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Existence results for the $\mathbf{A} - \varphi$ magnetodynamic formulation of the Maxwell system

Serge Nicaise*

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Abstract

The A/φ magnetodynamic Maxwell system given in its potential and space/time formulation is a popular model considered in the engineering community. We establish existence of strong solutions with the help of the theory of Showalter on degenerated parabolic problems; using energy estimates, existence of weak solutions are also deduced.

Key Words : Maxwell equations, potential formulation, degenerate parabolic problems.

AMS (MOS) subject classification 35H65; 35Q60; 78A25.

1 Introduction

Let $T > 0$ and $\Omega \subset \mathbb{R}^3$ be an open connected bounded domain with a lipschitz boundary Γ that is also connected. In this work, we consider the Maxwell system given in $\Omega \times [0, T]$ by :

$$\operatorname{curl} \mathbf{E} = -\partial_t \mathbf{B}, \quad (1)$$

$$\operatorname{curl} \mathbf{H} = \partial_t \mathbf{D} + \mathbf{J}, \quad (2)$$

with initial and boundary conditions to be specified. Here, \mathbf{E} stands for the electrical field, \mathbf{H} for the magnetic field, \mathbf{B} for the magnetic flux density, \mathbf{J} for the current flux density (or eddy current) and \mathbf{D} for the displacement flux density. In the low frequency regime, the quasistatic approximation can be applied, which consists in neglecting the temporal variation of the displacement flux density with respect to the current density [1], so that the propagation phenomena are not taken into account. Consequently, equation (2) reduces to

$$\operatorname{curl} \mathbf{H} = \mathbf{J}. \quad (3)$$

The current density \mathbf{J} can be decomposed in two terms such that $\mathbf{J} = \mathbf{J}_s + \mathbf{J}_e$. \mathbf{J}_s is a known distribution current density generally generated by a coil. \mathbf{J}_e represents the unknown eddy current. Both equations (1) and (3) are linked by the material constitutive laws :

$$\mathbf{B} = \mu \mathbf{H}, \quad (4)$$

$$\mathbf{J}_e = \sigma \mathbf{E}, \quad (5)$$

where μ stands for the magnetic permeability and σ for the electrical conductivity of the material. Figure 1 displays the domain configuration we are interested in. It is composed of an open connected conductor domain $\Omega_c \subset \Omega$ which boundary $B = \partial\Omega_c$ supposed to be lipschitz, connected and such that $B \cap \Gamma = \emptyset$. In Ω_c , the electrical conductivity σ is not equal to zero so that eddy currents can be created. The domain $\Omega_e = \Omega \setminus \overline{\Omega_c}$ is defined as the part of Ω where the electrical conductivity σ is identically equal to zero. Boundary conditions associated with the previous system are given by $\mathbf{B} \cdot \mathbf{n} = 0$ on Γ and $\mathbf{J}_e \cdot \mathbf{n} = 0$ on B , where \mathbf{n} denotes the unit

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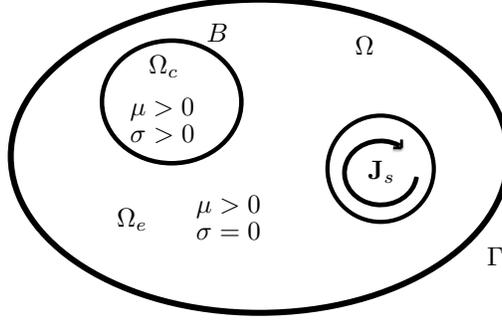


Figure 1: Domains configuration.

outward normal to Ω and Ω_c respectively. In order to solve the problem with the quasistatic approximation, a formulation which is able to take into account the eddy current in Ω_c and which verifies in Ω_e Maxwell's equations must be developed. This can be obtained by choosing the potential formulation often used for electromagnetic problems [10]. Indeed from the fact that $\operatorname{div} \mathbf{B} = 0$ in Ω and that its boundary is connected, by Theorem 3.12 of [3], a magnetic vector potential \mathbf{A} can be introduced such that

$$\mathbf{B} = \operatorname{curl} \mathbf{A} \text{ in } \Omega, \quad (6)$$

with the boundary condition $\mathbf{A} \times \mathbf{n} = 0$ on Γ allowing to guarantee $\mathbf{B} \cdot \mathbf{n} = 0$ on Γ . Like \mathbf{B} , the vector potential \mathbf{A} exists in the whole domain Ω . To ensure the uniqueness of the potential, it is necessary to impose a gauge condition. The most popular one is $\operatorname{div} \mathbf{A} = 0$ (the so-called Coulomb gauge). Moreover, from equations (1) and (6), an electrical scalar potential φ can be introduced in Ω_c so that the electrical field takes the form :

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi \text{ in } \Omega_c. \quad (7)$$

As for the vector potential, it must be gauged. To obtain uniqueness, the averaged value of the potential φ on Ω_c is taken equal to zero. From (4),(5),(6) and (7), equation (3) leads to the so-called $\mathbf{A} - \varphi$ formulation :

$$\operatorname{curl} (\mu^{-1} \operatorname{curl} \mathbf{A}) + \sigma (\partial_t \mathbf{A} + \nabla \varphi) = \mathbf{J}_s. \quad (8)$$

The great interest of this formulation relies in its effectivity in both domain Ω_c and Ω_e . Indeed, in Ω_e where σ is zero the second term vanishes and the $\mathbf{A} - \varphi$ formulation becomes the classical \mathbf{A} formulation used in the magnetostatic case.

Our main goal is to prove existence results for problem (8) completed with appropriated boundary conditions. More particularly, we have in mind to derive a weak formulation that will be used for the numerical resolution of (8) by the Finite Element Method in the context of electromagnetic problems [7].

Concerning the harmonic formulation of some Maxwell problems, several contributions have been proposed in the last decade. In that case, since no time derivatives are involved, we have only to deal with spatial problems for which existence results are easier to obtain (see for instance Theorem 2.1 in [6]).

Recent contributions on evolution Maxwell equations of degenerate parabolic type can be found in [2, 4, 5, 8, 9, 11]. A characteristic feature of these papers is the presence of conducting and nonconducting regions in the spatial domain. While [5] and [9] consider the model in bounded regions and [5] also sketches a quasilinear system, in [2] and [4] the problem is discussed in the whole space. The paper [8] deals with induction heating and considers a coupled system of the evolution Maxwell and heat equations. Finally in [11], an integrodifferential system was studied that accounts for the magnetic induction law that couples the given electrical voltage with the induced electrical current in the induction coil.

Let us finish this introduction by some notation used in the whole paper. On a given domain D , the $L^2(D)$ norm is denoted by $\|\cdot\|_D$, and the corresponding $L^2(D)$ inner product by $(\cdot, \cdot)_D$. The usual norm and semi-norm on $H^1(D)$ are respectively denoted by $\|\cdot\|_{1,D}$ and $|\cdot|_{1,D}$. In the case $D = \Omega$, we drop the index Ω . Recall that $H_0^1(D)$ is the subspace of $H^1(D)$ with vanishing trace on ∂D . Finally, the notation $a \lesssim b$ and

$a \sim b$ means the existence of positive constants C_1 and C_2 , which are independent of the quantities a and b under consideration such that $a \leq C_2 b$ and $C_1 b \leq a \leq C_2 b$, respectively.

The paper is organized as follows. In section 2, the strong and weak formulations of the problem are presented and the existence result is stated. Then, section 3 is devoted to the proof of some preliminary results in order to apply a result of Showalter on degenerated parabolic problems. Finally, in section 4 we prove energy estimates and the main result of our paper.

2 Formulation of the problem and the main result

Assuming that $\operatorname{div} \mathbf{J}_s = 0$, the \mathbf{A} - φ formulation of the magnetodynamic problem with given initial conditions on \mathbf{A} can be written as

$$\operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{A}) + \sigma(\partial_t \mathbf{A} + \nabla \varphi) = \mathbf{J}_s \text{ in } \Omega \times (0, T), \quad (9)$$

$$\operatorname{div}(\sigma(\partial_t \mathbf{A} + \nabla \varphi)) = 0 \text{ in } \Omega_c \times (0, T), \quad (10)$$

$$\mathbf{A} \times \mathbf{n} = 0 \text{ on } \Gamma \times (0, T), \quad (11)$$

$$\sigma(\partial_t \mathbf{A} + \nabla \varphi) \cdot \mathbf{n} = 0 \text{ on } B \times (0, T), \quad (12)$$

$$A(t=0, \cdot) = 0 \text{ in } \Omega_c. \quad (13)$$

We suppose that $\mu \in L^\infty(\Omega)$ and that there exists $\mu_0 \in \mathbb{R}_+^*$ such that $\mu > \mu_0$ in Ω . We also assume that $\sigma \in L^\infty(\Omega)$, $\sigma|_{\Omega_e} \equiv 0$, and that there exists $\sigma_0 \in \mathbb{R}_+^*$ such that $\sigma > \sigma_0$ in Ω_c . At last, we recall the Gauge conditions. Like mentioned in section 1, we choose the Coulomb one $\operatorname{div} \mathbf{A} = 0$ in Ω , and we ask for the averaged value of φ in Ω_c to be equal to zero.

We now define

$$\begin{aligned} X(\Omega) &= H_0(\operatorname{curl}, \Omega) = \left\{ \mathbf{A} \in L^2(\Omega); \operatorname{curl} \mathbf{A} \in L^2(\Omega) \text{ and } \mathbf{A} \times \mathbf{n} = \mathbf{0} \text{ on } \Gamma \right\}, \\ X_N(\Omega) &= \{ \mathbf{A} \in X(\Omega) : \operatorname{div} \mathbf{A} \in L^2(\Omega) \}, \\ X^0(\Omega) &= \{ \mathbf{A} \in X(\Omega) : \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega \}, \\ \widetilde{H}^1(\Omega_c) &= \left\{ \varphi \in H^1(\Omega_c); \int_{\Omega_c} \varphi \, d\mathbf{x} = 0 \right\}, \end{aligned}$$

equipped with their usual norm

$$\begin{aligned} \|\mathbf{A}\|_{X(\Omega)}^2 &= \|\mathbf{A}\|^2 + \|\operatorname{curl} \mathbf{A}\|^2, \forall \mathbf{A} \in X(\Omega), \\ \|\mathbf{A}\|_{X^0(\Omega)}^2 &= \|\mathbf{A}\|^2 + \|\operatorname{curl} \mathbf{A}\|^2, \forall \mathbf{A} \in X^0(\Omega), \\ \|\mathbf{A}\|_{X_N(\Omega)}^2 &= \|\mathbf{A}\|_{X(\Omega)}^2 + \|\operatorname{div} \mathbf{A}\|^2, \forall \mathbf{A} \in X_N(\Omega), \\ \|\varphi\|_{\widetilde{H}^1(\Omega_c)} &= |\varphi|_{1, \Omega_c}, \forall \varphi \in \widetilde{H}^1(\Omega_c). \end{aligned}$$

Similarly, we set

$$H(\operatorname{div} = 0, \Omega) = \{ \mathbf{A} \in L^2(\Omega)^3 : \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega \},$$

that is a closed subspace of $L^2(\Omega)^3$.

Note that, here and below, $\operatorname{div} \mathbf{A} = 0$ in Ω means equivalently that

$$(\mathbf{A}, \nabla \xi) = 0 \quad \forall \xi \in H_0^1(\Omega).$$

The variational (or weak) formulation associated with (9)-(13) is obtained in a usual way, multiplying (9) by a test function $\mathbf{A} \in X^0(\Omega)$ (resp. (10) by a test function $\varphi' \in \widetilde{H}^1(\Omega_c)$), integrating the results in Ω , formal integrations by parts and taking the sum we find

$$(\mu^{-1} \operatorname{curl} \mathbf{A}, \operatorname{curl} \mathbf{A}') + (\sigma (\partial_t \mathbf{A} + \nabla \varphi), \mathbf{A}' + \nabla \varphi')_{\Omega_c} = (\mathbf{J}_s, \mathbf{A}'), \forall (\mathbf{A}', \varphi') \in X^0(\Omega) \times \widetilde{H}^1(\Omega_c), \forall \text{ a.a. } t \in (0, T). \quad (14)$$

An existence result for this problem can be stated as follows

Theorem 2.1. *Let us assume that $\mathbf{J}_s \in H^1((0, T); H(\operatorname{div} = 0, \Omega))$ and set $\mathbf{J}_{s,0} = \mathbf{J}_s(t=0)$. Assume that*

$$\mathbf{J}_{s,0} \cdot \mathbf{n} = 0 \text{ on } B, \quad (15)$$

and that there exists $\mathbf{A}_0 \in X^0(\Omega)$ satisfying

$$\mathbf{A}_0 = 0 \text{ in } \Omega_c,$$

and

$$(\mu^{-1} \operatorname{curl} \mathbf{A}_0, \operatorname{curl} \mathbf{A}') = (\mathbf{J}_{s,0}, \mathbf{A}')_{\Omega_c}, \forall \mathbf{A}' \in X_N(\Omega).$$

Then problem (14) has a unique solution (\mathbf{A}, φ) in $H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c))$ with $\mathbf{A}(t=0) = \mathbf{A}_0$.

Proof. The proof is postponed to section 4. ■

By the uniqueness of the solution, this local existence result directly allows to obtain a global one.

3 Preparations for the application of a theorem by Showalter

Our results on existence and uniqueness rely on the following theorem:

Theorem 3.1 ([12], Theorem V4.B). *Let V_m be a seminorm space obtained from a symmetric and non-negative sesquilinear form $m(\cdot, \cdot)$, and let $\mathcal{M} \in \mathcal{L}(V_m, V'_m)$ be the corresponding operator given by $\mathcal{M}x(y) = m(x, y)$, for all $x, y \in V_m$. Let V be a Hilbert space which is dense and continuously embedded into V_m . Let \mathcal{A} be a continuous, sesquilinear and elliptic form on V and denote by \mathcal{A} the corresponding isomorphism from V onto V' . Let $D = \{u \in V : \mathcal{A}u \in V'_m\}$. Then for any $f \in C^1([0, \infty), V'_m)$ and $y_0 \in V_m$, there exists a unique solution y to*

$$\begin{cases} (\mathcal{M}y)_t(t) + \mathcal{A}y(t) = f(t) & \text{in } V'_m, \quad \forall t > 0, \\ \mathcal{M}y(0) = \mathcal{M}y_0 & \text{in } V'_m, \end{cases} \quad (16)$$

with the regularity

$$\mathcal{M}y \in C([0, \infty), V'_m) \cap C^1((0, \infty), V'_m)$$

and such that

$$y(t) \in D, \forall t > 0.$$

In order to apply this theorem, we show that problem (9)-(13) fits in the associated framework. Before we need some preliminary results. First for $\mathbf{A} \in L^2(\Omega)^3$, we consider the unique solution $\varphi_{\mathbf{A}} \in \widetilde{H}^1(\Omega_c)$ of

$$\int_{\Omega_c} \sigma \nabla \varphi_{\mathbf{A}} \cdot \nabla \bar{\chi} \, dx = - \int_{\Omega_c} \sigma \mathbf{A} \cdot \nabla \bar{\chi} \, dx, \forall \chi \in \widetilde{H}^1(\Omega_c). \quad (17)$$

Such a solution exists by Lax-Milgram lemma and furthermore by Cauchy-Schwarz's inequality, we have

$$\|\sigma^{1/2} \nabla \varphi_{\mathbf{A}}\|_{\Omega_c} \leq \|\sigma^{1/2} \mathbf{A}\|_{\Omega_c}. \quad (18)$$

From this problem, we deduce that the field

$$\sigma(\mathbf{A} + \nabla \varphi_{\mathbf{A}})$$

is divergence free in Ω_c , i.e.,

$$\operatorname{div}(\sigma(\mathbf{A} + \nabla \varphi_{\mathbf{A}})) = 0 \text{ in } \Omega_c, \quad (19)$$

and satisfies the boundary condition

$$\sigma(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega_c. \quad (20)$$

As the boundary of Ω_c is connected, by Theorem 3.12 of [3], there exists a vector potential $\mathbf{B}_{\mathbf{A}} \in X^0(\Omega_c)$ such that

$$\sigma(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) = \text{curl } \mathbf{B}_{\mathbf{A}} \text{ in } \Omega_c. \quad (21)$$

Note that the previous properties also show that for any H^1 extension $\tilde{\varphi}_{\mathbf{A}}$ of $\varphi_{\mathbf{A}}$ to Ω , one has

$$\text{div}(\sigma(\mathbf{A} + \nabla\tilde{\varphi}_{\mathbf{A}})) = 0 \text{ in } \Omega.$$

Recall that for $\mathbf{B} \in X^0(\Omega_c)$, we have

$$\|\text{curl } \mathbf{B}\|_{\Omega_c} \sim \|\mathbf{B}\|_{X_N(\Omega_c)}.$$

Hence

$$\sigma^{-1} \text{curl } X^0(\Omega_c) := \{\sigma^{-1} \text{curl } \mathbf{B} : \mathbf{B} \in X^0(\Omega_c)\}$$

is a closed subspace of $L^2(\Omega_c)^3$.

Note that (21) shows that $\sigma^{-1} \text{curl } \mathbf{B}_{\mathbf{A}}$ is the orthogonal projection of \mathbf{A} on $\sigma^{-1} \text{curl } X^0(\Omega_c)$ for the inner product

$$(\mathbf{A}, \mathbf{B})_{\Omega_c, \sigma} := \int_{\Omega_c} \sigma \mathbf{A} \cdot \bar{\mathbf{B}} \, dx, \forall \mathbf{A}, \mathbf{B} \in L^2(\Omega_c)^3. \quad (22)$$

Now we consider the mapping

$$\mathcal{M}_c : L^2(\Omega_c)^3 \rightarrow \text{curl } X^0(\Omega_c) \subset L^2(\Omega_c)^3 : \mathbf{A} \rightarrow \sigma(\mathbf{A} + \nabla\varphi_{\mathbf{A}}). \quad (23)$$

By (18), this mapping is linear and continuous. Furthermore it is symmetric and non negative (for the standard inner product). Indeed for $\mathbf{A}, \mathbf{B} \in L^2(\Omega_c)^3$, we first show that

$$\int_{\Omega_c} \sigma \nabla\varphi_{\mathbf{A}} \cdot \bar{\mathbf{B}} \, dx = - \int_{\Omega_c} \sigma \nabla\varphi_{\mathbf{A}} \cdot \nabla\bar{\varphi}_{\mathbf{B}} \, dx. \quad (24)$$

Indeed by (17), we have

$$\begin{aligned} \int_{\Omega_c} \sigma \nabla\varphi_{\mathbf{A}} \cdot \bar{\mathbf{B}} \, dx &= \overline{\int_{\Omega_c} \sigma \mathbf{B} \nabla\bar{\varphi}_{\mathbf{A}} \, dx} \\ &= - \int_{\Omega_c} \sigma \nabla\varphi_{\mathbf{B}} \nabla\bar{\varphi}_{\mathbf{A}} \, dx \\ &= - \int_{\Omega_c} \sigma \nabla\varphi_{\mathbf{A}} \nabla\bar{\varphi}_{\mathbf{B}} \, dx. \end{aligned}$$

This proves (24).

With this identity we directly deduce that

$$(\mathcal{M}_c \mathbf{A}, \mathbf{B})_{\Omega_c} = (\mathbf{A}, \mathcal{M}_c \mathbf{B})_{\Omega_c},$$

and the symmetry of \mathcal{M}_c is proved.

For the non negativity, we first notice that (17) is equivalent to

$$\int_{\Omega_c} \mathcal{M}_c \mathbf{A} \cdot \nabla\bar{\chi} \, dx = 0, \forall \chi \in \widetilde{H}^1(\Omega_c). \quad (25)$$

This identity directly implies that

$$\begin{aligned} (\mathcal{M}_c \mathbf{A}, \mathbf{A})_{\Omega_c} &= \int_{\Omega_c} \sigma(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) \cdot \bar{\mathbf{A}} \, dx \\ &= \int_{\Omega_c} \sigma(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) \cdot \overline{(\mathbf{A} + \nabla\varphi_{\mathbf{A}})} \, dx \\ &= \|\sigma^{1/2}(\mathbf{A} + \nabla\varphi_{\mathbf{A}})\|_{\Omega_c}^2 \\ &\sim \|\mathcal{M}_c \mathbf{A}\|_{\Omega_c}^2. \end{aligned}$$

We now equip $L^2(\Omega)^3$ with the semi-inner product

$$m(\mathbf{A}, \mathbf{B}) := (\mathcal{M}_c(\mathbf{A}|_{\Omega_c}), \mathbf{B})_{\Omega_c},$$

that is denoted by V_m (and is not closed). Before going on, we directly see that m is a symmetric and non-negative sesquilinear form on V_m since

$$m(\mathbf{A}, \mathbf{B}) := (\mathcal{M}_c(\mathbf{A}|_{\Omega_c}), \mathbf{B}|_{\Omega_c})_{\Omega_c}.$$

Furthermore let \mathcal{M} be the linear and continuous operator from V_m into its dual V'_m defined by

$$(\mathcal{M}\mathbf{A})(\mathbf{B}) = m(\mathbf{A}, \mathbf{B}), \forall \mathbf{A}, \mathbf{B} \in V_m.$$

Now recall that (see for instance [12]) the dual space V'_m of V_m is a Hilbert space that is characterized in the next lemma.

Lemma 3.2.

$$V'_m = \text{curl } X^0(\Omega_c).$$

In other words, $l \in V'_m$ if and only if there exist $\mathbf{B} \in X^0(\Omega_c)$ such that

$$l(\mathbf{A}) := \int_{\Omega_c} \text{curl } \mathbf{B} \cdot \bar{\mathbf{A}} \, dx, \forall \mathbf{A} \in L^2(\Omega)^3,$$

and

$$\|l\|_{V'_m} \sim \|\text{curl } \mathbf{B}\|_{\Omega_c}.$$

Proof. The inclusion $\text{curl } X^0(\Omega_c) \subset V'_m$ is direct since for $\mathbf{B} \in X^0(\Omega_c)$ and $\mathbf{A} \in V_m$ we can define

$$l_{\mathbf{B}}(\mathbf{A}) := \int_{\Omega_c} \text{curl } \mathbf{B} \cdot \bar{\mathbf{A}} \, dx,$$

that is a continuous linear form on V_m since

$$l_{\mathbf{B}}(\mathbf{A}) = \int_{\Omega_c} \sigma^{-1/2} \text{curl } \mathbf{B} \cdot \sigma^{1/2}(\bar{\mathbf{A}} + \nabla \bar{\varphi}_{\mathbf{A}}) \, dx.$$

Therefore,

$$|l_{\mathbf{B}}(\mathbf{A})| \lesssim \|\sigma^{-1/2} \text{curl } \mathbf{B}\|_{\Omega_c} \|\sigma^{1/2}(\mathbf{A} + \nabla \varphi_{\mathbf{A}})\|_{\Omega_c} \lesssim \|\mathbf{B}\|_{X^0(\Omega_c)} m(\mathbf{A}, \mathbf{A})^{1/2}.$$

For the converse inclusion, let us fix $l \in V'_m$, which means that

$$|l(\mathbf{A})| \lesssim m(\mathbf{A}, \mathbf{A})^{1/2}, \forall \mathbf{A} \in V_m. \quad (26)$$

We first notice that for $\chi \in \widetilde{H}^1(\Omega_c)$, we have

$$\mathcal{M}\nabla\chi = 0,$$

hence $m(\nabla\chi, \nabla\chi) = 0$ and therefore

$$l(\nabla\chi) = 0.$$

This identity implies that

$$l(\mathbf{A}) = l(\mathbf{A} + \nabla\varphi_{\mathbf{A}}) = l(\sigma^{-1} \text{curl } \mathbf{B}_{\mathbf{A}}). \quad (27)$$

As said before we have

$$\sigma^{-1} \text{curl } X^0(\Omega_c) \subset V_m,$$

and for any $\mathbf{C} \in X^0(\Omega_c)$, we have

$$m(\sigma^{-1} \text{curl } \mathbf{C}, \sigma^{-1} \text{curl } \mathbf{C}) = \|\sigma^{-1/2} \text{curl } \mathbf{C}\|_{\Omega_c}^2 \sim \|\mathbf{C}\|_{X^0(\Omega_c)}^2.$$

Therefore the mapping

$$l_0 : \mathbf{C} \rightarrow l(\sigma^{-1} \operatorname{curl} \mathbf{C}),$$

is linear and continuous from $X^0(\Omega_c)$ to \mathbb{C} . Hence since $X^0(\Omega_c)$ is a Hilbert space for the inner product

$$(\mathbf{B}, \mathbf{C}) := \int_{\Omega_c} \sigma^{-1} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \bar{\mathbf{C}} \, dx, \forall \mathbf{B}, \mathbf{C} \in X^0(\Omega_c),$$

there exists $\mathbf{B} \in X^0(\Omega_c)$ such that

$$l_0(\mathbf{C}) = \int_{\Omega_c} \sigma^{-1} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \bar{\mathbf{C}} \, dx, \forall \mathbf{C} \in X^0(\Omega_c).$$

By (27), we deduce that

$$l(\mathbf{A}) = \int_{\Omega_c} \sigma^{-1} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \bar{\mathbf{B}}_{\mathbf{A}} \, dx, \forall \mathbf{A} \in L^2(\Omega)^3,$$

and by (21), we conclude that

$$l(\mathbf{A}) = \int_{\Omega_c} \operatorname{curl} \mathbf{B} \cdot (\bar{\mathbf{A}} + \nabla \bar{\varphi}_{\mathbf{A}}) \, dx = \int_{\Omega_c} \operatorname{curl} \mathbf{B} \cdot \bar{\mathbf{A}} \, dx, \forall \mathbf{A} \in L^2(\Omega)^3.$$

The first step shows that the mapping

$$\operatorname{curl} X^0(\Omega_c) \rightarrow V'_m : \operatorname{curl} \mathbf{B} \rightarrow l_{\mathbf{B}},$$

is linear and continuous. Moreover the second step shows this mapping is surjective hence by the closed graph theorem, it is an isomorphism. ■

Before going on let us notice that for any $\mathbf{A} \in L^2(\Omega)^3$, we see that

$$\mathcal{M}\mathbf{A} = \sigma(\mathbf{A}|_{\Omega_c} - \nabla \varphi_{\mathbf{A}|_{\Omega_c}}),$$

and therefore by the previous consideration, we have

$$\|\mathcal{M}\mathbf{A}\|_{V'_m}^2 \sim \|\mathcal{M}_c \mathbf{A}|_{\Omega_c}\|_{\Omega_c}^2 \sim (\mathcal{M}\mathbf{A}, \mathbf{A}). \quad (28)$$

At this stage in order to apply Theorem 3.1, we define

$$a(\mathbf{A}, \mathbf{A}') = \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \bar{\mathbf{A}}' \, dx + \int_{\Omega} \operatorname{div} \mathbf{A} \operatorname{div} \bar{\mathbf{A}}' \, dx, \forall \mathbf{A}, \mathbf{A}' \in X_N(\Omega),$$

that is clearly continuous and coercive on $X_N(\Omega)$ due its compact embedding into $L^2(\Omega)^3$.

Furthermore $X_N(\Omega)$ is clearly dense in $L^2(\Omega)^3$ and continuously embedded into V_m . Therefore we have checked all assumptions of Theorem 3.1 (with $V = X_N(\Omega)$). Before stating a consequence of this theorem, we recall that

$$D := \{\mathbf{A} \in X_N(\Omega) : \exists l \in V'_m : a(\mathbf{A}, \mathbf{B}) = l(\mathbf{B}), \forall \mathbf{B} \in X_N(\Omega)\},$$

and show that elements of such a set are divergence free.

Lemma 3.3. *Any $\mathbf{A} \in D$ is divergence free in Ω .*

Proof. Fix $\mathbf{A} \in D$, then there exists $l \in V'_m$ such that

$$a(\mathbf{A}, \mathbf{B}) = l(\mathbf{B}), \forall \mathbf{B} \in X_N(\Omega). \quad (29)$$

For any $f \in L^2(\Omega)$, we take the unique solution $u_f \in H_0^1(\Omega)$ of

$$\Delta u_f = f \text{ in } \Omega,$$

or equivalently the unique solution of

$$\int_{\Omega} \nabla u_f \cdot \nabla \bar{v} \, d\mathbf{x} = - \int_{\Omega} f \bar{v} \, d\mathbf{x}, \forall v \in H_0^1(\Omega).$$

By taking $\mathbf{B} = \nabla u_f$ in (29) (allowed since $\nabla u_f \in X_N(\Omega)$), we find

$$\int_{\Omega} \operatorname{div} \mathbf{A} \bar{f} \, d\mathbf{x} = l(\nabla u_f) = 0.$$

Since f is arbitrary in $L^2(\Omega)$, we conclude that $\operatorname{div} \mathbf{A} = 0$ in Ω . \blacksquare

4 Some existence results

Let us now give a first consequence of Theorem 3.1.

Theorem 4.1. *Let $\mathbf{F} \in C^1([0, T], V'_m)$ and $\mathbf{B}_0 \in V_m$, then there exists a unique solution \mathbf{B} to*

$$\begin{cases} \partial_t(\mathcal{M}\mathbf{B})(t) + \mathbf{A}\mathbf{B}(t) = \mathbf{F}(t) & \text{in } V'_m, \forall t \in (0, T), \\ (\mathcal{M}\mathbf{B})(t=0) = \mathcal{M}\mathbf{B}_0 & \text{in } \Omega_c, \end{cases} \quad (30)$$

with the regularity $\mathcal{M}\mathbf{B} \in C([0, T], V'_m) \cap C^1((0, T], V'_m)$, and

$$\mathbf{B}(t) \in D, \forall t \in (0, T].$$

Furthermore one has (where $\langle \cdot, \cdot \rangle$ means the duality pairing between V'_m and V_m)

$$\langle \partial_t(\mathcal{M}\mathbf{B})(\cdot, t), \mathbf{Z} \rangle + \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \bar{\mathbf{Z}} \, dx = (\mathbf{F}(t), \mathbf{Z}), \forall \mathbf{Z} \in X_N(\Omega), \forall t \in (0, T), \quad (31)$$

and the next estimate holds

$$\|\mathbf{B}\|_{L^2(0, T; X^0(\Omega))} + \|\mathcal{M}\mathbf{B}\|_{C([0, T]; V'_m)} + \|\partial_t \mathcal{M}\mathbf{B}\|_{L^1((0, T]; X_N(\Omega)')} \lesssim \|\mathbf{F}\|_{L^2(0, T; V'_m)} + \|\mathbf{B}_0\|_{V_m}. \quad (32)$$

Proof. The existence and uniqueness result directly follows from Theorem 3.1, by taking an appropriate extension of \mathbf{F} in the whole $[0, \infty)$. Hence it remains to prove (31) and the estimate (32).

For that purpose, we closely follow the proof of Corollary 3.8 of [11]. As $X_N(\Omega) \subset V_m$, the first identity of (30) implies that

$$\langle \partial_t(\mathcal{M}\mathbf{B})(\cdot, t), \mathbf{Z} \rangle + a(\mathbf{B}, \mathbf{Z}) = (\mathbf{F}(t), \mathbf{Z}), \forall \mathbf{Z} \in X_N(\Omega), \forall t \in (0, T).$$

By Lemma 3.3 we directly arrive at (31).

To prove (32), we first take $\mathbf{Z} = \mathbf{B}$ in (31), to get

$$\frac{1}{2} \partial_t(\mathcal{M}\mathbf{B}, \mathbf{B}) + \|\mu^{-1/2} \operatorname{curl} \mathbf{B}\|^2 = (\mathbf{F}, \mathbf{B}).$$

Hence by Cauchy-Schwarz's inequality we get

$$\frac{1}{2} \partial_t(\mathcal{M}\mathbf{B}, \mathbf{B}) + \|\mu^{-1/2} \operatorname{curl} \mathbf{B}\|^2 \leq \|\mathbf{F}\| \|\mathbf{B}\|,$$

and with the estimate $\|\mathbf{B}\| \leq C \|\mu^{-1/2} \operatorname{curl} \mathbf{B}\|$, for some $C > 0$ and Young's inequality we obtain

$$\frac{1}{2} \partial_t(\mathcal{M}\mathbf{B}, \mathbf{B}) + \|\mu^{-1/2} \operatorname{curl} \mathbf{B}\|^2 \leq \frac{C^2}{2} \|\mathbf{F}\|^2 + \frac{1}{2} \|\mu^{-1/2} \operatorname{curl} \mathbf{B}\|^2.$$

This shows that

$$\partial_t(\mathcal{M}\mathbf{B}, \mathbf{B}) + \|\mu^{-1/2} \operatorname{curl} \mathbf{B}\|^2 \leq C^2 \|\mathbf{F}\|^2.$$

Integrating this estimate in $t \in (\eta, u)$, with $\eta, u \in (0, T]$ arbitrary but such that $\eta < u$, we get

$$(\mathcal{M}\mathbf{B}(\cdot, u), \mathbf{B}(\cdot, u)) + \int_{\eta}^u \|\mu^{-1/2} \operatorname{curl} \mathbf{B}(\cdot, t)\|^2 dt \leq C^2 \int_{\eta}^u \|\mathbf{F}(\cdot, t)\|^2 dt + (\mathcal{M}\mathbf{B}(\cdot, \eta), \mathbf{B}(\cdot, \eta)).$$

Letting η tends to zero and reminding the regularity $\mathcal{M}\mathbf{B} \in C([0, T], V'_m)$ (see also the estimates (28)), we find that

$$(\mathcal{M}\mathbf{B}(\cdot, u), \mathbf{B}(\cdot, u)) + \int_0^u \|\mu^{-1/2} \operatorname{curl} \mathbf{B}(\cdot, t)\|^2 dt \lesssim \int_0^T \|\mathbf{F}(\cdot, t)\|^2 dt + \|\mathbf{B}_0\|_{V_m}^2. \quad (33)$$

In a first step by taking $u = T$, this shows that

$$\|\mathbf{B}\|_{L^2(0, T; X^0(\Omega))} \lesssim \|\mathbf{F}\|_{L^2(0, T; L^2(\Omega))} + \|\mathbf{B}_0\|_{V_m}. \quad (34)$$

In a second step, for any $u \in (0, T]$, we also have

$$|(\mathcal{M}\mathbf{B}(\cdot, u), \mathbf{B}(\cdot, u))| \lesssim \int_0^T \|\mathbf{F}(\cdot, t)\|^2 dt + \|\mathbf{B}_0\|_{V_m}^2 + \left| \int_0^u \|\mu^{-1/2} \operatorname{curl} \mathbf{B}(\cdot, t)\|^2 dt \right|,$$

and therefore by (34)

$$|(\mathcal{M}\mathbf{B}(\cdot, u), \mathbf{B}(\cdot, u))| \lesssim \int_0^T \|\mathbf{F}(\cdot, t)\|^2 dt + \|\mathbf{B}_0\|_{V_m}^2, \forall u \in [0, T]. \quad (35)$$

It remains to estimate the last term of the left-hand side of (32).

For that purpose, we come back to (31) that can be equivalently written

$$\langle \partial_t(\mathcal{M}\mathbf{B})(\cdot, t), \mathbf{Z} \rangle = - \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \mathbf{Z} dx + \int_{\Omega} \mathbf{F}(x, t) \cdot \mathbf{Z}(x) dx, \forall \mathbf{Z} \in X_N(\Omega).$$

Now notice that it implies

$$|\langle \partial_t(\mathcal{M}\mathbf{B})(\cdot, t), \mathbf{Z} \rangle| \leq \left| \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \mathbf{Z} dx \right| + \left| \int_{\Omega} \mathbf{F}(x, t) \cdot \mathbf{Z}(x) dx \right|, \forall \mathbf{Z} \in X_N(\Omega).$$

Integrating this identity in $t \in (0, T)$ and applying Cauchy-Schwarz's inequality we get

$$\int_0^T |\langle \partial_t(\mathcal{M}\mathbf{B})(\cdot, t), \mathbf{Z} \rangle| dt \leq T(\|\mathbf{F}\|_{L^2(0, T; L^2(\Omega))} + \|\mu^{-1} \operatorname{curl} \mathbf{B}(\cdot, t)\|_{L^2(0, T; L^2(\Omega))}) \|\mathbf{Z}\|_{X_N(\Omega)},$$

for all $\mathbf{Z} \in X_N(\Omega)$. By (34), this implies that

$$\int_0^T |\langle \partial_t(\mathcal{M}\mathbf{B})(\cdot, t), \mathbf{Z} \rangle| dt \lesssim (\|\mathbf{F}\|_{L^2(0, T; L^2(\Omega))} + \|\mathbf{B}_0\|_{V_m}) \|\mathbf{Z}\|_{X_N(\Omega)},$$

for all $\mathbf{Z} \in X_N(\Omega)$. This estimate leads to

$$\|\partial_t(\mathcal{M}\mathbf{B})\|_{L^1(0, T; X_N(\Omega)')} \lesssim \|\mathbf{F}\|_{L^2(0, T; L^2(\Omega))} + \|\mathbf{B}_0\|_{V_m}, \quad (36)$$

due to the definition of the norm $X_N(\Omega)'$.

The proof of the estimate (49) is then complete. ■

Now we can prove an existence result of a strong solution to problem (9)-(13).

Theorem 4.2. *Let us suppose that $\mathbf{J}_s \in C^3([0, T], H(\operatorname{div} = 0, \Omega))$ with $\mathbf{J}_{s,0} = \mathbf{J}_s(t = 0)$ satisfying the assumptions of Theorem 2.1. Then, there exists one and only one solution $(\mathbf{A}, \varphi) \in H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c))$ to problem (9)-(13) with the (additional) initial condition*

$$\mathbf{A}(t = 0) = \mathbf{A}_0. \quad (37)$$

Proof. Assume that a solution (\mathbf{A}, φ) of problem (9)-(13) exists and is sufficiently regular, then comparing (10) and (13) with (19) and (20), we can say that $\varphi = \partial_t \varphi_{\mathbf{A}}$, $\varphi_{\mathbf{A}}$ being the unique solution of (17). With this property, problem (9)-(13) is then (formally) equivalent to

$$\begin{cases} \partial_t(\mathcal{M}\mathbf{A}) + \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{A}) = \mathbf{J}_s & \text{in } \Omega \times (0, T), \\ \mathbf{A}(t=0) = 0 & \text{in } \Omega_c. \end{cases} \quad (38)$$

Now in order to apply Theorem 3.1 (or equivalently Theorem 4.1), we need that the above right-hand side \mathbf{J}_s is in V'_m (in particular it should be zero in Ω_e) which is not the case in physical applications. Hence we perform an elliptic lifting, namely for all $t \in [0, T]$, we consider the unique solution $\mathbf{L}_j(t) \in X^0(\Omega)$ of

$$\begin{cases} \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{L}_j(t)) = \mathbf{J}_s(t) & \text{in } \Omega, \\ \mathbf{L}_j(t) \times \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \quad (39)$$

or in a weak form

$$\int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{L}_j(t) \cdot \operatorname{curl} \bar{\mathbf{W}} \, dx = \int_{\Omega} \mathbf{J}_s(t) \cdot \bar{\mathbf{W}} \, dx, \forall \bar{\mathbf{W}} \in X^0(\Omega). \quad (40)$$

This last problem has a unique solution due to Lax-Milgram lemma. Now any $\mathbf{W} \in X_N(\Omega)$ can be written in the form

$$\mathbf{W} = \mathbf{W}_0 + \nabla\psi,$$

with $\psi \in H_0^1(\Omega)$ and $\mathbf{W}_0 \in X^0(\Omega)$ and since

$$\int_{\Omega} \mathbf{J}_s \cdot \nabla\psi \, dx = 0, \forall \psi \in H_0^1(\Omega),$$

(40) implies that

$$\int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{L}_j \cdot \operatorname{curl} \bar{\mathbf{W}} \, dx = \int_{\Omega} \mathbf{J}_s \cdot \bar{\mathbf{W}} \, dx, \forall \bar{\mathbf{W}} \in X_N(\Omega), \quad (41)$$

and therefore the unique solution $\mathbf{L}_j \in X^0(\Omega)$ of (40) is a solution of (39).

From (40), we also have

$$\|\mathbf{L}_j\|_{X^0(\Omega)} \lesssim \|\mathbf{J}_s\|_{L^2(\Omega)^3}.$$

Furthermore due to this estimate, if $\mathbf{J}_s \in C^k([0, T], H(\operatorname{div} = 0, \Omega))$, for some $k \in \mathbb{N}$, then \mathbf{L}_j will be in $C^k([0, T], X^0(\Omega))$ with

$$\|\mathbf{L}_j\|_{C^k([0, T], X^0(\Omega))} \lesssim \|\mathbf{J}_s\|_{C^k([0, T], H(\operatorname{div} = 0, \Omega))}. \quad (42)$$

At this stage, by setting

$$\mathbf{B} = \mathbf{A} - \mathbf{L}_j,$$

we arrive at the problem

$$\begin{cases} \partial_t \mathcal{M}\mathbf{B} + \operatorname{curl}(\mu^{-1} \operatorname{curl} \mathbf{B}) = -\partial_t \mathcal{M}\mathbf{L}_j & \text{in } \Omega, \\ \mathbf{B}(t=0) = -\mathbf{L}_j(t=0) & \text{in } \Omega_c. \end{cases} \quad (43)$$

Now as $\mathbf{J}_s \in C^2([0, T], H(\operatorname{div} = 0, \Omega))$, $\mathcal{M}\mathbf{L}_j \in C^2([0, T], V'_m)$ (because the range of \mathcal{M} is exactly V'_m). Hence applying Theorem 4.1, we find a unique (strong) solution of

$$\begin{cases} \partial_t(\mathcal{M}\mathbf{B}) + \mathcal{A}\mathbf{B} = -\partial_t \mathcal{M}\mathbf{L}_j & \text{in } V'_m, t \in (0, T], \\ (\mathcal{M}\mathbf{B})(t=0) = -\mathcal{M}\mathbf{L}_j(t=0) & \text{in } \Omega_c, \end{cases} \quad (44)$$

with the regularity $\mathcal{M}\mathbf{B} \in C([0, T], V'_m) \cap C^1((0, T], V'_m)$, and

$$\mathbf{B}(t) \in D, \forall t \in (0, T].$$

Furthermore for any $t > 0$ we have

$$\langle \partial_t(\mathcal{M}\mathbf{B})(\cdot, t), \mathbf{Z} \rangle + \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{B} \cdot \operatorname{curl} \bar{\mathbf{Z}} \, dx = -(\partial_t(\mathcal{M}\mathbf{L}_j)(\cdot, t), \bar{\mathbf{Z}}), \forall \mathbf{Z} \in X_N(\Omega). \quad (45)$$

Now in order to come back to the original problem, we need more regularity for \mathbf{B} in order that

$$\partial_t(\mathcal{M}\mathbf{B}) = \mathcal{M}\partial_t\mathbf{B}.$$

This is the reason of the assumptions on $\mathbf{J}_s \in C^3([0, T], H(\operatorname{div} = 0, \Omega))$ and on $\mathbf{J}_s(t = 0)$ that imply $\mathcal{M}\mathbf{L}_j \in C^3([0, T], V'_m)$ with $\mathcal{M}(\partial_t\mathbf{L}_j)(t = 0) = \partial_t(\mathcal{M}\mathbf{L}_j)(t = 0)$, as well as $\mathbf{J}_{s,0}\mathbf{1}_{\Omega_c} = \mathcal{M}\theta$, for some $\theta \in V_m$. Hence again by applying Theorem 4.1, we find a unique (strong) solution of

$$\begin{cases} \partial_t(\mathcal{M}\mathbf{C}) + \mathcal{A}\mathbf{C} = -\partial_t^2\mathcal{M}\mathbf{L}_j & \text{in } V'_m, t \in (0, T], \\ (\mathcal{M}\mathbf{C})(t = 0) = -\partial_t\mathcal{M}\mathbf{L}_j(t = 0) + \mathbf{J}_{s,0}\mathbf{1}_{\Omega_c} & \text{in } \Omega_c, \end{cases} \quad (46)$$

with the regularity $\mathcal{M}\mathbf{C} \in C([0, T], V'_m) \cap C^1((0, T], V'_m)$, and

$$\mathbf{C}(t) \in D, \forall t \in (0, T].$$

Now we define a primitive of \mathbf{C} by

$$\mathbf{P}(t) = \int_0^t \mathbf{C}(s) ds - \mathbf{L}_j(t = 0) + \mathbf{A}_0,$$

that has a meaning due to (32).

Integrating the first identity of (46) between 0 and t (meaningful due to (32)), we find that

$$\mathcal{M}\mathbf{C} + \mathcal{A}\mathbf{P} = -\partial_t\mathcal{M}\mathbf{L}_j \text{ in } V'_m,$$

reminding that $\mathcal{A}\mathbf{A}_0 = \mathbf{J}_{s,0}\mathbf{1}_{\Omega_c}$. But the estimate (32) implies that $\mathcal{M}\mathbf{P}$ belongs $C^1(0, T], V'_m)$ and

$$\partial_t\mathcal{M}\mathbf{P} = \mathcal{M}\mathbf{C}.$$

Hence we have find that

$$\partial_t(\mathcal{M}\mathbf{P}) + \mathcal{A}\mathbf{P} = -\partial_t\mathcal{M}\mathbf{L}_j \text{ in } V'_m.$$

As $\mathbf{P}(t = 0) = -\mathbf{L}_j(t = 0) + \mathbf{A}_0$ and as $\mathbf{A}_0 = 0$ in Ω_c , we obtain that $\mathcal{M}\mathbf{P}(t = 0) = -\mathcal{M}\mathbf{L}_j(t = 0)$. In conclusion we see that \mathbf{P} is solution of the same problem as \mathbf{B} and therefore by the uniqueness of the solution of (44) we find that

$$\mathbf{B} = \mathbf{P}.$$

This also guarantees that \mathbf{B} is differentiable and by the estimate (32) that

$$\partial_t\mathbf{B} = \mathbf{C} \in L^2(0, T; X^0(\Omega)).$$

Hence we can consider $\nabla\varphi_{\partial_t\mathbf{B}}$ and by the estimate (18), we get

$$\nabla\varphi_{\partial_t\mathbf{B}} = \partial_t\nabla\varphi_{\mathbf{B}}.$$

Once the existence and regularity of \mathbf{B} are guaranteed, we can set $\mathbf{A} = \mathbf{B} + \mathbf{L}_j$ that satisfies $\mathcal{M}\mathbf{A} \in C([0, T], V'_m) \cap C^1([0, T], V'_m)$ (as $\partial_t(\mathcal{M}\mathbf{A}) = \mathcal{M}(\partial_t\mathbf{A}) = \mathcal{M}(\mathbf{C} - \partial_t\mathbf{L}_j)$), and

$$\mathbf{A}(t) \in X^0(\Omega), \forall t \in (0, T].$$

Notice that \mathbf{A} satisfies the initial condition (37) because $\mathbf{A}(t = 0) = \mathbf{P}(t = 0) + \mathbf{L}_j(t = 0)$. From (41) and (45), it also satisfies

$$\langle \partial_t(\mathcal{M}\mathbf{A})(\cdot, t), \mathbf{Z} \rangle + \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{A}(t) \cdot \operatorname{curl} \bar{\mathbf{Z}} dx = \int_{\Omega} \mathbf{J}_s(x, t) \cdot \mathbf{Z}(x) dx, \forall \mathbf{Z} \in X_N(\Omega), \forall t \in (0, T). \quad (47)$$

Finally as \mathbf{A} is differentiable with

$$\partial_t(\mathcal{M}\mathbf{A}) = \mathcal{M}(\partial_t\mathbf{A}) = \sigma(\partial_t\mathbf{A} - \nabla\varphi_{\partial_t\mathbf{A}}),$$

(47) becomes

$$\int_{\Omega_c} (\mathcal{M}\partial_t \mathbf{A})(x, t) \cdot \bar{\mathbf{Z}}(x) dx + \int_{\Omega} \mu^{-1} \operatorname{curl} \mathbf{A} \cdot \operatorname{curl} \bar{\mathbf{Z}} dx = \int_{\Omega} \mathbf{J}_s(x, t) \cdot \bar{\mathbf{Z}}(x) dx, \forall \bar{\mathbf{Z}} \in X_N(\Omega),$$

and by (25) we get

$$(\mu^{-1} \operatorname{curl} \mathbf{A}, \operatorname{curl} \mathbf{A}')_{\Omega} + (\mathcal{M}\partial_t \mathbf{A}, \mathbf{A}' + \nabla \varphi')_{\Omega_c} = (\mathbf{J}_s, \mathbf{A}')_{\Omega}, \forall \mathbf{A}' \in X_N(\Omega), \varphi' \in H^1(\Omega_c). \quad (48)$$

This proves that the pair $(\mathbf{A}, \varphi_{\partial_t \mathbf{A}})$ is solution of (9)-(13) by taking different tests functions, namely $(\mathbf{A}', 0)$ with $\mathbf{A}' \in \mathcal{D}(\Omega)^3$, $(0, \varphi)$ with $\varphi \in \mathcal{D}(\Omega_c)$ and finally $(0, \varphi)$ with $\varphi \in C^\infty(\bar{\Omega}_c)$.

Note also that (48) implies that (14) holds. \blacksquare

For the existence of weak solutions, we first keep smooth \mathbf{J}_s and prove the next a priori estimates.

Lemma 4.3. *Under the assumptions of Theorem 4.2, it holds*

$$\|\mathbf{A}\|_{L^2(0, T; X^0(\Omega))} + \|\mathcal{M}\mathbf{A}\|_{C([0, T]; V'_m)} + \|\partial_t \mathcal{M}\mathbf{A}\|_{L^1((0, T); X_N(\Omega)')} \lesssim \|\mathbf{J}_s\|_{L^2(0, T; L^2(\Omega)^3)}, \quad (49)$$

$$\|\partial_t \mathbf{A}\|_{L^2(0, T; X^0(\Omega))} + \|\mathcal{M}\partial_t \mathbf{A}\|_{C([0, T]; V'_m)} \lesssim \|\mathbf{J}_s\|_{H^1(0, T; L^2(\Omega)^3)} + \|\theta\|_{V_m}, \quad (50)$$

where $\theta \in V_m$ is such that $\mathbf{J}_{s,0} \mathbf{1}_{\Omega_c} = \mathcal{M}\theta$.

Proof. The proof of the first estimate is exactly the same as the one of the estimate (32) by using (47) instead of (31).

The proof of (50) is similar by noticing that $\partial_t \mathbf{A}$ is solution of

$$\begin{cases} \partial_t(\mathcal{M}\partial_t \mathbf{A}) + \operatorname{curl}(\mu^{-1} \operatorname{curl} \partial_t \mathbf{A}) = \partial_t \mathbf{J}_s & \text{in } \Omega, \\ \mathcal{M}\partial_t \mathbf{A}(t=0) = \mathcal{M}\theta & \text{in } \Omega_c. \end{cases}$$

\blacksquare

We are ready to prove our main result.

Proof. of Theorem 2.1.

For $\mathbf{J}_s \in H^1((0, T); H(\operatorname{div} = 0, \Omega))$, we can consider

$$\tilde{\mathbf{J}}_s = \mathbf{J}_s - \mathbf{J}_{s,0},$$

that is still in $H^1((0, T); H(\operatorname{div} = 0, \Omega))$ but is zero at $t = 0$. Hence there exists a sequence $\tilde{\mathbf{J}}_s^{(n)} \in C_l^\infty([0, T]; H(\operatorname{div} = 0, \Omega)) = \{\mathbf{F} \in C^\infty([0, T]; H(\operatorname{div} = 0, \Omega)) : \mathbf{F} = 0 \text{ in a neighbourhood of } t = 0\}$ such that

$$\tilde{\mathbf{J}}_s^{(n)} \rightarrow \tilde{\mathbf{J}}_s \text{ in } H^1((0, T); H(\operatorname{div} = 0, \Omega)) \text{ as } n \rightarrow \infty.$$

Setting

$$\mathbf{J}_s^{(n)} = \tilde{\mathbf{J}}_s^{(n)} + \mathbf{J}_{s,0}$$

we get a sequence in $C^\infty([0, T]; H(\operatorname{div} = 0, \Omega))$ satisfying the assumptions of Theorem 4.2 and such that

$$\mathbf{J}_s^{(n)} \rightarrow \mathbf{J}_s \text{ in } H^1((0, T); H(\operatorname{div} = 0, \Omega)) \text{ as } n \rightarrow \infty.$$

Hence by the previous arguments, for all n there exists a (strong) solution \mathbf{A}_n of

$$(\mu^{-1} \operatorname{curl} \mathbf{A}_n, \operatorname{curl} \mathbf{A}')_{\Omega} + (\mathcal{M}\partial_t \mathbf{A}_n, \mathbf{A}' + \nabla \varphi')_{\Omega_c} = (\mathbf{J}_s^{(n)}, \mathbf{A}')_{\Omega}, \forall \mathbf{A}' \in X_N(\Omega), \varphi' \in \widetilde{H}^1(\Omega_c), \quad (51)$$

that satisfies the a priori estimate (owing to (49) and (50)):

$$\|\mathbf{A}_n - \mathbf{A}_m\|_{H^1(0, T; X^0(\Omega))} + \|\mathcal{M}\partial_t(\mathbf{A}_n - \mathbf{A}_m)\|_{C([0, T]; V'_m)} \lesssim \|\mathbf{J}_s^{(n)} - \mathbf{J}_s^{(m)}\|_{H^1(0, T; L^2(\Omega)^3)},$$

for all n, m . Hence there exist $\mathbf{A} \in H^1(0, T; X^0(\Omega))$ and $\Theta \in C([0, T]; V'_m)$ such that

$$\mathbf{A}_n \rightarrow \mathbf{A} \text{ in } H^1(0, T; X^0(\Omega)) \text{ as } n \rightarrow \infty, \quad (52)$$

$$\mathcal{M}\partial_t \mathbf{A}_n \rightarrow \Theta \text{ in } C([0, T]; V'_m), \text{ as } n \rightarrow \infty. \quad (53)$$

These two properties imply that

$$\sigma(\partial_t \mathbf{A} + \nabla \varphi_{\partial_t \mathbf{A}}) = \Theta. \quad (54)$$

Indeed (52) directly implies that

$$\partial_t \mathbf{A}_n \rightarrow \partial_t \mathbf{A} \text{ in } L^2(0, T; X^0(\Omega)). \quad (55)$$

Hence

$$\sigma(\partial_t \mathbf{A}_n + \nabla \varphi_{\partial_t \mathbf{A}_n}) \rightarrow \sigma(\partial_t \mathbf{A} + \nabla \varphi_{\partial_t \mathbf{A}}) \text{ in } L^2(0, T; V'_m),$$

hence (54) holds.

The conclusion now follows by taking the limit in (51) and setting $\varphi = \varphi_{\partial_t \mathbf{A}}$. ■

Let us finish this paper by showing that the test functions in (14) can be ungauged.

Theorem 4.4. *If $\mathbf{J}_s \in H^1((0, T); H(\operatorname{div} = 0, \Omega))$, then any solution $(\mathbf{A}, \varphi) \in H^1(0, T; X^0(\Omega)) \times L^2(0, T; \widetilde{H}^1(\Omega_c))$ of (14) also satisfies*

$$\begin{aligned} & (\mu^{-1} \operatorname{curl} \mathbf{A}, \operatorname{curl} \mathbf{A}') + (\sigma(\partial_t \mathbf{A} + \nabla \varphi), \mathbf{A}' + \nabla \varphi')_{\Omega_c} \\ & = (\mathbf{J}_s, \mathbf{A}'), \forall (\mathbf{A}', \varphi') \in X(\Omega) \times \widetilde{H}^1(\Omega_c). \end{aligned} \quad (56)$$

Proof. In (14), we first take $\mathbf{A}' \equiv 0$ to deduce that

$$(\sigma(\partial_t \mathbf{A} + \nabla \varphi), \nabla \varphi')_{\Omega_c} = 0, \forall \varphi' \in H^1(\Omega_c).$$

In a second step taking any $\psi \in H_0^1(\Omega)$, as \mathbf{J}_s is divergence free we get

$$(\sigma(\partial_t \mathbf{A} + \nabla \varphi), \nabla \psi)_{\Omega_c} = (\mathbf{J}_s, \nabla \psi).$$

We conclude by using the Helmholtz decomposition of $\mathbf{A}' \in X(\Omega)$ into $\mathbf{A}' = \mathbf{B}' + \nabla \psi$ with $\psi \in H_0^1(\Omega)$ and $\mathbf{B}' \in X^0(\Omega)$. ■

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