $m(A(I)) \geq \rho \text{opt}(I)$  for a maximization problem), where  $m(x)$  denotes the value of the solution  $x$  of  $I$ , and  $\text{opt}(I)$  the optimum value of  $I$ .

While MinTSP is not  $2^{p(n)}$ -approximable where  $n = |V|$ , for any polynomial  $p$ , if  $P \neq NP$ , MaxTSP is in APX : the well known  $3/4$ -approximation algorithm by Serdyukov [29] has recently been slightly improved up to  $61/81$  in [8], or even  $25/33$  using a randomized algorithm [22]. Many classical subcases have been studied, the most famous being the so-called metric case, restriction where the weights satisfy the triangle inequality. Using this assumption, Christofides devised in [9] a  $3/2$ -approximation algorithm for MinMetricTSP, and this is up to now the best ratio obtained. Dealing with MaxMetricTSP, the  $3/4$ -ratio that holds for the general case can be improved up to  $17/20$  [8], or even  $7/8$  using a randomized algorithm [23]. Note that all these problems do not admit approximation schemes if  $P \neq NP$  [28].

In this article, we further study the approximation properties of symmetric TSP, but using another measure of the quality of a solution called the *differential ratio*. The differential ratio of a solution  $x$  of value  $m(x)$  is defined as  $\delta(x) = \frac{m(x) - \text{wor}(I)}{\text{opt}(I) - \text{wor}(I)}$ , where  $\text{opt}(I)$  is the value of an optimum solution, and  $\text{wor}(I)$  is the value of a worst solution. For instance, if one studies MaxTSP, then a worst solution is a minimum length tour. In other words,

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this ratio measures the relative position of  $m(x)$  in the interval  $[wor(I), opt(I)]$  containing all feasible values (the definition can be rephrased for a maximization problem as : the solution  $x$  is  $\delta$ -approximate if  $m(x) \geq \delta opt(I) + (1 - \delta)wor(I)$ ). Of course,  $\delta \in [0, 1]$  (0 for  $wor(I)$  and 1 for  $opt(I)$ ), and the closer to 1 the better the solution. The main property of this ratio is to be *stable under affine transformation of the objective function* (see [14] for a mathematical and operational justification of the ratio). Introduced in [2, 3], this ratio has been first used for studying mathematical programming problems, where the standard ratio is not suitable when very common operations such as “removing a constant” are performed, see for instance [31]. Afterwards, this approach has been considered for the main combinatorial optimization problems, leading to the development of new techniques and interesting results (see for instance [5] for vehicle routing, [20] for several results on graph problems, [10, 15, 17] for MinColoring, [21] for several weighted versions of graph partitioning, [12, 13] for Bin Packing, [7, 16] for satisfiability, [11, 6] for Set Cover, and very recently [18] for weighted Set Cover, etc.). A survey of many results about differential approximation can be found in the book chapter [4].

Dealing with symmetric TSP, we shall point out two major differences when using the differential ratio instead of the standard one. First, the dissymmetry between maximizing and minimizing completely disappears. More precisely, using an affine transformation of weights ( $w(e) \rightarrow w'(e) = M - w(e)$ , for a sufficiently large  $M$ , as for instance the heaviest weight plus 1), one can easily see that solving MinTSP (resp. MaxTSP) with the initial weights is equivalent to solve MaxTSP (resp. MinTSP) on the transformed weights. Indeed, the value of any tour  $T$  verifies  $w'(T) = nM - w(T)$ . Since the differential ratio is stable under affine transformation, this means that a  $\delta$ -approximation algorithm for MinTSP (resp. MaxTSP) can be immediately derived from a  $\delta$ -approximation algorithm for MaxTSP (resp. MinTSP).

The other difference, maybe rather surprising, is the equivalence between the metric case and the general case. While considering a metric distance is a rather important assumption when using the standard ratio, *TSP and MetricTSP are equivalent* when using the differential ratio. Indeed, again, one only has to affinely transform weights  $w(e) \rightarrow w_M + w(e)$ , where  $w_M$  is the weight of an heaviest edge, to get an equivalent metric instance of symmetric TSP.

To sum up, dealing with differential approximation ratios, MinTSP, MaxTSP, MinMetricTSP and MaxMetricTSP are all equivalent. These problems have been tackled several time from a differential approximation point of view. The best ratio obtained so far is  $2/3$  ([20, 25]), which can be improved up to  $3/4$  when the weights are restricted to be 1 or 2, [27] (note that in this case the best ratio known for the standard ratio is  $7/6$ , see [28]). Let us also mention that classical optimization strategies have been studied, such as the well known local 2-opt which has been shown in [26] to be a  $1/2$ -differential approximation (while not being a constant standard approximation algorithm even for MinMetricTSP). Note that, as well as in the standard approximation framework, these problems do not admit differential approximation schemes. Finally, dealing with asymmetric TSP, the best differential ratio obtained so far is  $1/2$  [20].

In this article, we improve these results by showing that symmetric TSP (*i.e.* MinTSP, MaxTSP, MinMetricTSP and MaxMetricTSP) is differential approximable within an asymptotic ratio of  $3/4$  (more precisely within a ratio of  $3/4 - O(1/n)$ ). Note that this is a noticeable improvement respect to  $2/3$  also because this is very close to the best ratio known for

MaxTSP (61/81). Since for a maximization problem the differential ratio is smaller than the standard one ( $m(x) \geq \delta \text{opt}(x) + (1 - \delta)\text{wor}(x)$  implies  $m(x) \geq \delta \text{opt}(x)$ , when solution values are nonnegative), the gap is now almost as small as it can be.

Let us already mention that, carrying on with this line of research, the study of symmetric TSP in the geometric case seems to be of particular interest. Indeed, when vertices are points in the plane (and the weight is the Euclidean distance), then it has been shown that both MaxTSP and MinTSP admit an approximation scheme (see resp. [30] and [1]). The existence of a differential approximation scheme is undoubtedly a very interesting and challenging question that would deserve further research.

In the following, we denote by  $\text{opt}(I)$ ,  $\text{apx}(I)$  and  $\text{wor}(I)$  the value of an optimal, an approximate and a worst solution respectively for an instance  $I$ . Due to the equivalence between MaxTSP and MinTSP, the results, only proven for MaxTSP, are obviously also valid for MinTSP. The proof of the result of the paper consists of two parts : in Section 2 we devise a 3/4-differential algorithm when the number of vertices is even. In Section 3, we show that the general case reduces to the previous subcase obtaining asymptotically the same ratio ( $3/4 - O(1/n)$  in our case).

## 2 Approximation for even instances

In this section, we assume that the number of vertices is even (ie  $|V| = 2n$ ), and provide a 3/4-differential approximation for symmetric TSP.

The method used is based on the computation of a maximum weight 2-matching  $E_2 = \{C_1, \dots, C_p\}$  of  $I = (K_{2n}, w)$ , which can be done in polynomial time, [19]. We separate two cases depending on the existence of a cycle of size 3 in  $E_2$ .

**Case 1:** There exists  $j \in \{1, \dots, p\}$  such that  $|C_j| = 3$ . Wlog, assume that  $j = p$  and  $C_p = \{v_1, v_2, v_3\}$ .

We present a heuristic which is an adaptation of the Serdyukov's algorithm working for MaxTSP, [29].

Let us first remind this algorithm and the result that can be derived from it (Lemma 2.1). This method consists in computing a maximum weight perfect matching  $E_1$  of  $I$  and moving one edge of each cycle  $C_i$  of  $E_2$  to  $E_1$  in such a way that we do not create any cycle (see Figure 1 for an illustration). At the end of the process, we obtain two collections of paths  $\mathcal{P}_1$  (containing  $E_1$ ) and  $\mathcal{P}_2$  such that:

**Lemma 2.1** *The collection of paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$  satisfy the following properties:*

- (i)  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are two collections of vertex disjoint paths such that the vertex sets of these collections of paths are exactly  $V(K_{2n})$ .
- (ii) If  $V_i$  are the endpoints of the paths of  $\mathcal{P}_i$  for  $i = 1, 2$ , then  $V_1 \cup V_2 = V(K_{2n})$  and  $V_1 \cap V_2 = \emptyset$ .
- (iii) Each path  $P$  of  $\mathcal{P}_1$  alternates between edge of  $E_1$  and  $E_2$  and the end edges of  $P$  are in  $E_1$ .

**Proof:** By construction, (i) is true for  $\mathcal{P}_2$ .

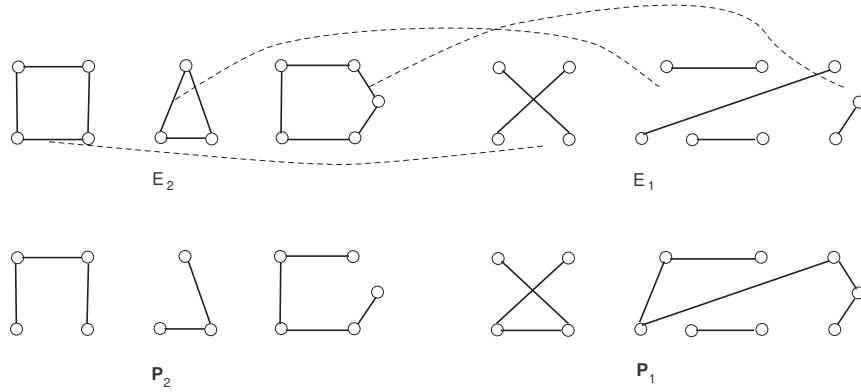


Figure 1: The two partition into paths  $\mathcal{P}_1$  and  $\mathcal{P}_2$ .

Let  $\mathcal{P}_2^j$  be the set of the  $j$  paths built from  $C_1, \dots, C_j$ , *i.e.* after iteration  $j$  (and similarly  $\mathcal{P}_1^j$  the collection of paths built from  $E_1$  after iteration  $j$ ). In particular,  $\mathcal{P}_k^p = \mathcal{P}_k$ ,  $k = 1, 2$ . We will prove the result by induction.

At the beginning (before moving any edge), *(ii)* and *(iii)* are true for  $\mathcal{P}_1^0$  and  $\mathcal{P}_2^0$ , and *(i)* is true for  $\mathcal{P}_1^0$ . Suppose this is true after iteration  $j - 1$ , and proceed the  $j^{\text{th}}$  iteration as follows: choose any vertex  $v$  in  $C_j$ , and consider the two edges  $e_1 = [v, a]$  and  $e_2 = [v, b]$  incident to  $v$  in  $C_j$ .  $v$  cannot be an internal vertex of a path of  $\mathcal{P}_1^{j-1}$  since otherwise  $v$  would be incident to 3 edges of  $E_2$  (using *(iii)*), contradiction. Using *(i)*, we obtain that  $v$  is an endpoint of a path  $P$  of  $\mathcal{P}_1^{j-1}$ . Thus, since  $a \neq b$ , at least one of these two vertices (assume that it is  $a$ ) is not the other endpoint of  $P$ . For the same reason,  $a$  is also the endpoint of another path  $P'$  of  $\mathcal{P}_1$ . When we move  $e_1$ :

- properties *(i)* (for  $\mathcal{P}_1^j$ ) and *(iii)* still hold;
- property *(ii)* also, since now  $v$  and  $a$  are new endpoints of a path in  $\mathcal{P}_2^j$ , but are no more endpoints of paths in  $\mathcal{P}_1^j$ .

■

Now, we describe our method. It uses the fact that we assume the existence of a triangle  $C_p = \{v_1, v_2, v_3\}$  in  $E_2$  in order to apply two times the previous construction, thus producing 4 collections of paths  $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}'_1, \mathcal{P}'_2$ , as follows.

We apply once the construction and get a first couple  $\mathcal{P}_1, \mathcal{P}_2$ . Then remark that, at each iteration in the construction, we can choose the vertex  $v$  incident to the two edges candidate to move from  $E_2$  to  $E_1$ . Then, wlog., assume that when applying the first construction the edge  $[v_1, v_2]$  has moved from  $C_p \in E_2$  to  $E_1$ . To get  $\mathcal{P}'_1, \mathcal{P}'_2$ , we apply exactly the same construction as previously, except for the last cycle. For  $C_p$ , we choose to move one of the two edges  $[v_1, v_3]$  or  $[v_2, v_3]$  incident to  $v_3$  (instead of  $[v_1, v_2]$ ). Wlog, assume that it is  $[v_1, v_3]$ ; using arguments of the proof of Lemma 2.1, we obtain two other collections of paths  $\mathcal{P}'_1$  (containing  $E_1$ ) and  $\mathcal{P}'_2$  satisfying properties *(i)*, *(ii)* and *(iii)*.

Moreover, if  $V'_i$  denotes the endpoints of  $\mathcal{P}'_i$  for  $i = 1, 2$ , then we can observe that  $v_3 \in V_1, v_2 \in V'_1$  and  $V_1 \setminus \{v_3\} = V'_1 \setminus \{v_2\}$ . Thus, it is possible to complete  $\mathcal{P}_1$  into a tour  $T_1$

and  $\mathcal{P}'_1$  into a tour  $T'_1$  such that the added edges form an hamiltonian path  $HP_1$  on vertices  $V_1 \cup \{v_2\}$  and with endpoints  $v_2, v_3$ .

Similarly, we have  $v_2 \in V_2, v_3 \in V'_2$  and  $V_2 \setminus \{v_2\} = V'_2 \setminus \{v_3\}$ . Thus, we can also add some edges to  $\mathcal{P}_2$  (resp.,  $\mathcal{P}'_2$ ) in order to obtain a tour  $T_2$  (resp.,  $T'_2$ ) in such a way that the added edges form an hamiltonian path  $HP_2$  on vertices  $V_2 \cup \{v_3\}$  and with endpoints  $v_2, v_3$ .

An illustration of this construction is given in Figure 2. For completeness, let us also give a formal proof of this claim for  $HP_2$ . Let  $V_2 = \{a_i, b_i : i = 1, \dots, p\}$  be the endpoints of the paths of  $\mathcal{P}_2$ , where  $[a_i, b_i]$  is the edge that has moved from  $E_2$  to  $E_1$  at iteration  $i$ . In particular, we have  $a_p = v_1$  and  $b_p = v_2$ . Similarly, let  $V'_2 = \{a'_i, b'_i : i = 1, \dots, p\}$  be the endpoints of the paths of  $\mathcal{P}'_2$  with  $a'_p = v_1 = a_p, b'_p = v_3$  and  $a'_i = a_i, b'_i = b_i$  for  $i = 1, \dots, p-1$ . The two tours  $T_2$  and  $T'_2$  depend on the parity of  $p$  and can be described as follow.

- Assume first that  $p$  is odd. We build  $T_2 = \mathcal{P}_2 \cup \{[a_i, b_{i+1}] : i = 1, \dots, p\}$  and  $T'_2 = \mathcal{P}'_2 \cup \{[b'_i, a'_{i+1}] : i = 1, \dots, p\}$ , with  $b_{p+1} = b_1$  and  $a'_{p+1} = a'_1$ . Since  $a_p = a'_p, b_p = v_2, b'_p = v_3$ , and  $a'_i = a_i, b'_i = b_i$  for  $i = 1, \dots, p-1$ , we deduce that the added edges  $HP_2 = \{[a_i, b_{i+1}], [b'_i, a'_{i+1}] : i = 1, \dots, p\}$  form an hamiltonian path from  $b'_p$  to  $b_p$  described by the sequence  $HP_2 = (b'_p, a_1, b_2, \dots, a_p, b_1, a_2, b_3, \dots, b_p)$ .
- Now, if  $p$  is even, then we only modify  $T'_2$  and define  $T'_2 = \mathcal{P}'_2 \cup \{[b'_i, a'_{i+1}] : i = 1, \dots, p-1\} \cup \{[a'_p, a'_1], [b'_p, b'_{p-1}]\}$ . As previously, one can easily check that the added edges  $HP_2 = \{[a_i, b_{i+1}] : i = 1, \dots, p\} \cup \{[b'_i, a'_{i+1}] : i = 1, \dots, p-1\} \cup \{[a'_1, a'_p], [b'_p, b'_1]\}$  form an hamiltonian path from  $b'_1$  to  $b_1$  described by the sequence  $HP_2 = (b'_p, b_1, a_2, b_3, \dots, a_p, a_1, b_2, a_3, \dots, a_{p-1}, b_p)$ .

In conclusion, we get 4 tours  $T_1, T'_1, T_2$  and  $T'_2$ . By taking the solution of maximum weight with cost  $apx(I)$ , we obtain:

$$4apx(I) \geq \sum_{i=1}^2 (w(T_i) + w(T'_i)) = 2(w(E_1) + w(E_2)) + (w(HP_1) + w(HP_2)) \quad (1)$$

On the one hand, we have  $w(E_2) \geq opt(I)$  and  $w(E_1) \geq opt(I)/2$ , and on the other hand since  $HP_1 \cup HP_2$  is a tour on  $K_{2n}$ , we get  $w(HP_1) + w(HP_2) \geq wor(I)$ . Plugging these inequalities with inequality (1), we deduce:

$$apx(I) \geq \frac{3}{4}opt(I) + \frac{1}{4}wor(I) \quad (2)$$

**Case 2:** For all  $j \in \{1, \dots, p\}$  we have  $|C_j| \geq 4$ . In this case, we extend the method proposed in [20, 25]. We study 2 subcases depending on the parity of  $p$ .

**Case 2.1:** If  $p$  is odd. Obviously, we can assume  $p > 1$ . For each cycle  $C_i$  of the 2 matching  $E_2 = \{C_1, \dots, C_p\}$ , we consider 4 consecutive edges  $A_i = \{[a_i, b_i], [b_i, c_i], [c_i, d_i], [d_i, f_i]\}$ <sup>1</sup>(with eventually  $f_i = a_i$  if  $|C_j| = 4$ ), and we produce 4 solutions by starting from  $E_2$ : the first solution  $T_a$  deletes the edge  $[a_i, b_i]$  for each cycle  $C_i$  with  $i = 1, \dots, p$ , and for each

<sup>1</sup>we denote the fourth vertex by  $f$  in order to avoid confusion with edges, denoted  $e$  in this article

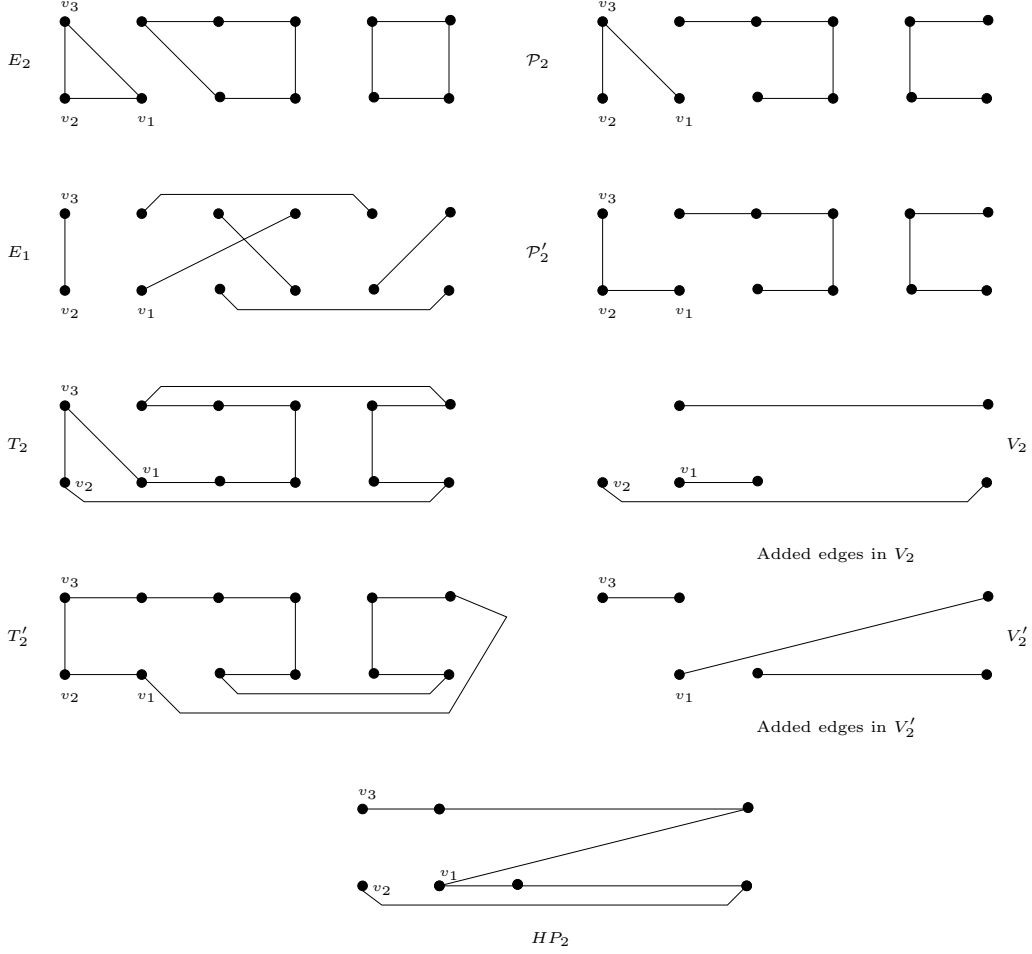


Figure 2: The construction of tours  $T_2$  and  $T'_2$  and the hamiltonian path  $HP_2$ .

$i \in \{1, \dots, p-1\}$  adds the edge  $[a_i, a_{i+1}]$  if  $i$  is odd, adds the edge  $[b_i, b_{i+1}]$  if  $i$  is even and finally adds the edge  $[a_p, b_1]$ . The 3 other solutions  $T_b, T_c$  and  $T_d$  are described similarly by deleting edges  $[b_i, c_i], [c_i, d_i]$  and  $[d_i, f_i]$  respectively. In particular, for the last solution  $T_d$ , we have added for  $i \in \{1, \dots, p-1\}$  the edges  $[d_i, d_{i+1}]$  if  $i$  is odd,  $[f_i, f_{i+1}]$  if  $i$  is even and finally edge  $[d_p, f_1]$ .

In the multigraph of these 4 solutions, that is  $(V, T_a + T_b + T_c + T_d)$ , each edge  $e$  of  $A = \cup_{i=1}^p A_i$  appears exactly 3 times whereas the other edges of  $E_2$  appears exactly 4 times. On the other hand, the edges of  $B = (T_a \cup T_b \cup T_c \cup T_d) - E_2$  appears one time.

Thus, by considering the best of the 4 solutions we produced, we get:  $4apx(I) \geq 3w(A) + 4w(E_2 - A) + w(B) = 3w(E_2) + w(B + E_2 - A) \geq 3opt(I) + w(B + E_2 - A)$ .

Now, remark that  $B + E_2 - A$  is a tour of  $I$ . Indeed,  $E_2 - A$  contains paths  $(f_i, g_i, \dots, a_i)$  for each  $i = 1, \dots, p$ . In  $B$ , we get a path  $P = (a_p, b_1, b_2, \dots, b_p, c_1, \dots, c_p, d_1, \dots, d_p, f_1)$ , and edges  $[a_i, a_{i+1}]$  ( $i$  odd) or  $[f_i, f_{i+1}]$  ( $i$  even). These edges and the paths of  $E_2 - A$  create a path from  $f_1$  to  $a_p$ , which constitutes together with  $P$  a tour. Hence,  $w(B + E_2 - A) \geq wor(I)$ . We get:

$$apx(I) \geq \frac{3}{4}opt(I) + \frac{1}{4}wor(I) \quad (3)$$



**Case 2.2:** If  $p$  is even, then the previous construction does not produce a tour. We adapt it as follows. As previously, we consider 4 consecutive edges  $A_i = \{[a_i, b_i], [b_i, c_i], [c_i, d_i], [d_i, f_i]\}$  in cycle  $C_i$  of the 2-matching  $E_2$ , except for the last cycle  $C_p$  where we replaced edge  $[d_p, f_p]$  by  $[z_p, a_p]$  with  $z_p$  is the other neighbor of  $a_p$  in  $C_p$  (eventually,  $z_p = d_p$  if  $|C_p| = 4$ ). Thus,  $A_p = \{[z_p, a_p], [a_p, b_p], [b_p, c_p], [c_p, d_p]\}$ .

Moreover, for  $C_2$ , we do not choose consecutive vertices  $a_2, b_2, c_2, d_2, f_2$  at random. We choose them such that:

$$w([a_1, b_2]) + w([a_2, b_1]) \leq w([a_1, a_2]) + w([b_1, b_2]) \quad (4)$$

Actually, this is always possible since otherwise for all  $e = [x, y] \in C_2$  we would get  $w([a_1, y]) + w([a_2, x]) < w([a_1, x]) + w([b_1, y])$  (here, we assume that each edge  $e = [x, y]$  is considered as a directed edge where the orientation is given when one walks around  $C_2$ ). Summing up the previous inequality for each edge  $e \in C_2$ , we obtain the inequality  $\sum_{v \in V(C_2)} (w([a_1, v]) + w([b_1, v])) > \sum_{v \in V(C_2)} (w([a_1, v]) + w([b_1, v]))$ , contradiction.

Then, we produce 4 tours  $T_a, T_b, T_c$  and  $T_d$  as follows: first,  $T_a$  deletes from  $E_2$  edges  $[a_i, b_{i+1}]$  if  $i < p$  and  $[z_p, a_p]$ ; then, solution  $T_a$  adds edges  $[a_i, a_{i+1}]$  if  $i < p$  is odd, adds the edge  $[b_i, b_{i+1}]$  if  $i < p$  is even and finally adds the edge  $[z_p, b_1]$ . The other tours  $T_b, T_c$  and  $T_d$  are constructed similarly. In particular,  $T_d$  deletes edges  $[d_i, f_i]$  if  $i < p$  and  $[z_p, a_p]$ , adds for  $i \in \{1, \dots, p-1\}$  edges  $[d_i, d_{i+1}]$  if  $i$  is odd,  $[f_i, f_{i+1}]$  if  $i$  is even and edge  $[c_p, f_1]$ .

As previously, in the multigraph  $(V, T_a + T_b + T_c + T_d)$ , each edge  $e$  of  $A = \cup_{i=1}^p A_i$  appears exactly 3 times whereas the other edges of  $E_2$  appears exactly 4 times. On the other hand, the edges of  $B = (T_a \cup T_b \cup T_c \cup T_d) - E_2$  appears one time. Thus, by considering the best of these 4 solutions, we get:

$$4\text{apx}(I) \geq 3w(A) + 4w(E_2 - A) + w(B) = 3w(E_2) + w(B + E_2 - A) \quad (5)$$

However, now  $B + E_2 - A$  is not a tour of  $I$ , but a 2-matching constituted by two cycles. The first one is  $(b_1, b_2, \dots, b_p, d_1, \dots, d_p, g_1, \dots, g_p, b_1)$ , constituted of edges in  $B$ ; the second one is constituted by the path  $(a_p, c_1, \dots, c_p, f_1)$  of  $B$ , the paths  $(f_i, g_i, \dots, z_i, a_i)$  of  $E_2 - A$ , and edges  $[a_i, a_{i+1}]$  ( $i$  odd) or  $[f_i, f_{i+1}]$  ( $i$  even).

But using inequality (4), one can flip edges  $[a_1, a_2], [b_1, b_2]$  by edges  $[a_1, b_2], [a_2, b_1]$  without increasing the global weight and one obtain a tour  $T$  such that  $\text{wor}(I) \leq w(T) \leq w(B + E_2 - A)$ . In conclusion, using this inequality and inequality (5), we obtain:

$$\text{apx}(I) \geq \frac{3}{4}\text{opt}(I) + \frac{1}{4}\text{wor}(I) \quad (6)$$

Combining the results obtained in cases 1 (equation (2)) and 2 (equation (6)), we obtain the following result.

**Theorem 2.2** *When the number of vertices is even, symmetric TSP is 3/4-differential approximable.*

### 3 General case

In the previous section, we dealt with even instances. Here, we show that one can solve symmetric TSP also on odd instances within an asymptotic differential ratio of 3/4.

**Theorem 3.1** *In the general case, symmetric TSP can be differential approximated with ratio  $3/4 - O(1/n)$ .*

**Proof:** From the discussion above, we have to deal with instances the number of vertices of which is odd. In this case, we find a  $(3/4 - O(1/n))$ -approximate solution using the previous result on even instances. Let  $n$  odd,  $I = (K_n, w)$  an instance of symmetric TSP and denote  $V = \{v_1, \dots, v_n\}$  the set of vertices.

We find an approximate solution as follows: for each  $i \in \{1, \dots, n\}$ , we consider the sub-instance  $I_i$  on the subgraph induced by  $V \setminus \{v_i\}$ . On this instance, we apply our approximation method given above and get a tour  $T_i$ . Then, we insert  $v_i$  in the best position in  $T_i$ , thus producing a tour  $T'_i$  on  $I$ . Finally, we take the best tour  $T$  among these  $n$  tours  $T'_i$ , *i.e.*  $apx(I) = w(T) = \max_{i=1, \dots, n} w(T'_i)$ .

Note that, when inserting vertex  $v_i$  in  $T_i$  between two vertices  $v_j$  and  $v_k$  (consecutive in  $T_i$ ), we get a tour of value  $w(T_i) + w([v_j, v_i]) + w([v_i, v_k]) - w([v_j, v_k])$ . Since we take the best of these nodes, by considering the  $n - 1$  possible insertions, we get:

$$\begin{aligned} (n-1)w(T'_i) &\geq (n-1)w(T_i) + 2 \sum_{k, k \neq i} w([v_i, v_k]) - w(T_i) \\ &\geq (n-2)w(T_i) + 2 \sum_{k, k \neq i} w([v_i, v_k]) \end{aligned}$$

Since we take the best tour among the  $T'_i$ 's, we get:

$$n(n-1)apx(I) \geq (n-2) \sum_{i=1}^n w(T_i) + 2S \quad (7)$$

where  $S = \sum_{i=1}^n \sum_{k, k \neq i} w([v_i, v_k])$  is twice the total weight of all edges in the graph.

Similarly, by inserting  $v_i$  in any position in a worst tour on  $I_i$ , we get a tour on  $I$ . The worst solution on  $I$  is of course worse than each of these solutions, *i.e.*:

$$\begin{aligned} (n-1)wor(I) &\leq (n-1)wor(I_i) + 2 \sum_{k, k \neq i} w([v_i, v_k]) - wor(I_i) \\ &\leq (n-2)wor(I_i) + 2 \sum_{k, k \neq i} w([v_i, v_k]) \end{aligned}$$

Hence:

$$n(n-1)wor(I) \leq (n-2) \sum_{i=1}^n wor(I_i) + 2S \quad (8)$$

Finally, consider an optimum solution  $(v_1^*, v_2^*, \dots, v_n^*)$  on  $I$ . By deleting  $v_i^*$  in this tour, we get a tour on  $I_i$  the value of which is  $opt(I) - w([v_i^*, v_{i-1}^*]) - w([v_i^*, v_{i+1}^*]) + w([v_{i-1}^*, v_{i+1}^*]) \leq opt(I_i)$ . By considering each of the possible deletion, we get (obviously  $v_0^*$  means  $v_n^*$  and  $v_{n+1}^*$  means  $v_1^*$ ):

$$n \times opt(I) - 2 \sum_{i=1}^n w([v_i^*, v_{i+1}^*]) + \sum_{i=1}^n w([v_{i-1}^*, v_{i+1}^*]) \leq \sum_{i=1}^n opt(I_i)$$

Since  $n$  is odd,  $\sum_{i=1}^n w([v_{i-1}^*, v_{i+1}^*])$  is the value of a tour, hence at least  $wor(I)$ . Then:

$$(n-2) \times opt(I) + wor(I) \leq \sum_{i=1}^n opt(I_i) \quad (9)$$

Now, using equations (7), (8), (9) and the fact that  $w(T_i) \geq (3opt(I_i) + wor(I_i))/4$ , we get:

$$\begin{aligned} 4n(n-1)apx(I) &\geq 3(n-2) \sum_{i=1}^n opt(I_i) + (n-2) \sum_{i=1}^n wor(I_i) + 8S \\ &\geq 3(n-2)^2 opt(I) + 3(n-2)wor(I) + n(n-1)wor(I) + 6S \end{aligned}$$

Finally, recall that  $S$  is twice the total weight of the  $n(n-1)/2$  edges of the graph. But by symmetry, the medium value of all the tours on the graph is equal to  $n$  times the medium value of the edges, *i.e.*  $n \times S/(n(n-1))$ . This medium value of the tours is of course greater than the worst value. Hence,  $wor(I) \leq S/(n-1)$ . This leads to:

$$4(n^2 - n)apx(I) \geq 3(n^2 - 4n + 4)opt(I) + (n^2 + 8n - 12) wor(I)$$

This is  $apx(I) = (3/4 - \alpha(n))opt(I) + (3/4 + \alpha(n))wor(I)$ , where  $\alpha(n) = (9n - 12)/(4n^2 - 4n) = O(1/n)$  (remark that  $4(n^2 - n) = 3(n^2 - 4n + 4) + (n^2 + 8n - 12)$ ). ■

Let us remark that Theorem 3.1 also holds for any  $\rho$ -differential approximation of symmetric TSP: any  $\rho$ -differential approximation algorithm of symmetric TSP on even instances can be polynomially converted in a  $\rho(1 - \alpha(n))$ -differential approximation of symmetric TSP (working on any instance) where we recall that  $\alpha(n) = (9n - 12)/(4n^2 - 4n) = O(1/n)$ .

An interesting open question is to know whether one can improve the differential ratio of 1/2 for asymmetric TSP given in [20] using similar ideas.

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